Technical Notes and Correspondence

Matrix Partial Fraction Expansions of Rational Matrices

YEN-LEI LING AND BOR-CHYUN WANG

Abstract—**A general and rigorous derivation is made of a new formula, established recently in the literature, for matrix partial fraction expansion of a rational matrix. The procedure, by use of a minimal Jordan realization of the rational matrix, provides as a byproduct, general expressions for residue matrices in terms of products of columns of the output matrix and rows of the input matrix of the realization.**

List of Symbols and Abbreviations

A', A^{-1}	transpose and inverse of matrix A.
$A \otimes B$	Kronecker product of matrix A and matrix B.
$D_j(\lambda^k)$	$(1)/(j!)(d^j\lambda^k)/(d\lambda^j)$, $D_0(\lambda^k) = \lambda^k$.
deg $G(s)$	degree of characteristic polynomial of rational $G(s)$.
F	complex field C or real field R.
h_j	Markov's parameters CA^jB.
I_n	$n \times n$ unit matrix.
$K^{m\times n}$	ring of $m \times n$ matrices over K, K is F, $R(s)$ or $R[s]$.
K^n	vector space of $n \times 1$ column vectors with elements in K.
θ_n, $\theta_{m\times n}$	zero n-vector and zero $m \times n$ matrix.
R	real field.
PFE	partial fraction expansion.
$R(s)$	field of rational functions in s with coefficients in R.
$R[s]$	ring of polynomials in s with coefficients in R.
$S(d)$	Stanley matrix associated with polynomial $d(s)$.

I. Introduction

Matrix PFE is frequently used in linear system theory, e.g., to obtain inverse Laplace and Z-transforms, and in state-space realizations of rational matrices. The problem of matrix PFE is phrased as follows.

Given a strictly proper rational matrix $G(s) \in R(s)^{q\times p}$

$$G(s)=N(s)/d(s),\ N(s) \in R[s]^{q\times p} \tag{1}$$

where

$$N(s)=N_1s^{t-1}+N_2s^{t-2}+\cdots+N_{t-1}s+N_t \tag{2}$$

is the numerator matrix,

$$d(s)=s^t+d_1s^{t-1}+\cdots+d_{t-1}s+d_t \tag{3a}$$

$$=\prod_{i=1}^{m}(s-\lambda_i)^{t_i} \tag{3b}$$

is the denominator polynomial, and

$$d_i \in R,\ N_i \in R^{q\times p},\ i=1,\cdots,t,\ \lambda_j \in F, j=1,\cdots,m,\ \sum_{i=1}^{m} t_i=t; \tag{4}$$

find the residue matrices $R_{ik} \in F^{q\times p}$, $i = 1, \cdots, m$, $k = 1, \cdots, t_i$, in terms of the coefficient matrices N_i, $i = 1, \cdots, t$, such that

$$G(s)=\sum_{i=1}^{m}\sum_{k=1}^{t_i}\frac{R_{ik}}{(s-\lambda_i)^k}. \tag{5}$$

The *fundamental* method for matrix PFE is the same as that used in PFE of rational functions, i.e., by assuming residues and equating corresponding terms. Other methods available in the literature are based on Lagrange interpolation [1], [2] or Taylor series expansion [3, p. 309]. But they are complicated for large dimension and high-order systems. Recently, as the solution of the matrix PFE problem, a new formula, namely

$$[R'_{11}R'_{12}\cdots R'_{1t_1}R'_{21}R'_{22}\cdots R'_{2t_2}\cdots R'_{m1}\cdots R'_{mt_m}]' = [(S(d)\,W)^{-1}\otimes I_q][N'_1N'_2\cdots N'_t]' \tag{6}$$

has been given [4], [5], where

$$S(d)=\begin{bmatrix} 1 & & & & \bigcirc \\ d_1 & 1 & & & \\ d_2 & d_1 & 1 & & \\ \cdots & & & & \\ d_{t-1} & d_{t-2} & \cdots & d_1 & 1 \end{bmatrix} \tag{7}$$

