Discrete Mathematics 65 (1987) 53–63
North-Holland

A CLASS OF ADDITIVE MULTIPLICATIVE GRAPH FUNCTIONS*

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Received 10 May 1985
Revised 14 August 1986

For a fixed graph G, the capacity function for G, P_G, is defined by $P_G(H) = \lim_{n\to\infty}[\gamma_G(H^n)]^{1/n}$, where $\gamma_G(H)$ is the maximum number of disjoint G's in H. In [2], Hsu proved that P_{K_2} can be viewed as a lower bound for multiplicative increasing graph functions. But it was not known whether P_{K_2} is multiplicative or not. In this paper, we prove that P_G is multiplicative and additive for some graphs G which include K_2. Some properties of P_G are also discussed in this paper.

1. Definition and introduction

$G = (V, E)$ is called a graph if V is a finite set and E is a subset of $\{[a, b] \mid a \neq b, [a, b]$ is an unordered pair of $V\}$. We say that $V = V(G)$ is the vertex set of G, $E = E(G)$ is the edge set of G.

Let $G = (X, E)$, $H = (Y, F)$ be two graphs. The sum of G and H is the graph $G + H = (W, B)$ with $W = X_1 \cup Y_1$, $B = E_1 \cup F_1$, where $G_1 = (X_1, E_1) \cong G$, $H_1 = (Y_1, F_1) \cong H$ and $X_1 \cap Y_1 = \emptyset$; the product of G and H is the graph $G \times H = (Z, K)$, where $Z = X \times Y$, the Cartesian product of X and Y, and $K = \{[(x_1, y_1), (x_2, y_2)] \mid [x_1, x_2] \in E$ and $[y_1, y_2] \in F\}$. We let G^k denote $G \times G \times \cdots \times G$ (k times). A real-valued function f, defined on the set of all graphs $\mathcal{G}$, is *additive* if it satisfies $f(G + H) = f(G) + f(H)$ for any G, $H \in \mathcal{G}$; f is *pseudo additive* if $f(G) \neq 0$ and $f(H) \neq 0$, then $f(G + H) = f(G) + f(H)$ for any G, $H \in \mathcal{G}$; f is *multiplicative* if $f(G \times H) = f(G) \times f(H)$ for any G, $H \in \mathcal{G}$; f is *pseudo multiplicative* if $f(G) \neq 0$ and $f(H) \neq 0$, then $f(G \times H) = f(G) \times f(H)$ for any G, $H \in \mathcal{G}$; f is *increasing* if $f(G) \leq f(H)$ whenever G is a subgraph of H. We use MI to denote the set of all multiplicative increasing graph functions and AMI to denote the set of all additive multiplicative increasing graph functions. The classification of multiplicative increasing graph functions is still unsolved.

A graph G' is a *homomorphic image* of G if there exists a homomorphism $\psi : G \to G'$ which is onto and for every $[g_1', g_2'] \in E(G')$ there exists $[g_1, g_2] \in E(G)$ such that $\psi(g_i) = g_i'$, $i = 1, 2$.

For any graph G, $P(G) = \lim_{n\to\infty}[\gamma(G^n)]^{1/n}$, where $\gamma(G)$ is the maximum

* Work partially supported by the National Science Council of the Republic of China.

0012-365X/87/$3.50

number of disjoint edges in G. In [2], Hsu proved that if f is multiplicative, increasing and $f(K_2)=2$, then $f(G)\geqslant P(G)$ for all G. Thus P can be viewed as a lower bound for multiplicative increasing graph functions. In this paper, we prove that P is indeed multiplicative and then generalize our result to get a class of additive multiplictive increasing graph functions.

2. The capacity functions

For a fixed graph G, we define the capacity function for G, P_G, from $\mathcal{G}$ to R as $P_G(H)=\lim_{n\to\infty}[\gamma_G(H^n)]^{1/n}$, where $\gamma_G(H)$ is the maximum number of disjoint G's in H. Obviously, P_G is always increasing.

Theorem 2.1. (1) *If K is a subgraph of H, then $P_H\leqslant P_K$.*
(2) *$P_{H^k}=P_H$, where k is a positive integer.*

Proof. (1) Since K is a subgraph of H, for any graph G we have $\gamma_H(G)\leqslant\gamma_K(G)$. Then $\gamma_H(G^n)\leqslant\gamma_K(G^n)$ which implies $P_H\leqslant P_K$.

