Chapter 1

Introduction

THE discrete multitone (DMT) tranceivers have enjoyed great success in high speed data transmission such as digital subscriber loop (DSL). They also have founds applications in a wide range of other transmission wire or wireless channels. It is typically called DMT for wired DSL applications and OFDM (orthogonal frequency division multiplexing) for broadcasting or wireless applications, e.g., digital audio broadcasting (DAB) and digital video broadcasting (DVB).

The use of fast internet connections has grown rapidly over the last few years. As more people buy home computers and create home networks, the demand for broadband (high-speed) connections steadily increases. Two technologies, cable modems and Asymmetric Digital Subscriber Line (ADSL), currently dominate the industry. However, the most recent and high-speed generation of digital subscriber line (DSL) family is very high-speed DSL (VDSL).

1.1 Discrete Multitone Modulation

The discrete multitone (DMT) has attracted considerable attention as a practical and viable technology for high-speed data transmission over spectrally shaped noisy channels. Modems employing this technology are already available in the market. The

DMT-based modems have, in particular, been found very useful in transmitting high-speed data over digital subscriber lines (DSLs) [1]. DMT is a special multicarrier data transmission technique that uses the properties of the discrete Fourier transform (DFT) in an elegant way so as to achieve a computationally efficient realization. Fig. 1.1 depicts a block diagram of a DMT modem. In the transmitter, the data sequence is partitioned into a number of parallel streams. Each stream of data is modulated via a particular subcarrier. The modulated subcarriers are summed to obtain the transmit signal. The use of DFT in DMT allows an efficient realization of the subcarrier modulators in a parallel processing structure which benefits from the computational efficiency of the fast Fourier transform (FFT). A similar DFT-based structure is used for efficient realization of the subcarrier demodulators in the receiver part of the DMT modem. Inter-symbol interference (ISI) is a major problem associated with broadband

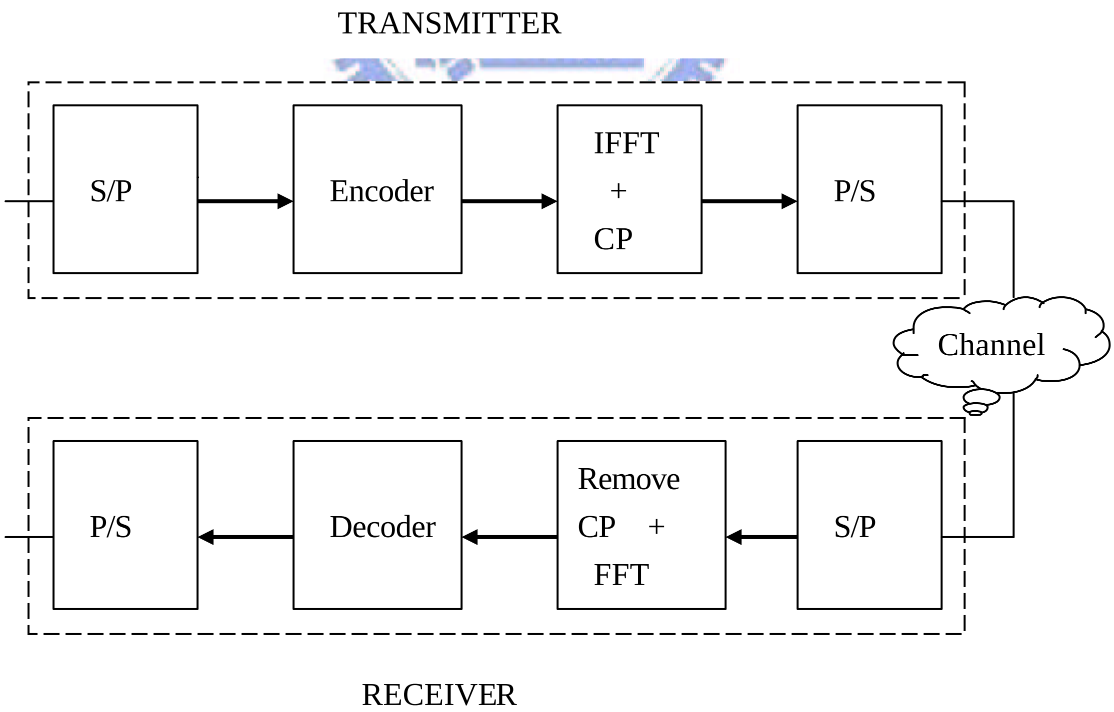

Fig. 1.1 Block diagram of a DMT transceiver

channels [1]. This undesirable effect is caused by the spectral shaping of the channel. In other words, variation of magnitude and phase responses of the channel over frequency causes neighboring symbols to interfere with each other at the receiver. ISI-free transmission can be achieved in a DMT system, by inserting a cyclic prefix of size ? onto each block of N transmitted samples. The length of the cyclic prefix should be at least equal to the duration of the channel impulse response minus one.

1.1.1 DMT Transmitter

In the transmitter, M bits of the input bit stream are buffered. These bits are then assigned to each of the $N/2$ subchannels using a bit loading algorithm. In DMT systems, bit loading algorithms assign the bits and available power to each subchannel according to the SNR in each subchannel, such that high SNR subchannels receive more bits than low SNR subchannels. The subchannels which are with very low SNR will not be used.

The next step is to map the assigned bits to subsymbols by using a modulation method, such as QAM in ADSL modems. These subsymbols are complex-valued in general and can be thought of being in the frequency domain. In the DSL system, we should transmit real samples in the twisted pair lines. In order to obtain real samples after IFFT, the $N/2$ subsymbols are duplicated with their conjugate symmetric counterparts. We call these obtained time domain samples a DMT symbol. After mapping, we perform the IFFT. To avoid ISI, guard interval is used. We insert a symbol with its last n samples, which is called a cyclic prefix. Therefore, one block consists of $N+n$ samples instead of N samples. However, this way decreases the transmission efficiency by a factor $\frac{N+n}{n}$. The prefix is selected as the last n

samples of the symbol to convert the linear convolution effect of the channel into circular convolution and to help the receiver perform symbol synchronization. Circular convolution can be implemented in the DFT domain by using the FFT.

1.2.2 DMT Receiver

The receiver does the inverse actions and steps in the transmitter. However, an important exception is the time domain equalizer (TEQ) [2]. The TEQ ensures that the equalized channel impulse response is shortened to be shorter than the length of the cyclic prefix. If the TEQ is well done, then the received complex subsymbols after the FFT are the multiplication of the transmitted subsymbols with the FFT of the shortened (equalized) channel impulse response. After demapping the subsymbols back to the corresponding bits using the QAM constellation, they are converted to serial bits.

1.2 VDSL system overview

The most recent and high-speed generation of digital subscriber line (DSL) family is very high-speed DSL (VDSL). This technology bridges the copper telecommunications infrastructure of today with the potentially all-fiber infrastructure of the distant future. We place the VDSL modems at the end of the fiber network and in the customer's premises respectively. When fiber terminates in a neighborhood, very high speeds are possible on the copper wiring spanning about 1.5 km (4500 ft) from the customer to the fiber end — potentially as high as 15 Mb/s total in both directions and 52 Mb/s for short distance (300 m or less) [3]. There are three standardization bodies — ITU-T Study Group 15/Question 4, ANSI T1E1.4, and

ETSI TM6 — are writing two standards. That is the multicarrier modulation (MCM) and signal-carrier modulation scheme. Figure 1.2 shows the VDSL system deployment.

