Multipurpose Deadbeat Controller Synthesis in m-Dimensional Discrete Multivariable Systems

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ABSTRACT: *In this paper, a more general deadbeat controller design algorithm for the* m-*dimensional discrete-time multivariable system is introduced. Since the internal stability requirement is achieved, this algorithm can easily handle any stable or unstable minimum or nonminimum phase* m-*dimensional system. A simple technique is employed to achieve the following design purposes in the* m-*dimensional discrete-time multivariable feedback systems: (i) input–output decoupling; (ii) reference signal tracking and disturbance rejection where reference signals and disturbances can be different at each channel; (iii) deadbeat response with minimum space interval. An example is given to illustrate the validity of the proposed method.*

I. Introduction

The problems of m-dimensional systems are significant in that they can be applied to m-dimensional filters, one-dimensional time varying filters, circuits with varying elements, nonlinear systems, and systems with partial differential equations (**1, 2, 3, 4, 5**). This paper refers to the general area of feedback control design techniques of linear time-invariant multidimensional (m-D) discrete-time multivariable systems. Although multidimensional linear systems have received extensive attention in the last few years (**1**), very little work has been done in the area of multidimensional deadbeat control systems. Kaczorek (**6**) has examined a kind of deadbeat response in 2-D systems, in the sense that the output deviation from a reference input and plant input vanish after a minimum 2-D space interval. Tzafestas (**7**) developed a 2-D deadbeat controller using state feedback, which results in a steady output, after a minimum space interval, and for an appropriate input sequence. Recently, Tzafestas and Theodorou (**8**) examined a multidimensional open-loop deadbeat control problem, where the input sequence is specified such that the system achieves deadbeat response. However, in their work the concerned systems are restricted to the single-input single-output open loop systems.

In this paper, a more general control design problem for discrete-time m-dimensional multivariable feedback system is considered, i.e. a comparatively general control algorithm in one-dimensional systems advanced by Youla *et al.* (**9**) is extended to m-dimensional multivariable systems. Since the internal stability requirement is achieved, this algorithm can easily handle any stable or unstable, minimum or nonminimum phase m-dimensional system. This control algorithm can simultaneously achieve the following design purposes in the m-dimensional discrete-time multivariable systems: (i) input–output decoupling; (ii) reference signal tracking and disturbance rejection where reference signals and disturbances can be different at each channel; and (iii) deadbeat response with minimum space interval.

II. The Model and Problem Formulation

Consider the linear time-invariant m-dimensional discrete-time multivariable system (Fig. 1) described as an input–output model as follows: the reference signals $r(d_1, d_2, \ldots, d_m)$, error signals $e(d_1, d_2, \ldots, d_m)$, control signals $u(d_1, d_2, \ldots, d_m)$, external disturbances $d(d_1, d_2, \ldots, d_m)$ and output signals $y(d_1, d_2, \ldots, d_m)$ are vectors in C^n. And the plant $P(d_1, d_2, \ldots, d_m) \in C^{n \times n}$, the controller $C(d_1, d_2, \ldots, d_m) \in C^{n \times n}$ where $d_i = z_i^{-1}$, $i = 1, 2, \ldots, m$ denote the delay operators of each dimension. We assume that the plant $P(d_1, d_2, \ldots, d_m)$ is a rational matrix and det $(P(d_1, d_2, \ldots, d_m)) \neq 0$. The design purposes are to synthesize a general parameterized controller $C(d_1, d_2, \ldots, d_m)$ in order to stabilize the m-dimensional feedback system of Fig. 1; and to introduce a simple technique to specify the parameters of the above general controller to achieve the following purposes: (i) input–output decoupling; (ii) reference signal tracking and disturbance rejection where reference signals and disturbances can vary at each channel; and (iii) deadbeat response with minimum space interval.

III. Stabilizing Controller Synthesis

Before considering the controller synthesis of m-dimensional discrete-time multivariable feedback systems, a control algorithm given by Youla *et al.* (**9**) on one-dimensional continuous-time multivariable systems will be extended to m-dimensional discrete-time multivariable feedback systems.