(2) If $V(H)=\{x_1,x_2,\ldots,x_u\}$, then the induced subgraph of $\{(x_1,\ldots,x_1),(x_2,\ldots,x_2),\ldots,(x_u,\ldots,x_u)\}$ in H^k with each x_i repeats k times is isomorphic to H. From (1), we have $P_{H^k}\leqslant P_H$. However, for any graph G we have $G^n\supseteq\gamma_H(G^n)H$. Then $G^{kn}\supseteq\gamma_H^k(G^n)H^k$. Thus $G^m\supseteq\gamma_H^k(G^{m/k})H^k$ and we get

$$P_{H^k}(G)=\lim_{m\to\infty}[\gamma_{H^k}(G^m)]^{1/m}\geqslant\lim_{m\to\infty}[\gamma_{H^k}(\gamma_H^k(G^{m/k})H^k)]^{1/m}$$

$$=\lim_{m\to\infty}[\gamma_H^k(G^{m/k})]^{1/m}=\lim_{m/k\to\infty}[\gamma_H(G^{m/k})]^{k/m}=P_H(G).$$

Therefore, $P_{H^k}=P_H$. □

From Theorem 2.1, we know that if two graphs G and H satisfies $H^n\subseteq G^t\subseteq H^m$ for some $n,t,m\in N$, then $P_G=P_H$. Thus some capacity functions are equal. However, the next theorem tells us that the graphs with different clique number or chromatic number will have different capacity functions. In [6], Mycielski described a method for constructing a graph with $\omega(G)=m$ and $\chi(G)=n$, where $2\leqslant m\leqslant n$ and $\omega(G)$ is the maximum clique size in G, $\chi(G)$ is the chromatic number of graph G.

Theorem 2.2. *If $(\omega(G),\chi(G))\neq(\omega(H),\chi(H))$, then $P_G\neq P_H$.*

Proof. If $\omega(G)\neq\omega(H)$, we assume that $\omega(G)<\omega(H)$. Since $\omega(K^m)=\omega(K)$ for any graph K and $m\in N$, we have $\gamma_H(G^m)=0$ for every m. Hence $P_H(G)=0$. On the other hand, for every integer m, since G is a subgraph of G^m, we have $\gamma_G(G^m)\geqslant 1$. Hence $P_G(G)\geqslant 1$. Therefore $P_H\neq P_G$. If $\chi(G)\neq\chi(H)$, then since

$\chi(K^m)=\chi(K)$ for every graph K and any $m \in N$, by the same argument as above, we have $P_H \neq P_G$. □

The statement "If $(\omega(G), \chi(G))=(\omega(H), \chi(H))$, then $P_G=P_H$" is not true. For example $(\omega(K_2), \chi(K_2))=(\omega(K_{1,2}), \chi(K_{1,2}))=(2,2)$. But $P_{K_2} \neq P_{K_{1,2}}$. This can be proved in [4]. Also in [4], Hsu et al. discuss when two graphs will have different capacity functions.

3. Uniform graphs

Let G, H be graphs with $V(G)=\{x_1, x_2, \ldots, x_u\}$ and $V(H)=\{y_1, y_2, \ldots, y_v\}$. Let $D=\{(a_1, a_2, \ldots, a_v) \mid 0 \leqslant a_i \leqslant 1, \sum_{i=1}^{v} a_i=1\}$. Let m be a positive integer and $\vec{z}=(z_1, z_2, \ldots, z_m)$ be a vertex in H^m. We call $\vec{a}=(a_1, a_2, \ldots, a_v)$ with $a_i=|\{j \mid z_j=y_i, 1 \leqslant j \leqslant m\}|/m$ the distribution of $\vec{z}$. For any graph H, we can define a u-ary relation $R_G(H)$ on D such that $(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_u) \in R_G(H)$ with $\vec{a}_i \in D$ if and only if either

(i) there exists a positive integer m such that in H^m we can find $\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_u \in V(H^m)$ with the distribution of $\vec{y}_i$ to be $\vec{a}_i$ for every i and the induced subgraph of $\{\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_u\}$ in H^m contains a subgraph isomorphic to G with $\vec{y}_i$ corresponding to x_i for every i, or

(ii) there exists a sequence in $R_G(H)$ of type (i), $\{(\vec{a}_{i,1}, \vec{a}_{i,2}, \ldots, \vec{a}_{i,u})\}_{i=1}^{\infty}$ such that $\lim_{i\to\infty}(\vec{a}_{i,1}, \vec{a}_{i,2}, \ldots, \vec{a}_{i,u})=(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_u)$. Let $I_G(H)=\{\vec{a} \in D(H) \mid (\vec{a}, \vec{a}, \ldots, \vec{a})$ (u times) is in $R_G(H)\}$. We say $\vec{a} \in I_G(H)$ is of type (i) if its corresponding vector in $R_G(H)$ is of type (i).

A graph G with u vertices is called *uniform* if for any graph H, $\sum_{i=1}^{u} \vec{a}_i/u$ is in $I_G(H)$ of type (i) whenever $(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_u)$ is in $R_G(H)$ of type (i).

Example. K_n and C_n are uniform but $K_{1,2}$ is not.

Proof. (1) Let $V(K_n)=\{x_1, x_2, \ldots, x_n\}$. For any graph H, if $(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n) \in R_{K_n}(H)$ is of type (i), then there exists an integer m such that in H^m we can find $\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_n \in V(H^m)$ with the properties that the distribution of $\vec{y}_i$ is $\vec{a}_i$ and that $\{\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_n\}$ induce a subgraph which contains a subgraph isomorphic to K_n. Now consider the n vertices in H^{mn}, $\vec{z}_1=(\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_n)$, $\vec{z}_2=(\vec{y}_2, \vec{y}_3, \ldots, \vec{y}_n, \vec{y}_1)$, $\ldots$, $\vec{z}_n=(\vec{y}_n, \vec{y}_1, \vec{y}_2, \ldots, \vec{y}_{n-1})$. The induced subgraph of $\{\vec{z}_1, \vec{z}_2, \ldots, \vec{z}_n\}$ in H^{mn} forms a K_n and the distribution of $\vec{z}_i$ is $\sum_{i=1}^{n} \vec{a}_i/n$ for all i. Hence $\sum_{i=1}^{n} \vec{a}_i/n$ is in $I_G(H)$ of type (i). Therefore K_n is uniform.

