Stability-equation Method for Multivariable Feedback Control Systems with Lead/Lag Compensators

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ABSTRACT: *The stability-equation method is extended to the analysis and design of multivariable control systems. The systems are first compensated by constant precompensating matrices, and then compensated by lead/lag compensators. The method can take into consideration integrity and asymptotic behaviour simultaneously. In addition, other considerations, such as high frequency alignment and low-interaction at high frequency, can be analysed. Numerical examples are given and comparisons made with other methods in the current literature.*

Nomenclature

a_{ij}	parameters for integrity analysis
b_{ij}	parameters for asymptotic behaviour analysis
D_g	determinant of $G(S)$
D_p	determinant of $P(S)$
E_{ss}	steady-state error
$F(S)$	characteristic polynomial
$F_e(S)$	even stability-equation
$F_o(S)$	odd stability-equation
$G(S)$	plant transfer function matrix
$g_{ij}(S)$	elements of plant transfer function matrix
i,j	integers
$K(S)$	controller gain matrix
k_1, k_2	diagonal elements of controller gain matrix
$P(S)$	precompensating matrix
$P_c(S)$	closed-loop characteristic polynomial of $T(S)$
p_{ij}	elements of constant precompensating matrix $P(S)$
$P_o(S)$	open-loop characteristic polynomial of $G(S)$ or $G_1(S)$
S	Laplace operator
$T(S)$	closed-loop transfer function matrix

$u = n/m$

X	negative sum of roots of the characteristic polynomial
y	$y = -S^2$
y_{ei}	roots of even stability-equation
y_{oj}	roots of odd stability-equation

I. Introduction

It is known from many practical examples offered by Han (**1**, **2**) and other researchers, that the stability-equation method is highly capable of handling single-input single-output systems. The main purpose of this paper is to utilize the stability-equation method for the analysis and design of multivariable feedback control systems.

From the analysis of the asymptotic behaviour of a feedback control system, one can predict the closed-loop poles approximately, because they tend to approach the open-loop zeros when the open-loop gain is increasing. This characteristic has been extended to the design of multivariable feedback control systems by Kouvaritakis (**3–5**). Kouvaritakis has noted the effects in the direction of asymptotes of root loci to system characteristics. Following Kouvaritakis' approach, the advantage of the method we propose in this paper is that the locations of the closed-loop poles can be predicted approximately, while the stability of the system has already been checked, because consideration of stability is the general property of the stability-equation method.

On the other hand, while designing a controller for a multivariable feedback control system it is important to check the stability of the system when one or more transducers or actuators fail. The term "high integrity" is defined as when the system remains stable under all likely failures. This integrity consideration has been discussed in detail by Belletrutti and MacFarlane (**6**). It will be shown later in the paper that the proposed method can take into consideration both integrity and damping behaviour simultaneously. In addition, other considerations, such as high frequency alignments and low-interactions, can also be achieved.

The considered system is first compensated by the constant precompensating matrix to match the desirable specifications. If the constant precompensating matrix is not sufficient for compensation, then the system can be further compensated by the use of cascaded lead/lag compensators. Based upon the constant precompensating matrix, it will be shown that only a simple geometry transformation in the parameter plane is required for obtaining the suitable lead/lag compensators.

II. Main Approach for Choosing the Values of Parameters in Compensators

Assume that the system characteristic equation is $F(S)$ which can be decomposed into two parts concerning even and odd exponents of S; i.e.

$$F(S) = F_e(S) + F_o(S) = 0. \tag{1}$$

Let $y = -S^2$, then the stability-equations are (**1**, **2**)

$$f_e(y) = F_e(S) \tag{2}$$

and

$$f_o(y) = F_o(S)/S. \tag{3}$$

From (1), one has the following stability criterion.

Stability Criterion: a system with characteristic equation $F(s)$ is stable if the roots y_{ei} and y_{oj} $(i, j = 1, 2, \ldots)$ of the stability-equations $f_e(y) = 0$ and $f_o(y) = 0$, respectively, are all real and positive and are alternating in sequence.

For a system with two parameters (k_1 and k_2), the stability-equations can be written as

$$f_e(y) = \sum_{i=0}^{m} a_i y^i = 0 \tag{4}$$

and

$$f_o(y) = \sum_{j=0}^{n} b_j y^j = 0 \tag{5}$$

where the coefficients a_i's and b_j's are assumed in the form of

$$a_i = A_{ei} + B_{ei}k_1 + C_{ei}k_2 \tag{6}$$

and

$$b_j = A_{oj} + B_{oj}k_1 + C_{oj}k_2 \tag{7}$$

where A's, B's and C's are constants. By inserting Eqs (6) and (7) into Eqs (4) and (5), the result can be arranged as

$$\sum_{i=0}^{m} A_{ei}y^i + k_1 \sum_{i=0}^{m} B_{ei}y^i + k_2 \sum_{i=0}^{m} C_{ei}y^i = 0 \tag{8}$$

for the even stability-equation, and

$$\sum_{j=0}^{n} A_{oj}y^j + k_1 \sum_{j=0}^{n} B_{oj}y^j + k_2 \sum_{j=0}^{n} C_{oj}y^j = 0 \tag{9}$$

for the odd stability-equation. From these two equations the following two kinds of curves can be plotted:

(1) *The stability-boundary curves:* For a sufficient number of suitable values of y, the simultaneous solutions of Eqs (8) and (9) can be used to sketch a number of curves in a k_1 vs. k_2 plane. Then the curve for $y_{ei} = y_{oj}$ which constitutes the stability-boundary can be obtained.

(2) *The constant-y curves:* By assigning a sufficient number of values of y to Eqs (8) and (9) the constant-y curves for even and odd stability-equations can be plotted in the k_1 vs. k_2 plane.

These two kinds of curves offer the information for analysis and design. From (1) and (2), it has been shown that the differences among the magnitudes of the real roots (y_{ei} and y_{oi}) can be used as indications of damping characteristics; therefore, the proper values of parameters (k_1 and k_2) of a compensator can be chosen by inspecting the relative differences (spacings) among these constant-y curves. This is the main approach of this paper for choosing the values of parameters in compensators.

III. Precompensation with Constant Matrix

Consider the multivariable feedback control system shown in Fig. 1, where $P(s)$ and $K(s)$ denote the desired precompensating matrix and controller gain matrix, respectively. Assume at the time that

$$G(S) = \begin{bmatrix} g_{11}(S) & g_{12}(S) \\ g_{21}(S) & g_{22}(S) \end{bmatrix} \tag{10}$$

$$P(S) = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \tag{11}$$

and

$$K(S) = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \tag{12}$$

then the closed-loop system transfer matrix is (7)

$$T(S) = [I+G(S)P(S)K(S)]^{-1}G(S)P(S)K(S)$$

$$= \frac{\begin{bmatrix} k_1\bar{g}_{11}((S)+k_1k_2D_{\bar{g}} & k_2\bar{g}_{12}(S) \\ k_1\bar{g}_{21}(S) & k_2\bar{g}_{22}(S)+k_2k_1D_{\bar{g}} \end{bmatrix}}{1+k_1\bar{g}_{11}(S)+k_2\bar{g}_{22}(S)+k_1k_2D_{\bar{g}}} \tag{13}$$

where

$$\begin{bmatrix} \bar{g}_{11}(S) & \bar{g}_{12}(S) \\ \bar{g}_{21}(S) & \bar{g}_{22}(S) \end{bmatrix} = \begin{bmatrix} g_{11}(S) & g_{12}(S) \\ g_{21}(S) & g_{22}(S) \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

$$= \begin{bmatrix} p_{11}g_{11}(S)+p_{21}g_{12}(S) & p_{12}g_{11}(S)+p_{22}g_{12}(S) \\ p_{11}g_{21}(S)+p_{21}g_{22}(S) & p_{12}g_{21}(S)+p_{22}g_{22}(S) \end{bmatrix} \tag{14}$$