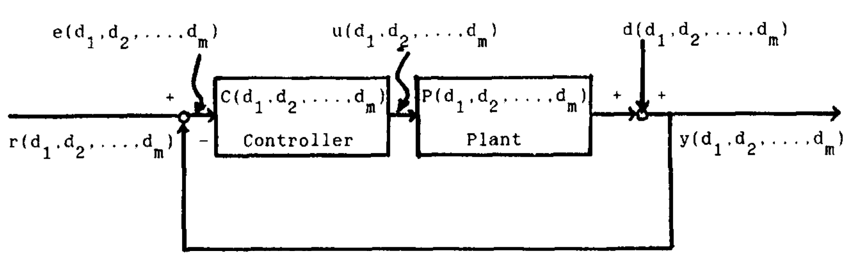

FIG. 1. m-Dimensional discrete-time multivariable feedback system.

Let (**2**)

$$P(d_1, d_2, \ldots, d_m) = A^{-1}(d_1, d_2, \ldots, d_m)B(d_1, d_2, \ldots, d_m)$$
$$= B_1(d_1, d_2, \ldots, d_m)A_1^{-1}(d_1, d_2, \ldots, d_m) \quad (1)$$

where $A(d_1, d_2, \ldots, d_m)$ and $B(d_1, d_2, \ldots, d_m)$ constitute any left-coprime polynomial matrix decomposition of $P(d_1, d_2, \ldots, d_m)$ and $B_1(d_1, d_2, \ldots, d_m)$ and $A_1(d_1, d_2, \ldots, d_m)$ constitute any right-coprime polynomial matrix decomposition of $P(d_1, d_2, \ldots, d_m)$ and select stable rational matrices (i.e. the matrices which are analytic in $|d_i| \leqslant 1$, $i = 1, 2, \ldots, m$) $X(d_1, d_2, \ldots, d_m)$, $Y(d_1, d_2, \ldots, d_m)$, $Y_1(d_1, d_2, \ldots, d_m)$ and $X_1(d_1, d_2, \ldots, d_m)$ such that [Bezout identity (**10**)]

$$\begin{bmatrix} -Y_1(d_1, d_2, \ldots, d_m) & X_1(d_1, d_2, \ldots, d_m) \\ A(d_1, d_2, \ldots, d_m) & B(d_1, d_2, \ldots, d_m) \end{bmatrix}$$
$$\cdot \begin{bmatrix} -B(d_1, d_2, \ldots, d_m) & X(d_1, d_2, \ldots, d_m) \\ A_1(d_1, d_2, \ldots, d_m) & Y(d_1, d_2, \ldots, d_m) \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & I_n \end{bmatrix}. \quad (2)$$

Since

$$U(d_1, d_2, \ldots, d_m)V(d_1, d_2, \ldots, d_m) = I \Rightarrow V(d_1, d_2, \ldots, d_m)U(d_1, d_2, \ldots, d_m) = I$$

we obtain

$$\begin{bmatrix} -B_1(d_1, d_2, \ldots, d_m) & X(d_1, d_2, \ldots, d_m) \\ A_1(d_1, d_2, \ldots, d_m) & Y(d_1, d_2, \ldots, d_m) \end{bmatrix}$$
$$\cdot \begin{bmatrix} -Y_1(d_1, d_2, \ldots, d_m) & X_1(d_1, d_2, \ldots, d_m) \\ A(d_1, d_2, \ldots, d_m) & B(d_1, d_2, \ldots, d_m) \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & I_n \end{bmatrix}. \quad (3)$$

Thus we achieve the following theorem.

Theorem I

The controller, which stabilizes the m-dimensional discrete-time multivariable system of Fig. 1 must be of the following form:

$$C(d_1, d_2, \ldots, d_m) = [Y(d_1, d_2, \ldots, d_m) + A_1(d_1, d_2, \ldots, d_m)K(d_1, d_2, \ldots, d_m)]$$
$$\cdot [X(d_1, d_2, \ldots, d_m) - B_1(d_1, d_2, \ldots, d_m)K(d_1, d_2, \ldots, d_m)]^{-1} \quad (4)$$

where $K(d_1, d_2, \ldots, d_m)$ is any $n \times n$ rational stable matrix (i.e. $K(d_1, d_2, \ldots, d_m)$ which must be analytic in $|d_i| \leqslant 1$, $i = 1, 2, \ldots, m$) and satisfy the constraint

$$\det [X(d_1, d_2, \ldots, d_m) - B_1(d_1, d_2, \ldots, d_m)K(d_1, d_2, \ldots, d_m)] \neq 0. \quad (5)$$

Proof. See Appendix A.