(2) Let C_n be the graph with vertices $x_0, x_1, \ldots, x_{n-1}$ such that x_i is adjacent to x_{i+1} (mod n). A similar proof as above for K_n shows that C_n is uniform.

(3) Let $G=H=K_{1,2}$, $V(G)=\{x_1, x_2, x_3\}$ and $E(G)=\{[x_1, x_2], [x_2, x_3]\}$. Obviously, $((1,0,0),(0,1,0),(0,0,1))$ is in $R_G(G)$ of type (i). For every m, let $\vec{y}_1=(y_{1,1}, y_{1,2}, \ldots, y_{1,m})$, $\vec{y}_2=(y_{2,1}, y_{2,2}, \ldots, y_{2,m})$ and $\vec{y}_3=$

$(y_{3,1}, y_{3,2}, \ldots, y_{3,m}) \in V(H^m)$ induce a $K_{1,2}$ with $\vec{y}_i$ corresponding to x_i. Observe that $y_{1,j} \neq x_2$ if and only if $y_{2,j} = x_2$. Thus $|\{j \mid y_{1,j} = x_2\}| + |\{j \mid y_{2,j} = x_2\}| = m$. This fact implies that $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is not in $I_G(G)$ of type (i). Hence $K_{1,2}$ is not uniform. □

In fact, every vertex transitive graph is uniform.

Theorem 3.1. *If G is vertex transitive, then G is uniform.*

Proof. Let $T(G)$ be the automorphism group for graph G with $|T(G)| = k$ and let $T_{ij}(G) = \{\pi \mid \pi \in T(G)$ and $\pi(x_i) = x_j\}$. Since G is vertex transitive, we have $|T_{ij}(G)| = |T_{kl}(G)|$ for any $1 \leqslant i, j, k, l, \leqslant u$, where $u = |V(G)|$. For any graph $H \in \mathscr{G}$, if $(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_u) \in R_G(H)$ of type (i) then $\exists m$, in H^m there exist u vertices $\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_u$ such that the induced subgraph of them in H^m contains a subgraph G_1 which is isomorphic to G. Since G is vertex transitive, G_1 is also vertex transitive. Let $T(G_1) = \{\pi_1, \pi_2, \ldots, \pi_k\}$. Then in H^{km}, there exist u vertices $(\pi_1(\vec{y}_1), \pi_2(\vec{y}_1), \ldots, \pi_k(\vec{y}_1))$, $(\pi_1(\vec{y}_2), \pi_2(\vec{y}_2), \ldots, \pi_k(\vec{y}_2)), \ldots,$ $(\pi_1(\vec{y}_u), \pi_2(\vec{y}_u), \ldots, \pi_k(\vec{y}_u))$ such that they are all of distribution $\sum_{i=1}^{u} \vec{a}_i/u$ and the induced subgraph of them in H^{km} contains a subgraph which is isomorphic to G. Thus $\sum_{i=1}^{u} \vec{a}_i/u \in I_G(H)$ of type (i). Thus G is uniform. □

Let $D = \{(a_1, a_2, \ldots, a_v) \mid a_i \geqslant 0, \sum_{i=1}^{v} a_i = 1\}$. Let $\mathscr{H}: D \to R$ be a function defined by $\mathscr{H}(\vec{a}) = \prod_{i=1}^{v} a_i^{-a_i}$, where $\vec{a} = (a_1, a_2, \ldots, a_v)$. Note that $\log_v \mathscr{H}$ is the entropy function. Hence the function $\mathscr{H}$ satisfies

$$(1) \quad \lim_{m \to \infty} \binom{m}{a_1 m, a_2 m, \ldots, a_v m}^{1/m} = \mathscr{H}(\vec{a}), \quad \text{where } a_i m \in I\ \forall i,$$

and

$$(2) \quad \mathscr{H}\left(\sum_{i=1}^{u} \vec{a}_i/u\right) \geqslant \min\{\mathscr{H}(\vec{a}_i) \mid i = 1, 2, \ldots, u\}.$$

Theorem 3.2. *If G is a uniform graph with $V(G) = \{x_1, x_2, \ldots, x_u\}$, then $P_G(H) = \max_{\vec{a} \in I_G(H)} \mathscr{H}(\vec{a})$.*