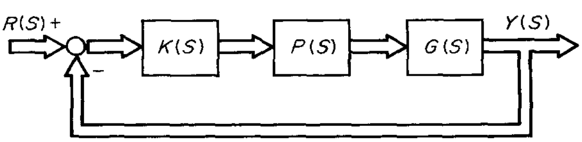


FIG. 1. Block diagram of a multivariable feedback control system.

and $D_{\bar{g}} = D_p D_g$; D_p and D_g are the determinants of $P(S)$ and $G(s)$, respectively. The closed-loop characteristic polynomial is (**7**)

$$P_c(S) = P_o(S)\,[1+k_1\bar{g}_{11}(S)+k_2\bar{g}_{22}(S)+k_1k_2D_pD_{\bar{g}}] = 0 \tag{15}$$

where $P_o(S)$ is the characteristic polynomial of the open-loop system.

According to the considerations of integrity and asymptotic behaviour, Eq. (15) is decomposed into the following two equations:

$$P_o(S)\,[1+k_1\bar{g}_{11}(S)]+k_2P_o(S)\,[\bar{g}_{22}(S)+k_1D_pD_g] = 0 \tag{16}$$

$$P_o(S)\,[1+k_2\bar{g}_{22}(S)]+k_1P_o(S)\,[\bar{g}_{11}(S)+k_2D_pD_g] = 0. \tag{17}$$

The first term of Eq. (16) represents the characteristic polynomial of loop-1 while the transducers or actuator element of loop-2 is fail (e.g. $k_2 = 0$ or the feedback loop is open); the roots of the second term of Eq. (16) represent the open-loop zeros of loop-2. Similarly, the first term of Eq. (17) represents the characteristic polynomial of loop-2 while the transducer or actuator element of loop-1 is fail (e.g. $k_1 = 0$); the roots of the second term represent the open-loop zeros of loop-1. Therefore, the analyses of the first terms of Eqs (16) and (17) represent the "consideration of integrity"; and the analyses of the second terms of Eqs (16) and (17) represent the "analysis of asymptotic behaviour".

From Eq. (13), it can be seen that if steady-state accuracy and low-interaction at low-frequency are to be achieved by "tight-feedback", then condition $D_{\bar{g}}(S \to 0) \gg \max\{|p_{i1}g_{ji}(S\to 0)/k_2|, |p_{i2}g_{ji}(S\to 0)/k_1|\}$ is required (**8**); thus, the value of $\min(D_p^{1/2}/|p_{ij}|)$ will be the larger the better for all $i,j = 1,2$. Note that the term "tight-feedback" is defined as large loop gains for the asymptotic behaviour.

For the consideration of high frequency alignment, the equivalence of high-order exponents of the second terms of Eqs (16) and (17) is required. This will be discussed in the examples given later.

Following all the equations given above, it can be seen that the proposed method is to do:

(1) parameter analysis of both $P_o(S)\,[\bar{g}_{11}(S)+k_2D_pD_g]$ and $P_o(S)\,[\bar{g}_{22}(S)+k_1D_pD_g]$ for the analysis of asymptotic behaviour, and

(2) parameter analysis of both $P_o(S)\,[1+k_1\bar{g}_{11}(S)]$ and $P_o(S)\,[1+k_2\bar{g}_{22}(S)]$ for the consideration of integrity.

In other words, one has the following four equations for doing parameter analysis:

$$P_o(S)\,[p_{11}g_{11}(S)+p_{21}g_{12}(S)+k_2D_pD_g] = 0 \tag{18}$$

$$P_o(S)\,[p_{12}g_{21}(S)+p_{22}g_{22}(S)+k_1D_pD_g] = 0 \tag{19}$$

$$P_o(S)\,[1+k_1p_{11}g_{11}(S)+k_1p_{21}g_{12}(S)] = 0 \tag{20}$$

$$P_o(S)\,[1+k_2p_{12}g_{21}(S)+k_2p_{22}g_{22}(S)] = 0. \tag{21}$$

Let $a_{ij} = k_jp_{ij}$ and $b_{ij} = p_{ij}/k_mD_p$ ($i,j = 1,2$ and $m = 1$ if $j = 2$ otherwise $m = 2$), then Eqs (18)–(21) become

$$P_o(S)\,[D_g+b_{11}g_{11}(S)+b_{21}g_{12}(S)] = 0 \tag{22}$$

$$P_o(S)\,[D_g+b_{12}g_{21}(S)+b_{22}g_{22}(S)] = 0 \quad (23)$$

$$P_o(S)\,[1+a_{11}g_{11}(S)+a_{21}g_{12}(S)] = 0 \quad (24)$$

$$P_o(S)\,[1+a_{12}g_{21}(S)+a_{22}g_{22}(S)] = 0. \quad (25)$$

Based upon the definitions of a_{ij} and b_{ij}, the relationships among them are found as

$$a_{1i}/a_{2i} = b_{1i}/b_{2i} = p_{1i}/p_{2i} \quad \text{for} \quad i = 1,2. \quad (26)$$

Because there are three objectives which must be considered simultaneously; i.e. (1) the asymptotic behaviour of each loop (loop gains are large), (2) the high integrity and (3) the low-interaction between every loop, it is required that the values of k_1 and k_2 must be large. Therefore, one should analyse these equations in the *a–a* planes as far from the origin as possible, and in the *b–b* planes as near the origin as possible.

Equations (22) and (23) show the asymptotic behaviours of loop-1 and loop-2 respectively, and Eqs (24) and (25) show the considerations of integrity of loop-1 and loop-2, respectively; therefore one can solve the problems concerning integrity and asymptotic behaviour simultaneously.

The application of the proposed method for analysis and design of multivariable feedback control system is explained along with the following example.

Example 1

Consider the system shown in Fig. 1. Assume that the plant transfer matrix is (8)

$$G(S) = \begin{bmatrix} \dfrac{1-S}{(1+S)^2} & \dfrac{2-S}{(1+S)^2} \\ \dfrac{1-3S}{3(1+S)^2} & \dfrac{1-S}{(1+S)^2} \end{bmatrix} \quad (27)$$

and that the precompensating matrix and the controller matrix are given as shown in Eqs (11) and (12), respectively. The characteristic polynomial is (7)

$$3S^2+9S^2+9S+3+k_1[p_{11}(-3S^2+3)+p_{21}(-3S^2+3S+6)] + k_2[p_{12}(-3S^2-2S+1)+p_{22}(-3S^2+3)]+k_1k_2D_p = 0. \quad (28)$$

From Eqs (22)–(25) the equations for parameter analysis are

$$(-3b_{11}-3b_{21})S^2+3b_{21}S+(3b_{11}+6b_{21}+1) = 0 \quad \text{(29-a)}$$

$$(-3b_{12}-3b_{22})S^2-2b_{12}S+(b_{12}+3b_{22}+1) = 0 \quad \text{(29-b)}$$

$$(S+1)\,[S^2+(2-a_{11}-a_{21})S+(1+2a_{21}+a_{11})] = 0 \quad \text{(29-c)}$$

$$(S+1)\,[3S^2+(6-3a_{12}-3a_{22})S+(3+a_{12}+3a_{22})] = 0. \quad \text{(29-d)}$$

By use of a computer, the constant-y curves can be plotted easily, as shown in Fig. 2(a)–(d), where the shaded curves are the stability boundaries. In these figures the constant-X curves which represent the negative sum of the characteristic roots of each equation are also plotted. Generally, the larger the value of X the better the damping characteristics of the system. By use of all these curves one can choose proper points in these planes to make the system have desirable characteristics (**1, 2**).