The key to VDSL is that the telephone companies are replacing many of their main feeds with fiber-optic cable. In fact, many phone companies are planning Fiber to the Curb (FTTC), which means that they will replace all existing copper lines right up to the point where your phone line branches off at your house. At the least, most companies expect to implement Fiber to the Neighborhood (FTTN). Instead of installing fiber-optic cable along each street, FTTN has fiber going to the main junction box for a particular neighborhood. The gateway takes care of the analog-digital-analog conversion problem that disables ADSL over fiber-optic lines. It converts the data received from the transceiver into pulses of light that can be transmitted over the fiber-optic system to the central office, where the data is routed to the appropriate network to reach its final destination. When data is sent back to your computer, the VDSL gateway converts the signal from the fiber-optic cable and sends it to the transceiver. All of this happens millions of times each second.

The subscriber loop suffers from the following main impairments [4]:

Attenuation of the twisted pair itself that depends on several parameters like the type of dielectric used, wire gauge, type of twisting, and length of the wire. Attenuation of the line usually increases with both frequency and the length of the wire. This results in potentially lower bit rate capacity when considering long loops and broadband signals like VDSL.

Bridged taps. The main impairments caused by bridged taps are big notches in the line transfer function. In practice, these notches will have finite depth due to the attenuation of the pairs.

Crosstalk. When a current flows through a wire, an electromagnetic field is

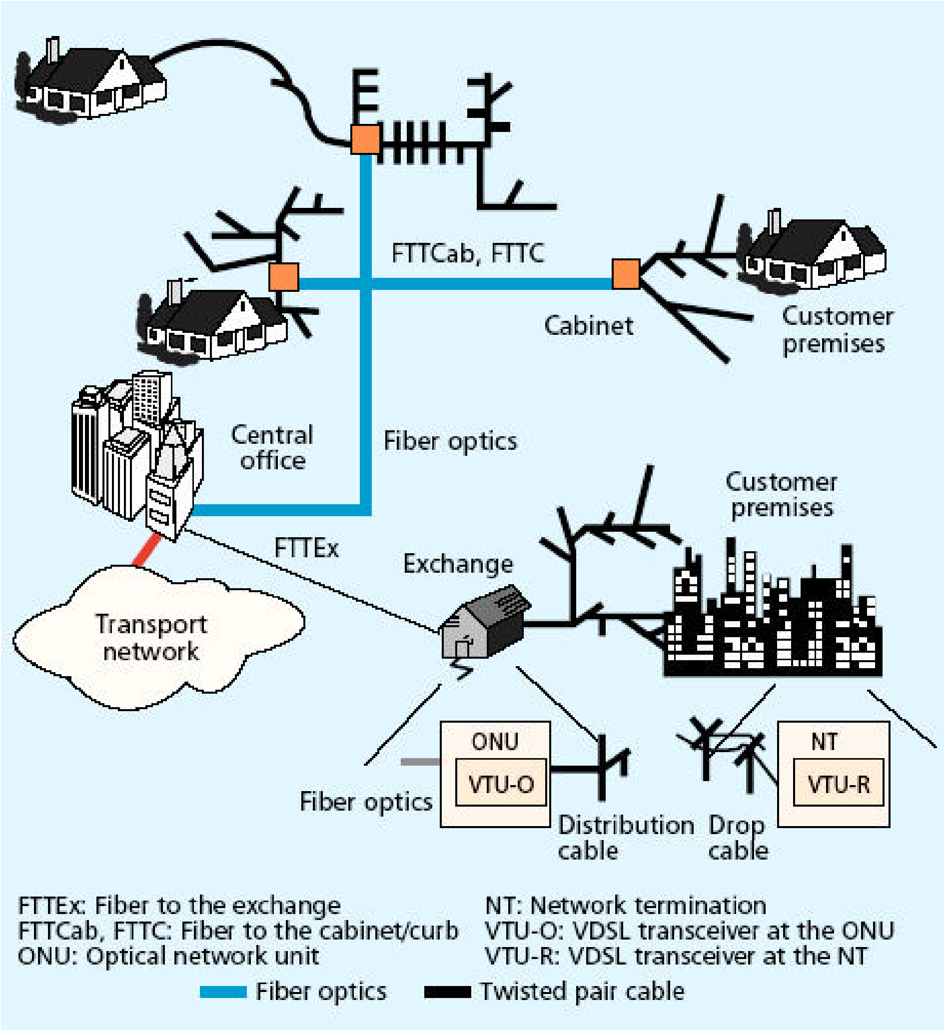

Fig. 1.2. Typical VDSL deployment scenarios

created which can interfere with signals on adjacent wires. As frequency increases, this effect becomes stronger. Each pair is twisted because this allows opposing fields in the wire pair to cancel each other. The tighter the twist, the more effective the cancellation and the higher the data rate supported by the cable. Maintaining this twist ratio is the single most important factor for a successful installation. If wires are not tightly twisted, the result is Near End Crosstalk (NEXT). The Far end crosstalk (FEXT) is caused by signals traveling in the same direction. Most of us have experienced a telephone call where we could hear another conversation faintly in the background. This is crosstalk. (Fig. 1.3)

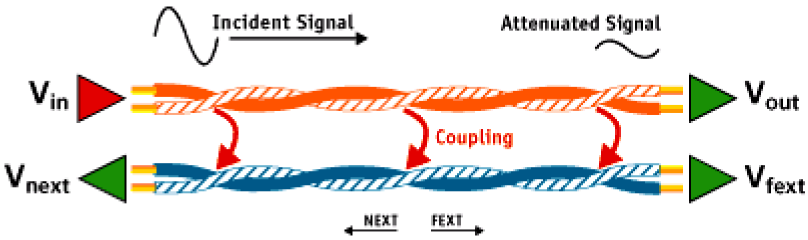

Fig. 1.3. NEXT and FEXT in a cable binder

The NEXT transfer function for 50-pair binder groups can be expressed as:

$$H_{NEXT} = -50\text{dB} + 15\log_{10}(f) + 6\log_{10}(N)$$

where f is the frequency normalized by a reference frequency f_0 = 1 MHz and N is the number of crosstalkers (i.e., interferers).

The FEXT transfer function for 50-pair binder groups can be expressed as:

$$H_{FEXT} = -45\text{dB} + 20\log_{10}(f) + 10\log_{10}(L) + 20\log_{10}(H_{line}(f)) + 6\log_{10}(N)$$

where L is line length normalized by a reference length L_0= 1 km, and $H_{line}(f)$ is the transfer function of the line.

There is an important thing is that NEXT is usually more harmful than FEXT because NEXT is not attenuated by the transfer function of the line.

Thermal noise or background noise. This noise is additive white Gaussian noise (AWGN) with fixed power spectral density (PSD) level of –140 dBm/Hz.

Impulsive noise. Impulsive noise can be caused by many electronic or electromechanical devices. It may also be a simple on/off hook of a POTS line close to the VDSL line.