Proof. Let $V(H) = \{y_1, y_2, \ldots, y_v\}$. For any $\vec{a} \in I_G(H)$, there exists a sequence $\{(\vec{a}_{i,1}, \vec{a}_{i,2}, \ldots, \vec{a}_{i,u})\}_{i=1}^{\infty}$ in $R_G(H)$ of type (i), such that $\lim_{i \to \infty}(\vec{a}_{i,1}, \vec{a}_{i,2}, \ldots, \vec{a}_{i,u}) = (\vec{a}, \vec{a}, \ldots, \vec{a})$ (u times). Since G is uniform, we have $\vec{b}_i = (b_{i,1}, b_{i,2}, \ldots, b_{i,v}) = \sum_{j=1}^{u} \vec{a}_{i,j}/u$ is in $I_G(H)$ of type (i) and $\lim_{i \to \infty} \vec{b}_i = \vec{a}$. Thus for every i, we can find $\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_u \in V(H^m)$ for some $m \in N$ such that $\vec{y}_j$ has distribution $\vec{b}_i\ \forall j$ and the induced subgaph of $\{\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_u\}$ in H^m contains a subgraph isomorphic to G with $\vec{y}_j$ corresponding to x_j. Now consider $S(m, \vec{b}_i) = \{\vec{y} \in V(H^m) \mid$ the distribution of $\vec{y}$ is $\vec{b}_i\}$. Then the induced subgraph

$H^m|_{S(m,\vec{b}_i)}$ is vertex transitive. Let $\{\vec{z}=\vec{z}_1, \vec{z}_2, \ldots, \vec{z}_u\} \subseteq S(m, \vec{b}_i)$ be such that its induced subgraph in H^m contains a subgraph $A_{\vec{z}}$ isomorphic to G with $\vec{z}_i$ corresponding to x_i. For every $\vec{w} \in S(m, \vec{b}_i)$, there exists a permutation $\pi_{\vec{w}} \in S_m$, the symmetric group on m letters, such that $\pi_{\vec{w}}(\vec{z}) = \vec{w}$. Thus the induced subgraph of $\{\vec{w} = \pi_{\vec{w}}(\vec{z}_1), \pi_{\vec{w}}(\vec{z}_2), \ldots, \pi_{\vec{w}}(\vec{z}_u)\} \subseteq S(m, \vec{b}_i)$ contains a subgraph $A_{\vec{w}}$ isomorphic to G with $\pi_{\vec{w}}(\vec{z}_i)$ corresponding to x_i. Thus $\mathcal{A} = \bigcup_{\vec{w}\in S(m,\vec{b}_i)} A_{\vec{w}}$ form a spanning subgraph of $H^m|_{S(m,\vec{b}_i)}$.

Now apply the following algorithm on $\mathcal{A}$.

Algorithm. Find an $A_{\vec{w}}$ in $\mathcal{A}$ then delete the edges of $A_{\vec{w}}$ and the edges of all $A_{\vec{w}'}$'s which are adjacent to $A_{\vec{w}}$ until $E(\mathcal{A})$ is empty.

The $A_{\vec{w}}$'s found in the above algorithm are surely mutually disjoint. Since any $A_{\vec{w}}$ can be adjacent to at most $u(u-1)$ $A_{\vec{w}'}$'s with $\vec{w}' \neq \vec{w}$, we get at least $|S(m, \vec{b}_i)|/(u(u-1)+1)$ disjoint G's in $\mathcal{A}$. Thus

$$\gamma_G(H^m) \geq \gamma_G(H^m|_{S(m,\vec{b}_i)}) \geq \gamma_G(\mathcal{A}) \geq |S(m, \vec{b}_i)|/(u(u-1)+1)$$
$$= \frac{1}{u^2-u+1}\binom{m}{mb_{i,1}, mb_{i,2}, \ldots, mb_{i,v}}$$

and it follows that

$$P_G(H) = \lim_{m\to\infty} [\gamma_G(H^m)]^{1/m} \geq \mathcal{H}(\vec{b}_i).$$

Since $\mathcal{H}$ is a continuous function, we get $\lim_{i\to\infty} \mathcal{H}(\vec{b}_i) = \mathcal{H}(\lim_{i\to\infty} \vec{b}_i) = \mathcal{H}(\vec{a})$. Thus $P_G(H) \geq \mathcal{H}(\vec{a})$ $\forall \vec{a} \in I_G(H)$. From the definition of $I_G(H)$ and D, we know that $I_G(H)$ is a bounded closed set. So there is a $\vec{c} \in I_G(H)$ such that $\mathcal{H}(\vec{c}) = \max\{\mathcal{H}(\vec{a}) \mid \vec{a} \in I_G(H)\}$. Thus

$$P_G(H) \geq \max_{\vec{a}\in I_G(H)} \mathcal{H}(\vec{a}). \tag{1}$$

Note that in H^m there are at most $c(m+v-1, m)$ different distributions and therefore there are at most m^{uv} different $(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_u)$ in $R_G(H)$, where $\vec{a}_i$ is the distribution of some vertex in H^m for all i. Let $\mathcal{M}$ be a set of disjoint G's in H^m with $|\mathcal{M}| = \gamma_G(H^m)$. Write $\mathcal{M} = \{[\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_u] \mid \vec{y}_i$ corresponds to $x_i\}$. We can define an equivalence relation on $\mathcal{M}: [\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_u] \sim [\vec{z}_1, \vec{z}_2, \ldots, \vec{z}_u]$ if and only if the distribution of $\vec{y}_i$ is the same as that of $\vec{z}_i$ for all i. By the Pigeonhole principle, there exists some $(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_u)$ such that the set $\mathcal{K} = \{[\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_u] \mid$ the distribution of $\vec{y}_i$ is $\vec{a}_i$ for all $i\}$ satisfies $|\mathcal{K}| \geq m^{-uv}\gamma_G(H^m)$. Therefore $m^{-uv}\gamma_G(H^m) \leq \min\{|S(m, \vec{a}_i)| \mid i = 1, 2, \ldots, u\}$, hence