Since the high-order exponents of the characteristic polynomial contribute to transient responses, the high-order equivalence of Eqs (28-a) and (28-d) will make the subsystem have similar transient responses; i.e. they may have the same rising times and settling times. It is impossible to achieve "strict equivalence" of the subsystems because the structure of controller is limited. But one can achieve the equivalent property to some degree by using a proper type of controller.

For multivariable feedback control systems, the interaction at high frequency is usually a serious problem. One can eliminate or reduce this kind of interaction by setting the high-order exponents of Eqs (29-a) and (29-d) equal to or near to zero.

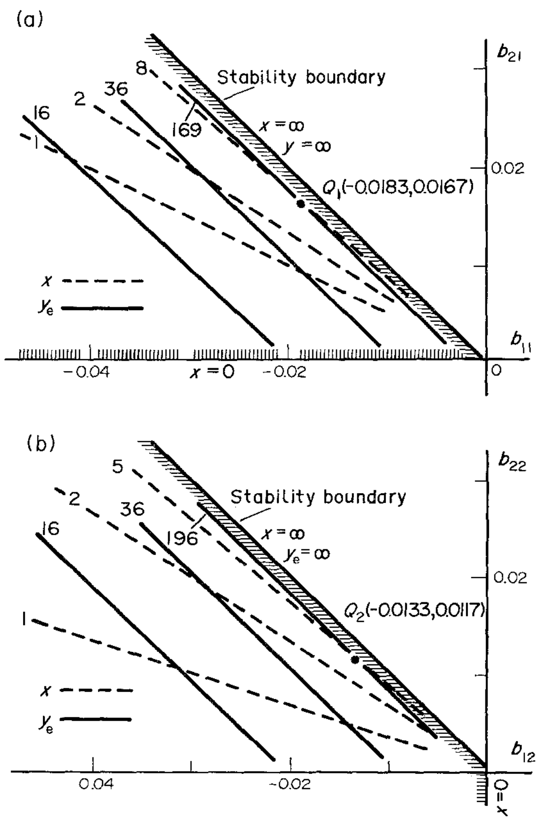
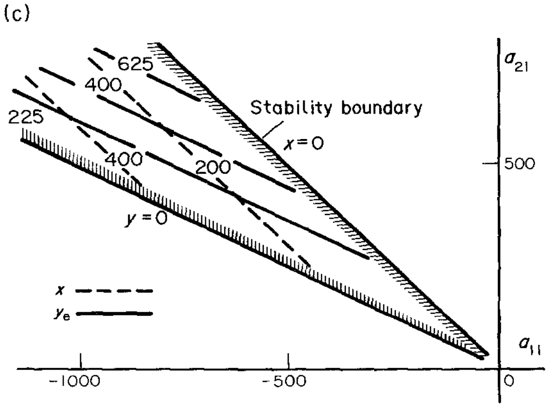


FIG. 2. Parameter analyses of: (a) Eq. (29-a) for small values of b_{11} and b_{21}; (b) Eq. (29-b) for small value of b_{12} and b_{22}; (c) Eq. (29-c) for large value of a_{11} and a_{21}; (d) Eq. (29-d) for large value of a_{12} and a_{22}.



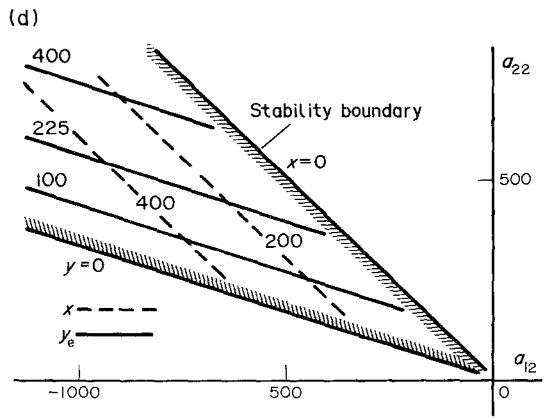


FIG. 2. *Continued*

From Eq. (13), the steady-state error (E_{ss1}) of loop-1 is

$$E_{ss1} = 1 - \frac{k_1\bar{g}_{11}(0)+k_1k_2(D_{\bar{g}}(0)}{1+k_1\bar{g}_{11}(0)+k_2\bar{g}_{22}(0)+k_1k_2D_{\bar{g}}(0)}$$

$$= \frac{1+k_2\bar{g}_{22}(0)}{1+k_1\bar{g}_{11}(0)+k_2\bar{g}_{22}(0)+k_1k_2D_{\bar{g}}(0)}$$

and the steady-state error (E_{ss2}) of loop-2 is

$$E_{ss2} = 1 - \frac{k_2\bar{g}_{22}(0)+k_1k_2D_{\bar{g}}(0)}{1+k_1\bar{g}_{11}(0)+k_2\bar{g}_{22}(0)+k_1k_2D_{\bar{g}}(0)}$$

$$= \frac{1+k_1\bar{g}_{11}(0)}{1+k_1\bar{g}_{11}(0)+k_2\bar{g}_{22}(0)+k_1k_2D_{\bar{g}}(0)}.$$

Since "tight-feedback" (with large loop gains) is used, the steady-state error (E_{ss})

of the overall systems is defined approximately by

$$E_{ss} = \max\,(|p_{ij}g_{ji}(0)|)/D_pD_g(0)\min\,(k_m) \quad i,j,m = 1,2 \tag{30-a}$$

and

$$E_{ss} = \max\,(|b_{ij}g_{ji}(0)|)/D_g(0) \quad i,j = 1,2. \tag{30-b}$$

According to the presentations given above, the following conditions are necessary for choosing a suitable precompensating matrix:

(a) From Eq. (29-a), for achieving fine damping properties, the result shown in Fig. 2(a) indicates that b_{21}/b_{12} should approach -1; correspondingly, from Eq. (29-b), Fig. 2(b) indicates that b_{22}/b_{12} should also approach -1.

(b) For achieving integrity against transducer failure in loop-2, Fig. 2(c) indicates that a_{21}/a_{11} should lie in between -1 and $-1/2$. Correspondingly, for integrity against transducer failure in loop-1, a_{22}/a_{12} should lie in between -1 and $-1/3$ as indicated in Fig. 2(d).

(c) High frequency alignment demands that $-3b_{11}-3b_{21} = -3b_{12}-3b_{22}$ and $3b_{21} = -2b_{12}$ for the equivalence of the high order exponents in Eqs (29-a) and (29-b).

(d) Low-interactions at high frequency requires that $-b_{11} = b_{21}-b_{12} = b_{22}$ in order to make the higher-order exponents of Eqs (29-a) and (29-b) as small as possible.

(e) Steady-state errors (E_{ss}) are defined approximately by Eqs (30-a) and (30-b).