is the $t \times t$ Stanley matrix [6] associated with $d(s)$,

$$\begin{aligned} W=[&v(\lambda_1)\ \ D_1(v(\lambda_1))\ \ D_2(v(\lambda_1))\ \cdots\ D_{t_1-1}\ (v(\lambda_1))\\ &\cdot v(\lambda_2)\ \ D_1(v(\lambda_2))\ \ D_2(v(\lambda_2))\ \cdots\ D_{t_2-1}\ (v(\lambda_2))\ \cdots\\ &\cdot v(\lambda_m)\ \ D_1(v(\lambda_m))\ \ D_2(v(\lambda_m))\ \cdots\ D_{t_m-1}\ (v(\lambda_m))] \end{aligned} \tag{8}$$

is the $t \times t$ generalized Vandermonde matrix, and

$$v(\lambda)=[1\ \ \lambda\ \ \lambda^2\ \ \cdots\ \ \lambda^{t-1}]', \tag{9}$$

$$D_j(\lambda^\alpha)=\frac{1}{j!}\frac{d^j\,\lambda^\alpha}{d\,\lambda^j}. \tag{10}$$

The derivation in [4], [5] is not in general form and is thus less rigorous. Instead, it is illustrated by simple examples, and is basically the same as the *fundamental* method.

A more rigorous derivation in a general form of formula (6) is presented in this note. The method is based on the fact that the residue matrices can be expressed in terms of products of columns of the output matrix and rows of the input matrix of the minimal Jordan realization of $G(s)$.

II. A Basic Formula

In (1), $G(s)$ is assumed irreducible, i.e., $d(s)$ and every $N_{ij}(s)$ is relatively prime. Suppose $\{A, B, C\}$ is the minimal Jordan realization [7,

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TABLE I
MINIMAL JORDAN REALIZATION $\{A, B, C\}$ OF $G(s)$

$(n\times n)A=\mathrm{diag}\,(A_1\ A_2\ \cdots\ A_m)$,

$(n_i\times n_i)A_i=\mathrm{diag}\,(A_{i1}\ A_{i2}\ \cdots\ A_{ir_i})$,

$r_i=$ number of Jordan blocks associated with λ_i,

$$\underset{(n_{ij}\times n_{ij})}{A_{ij}}=\begin{bmatrix}\lambda_i & 1 & & \bigcirc\\ & \lambda_i & 1 & \\ \bigcirc & & \ddots & \\ & & & \lambda_i\end{bmatrix},$$

$$\underset{(n\times p)}{B}=\begin{bmatrix}B_1\\ B_2\\ \cdot\\ \cdot\\ \cdot\\ B_m\end{bmatrix},\quad \underset{(n_i\times p)}{B_i}=\begin{bmatrix}B_{i1}\\ B_{i2}\\ \cdot\\ \cdot\\ \cdot\\ B_{ir_i}\end{bmatrix},\quad \underset{(n_{ij}\times p)}{B_{ij}}=\begin{bmatrix}b'_{ij,1}\\ b'_{ij,2}\\ \cdot\\ \cdot\\ \cdot\\ b'_{ij,n_{ij}}\end{bmatrix},$$

$(q\times n)C=[C_1\ C_2\ \cdots\ C_m]$,

$(q\times n_i)\ C_i=[C_{i1}\ C_{i2}\ \cdots\ C_{ir_i}]$,

$(q\times n_{ij})C_{ij}=[c_{ij,1}\ c_{ij,2}\ \cdots\ c_{ij,n_{ij}}]$

p. 240] of $G(s)$; then

$$C(sI-A)^{-1}B=G(s) \tag{11}$$

where the matrices $A\in F^{n\times n}$, $B\in F^{n\times p}$, and $C\in F^{q\times n}$ are of the forms shown in Table I, and $n=\deg G(s)$.