$$\gamma_G(H^m) \leq \max_{(\vec{a}1, \vec{a}_2, \ldots, \vec{a}_u)\in R_G(H)} \min\{(m^{uv}|S(m, \vec{a}_i)|) \mid i = 1, 2, \ldots, u\}.$$

This implies

$$
\begin{aligned}
P_G(H) &= \lim_{m\to\infty} [\gamma_G(H^m)]^{1/m} \\
&\leqslant \max_{(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_u)\in R_G(H)} \min\{\lim_{m\to\infty} (m^{uv}\,|S(m, \vec{a}_i)|)^{1/m} \mid i = 1, 2, \ldots, u\} \\
&= \max_{(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_u)\in R_G(H)} \min\{\mathscr{H}(\vec{a}_i) \mid i = 1, 2, \ldots, u\} \\
&\leqslant \max_{(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_u)\in R_G(H)} \mathscr{H}\Big(\sum_{i=1}^{u} \vec{a}_i/u\Big) \\
&\leqslant \max_{\vec{a}\in I_G(H)} \mathscr{H}(\vec{a}). \qquad (2)
\end{aligned}
$$

From (1) and (2) we have

$$P_G(H) = \max_{\vec{a}\in I_G(H)} \mathscr{H}(\vec{a}). \quad \square$$

4. Properties for uniform capacity functions

In this section, we discuss some properties for the capacity functions of uniform graphs.

Theorem 4.1. *If G is uniform, then P_G is pseudo-additive, i.e., if $P_G(H) \neq 0$ and $P_G(K) \neq 0$, then $P_G(H+K) = P_G(H) + P_G(K)$.*

Proof. Let $V(G) = \{x_1, x_2, \ldots, x_u\}$. Given any two graphs H and K with $|V(H)| = v$ and $|V(K)| = w$, since G is uniform, let $P_G(H) = g = \mathscr{H}(\vec{a})$ with $\vec{a} \in I_G(H)$ and $P_G(K) = h = \mathscr{H}(\vec{b})$ with $\vec{b} \in I_G(K)$.

Because G is uniform, there exists a sequence in $I_G(H)$ of type (i), $\{\vec{a}_i\}_{i=1}^{\infty}$, such that $\lim_{i\to\infty} \vec{a}_i = \vec{a}$ and there exists a sequence in $I_G(K)$ of type (i), $\{\vec{b}_i\}_{i=1}^{\infty}$, such that $\lim_{i\to\infty} \vec{b}_i = \vec{b}$. Therefore, for every i there exists a positive integer m such that in H^m we can find u vertices $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_u$ all with distribution $\vec{a}_i$ which form a G. Similarly, for every i there exists a positive integer l such that in K^l we can find u vertices $\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_u$ all with distribution $\vec{b}_i$ which form a G. Since g and h are real numbers, there exist sequences of rational numbers $\{g_i\}_{i=1}^{\infty}$ and $\{h_i\}_{i=1}^{\infty}$ such that $\lim_{i\to\infty} g_i = g$ and $\lim_{i\to\infty} h_i = h$. Thus for every i we can choose an integer t such that $p = tg_i/(g_i + h_i)$ and $q = th_i/(g_i + h_i)$ are integers.

Now in $(H+K)^{tlm}$ we can find u vertices $(\vec{x}_1, \vec{x}_1, \ldots, \vec{x}_1, \vec{y}_1, \vec{y}_1, \ldots, \vec{y}_1)$, $(\vec{x}_2, \vec{x}_2, \ldots, \vec{x}_2, \vec{y}_2, \vec{y}_2, \ldots, \vec{y}_2), \ldots, (\vec{x}_u, \vec{x}_u, \ldots, \vec{x}_u, \vec{y}_u, \vec{y}_u, \ldots, \vec{y}_u)$ such that each $\vec{x}_i$ repeats pl times and each $\vec{y}_i$ repeats qm times. We can easily check that these u vertices form a G and all with distribution

$$((pl)m\vec{a}_i, (qm)l\vec{b}_i)/(tml) = \Big(\frac{g_i}{g_i + h_i}\vec{a}_i, \frac{h_i}{g_i + h_i}\vec{b}_i\Big).$$

Thus

$$\left(\frac{g_i}{g_i+h_i}\vec{a}_i, \frac{h_i}{g_i+h_i}\vec{b}_i\right) \in I_G(H+K).$$