For a specified steady-state error (E_{ss}), the approximated values of the b_{ij}'s for analysis can be determined by Eq. (30-b) and the loop gains k_m are obtained by relating b_{ij} of Eq. (30-b) equal to p_{ij}/k_mD_p ($i,j = 1,2$ and $m = 1$ if $j = 2$, otherwise $m = 2$). If the steady-state error is specified to be within 5%, then Eq. (30-b) becomes $0.05 = 6b_{21}$; i.e. the analyses for b_{ij}'s around 0.01 should be examined. Following the conditions (a)–(d) and by inspecting the constant-y curves shown in Fig. 2(a)–(d), the proper choices of b_{11} and b_{21} are -0.0183 and 0.0167, respectively [point Q_1 in Fig. 2(a) for which $y_e = 169$ and $X = 8$], and proper choices of b_{12} and b_{22} are -0.0133 and 0.0117, respectively [point Q_2 in Fig. 2(b) for which $y_e = 196$ and $X = 5$]. Based upon the relationship defined by Eq. (26), one of the possible choices of $P(s)$ is

$$P(S) = \begin{bmatrix} -11 & -8 \\ 10 & 7 \end{bmatrix}. \tag{31}$$

The controller gain matrix for E_{ss} equal to 5% is selected to be

$$K(S) = \begin{bmatrix} 200 & 0 \\ 0 & 200 \end{bmatrix}. \tag{32}$$

The overall controller becomes

$$P(S)K(S) = \begin{bmatrix} -2200 & -1600 \\ 2000 & 1400 \end{bmatrix} \tag{33}$$

and the closed-loop system transfer function matrix is

$$T(S) = \frac{200}{\Delta(S)} \begin{bmatrix} S^2+10S+209 & S^2+7S+6 \\ S^2+7.33S+6.33 & S^2+5.33S+204.33 \end{bmatrix} \tag{34}$$

where $\Delta(S) = S^3+403S^2+3069.67S+42667.667$. The simulated results are shown in Fig. 3 which indicate that the damping characteristics of the system are acceptable. Naturally, the choice of the overall controller is not unique; the simulated results for other choices of $P(S)$ and $K(S)$ are given in Table I.

From the above presented example it can be seen that the stability boundary, constant-y curves and constant-X curves provided by the stability-equation method are useful for analysis and design.

IV. Precompensation with Lead/Lag Compensators

In the last section, while doing analysis and design for a system with a constant precompensating matrix the following assumptions are made:

(a) The problem is solvable by the strategy of making it "tight-feedback", i.e. high loop gains are generally required.

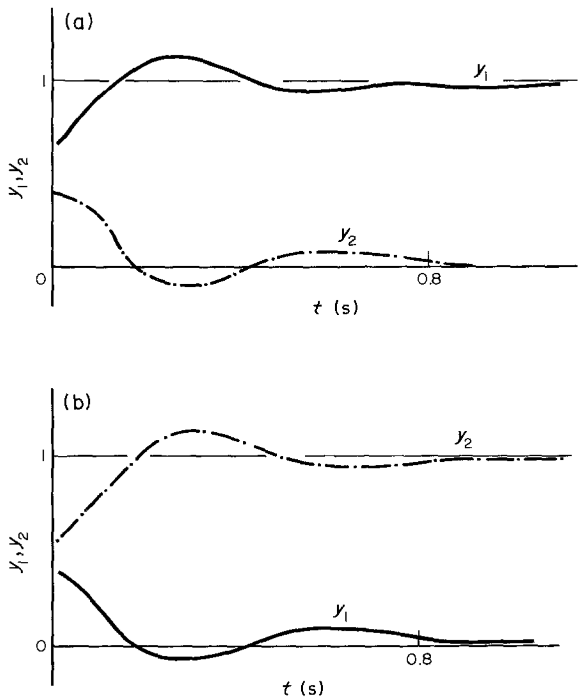


FIG. 3. Step responses of Example 1 for: (a) $r_1 = 1$ and $r_2 = 0$; (b) $r_1 = 0$ and $r_2 = 1$.

TABLE I
Accompanying table of performance for Example 1

$P(S)$	$k_1=k_2$	$c_1(t),\ r_1(t)=1,\ r_2(t)=0$				$c_2(t),\ r_1(t)=0,\ r_2(t)=1$			
		t_r (s)	t_s (s)	m_p (%)	acc. (%)	t_r (s)	t_s (s)	m_p (%)	acc. (%)
$\begin{matrix}-11 & -8\\ 10 & 7\end{matrix}$	200	0.18	0.44	14.3	2.1	0.22	0.49	14.3	4.3
	80	0.30	0.59	6.7	4.7	0.43	0.68	5.2	9.6
	40	0.51	0.38	1.7	8.2	0.60	0.77	1.2	16.9
$\begin{matrix}-11 & -5\\ 10 & 4\end{matrix}$	200	0.15	0.68	19.2	1.2	0.22	0.74	17.4	4.4
	80	0.17	0.72	17.4	1.4	0.26	0.77	15.2	5.2
	40	0.24	0.63	10.7	2.6	0.41	0.77	8.3	9.9
$\begin{matrix}-11 & -3\\ 10 & 2\end{matrix}$	200	0.13	0.84	25.2	0.2	0.19	0.87	23.0	3.0
	80	0.14	0.91	23.4	0.4	0.21	0.96	21.2	4.2
	40	0.19	0.86	17.8	0.8	0.33	0.70	13.9	7.9

t_r, rise time; acc, steady-state accuracy; t_s, settling time; m_p, % peak-overshot.

(b) The determinant of the plant transfer function matrix possesses only left-half plane zeros, and that the pole-zero excess values of transfer function is less than four.

Assumption (b) is due to the fact that if the number of pole-zero excess is four then the summation of the characteristic roots of Eq. (15) will be uncontrollable for any kind of constant precompensating matrix. It will be shown in this section that this limitation can be overcome by introducing lead/lag compensators to the constant precompensating matrix. In addition, the lead/lag compensators can reduce the controlling effects, e.g. the smaller loop gains can be used to obtain the same damping characteristics and steady-state accuracies as the system considered in Section III.

Consider the system shown in Fig. 1. Assume that the parameter k_2 is cascaded by $q_2(S) = (n/m)(S+m)/(S+n)$, then the equations to be analysed become

$$p_o(S)(S+n)[1+k_1p_{11}g_{11}(S)+k_1p_{21}g_{21}(S)] = 0 \tag{35}$$

$$p_o(S)(S+n)[1+k_2q_2(S)p_{12}g_{21}(S)+k_2q_2(S)p_{22}g_{22}(S)] = 0 \tag{36}$$

$$p_o(S)(S+n)[p_{11}g_{11}(S)+p_{21}g_{12}(S)+k_2q_2(S)D_pD_g] = 0 \tag{37}$$

$$p_o(S)(S+n)q_2(S)[p_{12}g_{21}(S)+p_{22}g_{12}(S)+k_1D_pD_g] = 0. \tag{38}$$

Therefore one has

$$(S+n)p_o(S)[1+a_{11}g_{11}(S)+a_{21}g_{12}(S)] = 0 \tag{39}$$

$$(S+m)p_o(S)[1+q_2(S)(a_{12}g_{21}(S)+a_{22}g_{22}(S))] = 0 \tag{40}$$

$$(S+n)p_o(S)\,[D_g+q_2^{-1}(S)\,(b_{11}g_{11}(S)+b_{21}g_{12}(S))]=0 \tag{41}$$

$$(S+m)p_o(S)\,[D_g+b_{12}g_{21}(S)+b_{22}g_{22}(S)]=0. \tag{42}$$

Since the terms $(S+m)$ and $(S+n)$ of Eqs (39)–(42) represent one zero at $-m$ and one zero at $-n$, respectively, the analyses of Eqs (39)–(42) can be simplified by neglecting these terms. Comparing these simplified equations to Eqs (22)–(25), the analyses of integrity for loop-1 [Eq. (39)] and asymptotic behaviour of loop-2 [Eq. (42)] are not changed. The analyses of Eqs (40) and (41) are interpreted in the following paragraphs.