Note that in submatrix A_i there is at least one Jordan block, say A_{i1}, of the maximum order t_i. Thus,

$$t_i=n_{i1}\geqslant n_{i2}\geqslant\cdots\geqslant n_{ir_i}. \tag{12}$$

The Jordan block A_{ij} has some useful properties

$$A_{ij}^{\alpha}=\begin{bmatrix}\lambda_i^{\alpha} & D_1(\lambda_i^{\alpha}) & D_2(\lambda_i^{\alpha}) & \cdots & D_{n_{ij}-1}(\lambda_i^{\alpha})\\ & \lambda_i^{\alpha} & D_1(\lambda_i^{\alpha}) & \cdots & D_{n_{ij}-2}(\lambda_i^{\alpha})\\ \bigcirc & & \cdots & & \lambda_i^{\alpha}\end{bmatrix}, \tag{13}$$

$$(sI_{n_{ij}}-A_{ij})^{-1}=\begin{bmatrix}\frac{1}{s-\lambda_i} & \frac{1}{(s-\lambda_i)^2} & \cdots & \frac{1}{(s-\lambda_i)^{n_{ij}}}\\ & \frac{1}{s-\lambda_i} & \cdots & \frac{1}{(s-\lambda_i)^{n_{ij}-1}}\\ \bigcirc & & \cdots & \frac{1}{s-\lambda_i}\end{bmatrix}. \tag{14}$$

Since $d(s)$ is the minimal polynomial of A, by [8, p. 325], we have

$$(sI_n-A)^{-1}=\frac{1}{d(s)}\sum_{j=1}^{t}\sum_{i=j}^{t}d_{t-i}A^{i-j}s^{j-1}. \tag{15}$$

After some algebraic manipulations and by use of the Markov parameters [9, p. 354]

$$h_j=CA^jB \tag{16}$$

and the Stanley matrix given by (7), the numerator matrix becomes

$$N_1s^{t-1}+N_2s^{t-2}+\cdots+N_{t-1}s+N_t$$

$$=\{[s^{t-1}\ \cdots\ s\ 1]\otimes I_q\}[N_1'\ N_2'\ \cdots\ N_t']' \tag{17a}$$

$$=C\sum_{j=1}^{t}\sum_{i=j}^{t}d_{t-i}A^{i-j}s^{j-1}B \tag{17b}$$

$$=\{([s^{t-1}\ \cdots\ s\ 1]S(d))\otimes I_q\}[h_0'\ h_1'\ \cdots\ h_{t-1}']'. \tag{17c}$$

We reach now a basic formula

$$[N_1'\ N_2'\ \cdots\ N_t']'=[S(d)\otimes I_q][h_0'\ h_1'\ \cdots\ h_{t-1}']'. \tag{18}$$

III. MATRIX PFE FOR GENERAL CASE OF MULTIPLE ROOTS

With (14), the residue matrices can be obtained in terms of products of columns of C and rows of B. First, note that

$$C(sI-A)^{-1}B=\sum_{i=1}^{m}C_i(sI_{n_i}-A_i)^{-1}B_i \tag{19a}$$

$$=\sum_{i=1}^{m}\sum_{j=1}^{r_i}C_{ij}(sI_{n_{ij}}-A_{ij})^{-1}B_{ij} \tag{19b}$$

$$=\sum_{i=1}^{m}\sum_{j=1}^{r_i}\sum_{k=1}^{n_{ij}}\frac{\hat{R}(ij,k)}{(s-\lambda_i)^k} \tag{19c}$$

where for $k\leqslant n_{ij}$,

$$\hat{R}(ij,k)=\sum_{\beta=1}^{n_{ij}-(k-1)}c_{ij,\beta}b'_{ij,\beta+k-1}. \tag{20}$$

By minimality of $\{A, B, C\}$ it follows that

$$\hat{R}(ij,n_{ij})=c_{ij,1}b'_{ij,n_{ij}}\neq\theta_{q\times p}. \tag{21}$$

Collecting $\hat{R}(ij,k)$'s corresponding to the same denominator and noting that $n_{ij}\leqslant t_i$, from (12), we get R_{ik} in (5)

$$R_{ik}=\sum_{j=1}^{r_i}\hat{R}(ij,k) \tag{22}$$

by setting $\hat{R}(ij,k)=\theta_{q\times p}$ for $k>n_{ij}$, rf., (14), (20).