Remarks

(i) From Theorem I, it is seen that the controller $C(d_1, d_2, \ldots, d_m)$ in Eq. (4) is a general parameterizing controller of the m-dimensional discrete-time multivariable feedback system, i.e. for any stabilizing controller, a stable m-dimensional matrix $K(d_1, d_2, \ldots, d_m)$ in Eq. (4) can be found to correspond with it (i.e. the stabilizing controller always exists).

(ii) Suppose the plant $P(d_1, d_2, \ldots, d_m)$ has both strictly proper right-coprime and left-coprime factorizations, then there exists a proper controller as in Eq. (4) to stabilize $P(d_1, d_2, \ldots, d_m)$ in the closed loop configuration of Fig. 1 (**11**).

(iii) In a one-dimensional case, Theorem I will be reduced to Lemma 3 in (**9**). From the above analysis, the main work of our design problem then is how to find an adequate stable m-dimensional matrix $K(d_1, d_2, \ldots, d_m)$ to achieve the design purposes.

Let us define the sensitivity function matrix $S(d_1, d_2, \ldots, d_m)$ as

$$S(d_1, d_2, \ldots, d_m) = [I + P(d_1, d_2, \ldots, d_m)C(d_1, d_2, \ldots, d_m)]^{-1} \tag{6}$$

then the transfer function matrix is given as

$$I - S(d_1, d_2, \ldots, d_m) = P(d_1, d_2, \ldots, d_m)C(d_1, d_2, \ldots, d_m) \cdot [I + P(d_1, d_2, \ldots, d_m)C(d_1, d_2, \ldots, d_m)]^{-1} \tag{7}$$

and we obtain the following lemma.

Lemma 1

If we choose $C(d_1, d_2, \ldots, d_m)$ in Eq. (4) with any stable matrix $K(d_1, d_2, \ldots, d_m)$ as a stabilizing controller of the m-dimensional multivariable feedback systems, then the sensitivity function matrix $S(d_1, d_2, \ldots, d_m)$ and the transfer function matrix $I - S(d_1, d_2, \ldots, d_m)$ must be of the following form:

(i) $$S(d_1, d_2, \ldots, d_m) = [X(d_1, d_2, \ldots, d_m) - B_1(d_1, d_2, \ldots, d_m) \cdot K(d_1, d_2, \ldots, d_m)]A(d_1, d_2, \ldots, d_m) \tag{8}$$

(ii) $$I - S(d_1, d_2, \ldots, d_m) = B_1(d_1, d_2, \ldots, d_m)[Y_1(d_1, d_2, \ldots, d_m) + K(d_1, d_2, \ldots, d_m)A(d_1, d_2, \ldots, d_m)]. \tag{9}$$

Proof. See Appendix B.

Remarks

(i) Since $C(d_1, d_2, \ldots, d_m)$ in Eq. (4) is the general stabilizing controller, Lemma 1 implies that if a m-dimensional closed-loop feedback system is asymptotically stable, then $S(d_1, d_2, \ldots, d_m)$ and $I - S(d_1, d_2, \ldots, d_m)$ must be of the form shown in Eqs. (8) and (9), respectively.

(ii) Since $S(d_1, d_2, \ldots, d_m)$ still contains $A(d_1, d_2, \ldots, d_m)$ as in Eq. (8) and $I - S(d_1, d_2, \ldots, d_m)$ still contains $B_1(d_1, d_2, \ldots, d_m)$ as in Eq. (9), it implies that there is not any pole-zero cancellation between $C(d_1, d_2, \ldots, d_m)$ and $P(d_1, d_2, \ldots, d_m)$ in $|d_i| \leqslant 1$, $i = 1, 2, \ldots, m$, i.e. the closed loop is internally stable (**9, 12**) (i.e. this controller can easily handle any stable or unstable, minimum or nonminimum phase system).

(iii) Under the above asymptotically stable constraints on $S(d_1, d_2, \ldots, d_m)$ and $I - S(d_1, d_2, \ldots, d_m)$ as in (8) and (9), respectively, we can choose an adequate $K(d_1, d_2, \ldots, d_m)$ to achieve design purposes on $S(d_1, d_2, \ldots, d_m)$ and $I - S(d_1, d_2, \ldots, d_m)$.

IV. Multipurpose Deadbeat Controller Design

In this section, a simple technique is employed to choose an adequate m-dimensional matrix $K(d_1, d_2, \ldots, d_m)$ in the stabilizing controller $C(d_1, d_2, \ldots, d_m)$ of Eq. (4) to achieve the following design purposes: (i) input–output decoupling; (ii) reference signal tracking and disturbance rejection where reference signals and disturbances can be different at each channel; and (iii) deadbeat response with minimum space interval.