Hence

$$\left(\frac{g}{g+h}\vec{a}, \frac{h}{g+h}\vec{b}\right) \in I_G(H+K)$$

and therefore

$$\begin{aligned} P_G(H+K) &\geq \mathcal{H}\left(\frac{g}{g+h}\vec{a}, \frac{h}{g+h}\vec{b}\right) = \prod_{i=1}^{v}\left(\frac{g}{g+h}a_i\right)^{-g/(g+h)a_i} \prod_{j=1}^{w}\left(\frac{h}{g+h}b_j\right)^{-h/(g+h)b_j} \\ &= \left(\frac{g}{g+h}\right)^{-g/(g+h)\sum_{i=1}^{v} a_i} \left(\prod_{i=1}^{v} a_i^{-a_i}\right)^{g/(g+h)} \left(\frac{h}{g+h}\right)^{-h/(g+h)\sum_{j=1}^{w} b_j} \left(\prod_{j=1}^{w} b_j^{-b_j}\right)^{h/(g+h)} \\ &= \left(\frac{g}{g+h}\right)^{-g/(g+h)} g^{g/(g+h)} \left(\frac{h}{g+h}\right)^{-h/(g+h)} h^{h/(g+h)} \\ &= g^{-g/(g+h)+g/(g+h)} h^{-h/(g+h)+h/(g+h)} \left(\frac{1}{g+h}\right)^{-g/(g+h)-h/(g+h)} \\ &= g+h. \end{aligned}$$

We have

$$P_G(H+K) \geq P_G(H) + P_G(K). \tag{3}$$

On the other hand, let $\vec{c} = (\vec{a}, \vec{b}) = (a_1, a_2, \ldots, a_v, b_1, b_2, \ldots, b_w) \in I(H+K)$ be such that $P_G(H+K) = \mathcal{H}(\vec{c})$. Let $p = \sum_{i=1}^{v} a_i$ and $q = \sum_{i=1}^{w} b_i$. Then $p+q=1$. Since $\vec{c} = (\vec{a}, \vec{b}) \in I_G(H+K)$ and G is uniform, there exists a sequence in $I_G(H+K)$ of type (i), $\{(\vec{a}_i, \vec{b}_i)\}_{i=1}^{\infty}$, such that $\lim_{i\to\infty}(\vec{a}_i, \vec{b}_i) = (\vec{a}, \vec{b})$. Assume that $\vec{a}_i = (a_{i,1}, a_{i,2}, \ldots, a_{i,v})$ and $\vec{b}_i = (b_{i,1}, b_{i,2}, \ldots, b_{i,w})$. Let $p_i = \sum_{j=1}^{v} a_{i,j}$ for every i. Then

$$\lim_{i\to\infty} p_i = \lim_{i\to\infty}\sum_{j=1}^{v} a_{i,j} = \sum_{j=1}^{v}\lim_{i\to\infty} a_{i,j} = \sum_{j=1}^{v} a_i = p.$$

Similarly, let $q_i = \sum_{j=1}^{w} b_{i,j}$ for every i, we have $\lim_{i\to\infty} q_i = q$.

Since $(\vec{a}_i, \vec{b}_i) \in I_G(H+K)$ of type (i), there exists an integer m such that in $(H+K)^m$ we can find $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_u$ such that they are all of distribution $(\vec{a}_i, \vec{b}_i)$ and they form a G. Without loss of generality, we may assume $\vec{x}_i = (z_{i,1}, z_{i,2}, \ldots, z_{i,n}, z_{i,n+1}, \ldots, z_{i,m})$ with $z_{i,j} \in V(H)$ if and only if $1 \leq j \leq n$. Let $\vec{x}_i = (z_{i,1}, z_{i,2}, \ldots, z_{i,n})$. Then we have u vertices $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_u$ all with distribution $\vec{a}_i/p_i$. If they form a G, then $\vec{a}_i/p_i \in I_G(H)$. Otherwise, $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_u$ form a homomorphic image of G. Since $P_G(H) \neq 0$, there exists an integer r such that in H^r we can find u vertices $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_u$ all with distribution $\vec{d}$ which form a G.

Let $s \in N$, then in H^{r+sn} we can find u vertices $(\vec{w}_1, \vec{x}_1, \vec{x}_1, \ldots, \vec{x}_1)$, $(\vec{w}_2, \vec{x}_2, \vec{x}_2, \ldots, \vec{x}_2), \ldots, (\vec{w}_u, \vec{x}_u, \vec{x}_u, \ldots, \vec{x}_u)$ with each $\vec{x}_i$ repeats s times. Then these u vertices form a G and all with distribution

$$\frac{\vec{d} + (s\vec{a}_i)/p_i}{s+1}.$$

Since

$$\lim_{s\to\infty} \frac{\vec{d} + (s\vec{a}_i)/p_i}{s+1} = \vec{a}_i/p_i,$$

we have $\vec{a}_i/p_i \in I_G(H)$. Thus $\vec{a}/p \in I_G(H)$. Similarly, we have $\vec{b}/q \in I_G(K)$.