The Markov's parameters h_α, $\alpha=0,1,2,\cdots,t-1$, in (18) can now be, by use of (13), (20), and (22), expressed in terms of R_{ik}'s and λ_i's

$$h_\alpha=\sum_{i=1}^{m}C_iA_i^{\alpha}B_i \tag{23a}$$

$$=\sum_{i=1}^{m}\sum_{j=1}^{r_i}C_{ij}A_{ij}^{\alpha}B_{ij} \tag{23b}$$

$$=\sum_{i=1}^{m}\sum_{j=1}^{r_i}\sum_{k=1}^{n_{ij}}\hat{R}(ij,k)D_{k-1}(\lambda_i^{\alpha}) \tag{23c}$$

$$=\sum_{i=1}^{m}\sum_{k=1}^{t_i}R_{ik}D_{k-1}(\lambda_i^{\alpha}) \tag{23d}$$

$$=\{[\lambda_1^{\alpha}\ D_1(\lambda_1^{\alpha})\ \cdots\ D_{t_1-1}(\lambda_1^{\alpha})\ \lambda_2^{\alpha}\ D_1(\lambda_2^{\alpha})\ \cdots\ D_{t_2-1}(\lambda_2^{\alpha})$$
$$\cdots\ \lambda_m^{\alpha}\ D_1(\lambda_m^{\alpha})\cdots\ D_{t_m-1}(\lambda_m^{\alpha})]\otimes I_q\}$$
$$\times[R_{11}'\ R_{12}'\ \cdots\ R_{1t_1}'\ R_{21}'\ R_{22}'\ \cdots\ R_{2t_2}'\ R_{m1}'\ \cdots\ R_{mt_m}']'. \tag{23e}$$

Then

$$[h_0' \; h_1' \; \cdots \; h_{t-1}']$$

$$= \begin{bmatrix} R_{11} \\ R_{12} \\ \cdot \\ \cdot \\ \cdot \\ R_{1t_1} \\ R_{21} \\ \cdot \\ \cdot \\ \cdot \\ R_{2t_2} \\ \cdot \\ \cdot \\ \cdot \\ R_m \\ \cdot \\ \cdot \\ \cdot \\ R_{mt_m} \end{bmatrix}' \left\{ \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{t-1} \\ 0 & 1 & D_1(\lambda_1^2) & \cdots & D_1(\lambda_1^{t-1}) \\ 0 & 0 & 1 & \cdots & D_2(\lambda_1^{t-1}) \\ & & & \cdots & \\ 0 & 0 & 0 & \cdots & D_{t_1-1}(\lambda_1^{t-1}) \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{t-1} \\ 0 & 1 & D_1(\lambda_2^2) & \cdots & D_1(\lambda_2^{t-1}) \\ & & & \cdots & \\ 0 & 0 & 0 & \cdots & D_{t_2-1}(\lambda_2^{t-1}) \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ 1 & \lambda_m & \lambda_m^2 & \cdots & \lambda_m^{t-1} \\ 0 & 1 & D_1(\lambda_m^2) & \cdots & D_1(\lambda_m^{t-1}) \\ & & & \cdots & \\ 0 & 0 & 0 & \cdots & D_{t_m-1}(\lambda_m^{t-1}) \end{bmatrix} \otimes I_q \right\} \tag{24a}$$

$$= [R_{11}' \; \cdots \; R_{mt_m}'](W' \otimes I_q). \tag{24b}$$

Substitute (24b) into (18), we have

$$[N_1' \; N_2' \; \cdots \; N_t']' = \{[S(d)\,W] \otimes I_q\}[R_{11}' \; \cdots \; R_{mt_m}']' \tag{25}$$

from which formula (6) results.