From Fig. 1, it is seen that the error signals $e(d_1, d_2, \ldots, d_m)$ are given as

$$e(d_1, d_2, \ldots, d_m) = S(d_1, d_2, \ldots, d_m)\,[r(d_1, d_2, \ldots, d_m) - d(d_1, d_2, \ldots, d_m)] \tag{10}$$

where reference signals $r(d_1, d_2, \ldots, d_m)$ and disturbances $d(d_1, d_2, \ldots, d_m)$ are rational vectors, and the denominator of $r(d_1, d_2, \ldots, d_m)$ and $d(d_1, d_2, \ldots, d_m)$ are known polynomials but the numerators of $r(d_1, d_2, \ldots, d_m)$ and $(d_1, d_2, \ldots, d_m)$ can be arbitrarily unknown polynomials [i.e. $r(d_1, d_2, \ldots, d_m)$ and $d(d_1, d_2, \ldots, d_m)$ can denote a class of reference signals and disturbances, respectively, not only the particular reference signal and disturbance].

Under the decoupling and deadbeat response constraints, the sensitivity function matrix $S(d_1, d_2, \ldots, d_m)$ can be chosen as a diagonal polynomial matrix of the following form:

$$S(d_1, d_2, \ldots, d_m) = \begin{bmatrix} s_1(d_1, d_2, \ldots, d_m) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & s_n(d_1, d_2, \ldots, d_m) \end{bmatrix} \tag{11}$$

where $s_j(d_1, d_2, \ldots, d_m)$, $j = 1, 2, \ldots, n$ are monic polynomials with zero-order (i.e. $s_j(0, 0, \ldots, 0) = 1$, $j = 1, 2, \ldots, n$).

Definition. Reference signal tracking and disturbance rejection with deadbeat response occur in the system of Fig. 1, if the error signal is a polynomial vector with a finite degree (i.e. the error signals vanish after a finite period of space interval).

From Eq. (10), in order to let the error signal be a polynomial vector, the sensitivity function matrix $S(d_1, d_2, \ldots, d_m)$ must cancel all the poles of reference signals $r(d_1, d_2, \ldots, d_m)$ and disturbances $d(d_1, d_2, \ldots, d_m)$.

To achieve these purposes, the sensitivity function matrix takes the following form:

$$S(d_1, d_2, \ldots, d_m) = \begin{bmatrix} H_1(d_1, d_2, \ldots, d_m)W_1(d_1, d_2, \ldots, d_m) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & H_n(d_1, d_2, \ldots, d_m)W_n(d_1, d_2, \ldots, d_m) \end{bmatrix} \tag{12}$$

where the monic polynomials $W_j(d_1, d_2, \ldots, d_m)$, $j = 1, 2, \ldots, n$ are with the desired zeros to cancel all the poles of reference signals and external disturbances in each channel for reference signal tracking and disturbance rejection. $H_j(d_1, d_2, \ldots, d_m)$, $j = 1, 2, \ldots, n$ are undetermined monic polynomials and will be determined by the asymptotical stability of feedback system [i.e. the analyticity of $K(d_1, d_2, \ldots, d_m)$ in $|d_i| \leqslant 1$ for $i = 1, 2, \ldots, m$].

From Lemma 1, since we choose $C(d_1, d_2, \ldots, d_m)$ as a stabilizing controller, $S(d_1, d_2, \ldots, d_m)$ must be of the form shown in Eq. (8), i.e.

$$\begin{aligned} S(d_1, d_2, \ldots, d_m) &= [X(d_1, d_2, \ldots, d_m) - B_1(d_1, d_2, \ldots, d_m) \\ &\quad \cdot K(d_1, d_2, \ldots, d_m)]A(d_1, d_2, \ldots, d_m) \\ &= \begin{bmatrix} H_1(d_1, d_2, \ldots, d_m)W_1(d_1, d_2, \ldots, d_m) & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & H_n(d_1, d_2, \ldots, d_m)W_n(d_1, d_2, \ldots, d_m) \end{bmatrix}. \end{aligned} \tag{13}$$