Radio frequency ingress (RFI). RFI noise is caused by an amateur radio (HAM) signal that is transmitted in close proximity of a VDSL transceiver.

Nowadays, the VDSL system that we use is mainly following the specification and standard of ANSI-T1E1.4 and ETSI-TM6. The normal specification is formulated by ITU-T. It will provide all coefficients in VDSL, including the power requirements, service categories, range of service, and the payload bit rates. At first, we can get the corresponding power according to the different standards and deployments from Table 1.1. The FTTEx (Fiber to The Exchange) in the table means the fiber end in the exchange. The transmission medium of system from the exchange to the client is the twisted pairs. Similarly, the FTTCab (Fiber to The Cabinet) means that the transmission medium of system from the cabinet to the client is the twisted pairs.

		FTTEx		FTTCab	
Standard	Impedance	upstream	downstream	upstream	downstream
ANSI	100O	11.5 dBm	11.5 dBm	11.5 dBm	14.5 dBm
ETSI	135O	14.5 dBm	14.5 dBm	11.5 dBm	11.5 dBm

Table 1.1. VDSL power requirement

Similarly, Table 1.2. offers the VDSL payload bits rates. In the table, “A” means Asymmetric and “S” represents for Symmetric.

	ETSI requirements		ANSI requirements	
	Downstream	Upstream	Downstream	Upstream
Class I (A4)	23.168 Mb/s	4.096 Mb/s	22 Mb/s	3 Mb/s
Class I (A3)	14.464 Mb/s	3.072 Mb/s	–	–
Class I (A2)	8.576 Mb/s	2.048 Mb/s	–	–
Class I (A1)	6.4 Mb/s	2.048 Mb/s	–	–
Class II (S5)	28.288 Mb/s	28.288 Mb/s	–	–
Class II (S4)	23.168 Mb/s	23.168 Mb/s	–	–
Class II (S3)	14.464 Mb/s	14.464 Mb/s	13 Mb/s	13 Mb/s
Class II (S2)	8.576 Mb/s	8.576 Mb/s	–	–
Class II (S1)	6.4 Mb/s	6.4 Mb/s	6 Mb/s	6 Mb/s

Table 1.2. VDSL payload bits rates

1.3 Organization of the Thesis

The rest of this thesis is organized as follows. In Chapter 2, we describe the system model and review some time domain equalization algorithms MMSE, MSSNR, MGSNR, MBR, and Min-ISI. In Chapter 3, we describe our proposed modified TEQ algorithm. Chapter 4 contains many simulation results and discussions. Finally, we draw conclusion in Chapter 5.

Chapter 2

System Model and TEQ Algorithm

In this chapter, we first present the general architecture of the System Model. Then we give the details of many time domain equalizer algorithms. Finally, we analyze each algorithm, summarize them and give a conclusion.

2.1 System model and equalizer

Most modem communications systems that operate near theoretical limits employ equalization in the receiver to optimize transmission. Often the equalization is done digitally by adaptive filters. This approach provides a very flexible way to

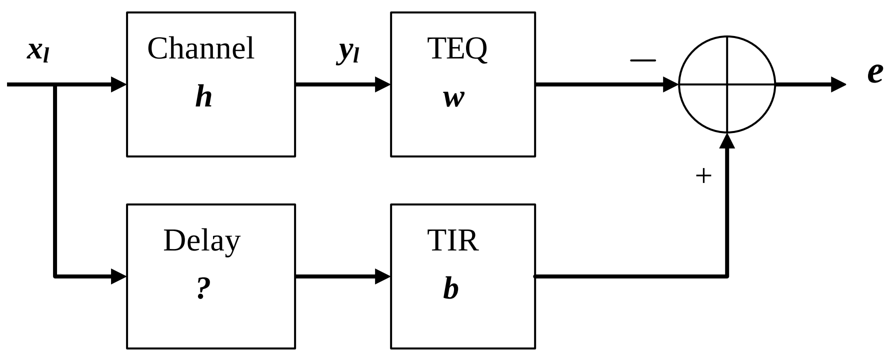

Fig. 2.1 Block diagram of the minimum mean-squared error (MMSE) equalizer

accommodate different types of noise environments. In most DSLs, the adaptive filters converge to optimal initial settings during a training period and then can be updated during normal operation of the system.

Any time a channel's frequency response is not flat over the range of frequencies being transmitted, intersymbol interference (ISI) can occur. Therefore, we need an equalizer to compensate the effect of the channel to reduce the ISI. The system model of the equalizer is as in Fig. 2.1. We look forward to finding a equalizer to make the total effect of channel and the equalizer be the same with the target impulse response (TIR) which is we choosing previously.

2.2 Time domain equalization algorithms

From the previous chapter, we know that when the length of channel impulse response is no longer than that of cyclic prefix ?, the DMT system is ISI free. However, the long cyclic prefix will reduce the channel throughput. As a result, for channels with long impulse responses such as DSL loops, a time-domain equalizer (TEQ) is typically added at the receiver to shorten the effective impulse response. It wants to shorten the length of effective channel to the cyclic prefix. Time domain equalizer design methods can be categorized into three major approaches: Minimizing Mean Squared Error (MSE) [5][6], Maximizing Shortening SNR (SSNR) [7], and Maximizing Channel Capacity. Following, we will introduce some existing TEQ design methods.

2.2.1 Minimum mean square error (MMSE)

Essentially, the first approach makes use of the Wiener principle of optimum filtering. The block diagram is as in Fig. 2.1

The underlying idea is the minimization of the mean square error $\boldsymbol{e}^2$ between the impulse response of channel $\boldsymbol{h}$ and the target impulse response $\boldsymbol{b}$ by using the TEQ equalizer, which is realized as an FIR digital filter with taps $\boldsymbol{w}$. Assume that the error is zero for any given input signal. That means the impulse responses of both branches are equal. In other words, the equalized channel impulse response (upper branch) would be equal to a delayed version of the TIR. Setting the number of taps of the TIR to a desired length forces the equalizer channel impulse response to have the same length.

The k-th sequence of output signal y_k can now be defined as follows:

$$y_k = Hx_k + n_k \tag{2.1}$$

where

$$y_k = \begin{bmatrix} y_{k0} \\ \vdots \\ y_{kS} \end{bmatrix}, x_k = \begin{bmatrix} x_{k0} \\ \vdots \\ x_{kS} \end{bmatrix}, n_k = \begin{bmatrix} n_{k0} \\ \vdots \\ n_{kS} \end{bmatrix} \tag{2.2}$$

and (2.1) can be written as

$$\begin{bmatrix} y_{k+N_w-1} \\ y_{k+N_w-2} \\ \vdots \\ y_k \end{bmatrix} = \begin{bmatrix} h_0 & \cdots & h_L & 0 & \cdots & 0 \\ 0 & h_0 & \cdots & h_L & 0 & \cdots \\ \vdots & \ddots & \ddots & & \ddots & \vdots \\ 0 & \cdots & 0 & h_0 & \cdots & h_L \end{bmatrix} \begin{bmatrix} x_{k+N_w-1} \\ x_{k+N_w-2} \\ \vdots \\ x_{k-L} \end{bmatrix} + \begin{bmatrix} n_{k+N_w-1} \\ n_{k+N_w-2} \\ \vdots \\ n_{k-L} \end{bmatrix} \tag{2.3}$$

Our objective is to minimize the MSE which is given as

$$MSE = e\{e_k^2\} = b^T R_{xx} b - b^T R_{xy} w - w^T R_{yx} b + w^T R_{xx} w \quad (2.4)$$

where $R_{xx} = e\{x_k x_k^T\}, R_{xy} = e\{x_k y_k^T\}, R_{yx} = e\{y_k x_k^T\}, R_{yy} = e\{y_k y_k^T\}$.