IV. Matrix PFE for Case of Simple Roots

In this simple case $t_i = 1$, then $n_{ij} = 1$, $m = t$, $n_i = r_i$

$$G(s) = N(s)/d(s) = \sum_{i=1}^{t} \frac{R_i}{s-\lambda_i} \tag{26}$$

where

$$d(s) = \prod_{i=1}^{t} (s-\lambda_i). \tag{27}$$

Formula (6) becomes

$$[R_1' \; R_2' \; \cdots \; R_t']' = [(S(d)\,V)^{-1} \otimes I_q][N_1' \; N_2' \; \cdots \; N_t']' \tag{28}$$

where

$$V = [v(\lambda_1) \; v(\lambda_2) \; \cdots \; v(\lambda_t)] \tag{29}$$

with $v(\lambda)$ given by (9), is the Vandermonde matrix.

As an alternative and interesting derivation, (28) can also be easily proved by considering $\{A, B, C\}$ as the Gilbert diagonal realization [9, p. 349] of $G(s)$, where

$$A = \text{diag}\,(\lambda_i I_{n_i},\; i = 1 \to t) \quad \lambda_i \in F, \tag{30}$$

$$B' = [B_1' \; B_2' \; \cdots \; B_t'],\; B_i \in F^{n_i \times p}, \tag{31}$$

$$C = [C_1 \; C_2 \; \cdots \; C_t],\; C_i \in F^{q \times n_i}, \tag{32}$$

$$\sum_{i=1}^{t} n_i = n, \tag{33}$$

$$R_i = C_i B_i; \qquad i = 1 \to t. \tag{34}$$

Note that

$$[h_0' \; h_1' \; \cdots \; h_{t-1}']' = [V \otimes I_q][B_1'C_1' \; B_2'C_2' \; \cdots \; B_t'C_t']' \tag{35a}$$

$$= [V \otimes I_q][R_1' \; R_2' \; \cdots \; R_t']'. \tag{35b}$$

Then (28) follows with (35b) substituting into (18).

V. Numerical Example [7, p. 251]

Let

$$G(s) = \begin{bmatrix} \dfrac{-s}{(s+1)^2} & \dfrac{1}{s+1} \\ \dfrac{2s+1}{s(s+1)} & \dfrac{1}{s+1} \end{bmatrix}$$

$$= \frac{1}{d(s)} \left(s^2 \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix} + s \begin{bmatrix} 0 & 1 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right)$$

$$= \frac{R_{11}}{s+1} + \frac{R_{12}}{(s+1)^2} + \frac{R_{21}}{s}$$

where $d(s) = s(s+1)^2 = s^3 + 2s^2 + s$. By (6), the residue matrices are obtained as follows:

$$\begin{bmatrix} R_{11} \\ R_{12} \\ R_{21} \end{bmatrix} = \left\{ \left(\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{array}{ccc} v(-1) & D_1[v(-1)] & v(0) \\ \downarrow & \downarrow & \downarrow \\ \end{array} \! \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & -2 & 0 \end{bmatrix} \right)^{-1} \otimes I_2 \right\}$$

$$\cdot \begin{bmatrix} -1 & 1 \\ 2 & 1 \\ 0 & 1 \\ 3 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

The matrices B and C of the minimal Jordan realization of $G(s)$ are

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{array}{l} \leftarrow b_{11,1}' \\ \leftarrow b_{11,2}' \\ \leftarrow b_{12,1}' \\ \leftarrow b_{21,1}' \end{array}, \qquad \begin{array}{c} C = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 \end{bmatrix}. \\ \begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ c_{11,1} & c_{11,2} & c_{12,1} & c_{21,1} \end{array} \end{array}$$