Let $k = \mathcal{H}(\vec{a}/p)$ and $l = \mathcal{H}(\vec{b}/q)$. Then

$$\begin{aligned} P_G(H+K) &= \mathcal{H}(\vec{c}) = \prod_{i=1}^{v} a_i^{-a_i} \prod_{j=1}^{w} b_j^{-b_j} \\ &= \prod_{i=1}^{v} (p(a_i/p))^{-p(a_i/p)} \prod_{j=1}^{w} (q(b_j/q))^{-q(b_j/q)} \\ &= p^{-p}\left[\prod_{i=1}^{v} (a_i/p)^{-a_i/p}\right]^p q^{-q}\left[\prod_{j=1}^{w} (b_j/q)^{-b_j/q}\right]^q \\ &= p^{-p}k^p q^{-q} l^q = p^{-p}k^p(1-p)^{-(1-p)} l^{(1-p)}. \end{aligned}$$

Consider the function $f(x) = x^{-x}k^x(1-x)^{-(1-x)}l^{(1-x)}$. Then $f'(x) = f(x)\ln(k(1-x)/(lx))$. Since $f'(x) = 0$ if and only if $x = k/(k+l)$ and $f'(x) > 0$ when $x < k/(k+l)$, $f'(x) < 0$ when $x > k/(k+l)$. Therefore $f(x)$ has a maximum value at $x = k/(k+l)$. Thus

$$\begin{aligned} P_G(H+K) &\leq (k/(k+l))^{-k/(k+l)} k^{k/(k+l)} (l/(k+l))^{-l/(k+l)} l^{l/(k+l)} \\ &= k + l \leq P_G(H) + P_G(K). \end{aligned} \tag{4}$$

From (3) and (4), we have $P_G(H+K) = P_G(H) + P_G(K)$. Hence P_G is pseudo additive. □

Corollary. *Let G be a uniform graph, and $P_G(H) \neq 0$, $P_G(K) \neq 0$. Then $P_G(H) = \mathcal{H}(\vec{a})$ with $\vec{a} \in I_G(H)$ and $P_G(K) = \mathcal{H}(\vec{b})$ with $\vec{b} \in I_G(K)$ if and only if $P_G(H+K) = \mathcal{H}(\vec{c})$, where*

$$\vec{c} = \left(\frac{P_G(H)}{P_G(H) + P_G(K)}\vec{a}, \frac{P_G(K)}{P_G(H) + P_G(K)}\vec{b}\right),$$

and $\vec{c} \in I_G(H+K)$.

Theorem 4.2. *If G is uniform, then P_G is pseudo multiplicative, i.e., if $P_G(H) \neq 0$ and $P_G(K) \neq 0$, then $P_G(H \times K) = P_G(H) \times P_G(K)$.*

Proof. Since

$$P_G(H^2) = \lim_{n\to\infty} [\gamma_G(H^{2n})]^{1/n} = \lim_{n\to\infty} \{[\gamma_G(H^{2n})]^{1/2n}\}^2 = P_G^2(H),$$

we have $P_G((H+K)^2) = P_G^2(H+K)$. Moreover, since $P_G(H) \neq 0$ and $P_G(K) \neq 0$, we have $G \subseteq H^r$ and $G \subseteq K^s$ for some $r, s \in N$. This implies $G \subseteq (H \times K)^{\max(r,s)}$.

Thus $P_G(H \times K) \neq 0$. From Theorem 4.1, we know P_G is pseudo additive. Hence

$$P_G((H+K)^2) = P_G(H^2 + 2H \times K + K^2) = P_G(H^2) + 2P_G(H \times K) + P_G(K^2)$$
$$= P_G^2(H) + 2P_G(H \times K) + P_G^2(K). \quad (5)$$

But

$$P_G((H+K)^2) = P_G^2(H+K) = [P_G(H) + P_G(K)]^2$$
$$= P_G^2(H) + 2P_G(H)P_G(K) + P_G^2(K). \quad (6)$$

Comparing (5) and (6) we obtain $P_G(H \times K) = P_G(H)P_G(K)$. Hence P_G is pseudo multiplicative. □

We have the following result which is similar to the Corollary of Theorem 4.1. The proof is omitted.

Corollary. *Assume G is uniform, and $P_G(H) \neq 0$, $P_G(K) \neq 0$. Then $P_G(H) = \mathcal{H}(\vec{a})$ with $\vec{a} = (a_1, a_2, \ldots, a_v) \in I_G(H)$ and $P_G(K) = \mathcal{H}(\vec{b})$ with $\vec{b} = (b_1, b_2, \ldots, b_w) \in I_G(K)$ if and only if $P_G(H \times K) = \mathcal{H}(\vec{c})$, where $\vec{c} \in I_G(H \times K)$ and $\vec{c} = (c_{1,1}, c_{1,2}, \ldots, c_{1,w}, c_{2,1}, \ldots, c_{2,w}, \ldots, c_{v,w})$ with $c_{i,j} = a_i b_j$.*

Let us construct a graph $KG_{n,k}$ as follows. The vertices of $KG_{n,k}$ are the n-subsets of $\{1, 2, \ldots, 2n+k\}$ and two of them are joined by an edge if and only if they are disjoint. These graphs are called Kneser's graphs. It is easy to see that Kneser's graph is vertex transitive. In [5], Lovász has proved that

$$\omega(KG_{n,k}) = \left[\frac{2n+k}{n}\right] \quad \text{and} \quad \chi(KG_{n,k}) = k+2.$$

Thus by Theorem 2.2, Theorem 3.1, Theorem 4.1 and Theorem 4.2, we have a lot of different capacity functions which are pseudo additive, pseudo multiplicative and increasing.