From the above equation, we obtain

$$\begin{aligned} K(d_1, d_2, \ldots, d_m) &= -B_1^{-1}(d_1, d_2, \ldots, d_m) \\ &\cdot \left\{ \begin{bmatrix} H_1(d_1, d_2, \ldots, d_m)W_1(d_1, d_2, \ldots, d_m) & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & H_n(d_1, d_2, \ldots, d_m)W_n(d_1, d_2, \ldots, d_m) \end{bmatrix} A^{-1}(d_1, d_2, \ldots, d_m) \right. \\ &\qquad \left. - X(d_1, d_2, \ldots, d_m) \right\}. \end{aligned} \tag{14}$$

Since $K(d_1, d_2, \ldots, d_m)$ must be analytic in $|d_i| \leqslant 1$, $i = 1, 2, \ldots m$ the m-dimensional polynomials $H_j(d_1, d_2, \ldots, d_m)$, $j = 1, 2, \ldots, n$ are therefore determined by the requirement of asymptotical stability of $K(d_1, d_2, \ldots, d_m)$ [i.e. we choose adequate coefficients of $H_j(d_1, d_2, \ldots, d_m)$, $j = 1, 2, \ldots, n$ to cancel the poles in $|d_i| \leqslant 1$, $i = 1, 2, \ldots, m$ on the right-hand side of Eq. (14) to guarantee the asymptotical stability of $K(d_1, d_2, \ldots, d_m)$]. After the m-dimensional polynomials $H_j(d_1, d_2, \ldots, d_m)$, $j = 1, 2, \ldots, n$ are specified by the requirement of analyticity of $K(d_1, d_2, \ldots, d_m)$ in $|d_i| \leqslant 1$, $i = 1, 2, \ldots, m$ and substitute $K(d_1, d_2, \ldots, d_m)$ into the controller $C(d_1, d_2, \ldots, d_m)$ as in Eq. (4), we obtain the corresponding deadbeat controller. From the above analysis, we obtain the following multipurpose deadbeat controller design algorithm in the m-dimensional discrete-time multivariable systems.

Algorithm

Step 1. Perform the left and right coprime factorization of the plant as in Eq. (1), and then from Eq. (2) we can solve the following equation:

$$A(d_1, d_2, \ldots, d_m)X(d_1, d_2, \ldots, d_m) + B(d_1, d_2, \ldots, d_m)Y(d_1, d_2, \ldots, d_m) = I_n \tag{15}$$

to obtain the stable rational matrices $X(d_1, d_2, \ldots, d_m)$ and $Y(d_1, d_2, \ldots, d_m)$.

Step 2. Choose the sensitivity function matrix to be of the form as in Eq. (12), where the monic polynomials $W_j(d_1, d_2, \ldots, d_m)$, $j = 1, 2, \ldots, n$ are with desired zeros to cancel all the poles of reference signals and external disturbances of each channel for reference signals tracking and disturbances rejection.

Step 3. Determine the m-dimensional polynomials $H_j(d_1, d_2, \ldots, d_m)$, $j = 1, 2, \ldots, n$ by the requirement of asymptotical stability of $K(d_1, d_2, \ldots, d_m)$ [i.e. choose adequate coefficients of $H_j(d_1, d_2, \ldots, d_m)$, $j = 1, 2, \ldots, n$ to cancel the poles in $|d_i| \leqslant 1$ on the right-hand side of Eq. (14)], then obtain the stable rational matrix $K(d_1, d_2, \ldots, d_m)$ as in (14).

Step 4. Substitute $K(d_1, d_2, \ldots, d_m)$ into Eq. (4), and obtain the corresponding multipurpose deadbeat controller as in (4).

Remarks

(i) If some of the zeros of $H_j(d_1, d_2, \ldots, d_m)$ are equal to the zeros of $W_j(d_1, d_2, \ldots, d_m)$, then the least common multiplier of $H_j(d_1, d_2, \ldots, d_m)$ and $W_j(d_1, d_2, \ldots, d_m)$ is chosen to be a factor of $s_j(d_1, d_2, \ldots, d_m)$, $j = 1, 2, \ldots, n$.

(ii) If the minimum number of free parameters of $H_j(d_1, d_2, \ldots, d_m)$, $j = 1, 2, \ldots n$ in (14) is just uniquely determined by the requirement of analyticity of $K(d_1, d_2, \ldots, d_m)$ in $|d_i| \leqslant 1$, then it is the solution of the minimum space interval deadbeat controller. As the number of free parameters of $H_j(d_1, d_2, \ldots, d_m)$, $j = 1, 2, \ldots, n$ increase, this leads to an over-parameterized solution, and we have more freedom to assign some coefficients of $H_j(d_1, d_2, \ldots, d_m)$, $j = 1, 2, \ldots, n$ (of course, this is not the case of minimum space interval deadbeat response again).