By (20) and (22), R_{11}, R_{12}, and R_{21} can also be expressed as:

$$R_{11} = \hat{R}(11, 1) + \hat{R}(12, 1) = c_{11,1}b_{11,1}' + c_{11,2}b_{11,2}' + c_{12,1}b_{12,1}'$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} [0 \quad 0] + \begin{bmatrix} 0 \\ 2 \end{bmatrix} [1 \quad 0] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} [-1 \quad 1] = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix},$$

$$R_{12} = \hat{R}(11, 2) + \hat{R}(12, 2)$$

$$= c_{11,1}\, b_{11,2}' + \theta_{2\times 2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} [1 \quad 0] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$R_{21} = \hat{R}(21, 1) = c_{21,1}b_{21,1}' = \begin{bmatrix} 0 \\ 1 \end{bmatrix} [1 \quad 0] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Remarks: Computations of the R_{ik}'s by (22), i.e., by summations of the products of the columns of C and the rows of B, provides an interesting double check only. Otherwise, it could be argued if there is any value in these computations, since in the Jordan realization problem [7] B and C are obtained using the result of the matrix PFE of $G(s)$.

VI. Conclusions

A completely different and rigorous derivation of a new formula for matrix partial fraction expansions of a rational matrix is presented. For the general case of multiple roots the proof is via the minimal Jordan realization of the rational matrix, while for the case of simple roots the proof would be easy via the Gilbert diagonal realization. In both cases the key steps are: 1) to express residue matrices in terms of products of columns of the output matrix and the rows of the input matrix of the realization; 2) to relate the Markov parameters with residue matrices and the roots. Study of possible applications in linear multivariable control design, such as Owen's dyadic approach [10] and modal control [11], is now under way.

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Coprime Fraction Computation of 2-D Rational Matrices

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Abstract—**This note presents a numerical method of computing a coprime fraction of a two-dimensional (2-D) rational matrix, not necessarily proper. It is achieved by searching the primary linearly dependent rows, in order from top to bottom, of the two generalized resultants. The procedure can be extended to the three- or higher dimensional case and the result can also be used to compute the greatest common divisor (GCD) of 2-D polynomial matrices without employing primitive factorizations which does not exist in the three- or higher dimensional case.**

I. Introduction

The computation of an irreducible fraction of a two-dimensional (2-D) rational function or matrix is important in the minimal implementation of digital signal processing and in the study of multidimensional system theory, and has been studied by a number of authors [1], [2]. These methods check whether or not the two polynomials are coprime and compute their greatest common divisor (GCD). They are carried out by considering $R[z_1, z_2]$ as $R[z_1][z_2]$ or $R[z_2][z_1]$. In this note, we bypass the computation of the GCD and compute directly the reduced rational matrix.

This paper is an extension of the 2-D scalar case in [3] to the 2-D matrix case. The extension, however, is highly nontrivial. Many properties which do not exist in the scalar case will appear in the matrix case. The 2-D matrix case is also drastically different from the 1-D matrix case. In this note, by coprimeness, we mean factor coprimeness [7].

Let $\boldsymbol{G}(z_1, z_2)$ be a 2-D $q \times p$ rational matrix, not necessarily proper, factored as

$$\boldsymbol{G}(z_1, z_2) = \boldsymbol{N}(z_1, z_2)\boldsymbol{D}^{-1}(z_1, z_2) = -\boldsymbol{A}^{-1}(z_1, z_2)\boldsymbol{B}(z_1, z_2) \tag{1}$$

where $\boldsymbol{N}(z_1, z_2)$, $\boldsymbol{D}(z_1, z_2)$, $\boldsymbol{A}(z_1, z_2)$, and $\boldsymbol{B}(z_1, z_2)$ are, respectively, $q \times p$, $p \times p$, $q \times q$, and $q \times p$ 2-D polynomial matrices. Equation (1) implies

$$[\boldsymbol{B}(z_1, z_2)\boldsymbol{A}(z_1, z_2)] \begin{bmatrix} \boldsymbol{D}(z_1, z_2) \\ \boldsymbol{N}(z_1, z_2) \end{bmatrix} = \boldsymbol{O}. \tag{2}$$

This is a set of linear homogeneous algebraic equations with elements in the 2-D polynomial commutative ring $R[z_1, z_2]$. Given $\boldsymbol{D}(z_1, z_2)$ and $\boldsymbol{N}(z_1, z_2)$, it can be shown that all solutions $[\boldsymbol{B}(z_1, z_2)\boldsymbol{A}(z_1, z_2)]$ of (2) form a free module over $R[z_1, z_2]$ of dimension q. Let V denote the module. A basis of V is the minimal set of generators (in this case, q generators) which generate V [4], [6]. Then it is easy to establish the following lemma.