Taking the gradient with respect to ***w*** and setting it to zero yields

$$b^T R_{xy} = w^T R_{yy} \quad (2.5)$$

Substituting then in (2.4) we obtain:

$$MSE = b^T [R_{xx} - R_{xy} R_{yy}^{-1} R_{yx}] b = b^T R_{x|y} b \quad (2.6)$$

Define

$$S = [0_{(v+1)x\Delta} \quad I_{(v+1)x(v+1)} \quad 0_{(v+1)x(N_w+L-\Delta-v-1)}]^T \quad (2.7)$$

where 0_{mxn} is a $m \times n$ matrix of zeros, I_{nxn} is an $n \times n$ identity matrix, and ? + 1 is the number of elements in ***b***. By defining

$$R_\Delta = S^T R_{x|y} S \quad (2.8)$$

the MSE can be written as

$$MSE = b^T R_\Delta b$$

To obtain a non-trivial solution (i.e. when vector ***w*** and ***b*** are non-zero) it is necessary to introduce limiting constraints. To get the unit-energy constraint ($b^T b = 1$) solution, the minimum is then calculated as follows:

$$L_{MSE}(b, l) = b^T R_\Delta b + l(b^T b - 1) \quad (2.9)$$

Taking the gradient with respect to b and setting it to zero, we obtain

$$R_\Delta b = l b \quad (2.10)$$

This is the eigen-problem, and b should be chosen as the eigenvector corresponding to the minimum eigenvalue of R_Δ to minimize the MSE. Thus,

$\mathbf{b_{opt}}$ = eigenvector of R_Δ corresponding to the minimum eigenvalue

Since the MMSE method in general cannot force the error to become exactly zero, some residual ISI will remain. To maximize channel capacity, the residual ISI should be placed in frequency bands with high channel noise. This ensures that the residual ISI would be small compared to the noise and the effect on the SNR would be negligible. The MMSE design method does not have a mechanism to shape the residual ISI in frequency. Therefore, it is not optimal in the sense of maximizing channel capacity.

Another drawback of the MMSE design method is the deep notches in the frequency response of the designed TEQ. The subchannels in which a notch appears cannot be used for data transmission because the gain in the subchannel is too small.

In Fig. 2.2, it shows a target impulse response (TIR) and shortened impulse response (SIR).

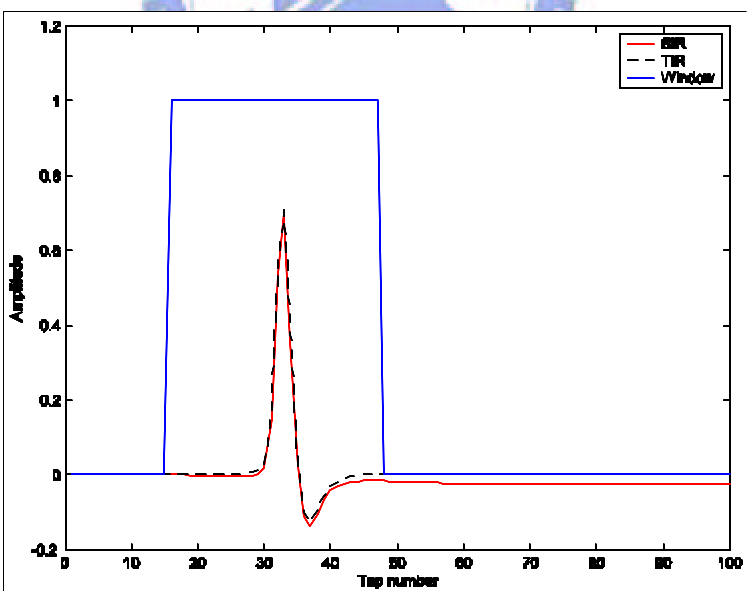

Fig. 2.2 A target impulse response (TIR) and shortened impulse response (SIR).

2.2.2 Maximum Shortening SNR (MSSNR)

Another approach to shorten the channel effect is the maximum shortening SNR method. Regardless of the choice of ***w***, it is generally not possible to shorten the impulse response perfectly. Some energy will lie outside the largest ?+1 consecutive sample of the shortening impulse response (SIR). As a result, the goal is to find a TEQ that minimizes the energy of the SIR outside the target window, while keeping the energy inside constant. They have a reasonable assumption that the channel impulse response is known. In DMT applications such as ADSL, the channel FFT coefficients are estimated for bit loading. The channel impulse response can be estimated from the FFT coefficients.

The samples of the SIR inside the target window can be written in matrix form as

$$\mathbf{h}_{win} = \begin{bmatrix} h_{\Delta+1} & h_{\Delta} & \cdots & h_{\Delta-N_w+2} \\ h_{\Delta+2} & h_{\Delta+1} & \cdots & h_{\Delta-N_w+3} \\ \vdots & & \ddots & \vdots \\ h_{\Delta+\nu+1} & h_{\Delta+\nu} & \cdots & h_{\Delta-N_w+\nu+2} \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_{N_w-1} \end{bmatrix} = \mathbf{H}_{win}\mathbf{w} \tag{2.11}$$

and the remaining samples outside the target window can be written as

$$\mathbf{h}_{wall} = \begin{bmatrix} h_0 & 0 & \cdots & 0 \\ h_1 & h_0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ h_{\Delta} & h_{\Delta-1} & \cdots & h_{\Delta-N_w+1} \\ h_{\Delta+\nu+2} & h_{\Delta+\nu+1} & \cdots & h_{\Delta-N_w+\nu+3} \\ \vdots & \vdots & & \vdots \\ h_{L-1} & h_{L-2} & \cdots & h_{L-Nw+1} \\ 0 & h_{L-1} & \cdots & h_{L-Nw+2} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & h_{L-1} \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_{N_w-1} \end{bmatrix} = \mathbf{H}_{wall}\mathbf{w} \tag{2.12}$$

Optimal shortening can be expressed as choosing ***w*** to minimize $h_{wall}^{T}h_{wall}$ while satisfying the constraint: $h_{win}^T h_{win} = 1$. Constraining the energy in the window ensures that the trivial solution is disallowed. The expression for the energy outside and inside the window can be written as

$$h_{win}^T h_{win} = w^T H_{win}^T H_{win} w = w^T B w \tag{2.13}$$

$$h_{wall}^T h_{wall} = w^T H_{wall}^T H_{wall} w = w^T A w \tag{2.14}$$

Optimal shortening can be considered as choosing w to minimize $w^T A w$ while satisfying a constraint of $w^T B w = 1$.