(1) Consider the analysis in the a_{12} vs. a_{22} plane. The equations with and without the lead/lag compensator $q_2(S)$ to be analysed are correspondingly

$$p_o(S)+p_o(S)\,[a_{12}g_{21}(S)+a_{22}g_{22}(S)]=0 \tag{43}$$

and

$$p_o(S)\,(S+n)+p_o(S)\,(S+m)\frac{n}{m}\,[a_{12}g_{21}(S)+a_{22}g_{22}(S)]=0. \tag{44}$$

The coefficients of the polynomial shown in Eq. (43) are generally in the form of

$$1 \quad r_1+s_1 \quad r_2+s_2 \quad r_3+s_3 \quad \ldots,$$

or

$$1 \quad r_1 \quad r_2+s_2 \quad r_3+s_3 \quad \ldots,$$

where r_i's and s_j's represent the first and second parts of the coefficients of the polynomial given in Eq. (43). Similarly, the coefficients for the polynomial given in Eq. (44) are in the form of

$$1 \quad r_1+n+us_1 \quad r_2+nr_1+us_2+ns_1 \quad r_3+nr_2+us_3+ns_2 \quad \ldots,$$

or

$$1 \quad r_1+n \quad r_2+nr_1+us_2 \quad r_3+nr_2+us_3+ns_2 \quad \ldots,$$

where $u = n/m$. Assume the feedback is tight, i.e. the a_{ij}'s are large and the r_i's are small in comparison to s_j's, then one has

$$1 \quad r_1+n+us_1 \quad us_2+ns_2 \quad us_3+ns_2 \quad \ldots,$$

or

$$1 \quad r_1+n \quad r_2+nr_1+us_2 \quad us_2+ns_2 \quad \ldots.$$

The coefficients of the system with and without the lead/lag compensator are shown in Table II, which will approximately be true if tight feedback is applied and the value of $m \times n$ is not too large compared to the value of u. From Table II, the relationships between the analyses with and without lead/lag compensators are shown in Table III, where u denotes the scaling factor.

(2) Consider the analysis in the b_{11} vs. b_{21} plane. The equations, with and without the lead/lag element $q_2(S)$, to be analysed are correspondingly

$$p_o(S)\,[b_{11}g_{11}(S)+b_{21}g_{12}(S)]+p_o(S)D_g=0 \tag{45}$$

and

$$p_o(S)\,(S+n)\,[b_{11}g_{11}(S)+b_{21}g_{12}(S)+p_o(S)\,(S+m)\frac{n}{m}D_g=0. \tag{46}$$

TABLE II
The coefficients of the even and odd stability equations

Cases	Even/odd	Coefficients
1	even (odd) odd (even)	$1 \quad s_2 \; s_4 \quad s_6 \quad \ldots$ $r_1+s_1 \quad s_3 \quad s_5 \quad s_7 \quad \ldots$
1 with $q_2(S)$	even (odd) odd (even)	$1 \quad us_2 \quad us_4 \quad us_6 \quad \ldots$ $r_1+n-m+us_1 \quad us_3 \quad us_5 \quad us_7 \quad \ldots$
2	even (odd) odd (even)	$1 \; s_2 \quad s_4 \quad s_6 \quad \ldots$ $r_1 \quad s_3 \quad s_5 \quad s_7 \quad \ldots$
2 with $q_2(S)$	even (odd) odd (even)	$1 \quad us_2 \quad us_4 \quad us_6 \; \ldots$ $r_1+n-m \quad us_3 \quad us_5 \quad us_7 \quad \ldots.$

Even, even stability equation; $u = n/m$; odd, odd stability equation.

Note that these equations are similar to Eqs (43) and (44) for doing analysis in the a_{12} vs. a_{22} plane. Performing the same procedure except that large values of a_{ij}'s are replaced by assuming small values of b_{ij}'s. The final resulted case for the coefficients of the polynomials given in Eqs (43) and (44) are the same as those for Eqs (45) and (46). The transformed results for the analyses in the b_{11} vs. b_{21} plane are given in Table IV.

TABLE III
The transformation of old analyses to new analyses in a_{12} vs. a_{22} plane

Case	With const precompensating matrix (old)	With lead/lag compensators (new)
1	$a_{12}-a_{22}$ plane $X = X_{\text{old}}$ $y_{\text{even(odd)}}$ $y_{\text{odd(even)}}$	$ua_{12}-ua_{22}$ plane ($u = n/m$) Const-X curves are reproduced and $X = X_{\text{new}} = X_{\text{old}}+n-m$ Const-$y_{e(o)}$ curves are reproduced Const-$y_{o(e)}$ curves are reproduced with the dominant (or highest) $y_{o(e)}$ values changed by a multiplicating factor $X_{\text{old}}/X_{\text{new}}$ at each corresponding point
2	$a_{12}-a_{22}$ plane $X = X_{\text{old}}$ a constant $y_{\text{even(odd)}}$ $y_{\text{odd(even)}}$	$ua_{12}-ua_{22}$ plane $X = X_{\text{new}} = X_{\text{old}}+n-m$ a constant Const-$y_{e(o)}$ curves are reproduced Const-$y_{o(e)}$ curves are reproduced with the dominant $y_{o(e)}$ values changed by a constant multiplicating factor of $X_{\text{old}}/X_{\text{new}}$

TABLE IV

The transformation of old analyses to new analyses in the b_{11} vs. b_{21} plane

With const. precompensating matrix (old)	With lead/lag compensators (new)
$b_{11}-b_{21}$ plane	$u'b_{11}-u'b_{21}$ plane ($u' = 1/u = m/n$)
$X = X_{old}$	$X = X_{new} = X_{old}+n-m$
$y_{even(odd)}$	Const-$y_{e(o)}$ curves are reproduced
$y_{odd(even)}$	Const-$y_{o(e)}$ curves are reproduced except for the dominant (or highest) $y_{o(e)}$ values changed by a multiplicating factor X_{old}/X_{new} at each corresponding point

From Tables III and IV, one can see that a simple geometry transformation is required for cascading lead/lag elements to the considered system. Similarly, if a lead/lag compensator $q_1(s)$ is cascaded to k_1, the same transformation can be derived in the a_{11} vs. a_{21} and b_{12} vs. b_{22} planes leaving the analyses in the a_{12} vs. a_{22} and b_{11} vs. b_{21} planes unchanged. This unchanged property and geometry transformation shown in Tables III and IV will be justified by introducing a known compensator to parameter k_2. The details are given in the following example.

Example 2. Precompensation with a lag compensator

Consider the analyses performed in Example 1 as shown in Figs. 2(a)–(d). If the lag compensator $q_2(S) = 0.2(S+1)/(S+0.2)$ is cascaded to parameter k_2, the results of the analyses are shown in Fig. 4(a)–(d). Comparing these figures to Fig. 2(a)–(d), it can be seen that the constant-y curves in the b_{12} vs. b_{22} and a_{11} vs. a_{21} planes are the same, and that the results of the analyses in the b_{11} vs. b_{21} and a_{12}

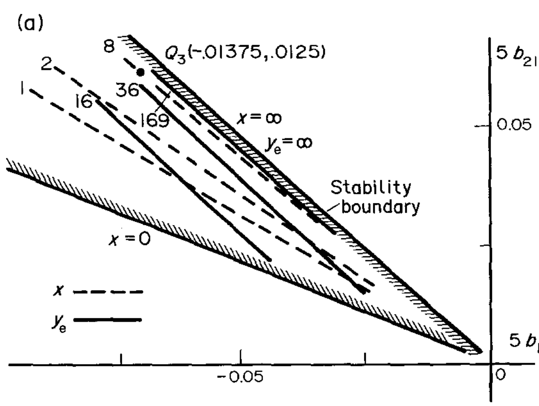


FIG. 4. Parameter analysis of Example 2 for: (a) small values of b_{11} and b_{21}; (b) small values of b_{12} and b_{22}; (c) large values of a_{11} and a_{21}; (d) large values of a_{12} and a_{22}.