Lemma 1 [8]: The left fraction $\boldsymbol{G}(z_1, z_2) = -\boldsymbol{A}^{-1}(z_1, z_2)\,\boldsymbol{B}(z_1, z_2)$ is coprime if and only if $[\boldsymbol{B}(z_1, z_2)\boldsymbol{A}(z_1, z_2)]$ is a basis of the module V.

Lemma 2 [8]: Consider a 2-D $q \times p$ rational matrix $\boldsymbol{G}(z_1, z_2) = -\bar{\boldsymbol{A}}^{-1}(z_1, z_2)\bar{\boldsymbol{B}}(z_1, z_2) = -\boldsymbol{A}^{-1}(z_1, z_2)\boldsymbol{B}(z_1, z_2)$. Let $\boldsymbol{O}(z_1, z_2)$ be any q different columns of $[\boldsymbol{B}(z_1, z_2)\boldsymbol{A}(z_1, z_2)]$ and let $\bar{\boldsymbol{O}}(z_1, z_2)$ be the corresponding q columns of $[\bar{\boldsymbol{B}}(z_1, z_2)\bar{\boldsymbol{A}}(z_1, z_2)]$. If $\boldsymbol{A}(z_1, z_2)$ and $\boldsymbol{B}(z_1, z_2)$ are left coprime, and if $\bar{\boldsymbol{O}}(z_1, z_2)$ is nonsingular, then

$$\delta_i|\boldsymbol{O}(z_1, z_2)| \le \delta_i|\bar{\boldsymbol{O}}(z_1, z_2)| \qquad i = 1, 2$$

where $|\cdot|$ denotes the determinant and $\delta_i f(z_1, z_2)$ denotes the highest degree of z_i in $f(z_1, z_2)$. The equalities hold if $\bar{\boldsymbol{A}}(z_1, z_2)$ and $\bar{\boldsymbol{B}}(z_1, z_2)$ are also left coprime.

Instead of solving (2) directly, we shall transform it into sets of linear homogeneous algebraic equations. Define

$$\boldsymbol{N}(z_1, z_2) = \sum_{j=0}^{N} \sum_{i=0}^{M} \boldsymbol{N}_{i,j} z_1^i z_2^j \tag{3a}$$

$$\boldsymbol{D}(z_1, z_2) = \sum_{j=0}^{N} \sum_{i=0}^{M} \boldsymbol{D}_{i,j} z_1^i z_2^j \tag{3b}$$

and

$$\boldsymbol{B}(z_1, z_2) = \sum_{j=0}^{L} \sum_{i=0}^{K} \boldsymbol{B}_{i,j} z_1^i z_2^j \tag{3c}$$

$$\boldsymbol{A}(z_1, z_2) = \sum_{j=0}^{L} \sum_{i=0}^{K} \boldsymbol{A}_{i,j} z_1^i z_2^j \tag{3d}$$

where $M = \max\,(\delta_1\boldsymbol{N}(z_1, z_2)\ \delta_1\boldsymbol{D}(z_1, z_2))$, $N = \max\,(\delta_2\boldsymbol{N}(z_1, z_2), \delta_2\boldsymbol{D}(z_1, z_2))$, and K and L are similarly defined. All the $\boldsymbol{B}_{i,j}$'s, $\boldsymbol{A}_{i,j}$'s, $\boldsymbol{N}_{i,j}$'s, and $\boldsymbol{D}_{i,j}$'s are real constant matrices, not necessarily all nonzeros. The substitution of (3) into (2) and equating the coefficients of various

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