The following development assumes that B is invertible. The solution when B is singular is more complicated. The matrix B also has to be positive definite in order to have a Cholesky decomposition.

$$\begin{aligned} B &= Q\Lambda Q^T = (Q\sqrt{\Lambda})(\sqrt{\Lambda}Q^T) \\ &= (Q\sqrt{\Lambda})(\sqrt{\Lambda}Q)^T = \sqrt{B}\sqrt{B}^T \end{aligned} \tag{2.15}$$

where Λ is a diagonal matrix formed from the eigenvalues of B, and Q are the orthonormal eigenvectors.

To satisfy the constraint, it defines

$$y = \sqrt{B}^T w \tag{2.16}$$

so that

$$y^T y = w^T \sqrt{B}\sqrt{B}^T w = w^T B w = 1 \tag{2.17}$$

From (2.16) we can derive that

$$w^T A w = y^T (\sqrt{B})^{-1} A (\sqrt{B})^{-1} y = y^T C y \tag{2.18}$$

It can transfer the problem to

$$\min_y \; y^T C y \quad \text{s.t.} \quad y^T y = 1 \tag{2.19}$$

The solution to this problem is when y equals to the eigenvector $l_{\min}$ corresponding to the minimum eigenvalue $l_{\min}$ of C.

We can obtain the resulting coefficient as

$$w_{opt} = (\sqrt{B}^T)^{-1} l_{\min} \tag{2.20}$$

The shortening SNR we want to maximize can be expressed as

$$SSNR = \frac{w^T Bw}{w^T Aw} \tag{2.21}$$

The MSSNR method minimizes the part of the SIR that causes ISI. If the energy outside the target window were zero, then the channel would be perfectly shortened and ISI would be totally eliminated. The solution which gives zero energy outside the target window is optimum also in the sense of maximum channel capacity since this is the case where ISI is totally canceled. However, the problem with the MSSNR design approach is the computation complexity due to the eigenvalue and Cholesky decompositions. Besides, in practice this optimum solution cannot be achieved. For this case, the MSSNR solution is not guaranteed to yield maximum channel capacity solution.

2.2.3 Maximum Geometric SNR (MGSNR) [8]

Although the mean square error (MSE) is the most popular equalization criterion since it is easy to analyze and it is a simple adaptive implementation, we will argue next that it is not the optimum equalization criterion in conjunction with the DMT system.

If we assume that all subchannels in the DMT system are equally spaced, memoryless, and independent. The total bits transmitted in one DMT symbol is defined by

$$b_{DMT} = \sum_{i=1}^{\bar{N}} b_i$$

$$= \sum_{i=1}^{\bar{N}} \log_2 (1 + \frac{SNR_i}{\Gamma_i}) \qquad (2.22)$$

where SNR_i is the signal-to-noise ratio of the ith tone and it is given by $SNR_i = \frac{S_{x,i}|H_i|^2}{R_{nn,i}}$ where $S_{x,i}$, $|H_i|^2$ and $R_{nn,i}$, are the input energy, channel gain of ith channel, and noise power spectral density, respectively. In addition, we shall assume a flat input energy distribution across the subchannels, in which case $SNR_i = \frac{S_x|H_i|^2}{R_{nn,i}}$.

Equation (2.22) can be expressed as follows

$$b_{DMT} = \bar{N} \log_2 (1 + \frac{SNR_{geom}}{\Gamma}) \qquad (2.23)$$

where the geometric SNR is defined by

$$SNR_{geom} = \Gamma \left\{ \left[\prod_{i=1}^{\bar{N}} (1 + \frac{SNR_i}{\Gamma}) \right]^{\frac{1}{\bar{N}}} - 1 \right\} \qquad (2.24)$$

This means that all of the subchannels act together like $\bar{N} = \frac{N}{2}$ AWGN channels, with each channel having an SNR equal to the SNR_{gemo}. Therefore, maximizing the SNR_{gemo} is equivalent to maximizing the channel capacity.

Furthermore, the "1+" and "-1" terms can be typically ignored, so that we can simplify equation (2.24) to

$$SNR_{geom} \approx \left[\prod_{i=1}^{\bar{N}} (SNR_i) \right]^{\frac{1}{\bar{N}}} \qquad (2.25)$$

This expression makes the name "geometric SNR" obvious. Besides, this approximation is valid if the SNR in each subchannel is larger than one, so that the “1” terms can be ignored. This assumption may be reasonable only if bandwidth optimization is used. That is, the channels without sufficient SNR to carry bits are not used.

For the equalized DMT, the geometric SNR is approximately given by

$$SNR_{gemo} \approx S_x \left[\prod_{i=1}^{\bar{N}} \left(\frac{|B_i|^2}{S_{n,i}|W_i|^2} \right) \right]^{\frac{1}{\bar{N}}} \tag{2.26}$$

where S_x is signal power, $S_{n,i}$ is the noise power, and B_i and W_i are the FFT coefficients of the TIR and the TEQ in the ith subchannel, respectively. In this case, the problem of maximizing $SNR_{geom.}$ can be converted to the maximization of

$$L(b) \overset{def}{=} \frac{1}{\bar{N}} \sum_{i=1}^{\bar{N}} \ln |B_i|^2 \tag{2.27}$$

This cost function also assumes that the noise at the output of the equalizer is independent of, which is not accurate, nevertheless, is simplilifies the analysis considerably.

Now

$$B_i \overset{def}{=} \sum_{k=0}^{N_b} b_k^* e^{-j\frac{2p}{N}ik}$$

$$\overset{def}{=} \mathrm{b}^* g_i^{(N_b+1)} \tag{2.28}$$

where $g_i^{(N_b+1)} \overset{def}{=} \left[1 \;\; e^{-j\frac{2p}{N}i} \;\; \cdots \;\; \mathrm{e}^{-j\frac{2p}{N}iN_b} \right]$

The assumption that b and w do not depend on each other is not accurate because once b_{opt} is calculated by maximizing (2.27), the optimum (in the MMSE sense) TIR w_{opt} is found using

$$w_{opt}^T = b_{opt}^T R_{xy} R_{yy}^{-1} \tag{2.29}$$

where R_{xy} and R_{yy} are the channel input-output cross-correlation and channel output autocorrelation matrices, respectively. This choice of TEQ taps ensures that the MSE is the minimum for the given TIR.

To maximize (2.27), a unit-energy constraint is placed on b to prevent an infinite-gain TIR. This constraint maximizes the cost function for $|B_i|^2 = 1 \ \forall i$, which implies a zero forcing equalization of the channel. This requires "full" equalization of the channel impulse response which could result in a large equalization MSE (especially for short TEQ lengths).