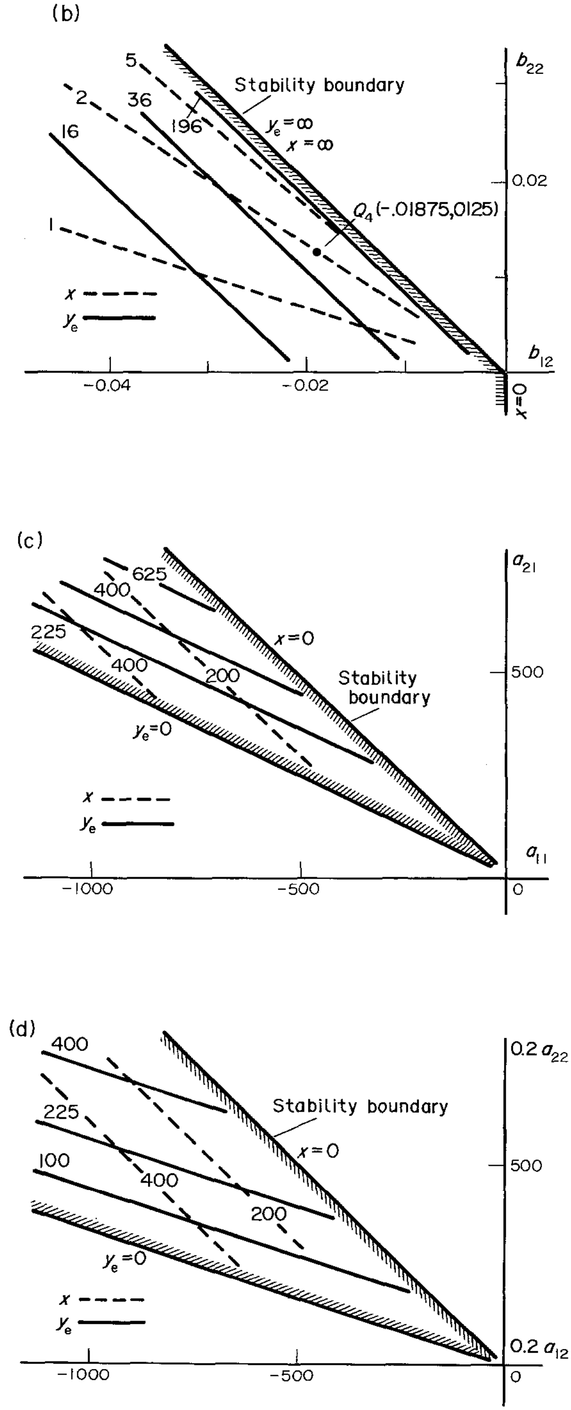


FIG. 4. *Continued*

vs. a_{22} planes are nearly the same except for the change of scale. The scaling factor is $5(n/m)$ with respect to those of the analyses without the lag compensator. These results justify the formulation of Tables II–IV.

By inspecting the constant-y and constant-X curves shown in Fig. 4(a)–(d), the following results can be obtained:

(a) Let b_{ij}'s be around $0.01 (E_{ss} = 5\%)$ and $b_{11}/b_{21} = -11/10$ [i.e. point $Q_3(-0.01375, 0.0125)$ in Fig. 4(a) which gives $X = 8$ and $y_e = 50$], where y_e shows the dominant characteristic of loop-1. It can be seen that the nice damping characteristics can be obtained because the large values of y_e and X.

(b) Since high-frequency alignment is under consideration, the analysis in the b_{12} vs. b_{22} plane for b_{ij}'s around 0.01 ($E_{ss} = 5\%$) shows that the choice of $b_{12}/b_{22} = -3/2$ with $y_e = 50$ and $X = 2$ [i.e. point Q_4 $(-0.01875, 0.0125)$ in Fig. 4(b)] is proper.

Therefore, the precompensating matrix is

$$P(S) = \begin{bmatrix} -11 & -15\dfrac{0.2(S+1)}{S+0.2} \\ 10 & 10\dfrac{0.2(S+1)}{S+0.2} \end{bmatrix}. \tag{47}$$

Since the steady-state error E_{ss} is specified to be 5%, the controller gain matrix can also be predicted as

$$K(S) = \begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix} \tag{48}$$

for the relationships among b_{ij}'s and p_{ij}'s are defined by Eq. (26).

The overall controller is

$$P(S)K(S) = \begin{bmatrix} -220 & -300\dfrac{S+1}{5S+1} \\ 200 & 200\dfrac{S+1}{5S+1} \end{bmatrix} \tag{49}$$

and the overall system transfer function matrix becomes

$$T(S) = \frac{20}{\Delta(S)}\begin{bmatrix} 5S^2+46S+275.67 & 5S^2+10S+5 \\ 5S^2+32.67S+6.33 & 5S^2+10S+271.67 \end{bmatrix} \tag{50}$$

where $\Delta(S) = 5S^3+211S^2+1127S+5614.33$. The simulated results as shown in Fig. 5 agree with the analysis.

The same problem has been solved by MacFarlane (**9**) utilizing the characteristic root-loci method in which a three-step controller design with a leading compensating element cascading k_1 was adapted. But in this paper, only a geometry transformation generated from the analyses of constant precompensating matrix

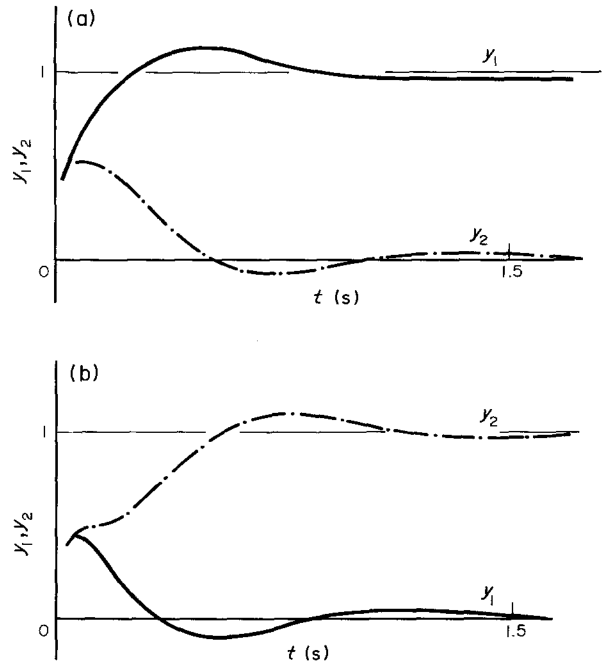


FIG. 5. Step responses of Example 2 for (a) $r_1 = 1$ and $r_2 = 0$; (b) $r_1 = 0$ and $r_2 = 1$.

is required, and the scaling factors are closely related to the parameters of the lag/lead compensator.