5. Primary uniform graphs

We say a graph is *primary* if for any homomorphic image G' of G, we have $P_{G'} \leq P_G$, i.e., $G \subseteq (G')^k$ for some $k \in N$. For example, C_{2k+1}, K_n and Petersen graph are primary.

Lemma 5.1. *If G is primary and H contains a homomorphic image G' of G, then $P_G(H) \neq 0$.*

Proof. Since G is primary, we have $G \subseteq (G')^k \subseteq H^k$ for some $k \in N$. Thus $P_G(H) \neq 0$. □

By Lemma 5.1, Theorem 4.1 and Theorem 4.2, we have the following corollary.

Corollary. *If G is primary and uniform, then* $P_G \in$ AMI.

6. Conclusions

For any graph H and any integer $m \geqslant 1$ let H_m be the induced subgraph of H such that $x \in V(H_m)$ if and only if x is in any m-clique of H. For any graph function f, we can define another graph function f_m by $f_m(H) = f(H_m)$ for any graph H.

Lemma 6.1. *If f is additive (respectively, multiplicative, increasing), then f_m is also additive (respectively, multiplicative, increasing).*

Proof. Assume that f is additive. Observe that $x \in V((G+H)_m)$ if and only if x is in G_m or x is in H_m. Therefore $(G+H)_m = G_m + H_m$. Then we have

$$f_m(G+H) = f((G+H)_m) = f(G_m + H_m) = f(G_m) + f(H_m) = f_m(G) + f_m(H).$$

Hence f_m is additive. Next assume that f is multiplicative. Since $(x, y) \in V((G \times H)_m)$ if and only if x is in G_m and y is in H_m, we have $(G \times H)_m = G_m \times H_m$. Then

$$f_m(G \times H) = f((G \times H)_m) = f(G_m \times H_m) = f(G_m)f(H_m) = f_m(G)f_m(H).$$

Hence f_m is multiplicative. Finally, if f is increasing, then for any G, $H \in \mathcal{G}$ with $G \subseteq H$, we have $G_m \subseteq H_m$. Hence $f_m(G) = f(G_m) \leqslant f(H_m) = f_m(H)$. Thus f_m is increasing. □

Note that the function defined by $P(G) = \lim_{m\to\infty}[\gamma(G^m)]^{1/m}$, where $\gamma(G)$ is the maximum number of disjoint edges in G is in fact P_{K_2}. In [2], Hsu discovered that P can be viewed as a lower bound for some multiplicative increasing graph functions. But it was not known whether P is multiplicative or not. Now we know K_2 is primary and uniform. Hence $P \in$ AMI.

The classification of the set of additive multiplicative increasing graph functions is still unsolved. But with Lemma 6.1, we have the following functions which are additive multiplicative increasing:

(1) $(h_H)_m$ defined in [2] with H connected,
(2) δ_m with δ defined in [2], and
(3) $(p_G)_m$ with G primary and uniform.

Moreover P_G can be viewed as a lower bound for additive multiplicative increasing graph functions. Indeed, if $f \in$ AMI and $f(G) = P_G(G)$, then we have

$$\begin{aligned} f(H) &= (f(H^m))^{1/m} \\ &\geqslant (f(\gamma_G(H^m)G))^{1/m} = (\gamma_G(H^m)f(G))^{1/m} \\ &= (\gamma_G(H^m)P_G(G))^{1/m} \quad \text{for any } H \in \mathcal{G}. \end{aligned}$$

Thus

$$f(H) \geqslant \lim_{m\to\infty} (\gamma_G(H^m)P_G(G))^{1/m} = \lim_{m\to\infty} (\gamma_G(H^m))^{1/m} = P_G(H).$$

In [4], Hsu et al. have proved that if G is bipartite, then P_G is equal to one of P_{K_1}, P_{2K_1}, P_{K_2} and $P_{K_{1,2}}$. Moreover, these four functions are all different. Also, it is proved that if $P_{K_{1,2}}(H) \neq 0$, then $P_{K_{1,2}}(H) = P_{K_2}(H)$. Actually, if G' is a homomorphic image of G and H is any graph such that $P_G(H) \neq 0$, then $P_G(H) = P_{G'}(H)$. In fact, we know P_{2K_1} is not in AMI. For example, $P_{2K_1}(K_1 + K_2) = 3$ but $P_{2K_1}(K_1) + P_{2K_1}(K_2) = 0 + 2$. The calculation of some capacity functions will also be discussed in [3].

It is interesting that for some H, γ_H is very difficult to calculate but the asymptotic behavior of it is good. In [1], it is proved that the 3-dimensional matching problem (3DM) is NP-hard. Let us take $H = K_3$ as an example. Since the 3DM can be reduced to the calcultion of γ_{K_3}, thus finding γ_{K_3} is NP-hard. However, we do know the asymptotic behavior of K_3 since P_{K_3} can be easily calculated and $P_{K_3} \in$ AMI.

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