Therefore, we shall add a constraint for the MSE of the TEQ to remain below a threshold value, call it MSE_{max}. This threshold has to be tuned if the channel, noise level, or signal power changes. Including the above constraints, Al-Dhahir and Cioffi state the optimum TIR problem as

$$\max_{b} \sum_{i=1}^{\bar{N}} \ln|B_i|^2 + l\,(b^* b - 1) \quad s.t. \quad b^T R_\Delta b \le MSE_{\max} \tag{2.30}$$

where

$$\mathbf{R}_\Delta = [\mathbf{0}_{(\nu+1)\times\Delta}\ \mathbf{I}_{\nu+1}\ \mathbf{0}_{(\nu+1)\times P}] \left(\frac{1}{S_x}\mathbf{I}_{N_w+L-1} + \mathbf{H}^T\mathbf{R}_{nn}^{-1}\mathbf{H}\right)^{-1} \begin{bmatrix} \mathbf{0}_{\Delta\times(\nu+1)} \\ \mathbf{I}_{\nu+1} \\ \mathbf{0}_{P\times(\nu+1)} \end{bmatrix}$$

Here, P = N_w + L -? -? -2, $\mathbf{0}_{mxn}$ is an $m\times n$ matrix of zeros, $\mathbf{I}_m$ is the $m\times m$ identity matrix, S_x is the average energy of the input symbols, $\mathbf{R}_{nn}$ is the $N_w \times N_w$ noise correlation matrix, and **H** is the $N_w \times (N_w + L - 1)$ channel convolution matrix. This is a nonlinear constrained optimization problem which does not have a closed-form solution for b. But it may be solved by numerical methods [9]. We can conclude that the drawbacks of the MGSNR TEQ method are that its derivation is based on a subchannel SNR definition SNR_i that does not include the effect of ISI and it requires a constrained nonlinear optimization method. In addition, it depends on the parameter $MSE_{\max}$ which has to be tuned for different channels.

2.2.4 Maximum Bit Rate Design (MBR)

The problem one comes across in optimal TEQ design is the lack of a mathematical foundation of the effect of a TEQ on channel capacity. Ideally, one would like to have the channel capacity as a function of the TEQ taps. The only parameter of channel capacity that might be affected by a TEQ is the SNR in each subchannel. The formal definition of SNR as the ratio of signal power to channel noise power does not provide a relationship between the SNR and TEQ taps since both the signal power and channel noise power are filtered by the same filter. However, in this paper [10], the authors propose a new definition of subchannel SNR. The idea is that it separates the received samples into three parts. The received samples consist of a desired signal, channel noise, and ISI. They define SNR as the ratio of the desired signal power to the channel noise plus ISI power.

There is an example in the paper, we restate it as follows:

It assumes that there is a DMT system with an FFT size of N = 4, and a cyclic prefix length of ? = 1. If we transmit two DMT symbols $a = [a_1\ a_2\ a_3\ a_4]$ and $b = [b_1\ b_2\ b_3\ b_4]$ over an equalized channel with impulse response $\tilde{h} = h * w$, The length of the equalized channel $\tilde{h} = \left[\tilde{h}_1\ \tilde{h}_2\ \tilde{h}_3\ \tilde{h}_4\right]$ is four, and we assume its delay to be ? = 1. As the result of the length of the equalized channel is longer than ? + 1, ISI will occur. Adding the cyclic prefix, the symbol becomes $\hat{a} = [a_4\ a_1\ a_2\ a_3\ a_4]$ and $\hat{b} = [b_4\ b_1\ b_2\ b_3\ b_4]$, which form the transmit sequence $x = \left[\hat{a}\ \hat{b}\right]$.

The received signal $y = x * \tilde{h} + \tilde{n}$ can be expressed as

$$
\begin{array}{r}
\text{delay} \rightarrow \\
\text{CP} \rightarrow \\
\\
\\
\\
\\
\text{CP} \rightarrow \\
\\
\\
\\
\\
\text{tail} \rightarrow \\
\\
\end{array}
\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \\ y_9 \\ y_{10} \\ y_{11} \\ y_{12} \\ y_{13} \end{bmatrix}
=
\begin{bmatrix}
\tilde{h}_1 a_4 & & & & & & \\
\tilde{h}_1 a_1 & + & \tilde{h}_2 a_4 & & & & \\
\tilde{h}_1 a_2 & + & \tilde{h}_2 a_1 & + & \tilde{h}_3 a_4 & & \\
\tilde{h}_1 a_3 & + & \tilde{h}_2 a_2 & + & \tilde{h}_3 a_1 & + & \tilde{h}_4 a_4 \\
\tilde{h}_1 a_4 & + & \tilde{h}_2 a_3 & + & \tilde{h}_3 a_2 & + & \tilde{h}_4 a_1 \\
\tilde{h}_1 b_4 & + & \tilde{h}_2 a_4 & + & \tilde{h}_3 a_3 & + & \tilde{h}_4 a_2 \\
\tilde{h}_1 b_1 & + & \tilde{h}_2 b_4 & + & \tilde{h}_3 a_4 & + & \tilde{h}_4 a_3 \\
\tilde{h}_1 b_2 & + & \tilde{h}_2 b_1 & + & \tilde{h}_3 b_4 & + & \tilde{h}_4 a_4 \\
\tilde{h}_1 b_3 & + & \tilde{h}_2 b_2 & + & \tilde{h}_3 b_1 & + & \tilde{h}_4 b_4 \\
\tilde{h}_1 b_4 & + & \tilde{h}_2 b_3 & + & \tilde{h}_3 b_2 & + & \tilde{h}_4 b_1 \\
 & & \tilde{h}_2 b_4 & + & \tilde{h}_3 b_3 & + & \tilde{h}_4 b_2 \\
 & & & & \tilde{h}_3 b_4 & + & \tilde{h}_4 b_3 \\
 & & \text{desired part} & & & & \tilde{h}_4 b_4
\end{bmatrix}
+
\begin{bmatrix} \tilde{n}_1 \\ \tilde{n}_2 \\ \tilde{n}_3 \\ \tilde{n}_4 \\ \tilde{n}_5 \\ \tilde{n}_6 \\ \tilde{n}_7 \\ \tilde{n}_8 \\ \tilde{n}_9 \\ \tilde{n}_{10} \\ \tilde{n}_{11} \\ \tilde{n}_{12} \\ \tilde{n}_{13} \end{bmatrix}
\qquad (2.31)
$$

where $\tilde{n}$ is the additive channel noise at the output of the equalizer.

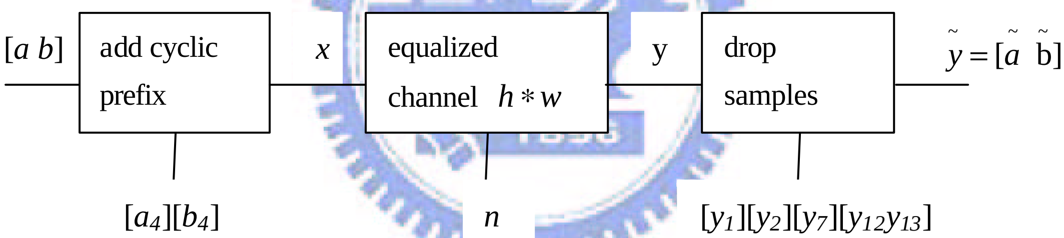

Fig. 2.3 Example: The two DMT symbols a and b are transmitted over an equalized channel. After removing the invalid samples and prefixes, the two received symbols are $\tilde{a}$ and $\tilde{b}$.