Example 3. Precompensation with lead compensators

Consider the system shown in Fig. 1. Assume that the plant transfer function matrix is **(10)**

$$G(S) = \begin{bmatrix} \dfrac{10}{S(S+1)(S+2)} & \dfrac{-1}{S(S+1)} \\ \dfrac{3.5}{S(S+2)} & \dfrac{6}{S(S+1)} \end{bmatrix}. \tag{51}$$

Since the pole-zero excess values of Det $[G(S)]$ is four, frequency compensation is necessary in general.

Performing parameter analyses without the lead/lag compensator, the equations to be analysed are

$$-b_{21}S^3+(10b_{11}-3b_{21})S^2+(3.5+10b_{11}-2b_{21})S+63.5 = 0 \tag{52-a}$$

$$(3.5b_{12}+6b_{22})S^3(7b_{12}+18b_{22})S^2+(3.5+3.5b_{12}+12b_{22})S+63.5 = 0 \tag{52-b}$$

$$S(S+1)[S^3+3S^2+(2-a_{21})S+(10a_{11}-2a_{21})] = 0 \tag{52-c}$$

and

$$S(S+1)\,[S^3+3S^2+(2+3.5a_{12}+6a_{22})S+(3.5a_{12}+12a_{22})] = 0. \quad (52\text{-d})$$

The parameter analyses are given in Fig. 6(a)–(d). Since the corrsponding pairs of planes ($a_{11}-a_{21}$ vs. $b_{11}-b_{21}$ planes and $a_{12}-a_{22}$ vs. $b_{12}-b_{22}$ planes) do not share proper common volumes, the integrity may be violated due to the requirement of fine damping characteristics. For example, the parameters b_{12} and b_{22} lying in the stability boundaries shown in Fig. 6(b) will violate the integrity when loop-1 is opened because the corresponding values of a_{12} and a_{22} are outside the stability boundaries shown in Fig. 6(d). Similarly, the selections of b_{11} and b_{21} (i.e. a_{11} and a_{21}) will violate the consideration of integrity when loop-2 is opened. Therefore, the frequency compensating elements are necessary in general, to enlarge the stable regions shown in Fig. 6(b) and (c). One may assume that

$$P(S) = \begin{bmatrix} 6 & 1 \\ -3.5 & 0 \end{bmatrix} \quad (53)$$

which is commonly chosen by every method in the current literature (**8**, **9**, **11**–**13**) where high frequency interactions are minimized by matrix transformation.

Consider the analysis in the b_{11} vs. b_{21} plane for $b_{11}/b_{21} = 6/-3.5$, the ratio of y_o/y_e is approximately at 1.1, and the value of X is 20. From the prediction indicated in Table IV and for a proper value of y_o/y_e, the frequency compensating element cascaded to k_2 may be chosen as

$$q_2(S) = 50(S+1)/(S+50). \quad (54)$$

Similarly, the analysis in the b_{12} vs. b_{22} plane demands a frequency compensating element cascaded to k_1 which is

$$q_1(S) = 100(S+1)/(S+100). \quad (55)$$

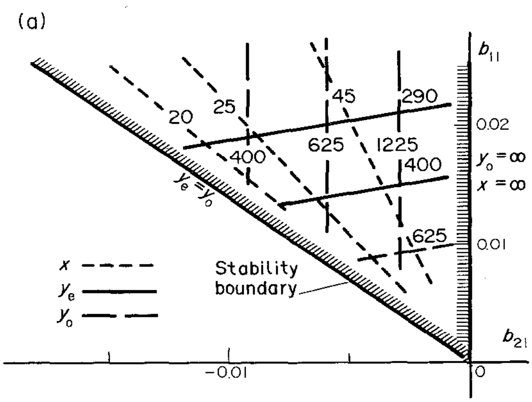


FIG. 6. Parameter analysis of: (a) Eq. (52-a) for small values of b_{21} and b_{11}; (b) Eq. (52-b) for small values of b_{22} and b_{12}; (c) Eq. (52-c) for large values of a_{21} and a_{11}; (d) Eq. (52-d) for large values of a_{22} and a_{12}.

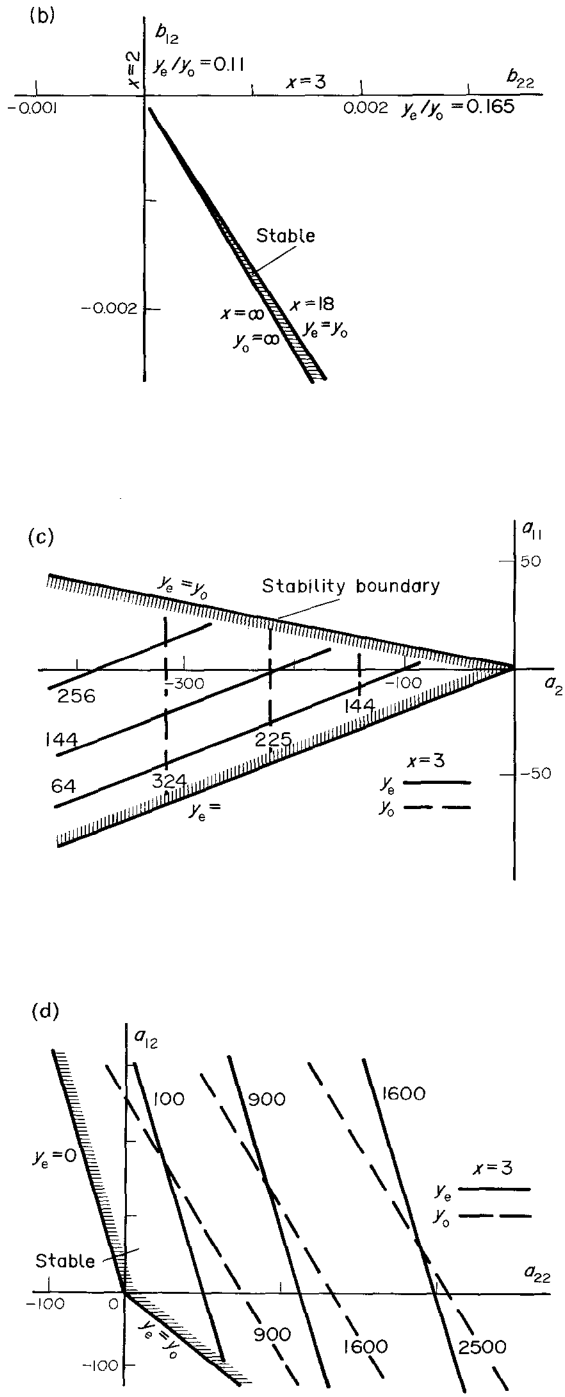


FIG. 6. *Continued*

The overall controller can be reformed as

$$P(S)K(S) = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} 100 & \frac{S+1}{S+100} & 0 \\ 0 & 50 & \frac{S+1}{S+50} \end{bmatrix}. \tag{56}$$

Performing the analyses with Eq. (56), the results are given in Fig. 7(a)–(d). From these figures one can see that the results agree with the prediction for Eq. (52) and with the compensators $q_1(S)$ and $q_s(S)$ which enlarge the stable regions. The analyses with and without lead/lag compensators are related by the relationship shown in Tables III and IV.