(iii) In the rectangular full rank plant case, $B_1^{-1}(d_1, d_2, \ldots, d_m)$ in (14) must be substituted by the pseudo inverse

$$B_1^{+}(d_1, d_2, \ldots, d_m) = [B_1^T(d_1, d_2, \ldots, d_m) B_1(d_1, d_2, \ldots, d_m)]^{-1} \cdot B_1^T(d_1, d_2, \ldots, d_m). \quad (16)$$

V. Example

In this example, we consider the 3-dimensional system (Fig. 1), where the reference signals and disturbances are different at each channel. Assume the plant can be described as

$$P(d_1, d_2, \ldots, d_m) = \begin{bmatrix} \dfrac{1}{d_1 + d_2 + d_3 - 5} & \dfrac{d_1 + d_2 + d_3 - 1}{d_1 + d_2 + d_3 - 5} \\ 0 & \dfrac{1}{d_1 + d_2 + d_3 - 1} \end{bmatrix} \quad (17)$$

and the disturbances are given as $d(d_1, d_2, d_3) = \left[\dfrac{x_1}{1 - d_1 d_2}, 0\right]^T$, where x_1 is an arbitrary scale.

How do we synthesize a controller to decouple the system, track the reference signals $r(d_1, d_2, d_3) = [x_0/(1-d_3), y_0/(1-d_1)]^T$ where x_0 and y_0 are arbitrary scales, and reject the disturbances and simultaneously achieve the deadbeat response with minimum space interval?

Solution. Follow our algorithm step by step.

Step 1. Performing the left and right coprime factorizations of $P(d_1, d_2, d_3)$, we obtain

$$\begin{aligned} P(d_1,d_2,d_3) &= A^{-1}(d_1,d_2,d_3)B(d_1,d_2,d_3) \\ &= B_1(d_1,d_2,d_3)A_1^{-1}(d_1,d_2,d_3) \end{aligned} \tag{18}$$

where

$$A(d_1,d_2,d_3) = \begin{bmatrix} d_1+d_2+d_3-5 & 0 \\ 0 & d_1+d_2+d_3-1 \end{bmatrix} \tag{19}$$

$$B(d_1,d_2,d_3) = \begin{bmatrix} 1 & d_1+d_2+d_3-1 \\ 0 & 1 \end{bmatrix} \tag{20}$$

$$A_1(d_1,d_2,d_3) = \begin{bmatrix} (d_1+d_2+d_3-5)^2 & 0 \\ 0 & (d_1+d_2+d_3-5)(d_1+d_2+d_3-1) \end{bmatrix} \tag{21}$$

$$B_1(d_1,d_2,d_3) = \begin{bmatrix} d_1+d_2+d_3-5 & (d_1+d_2+d_3-1)^2 \\ 0 & d_1+d_2+d_3-5 \end{bmatrix}. \tag{22}$$

Solving the following equation

$$A(d_1,d_2,d_3)X(d_1,d_2,d_3)+B(d_1,d_2,d_3)Y(d_1,d_2,d_3) = I_2, \tag{23}$$

we obtain

$$X(d_1,d_2,d_3) = \begin{bmatrix} \dfrac{1}{d_1+d_2+d_3-5} & \dfrac{4(d_1+d_2+d_3-1)}{(d_1+d_2+d_3-5)^2} \\ 0 & \dfrac{1}{d_1+d_2+d_3-5} \end{bmatrix} \tag{24}$$

and

$$Y(d_1,d_2,d_3) = \begin{bmatrix} 0 & 0 \\ 0 & \dfrac{-4}{d_1+d_2+d_3-5} \end{bmatrix}. \tag{25}$$

Step 2. In order to achieve the reference signal tracking and disturbance rejection with deadbeat response, the sensitivity function matrix $S(d_1, d_2, d_3)$ can be chosen in the following form:

$$S(d_1,d_2,d_3) = \begin{bmatrix} H_1(d_1,d_2,d_3)(1-d_1d_2)(1-d_3) & 0 \\ 0 & H_2(d_1,d_2,d_3)(1-d_1) \end{bmatrix} \tag{26}$$

where $H_j(d_1, d_2, d_3)$, $j = 1,2$ are to be determined by the analyticity of $K(d_1, d_2, d_3)$ in $|d_i| \leqslant 1$, $i = 1,2,3$.