We would like to investigate each element of y in the equation (2.31); at first, y_1 is invalid because the equalized channel has a delay of one. The y_2 is a cyclic prefix and is dropped. From the y_3 to y_6 is the samples correspond to the first received DMT symbol $\tilde{a}$. The y_7 is a cyclic prefix and is ignored. From the y_8 to y_{11} is the samples correspond to the first received DMT symbol $\tilde{b}$. From the y_{12} to y_{13} is the interference caused by the duration of the channel impulse response. If we want to demodulate the received DMT symbols correctly, the length of the channel should be at most ? + 1=2. Because the channel impulse response length in this example is four, the received symbols have an ISI component in addition to the desired signal component and the noise component

$$y = x * \tilde{h}^{signal} + x * \tilde{h}^{ISI} + \tilde{n} \tag{2.32}$$

where $\tilde{h}^{signal}$ is the equivalent signal path impulse response, and $\tilde{h}^{ISI}$ is the equivalent ISI path impulse response. “The desired signal component” can be written as follows:

$$x * \tilde{h}^{signal} = x * \begin{bmatrix} 0 & \tilde{h}_2 & \tilde{h}_3 & 0 \end{bmatrix} \tag{2.33}$$

where

$$\tilde{h}^{signal} = \tilde{h} \otimes g = \begin{bmatrix} \tilde{h}_1 & \tilde{h}_2 & \tilde{h}_3 & \tilde{h}_4 \end{bmatrix} \otimes [0 \ 1 \ 1 \ 0]$$
$$= \begin{bmatrix} 0 & \tilde{h}_2 & \tilde{h}_3 & 0 \end{bmatrix}$$

The symbol “$\otimes$” means element by element multiplication, and g is a window function.

“The ISI component”: This part is due to the extra nonzero taps in the channel impulse response:

$$x * \tilde{h}^{ISI} = x * \begin{bmatrix} \tilde{h}_1 & 0 & 0 & \tilde{h}_4 \end{bmatrix} \tag{2.34}$$

where

$$\tilde{h}^{ISI} = \tilde{h} \otimes (1 - g) = \begin{bmatrix} \tilde{h}_1 & \tilde{h}_2 & \tilde{h}_3 & \tilde{h}_4 \end{bmatrix} \otimes [1 \ 0 \ 0 \ 1]$$
$$= \begin{bmatrix} \tilde{h}_1 & 0 & 0 & \tilde{h}_4 \end{bmatrix}$$

“The output noise component”: The channel noise is only filtered by the equalizer, and it can be expressed as follows:

$$\tilde{n} = w * n = h^{noise} * n \tag{2.35}$$

We would like to generalize the previous example so that we can partition the received signal into the desired signal, ISI, and noise components. In the paper, h_k and w_k are channel impulse response and TEQ, respectively, and g_k is a target window .That is

$$g_k = \begin{cases} 1, & \Delta + 1 \le \mathrm{k} \le \Delta + n + 1 \\ 0, & \text{otherwise} \end{cases}$$

If $\tilde{h}_k = h_k * w_k$, then the equivalent signal path impulse response and ISI path impulse response can be represented as

$$h_k^{signal} = \tilde{h}_k\, g_k$$

$$h_k^{ISI} = \tilde{h}_k (1 - g_k) \tag{2.36}$$

Since the received signal consists of three components, they give a new definition of subchannel SNR. That is

$$\mathrm{SNR} = \frac{\text{siganl power}}{\text{noise power} + \text{ISI power}}$$

Using the equivalent path definitions, they define a new subchannel SNR (SNR_i^{NEW}) to incorporate both types of distortion as

$$SNR_i^{NEW} = \frac{S_{x,i}\left|H_i^{signal}\right|^2}{S_{n,i}\left|H_i^{noise}\right|^2 + S_{x,i}\left|H_i^{ISI}\right|^2} \tag{2.37}$$

where $S_{x,i}$, $S_{n,i}$, H_i^{signal}, H_i^{noise}, H_i^{ISI} are the transmitted signal power, channel noise power (before the equalizer), signal path gain, noise path gain, and the ISI path gain in the ith subchannel, respectively. When the channel is perfectly equalized to the desired length, the ISI path impulse response is equal to zero. In this case,

$$h_k^{signal} = \tilde{h}_k * w_k \rightarrow H_i^{signal} = W_i H_i$$

$$h_k^{noise} = w_k \rightarrow H_i^{noise} = W_i$$

$$h_k^{ISI} = 0 \rightarrow H_i^{ISI} = 0$$

The subchannel SNR ($SNR_i^{No\ ISI}$) can be represented as

$$SNR_i^{No\ ISI} = \frac{S_{x,i}\left|W_i\right|^2\left|H_i\right|^2}{S_{n,i}\left|W_i\right|^2} = \frac{S_{x,i}\left|H_i\right|^2}{S_{n,i}} \tag{2.38}$$

This is the matched filter bound we want to achieve. This is expected since the SNR should be a maximum when there is no ISI. To write the achievable channel capacity in terms of the TEQ tap values, they derive the subchannel SNRs as a function of the TEQ taps. The equivalent signal, ISI, and noise path impulse responses can be rewritten in matrix form as

$$h^{signal} = \mathrm{GH}w$$

$$h^{ISI} = \mathrm{DH}w$$

$$h^{noise} = \mathrm{F}w \tag{2.39}$$

where h^{signal}, h^{ISI}, and h^{noise} are length-N vectors representing the equivalent signal, ISI, and noise path impulse responses, respectively. The $N \times N_w$ matrix H is defined as the first N rows of the convolution matrix of the channel. The G and D are $N \times N$ diagonal matrices representing the window function g_k and $1-g_k$, respectively, which are defined as follows:

$$\mathbf{H} = \begin{bmatrix} h_0 & 0 & 0 & \cdots & 0 \\ h_1 & h_0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{N_w-1} & h_{N_w-2} & h_{N_w-3} & \cdots & h_0 \\ h_{N_w} & h_{N_w-1} & h_{N_w-2} & \cdots & h_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{N-1} & h_{N-2} & h_{N-3} & \cdots & h_{N-N_w} \end{bmatrix},$$

$$\mathbf{G} = \mathrm{diag}(\overbrace{\underbrace{0,\cdots,0}_{\Delta\,\text{zeros}},\underbrace{1,\cdots,1}_{\nu+1\,\text{ones}},0,\cdots,0}^{N\,\text{elements}})$$

and

$$\mathbf{D} = \mathrm{diag}(\overbrace{\underbrace{1,\cdots,1}_{\Delta\,\text{ones}},\underbrace{0,\cdots,0}_{\nu+1\,\text{zeros}},1,\cdots,1}^{N\,\text{elements}})$$

The F is a matrix with dimension $N \times N_w$ and is represented as

$$\mathbf{F} = \begin{bmatrix} \mathbf{I}_{N_w \times N_w} \\ \mathbf{0}_{(N-N_w)\times N_w} \end{bmatrix}$$

where **I** is the identity matrix and **0** is a matrix consists of zeros. The author also defines the FFT vector as

$$q_i = [1 \;\; e^{j2\pi i/N} \;\; e^{j2\pi 2i/N} \;\; \cdots \;\; e^{j2\pi(N-1)i/N}] \tag{2.40}$$

As a result, we can get the ith FFT coefficient of vector from the inner product of q_i^H and a N-point vector. Since (2.40), we can rewrite equation (2.39) as follows:

$$H_i^{signal} = q_i^H \mathrm{GH}w$$

$$H_i^{ISI} = q_i^H \mathrm{DH}w$$

$$H_i^{noise} = q_i^H \mathrm{F}w \tag{2.41}$$

Eventually, substituting (2.41) into (2.37) will get the result as

$$SNR_i^{NEW} = \frac{S_{x,i}\left|q_i^H \mathrm{GH}w\right|^2}{S_{n,i}\left|q_i^H \mathrm{F}w\right|^2 + S_{x,i}\left|q_i^H \mathrm{DH}w\right|^2} \tag{2.42}$$

We can observe that this new definition includes the ISI and a TEQ.