Following the presented results given above, the precompensated plant becomes

$$\bar{G}(S) = G(S)P(S)$$

$$= \frac{50}{S(S+2)} \begin{bmatrix} \frac{7S+134}{S+100} & \frac{10}{S+50} \\ \frac{-42}{S+100} & \frac{3.5S+3.5}{S+50} \end{bmatrix}. \tag{57}$$

The controller gain matrix $K(S)$ can be predicted from the following considerations:

(a) From Fig. 7(a), observing the b_{11} vs. b_{21} plane for $b_{11}/b_{21} = 6/-3.5$, a proper result is obtained for b_{11} around the value of 0.07 which offers the asymptotic characteristics of

$$y_e = 1000, \quad y_o = 4700 \quad \text{with} \quad X = 69,$$

i.e. point Q_5 $(-0.040, 0.0685)$ in Fig. 7(a).

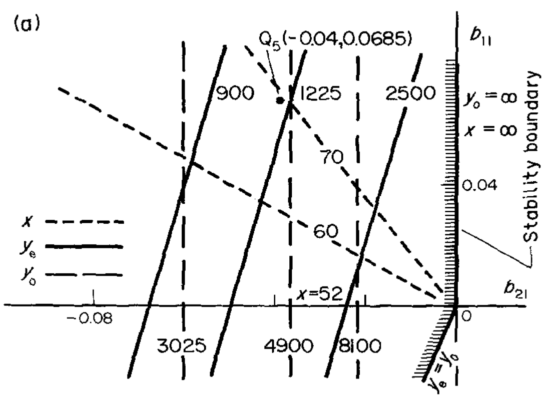


FIG. 7. Parameter analysis of Example 3 (a) for small values of: b_{21} and b_{11}; (b) for small values of b_{22} and b_{12}; (c) for large values of a_{21} and a_{11}; (d) for large values of a_{22} and a_{12}.

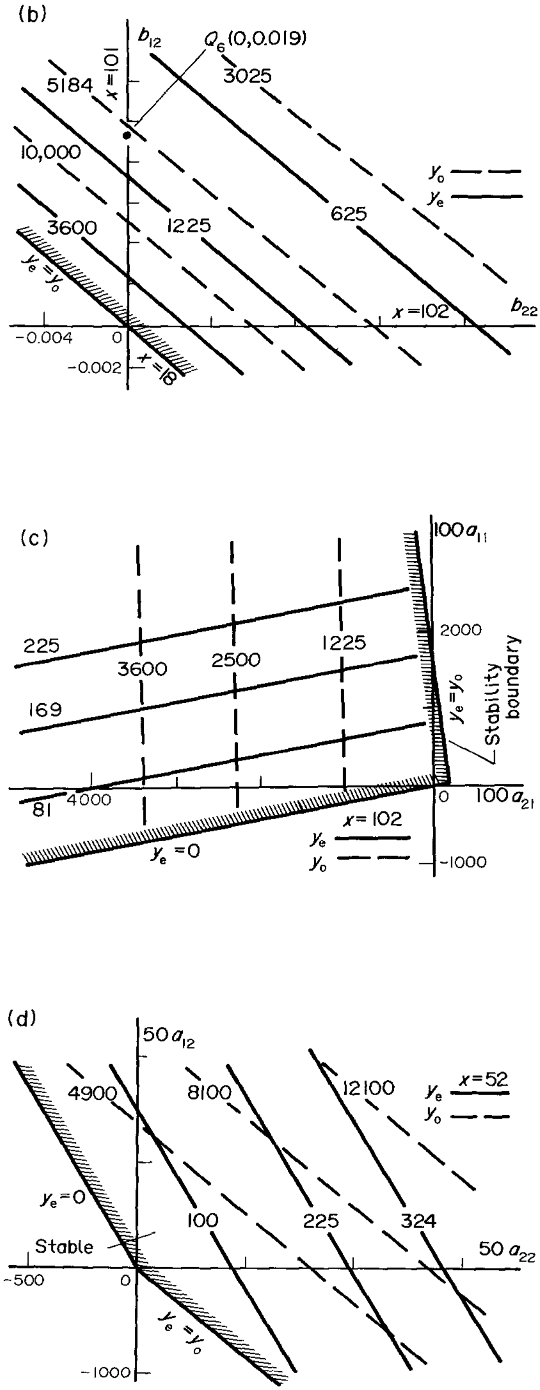


FIG. 7. *Continued*

(b) Consider Fig. 7(b) for $b_{22} = 0$. A proper result is obtained for b_{12} around the value of 0.02, i.e.

$$y_e = 1000, \quad y_o = 5200 \quad \text{with} \quad X = 101,$$

i.e. point $Q_6(0.0, 0.019)$ in Fig. 7(b) which is obviously aligned with the result of (a).

(c) Since the value of D_p is 3.5 and the values of p_{11} and p_{12} are at 6 and 1 respectively, the controller gain matrix is

$$K(S) = \begin{bmatrix} 15 & 0 \\ 0 & 25 \end{bmatrix}. \tag{58}$$

Therefore, the overall controller is

$$P(S)K(S) = \begin{bmatrix} 6 & 1 \\ -3.5 & 0 \end{bmatrix} \begin{bmatrix} 1500 & \dfrac{S+1}{S+100} & 0 \\ 0 & 1250 & \dfrac{S+1}{S+50} \end{bmatrix}. \tag{59}$$

The characteristic roots of the closed-loop system are found at

$$-1, -1, -29.068, -35.845 \pm j43.818, -25.621 \pm j61.780.$$

The step responses of the overall system are shown in Fig. 8.

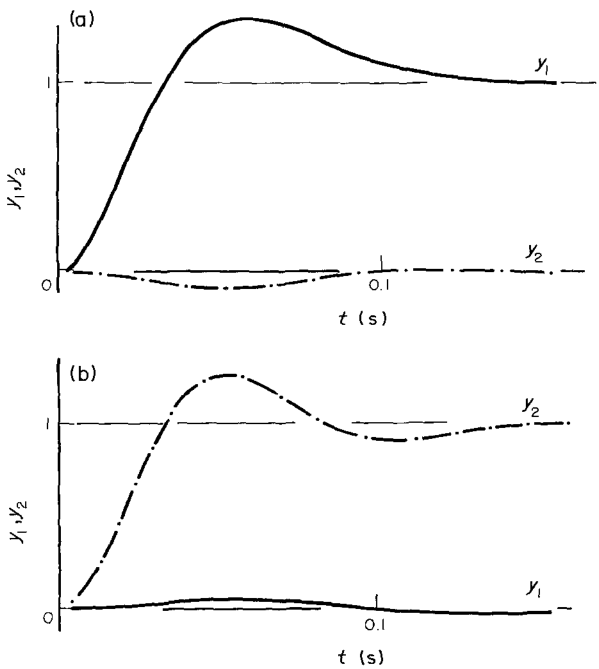


FIG. 8. Step responses of Example 3 for: (a) $r_1 = 1$ and $r_2 = 0$; (b) $r_1 = 0$ and $r_2 = 1$.

From the above given examples, the following remarks can be made:

(a) The design of multivariable feedback control systems by use of the stability-equation method is an interative and interactive procedure similar to the inverse Nyquist array method (**8**) and the characteristic root-locus method (**9**, **14**). However, the calculations necessary for achieving the closed-loop characteristics are easier in the stability-equation method.

(b) The stability-equation method can take into consideration the stability, integrity and performance of the system simultaneously. This advantage over the sequential-return-difference method (**15**, **16**) is attractive, since the sequential-return-difference method is a way of trial and error, i.e. the compensating element chosen for the first loop possesses consequential effects to the remaining loops which are generally unpredictable.

V. Conclusions

The stability-equation method has been extended and applied to the analysis and design of multivariable feedback control systems.

From the presented examples it can be seen that the system characteristics, such as stability, integrity and performance can be considered simultaneously; thus it is a useful tool for analysis and design.

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