Step 3. From Eq. (14), since

$$K(d_1,d_2,d_3) = -B_1^{-1}(d_1,d_2,d_3)[S(d_1,d_2,d_3)A^{-1}(d_1,d_2,d_3)-X(d_1,d_2,d_3)]$$

$$= \frac{1}{(d_1+d_2+d_3-5)^2}\begin{bmatrix} 1-H_1(d_1,d_2,d_3)(1-d_1d_2)(1-d_3) & -(d_1+d_2+d_3-1)(1-H_2(d_1,d_2,d_3)(1-d_1)) \\ 0 & 1-\dfrac{d_1+d_2+d_3-5}{d_1+d_2+d_3-1}H_2(d_1,d_2,d_3)(1-d_1) \end{bmatrix} \tag{27}$$

must be analytic in $|d_i| \leqslant 1$, $i = 1,2,3$, we obtain

$$H_1(d_1,d_2,d_3) = 1 \tag{28}$$

and

$$H_2(d_1,d_2,d_3) = d_1+d_2+d_3-1. \tag{29}$$

Substituting Eqs. (28) and (29) into Eq. (27), we obtain

$$K(d_1,d_2,d_3) = \frac{1}{(d_1+d_2+d_3-5)^2} \cdot \begin{bmatrix} 1-(1-d_1d_2)(1-d_3) & -(d_1+d_2+d_3-1)(1-(d_1+d_2+d_3-1)(1-d_1)) \\ 0 & 1-(d_1+d_2+d_3-5)(1-d_1) \end{bmatrix}. \tag{30}$$

Step 4. Substituting (30) into (4), we obtain the multipurpose deadbeat controller as

$$C(d_1,d_2,d_3) = [Y(d_1,d_2,d_3)+A_1(d_1,d_2,d_3)K(d_1,d_2,d_3)] \cdot [X(d_1,d_2,d_3)-B_1(d_1,d_2,d_3)K(d_1,d_2,d_3)]^{-1}$$

$$= \begin{bmatrix} \dfrac{(d_1+d_2+d_3-5)(1-(1-d_1d_2)(1-d_3))}{(1-d_1d_2)(1-d_3)} & \dfrac{-(d_1+d_2+d_3-1)(1-(d_1+d_2+d_3-1)(1-d_1))}{1-d_1} \\ 0 & \dfrac{1-(d_1+d_2+d_3-1)(1-d_1)}{1-d_1} \end{bmatrix}. \quad (31)$$

Check: From Eq. (6), we have

$$S(d_1,d_2,d_3) = [I+P(d_1,d_2,d_3)C(d_1,d_2,d_3)]^{-1}$$

$$= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \dfrac{1}{d_1+d_2+d_3-5} & \dfrac{d_1+d_2+d_3-1}{d_1+d_2+d_3-5} \\ 0 & \dfrac{1}{d_1+d_2+d_3-1} \end{bmatrix} \cdot \begin{bmatrix} \dfrac{(d_1+d_2+d_3-5)(1-(1-d_1d_2)(1-d_3))}{(1-d_1d_2)(1-d_3)} & \dfrac{-(d_1+d_2+d_3-1)(1-(d_1+d_2+d_3-1)(1-d_1))}{1-d_1} \\ 0 & \dfrac{1-(d_1+d_2+d_3-1)(1-d_1)}{1-d_1} \end{bmatrix} \right\}^{-1}$$

$$= \begin{bmatrix} (1-d_1d_2)(1-d_3) & 0 \\ 0 & (d_1+d_2+d_3-1)(1-d_1) \end{bmatrix} \quad (32)$$

and the error signals

$$e(d_1,d_2,d_3) = S(d_1,d_2,d_3)[r(d_1,d_2,d_3)-d(d_1,d_2,d_3)]$$
$$= [(x_0-x_1)+x_1d_3-x_0d_1d_2, y_0(-1+d_1+d_2+d_3)]^T, \quad (33)$$

i.e. the system can track reference signals and reject disturbances and achieve deadbeat response with minimum space interval.

VI. Conclusion

In this paper, a very simple and direct algorithm is introduced to synthesize a more general deadbeat controller which simultaneously ensures a variety of purposes in the m-dimensional discrete-time multivariable systems. Internal stability is achieved so that this algorithm can easily handle any stable or unstable, minimum or nonminimum phase system.