To obtain the optimal TEQ which maximizes b_{DMT} given by (2.22), they expand the absolute value quantities in (2.42) as

$$SNR_i^{NEW} = \frac{w^T \mathrm{H}^\mathrm{T} \mathrm{G}^\mathrm{T} q_i S_{x,i} q_i^H \mathrm{GH}w}{w^T \mathrm{F}^\mathrm{T} q_i S_{n,i} q_i^H \mathrm{F}w + w^T \mathrm{H}^\mathrm{T} \mathrm{D}^\mathrm{T} q_i S_{x,i} q_i^H \mathrm{DH}w}$$

(2.43)

$$= \frac{w^T A_i w}{w^T B_i w} \tag{2.44}$$

where

$$A_i = \mathrm{H}^\mathrm{T} \mathrm{G}^\mathrm{T} q_i S_{x,i} q_i^H \mathrm{GH}$$

$$B_i = \mathrm{F}^\mathrm{T} q_i S_{n,i} q_i^H \mathrm{F} + \mathrm{H}^\mathrm{T} \mathrm{D}^\mathrm{T} q_i S_{x,i} q_i^H \mathrm{DH}$$

By substituting (2.44) into (2.22), we can obtain

$$b_{DMT} = \sum_{i=1}^{\bar{N}} \log_2\left(1 + \frac{1}{\Gamma}\frac{w^T A_i w}{w^T B_i w}\right) \quad \text{bits/symbol} \tag{2.45}$$

which gives the achievable capacity as a function of the TEQ taps w. This is a nonlinear optimization problem like the geometric TEQ method. We can conclude that the MBR TEQ includes the effect of ISI as part of the proposed subchannel SNR model and it does not make unrealistic assumptions to obtain the achievable capacity as a function of equalizer taps. Besides, we can obtain the TEQ taps directly from the optimization, unlike the geometric TEQ method which calculates the equalizer by using (2.29) after the TIR is obtained from the optimization.

2.2.5 Minimum Inter-symbol Interference (Min-ISI)

The previous method MBR TEQ is optimum in the sense of channel capacity. However, a nonlinear optimization method is required to calculate the optimum TEQ which makes the MBR design method impractical for a low-cost real-time implementation. Therefore, there is a near-optimum design method. It declares that it will reach about 99% of the channel capacity of the optimum method but does not require a nonlinear optimization. Therefore, the authors develop fast algorithms for the minimum-ISI method. The minimum-ISI [10][11] method is based on the observation that the only effect that TEQ has on channel capacity is the way it distributes ISI power over frequency. Minimizing the sum of the ISI power over all of the subchannels would reduce ISI but does not optimize the distribution of ISI power over frequency. In high noise regions, ISI is dominated by the noise and its effect on SNR can be ignored. If the same amount of ISI were placed in low noise frequency bands, then ISI would be reduced dramatically. The capacity of a discrete time multitone system is the sum of capacities of the AWGN subchannels. The capacity of AWGN channel is a logarithm function of its SNR. As a result, the capacity of the multicarrier channel is a sum of logarithms which is a nonlinear function. To avoid nonlinearity, hence nonlinear optimization, they avoid using capacity as the objective function.

The idea behind the min-ISI method can be explained from (2.43). Both the numerator and the denominator of (2.43) are power terms. Since a power term is always non-negative, minimizing the distortion power in each subchannel is equivalent to minimizing the sum of the distortion powers over all subchannels

$$p_d(w) = \sum_{i \in S} (w^T \mathrm{F}^\mathrm{T} q_i S_{n,i} q_i^H \mathrm{F} w + w^T \mathrm{H}^\mathrm{T} \mathrm{D}^\mathrm{T} q_i S_{x,i} q_i^H \mathrm{DH} w)$$

Now, we normalize $P_d(w)$ by $S_{n,i}$ and we will get

$$p_d^{norm}(w) = \sum_{i \in S} (w^T \mathrm{F}^{\mathrm{T}} q_i q_i^H \mathrm{F} w) + \sum_{i \in S} (w^T \mathrm{H}^{\mathrm{T}} \mathrm{D}^{\mathrm{T}} q_i \frac{S_{x,i}}{S_{n,i}} q_i^H \mathrm{DH} w) \quad (2.46)$$

where S is the set of used subchannels and $q_i^H \mathrm{F} w$ is the ith N-point FFT coefficient of w. According to Parseval's theorem, the square sum of the N-point FFT coefficients of w is equal to the square sum of the coefficient of w.

$$p_d^{norm}(w) = w^T w + w^T \mathrm{H}^T \mathrm{D}^{\mathrm{T}} \sum_{i \in S} (q_i \frac{S_{x,i}}{S_{n,i}} q_i^H) \mathrm{DH} w \quad (2.47)$$

We can observe the first term does not affect the minimization of (2.47) for a constant norm w (the optimal can always be scaled to force). Therefore, we can minimize the second directly. While minimizing the distortion power, a constraint is required to prevent the minimization of the signal power as well. So, we define the TEQ design problem as

$$\arg\min_w \left(w^T \mathrm{H}^T \mathrm{D}^{\mathrm{T}} \sum_{i \in S} (q_i \frac{S_{x,i}}{S_{n,i}} q_i^H) \mathrm{DH} w \right) \quad (2.48)$$

By the constraint

$$\left\| h^{signal} \right\|^2 = w^T \mathrm{H}^{\mathrm{T}} \mathrm{G}^{\mathrm{T}} \mathrm{GH} w = w^T Y w = 1$$

This ensures that the output signal power is equal to the input signal power. Eventually, the optimization problem for minimum ISI becomes

$$\arg\min_w \left(w^T X w \right) \quad \text{s.t.} \quad w^T Y w = 1 \quad (2.49)$$

where

$$X = \mathrm{H}^T \mathrm{D}^{\mathrm{T}} \sum_{i \in S} (q_i \frac{S_{x,i}}{S_{n,i}} q_i^H) \mathrm{DH}$$

In (2.48), the weighting by $\frac{S_{x,i}}{S_{n,i}}$ amplifies the objective function (which measures the ISI) in the subchannels with low noise power (high SNR). A small amount of ISI power in subchannels with low noise power can reduce the SNR in that subchannel dramatically, which in turn would reduce the bit rate. However, in subchannels with low SNR, the noise power is large enough to dominate the ISI power; hence, the effect of the ISI power on the SNR is negligible.