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Appendix A. Proof of Theorem I

If we assume the stabilizing controller C is of the following form:

$$C = ND^{-1} \tag{A.1}$$

where N and D are coprime m-dimensional polynomial matrices, then the system input–output transfer function matrix of Fig. 1 is of the following form:

$$\begin{aligned} T_{yr} &= A^{-1}BND^{-1}[I+A^{-1}BND^{-1}]^{-1} \\ &= I-D[AD+BN]^{-1}A \end{aligned}$$

[from Eq. (1) and $(W'+X'Y'Z')^{-1} = W'^{-1}-W'^{-1}X'(Y'^{-1}+Z'W'^{-1}X')^{-1}Z'W'^{-1}$]. (A.2)

As D and A are polynomial matrices, then C will be a stabilizing controller if

$$AD+BN = L \tag{A.3}$$

has a stable inverse (i.e. L is a minimum phase matrix): The synthesis of the stabilizing controller $C = ND^{-1}$ becomes how to choose m-dimensional polynomial matrices D and N such that L has a stable inverse.

Since A and B are coprime, it follows that homogeneous solution for N and D in the algebra Eq. (A3) is given as (**13**)

$$D_h = -B_1 Q \tag{A.4}$$

$$N_h = A_1 Q \tag{A.5}$$

which satisfy the homogeneous equation (since $BA_1 = AB_1$):

$$AD+BN = 0 \tag{A.6}$$

for any m-dimensional polynomial matrix Q.

From Eq. (2), we obtain the following equation:

$$AX+BY = I_n. \tag{A.7}$$

Multiplying the above equation from the right by L, we obtain

$$AXL+BYL = L. \tag{A.8}$$

By comparing (A3) with (A8), we can obtain a particular solution of (A3) as, (**13**),

$$D_p = XL \tag{A.9}$$

$$N_p = YL. \tag{A.10}$$

From the above analysis, the solution of D and N in (A3) is given by

$$D = -B_1 Q + XL \tag{A.11}$$

$$N = A_1 Q + YL \tag{A.12}$$

where Q is any m-dimensional polynomial matrix and L is any m-dimensional polynomial matrix which has a stable inverse.

As a result, the stabilizing controller C must be of the following form:

$$C = ND^{-1} = [YL+A_1 Q][XL-B_1 Q]^{-1}. \tag{A.13}$$

Since L has a stable inverse, the stabilizing controller takes the following form:

$$C = [Y+A_1 K][X-B_1 K]^{-1} \tag{A.14}$$

where $K = QL^{-1}$ is any stable m-dimensional rational matrix.

Let us choose the controller in Fig. 1, as in (4), then

$$\begin{aligned}
T_{yr} &= PC[I+PC]^{-1} \\
&= B_1 A_1^{-1}[Y+A_1 K][X-B_1 K]^{-1}\{I+B_1 A_1^{-1}[Y+A_1 K][X-B_1 K]^{-1}\}^{-1} \\
&\qquad\qquad \text{[from Eqs. (1) and (4)]} \\
&= B_1 A_1^{-1}[Y+A_1 K][X+A^{-1}BY]^{-1} \\
&= B_1 A_1^{-1}[Y+A_1 K]A \qquad \text{[from (2) and } X+A^{-1}BY = A^{-1}] \\
&= B_1[A_1^{-1} YA+KA] \\
&= B_1[Y_1+KA] \qquad \text{[from (2) and } A_1 Y_1 = YA].
\end{aligned} \tag{A.15}$$

Since B_1 and A are m-dimensional polynomial matrices and K and Y_1 are stable rational matrices analytic in $|d_i| \leqslant 1$, $i = 1, 2, \ldots, m$, hence T_{yr} is stable. Q.E.D.

Appendix B. Proof of Lemma 1

(i) $S = \{I + B_1 A_1^{-1}[Y + A_1 K][X - B_1 K]^{-1}\}^{-1}$ [from Eqs. (1) and (4)]

$$= \{(X - B_1 K + B_1 A_1^{-1}[Y + A_1 K])[X - B_1 K]^{-1}\}^{-1}$$

$$= \{[X + B_1 A_1^{-1} Y][X - B_1 K]^{-1}\}^{-1}$$

$$= \{A^{-1}[X - B_1 K]^{-1}\}^{-1} \quad \text{[from Eq. (3) and } A^{-1}B = B_1 A_1^{-1}\text{]}$$

$$= [X - B_1 K]A. \tag{B.1}$$

(ii) The proof is obvious from (8) and $B_1 Y_1 + XA = I_n$ in (3). Q.E.D.