TABLE I
SOLUTIONS FOR PARAMETERIZED INITIAL CONDITIONS

α	x_1	x_2	v_1	v_2	c_1	c_2	c_3	c_4	T
0.0	18.028	0.000	-4.123	0.000	0.194	0.000	0.199	0.000	6.177
0.1	17.997	1.059	-4.087	0.545	0.193	0.036	0.213	0.080	6.215
0.2	17.903	2.115	-3.979	1.081	0.184	0.100	0.239	0.288	6.380
0.3	17.748	3.164	-3.801	1.597	0.140	0.208	0.190	0.777	6.813
0.4	17.531	4.201	-3.557	2.086	0.061	0.299	0.000	1.405	7.622
0.5	17.254	5.224	-3.250	2.537	0.014	0.314	-0.128	1.745	8.631
0.6	16.917	6.229	-2.886	2.945	0.003	0.295	-0.128	1.856	9.625
0.7	16.522	7.213	-2.471	3.300	0.007	0.272	-0.058	1.882	10.528
0.8	16.070	8.171	-2.013	3.598	0.016	0.252	0.042	1.876	11.330
0.9	15.562	9.101	-1.520	3.833	0.027	0.236	0.156	1.855	12.033
1.0	15.000	10.000	-1.000	4.000	0.039	0.221	0.276	1.826	12.641

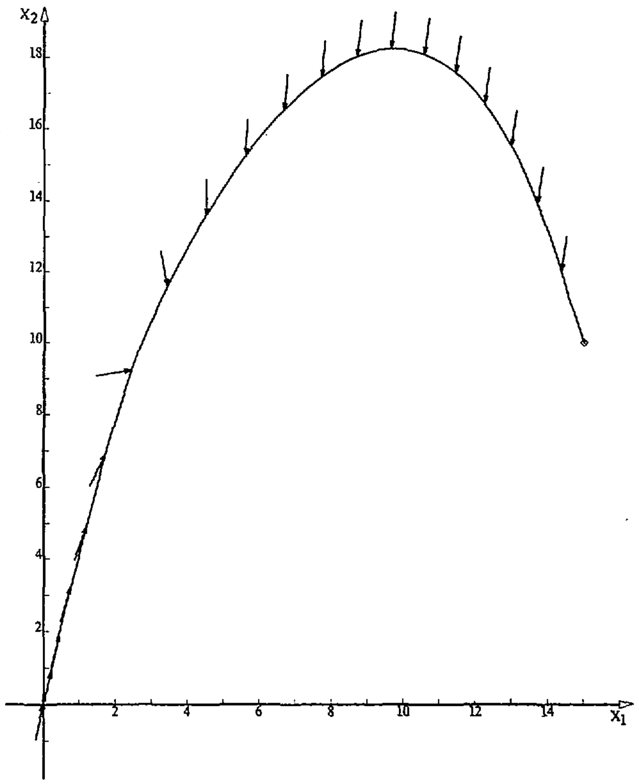

Fig. 1. Time optimal trajectory and control vector.

The reduction of the problem formulated in this note to a set of nonlinear equations makes it feasible to compute the optimal control in real time. Thus, the simple dynamic model (1) could be used as the basis for a feedback control scheme in which the control is recomputed at each sampling instant using the current state. This *open-loop feedback* approach to real-time steering control is considered in [10], [11]. The results of this correspondence are currently being incorporated as a part of the feedback algorithm proposed in [11].

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Discrete Optimal Control with Eigenvalue Assigned Inside a Circular Region

TSU-TIAN LEE AND SHIOW-HARN LEE

Abstract—**A discrete-time optimal control that guarantees that all the closed-loop poles will lie inside a circle centered at $(\beta, 0)$ with radius α is formulated. It is shown how the exposed problem can be reduced to a standard discrete-time linear quadratic regulator problem. Furthermore, a quantitative measure of the robustness of linear quadratic state feedback design in the presence of a perturbation is obtained. Bounds are derived for allowable nonlinear perturbations such that the resultant closed loop is stable.**

I. INTRODUCTION

For a continuous-time system which is stabilizable and detectable, Anderson and Moore [1] have shown how it is possible to minimize a quadratic performance index and, at the same time, to ensure that the closed-loop system will have poles with real parts all less than some real number α. Similarly, Franklin and Powell [2] have derived a state variable feedback law that minimizes a discrete-time quadratic performance index and, meanwhile, ensures that the closed-loop system has poles all less than $\alpha \leq 1$. The aim of this note is to formulate a discrete-time quadratic minimization problem in such a way as to give rise to a linear state variable feedback law guaranteeing that closed-loop poles all lie inside a circle centered at $(\beta, 0)$ with radius α, where $\alpha + |\beta| \leq 1$. Moreover, it is known that the stability of a discrete-time linear quadratic regulator is guaranteed. But the behavior of such regulated discrete-time systems to nonlinear perturbations is unknown. We have, therefore, derived bounds for allowable nonlinear perturbations such that the resultant closed-loop is stable.

II. OPTIMIZATION WITH PRESCRIBED CLOSED-LOOP POLES INSIDE A CIRCULAR REGION

Consider a linear time-invariant discrete-time controllable system

$$X(p+1) = AX(p) + BU(p), \qquad X(0) = X_0 \tag{1}$$

where X is an $n \times 1$ state vector, U is an $r \times 1$ control vector, and A and B are $n \times n$ and $n \times r$ constant matrices, respectively.

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The problem considered in this section is to formulate an optimal control problem such that the optimal control that minimizes the specified performance index will at the same time place the closed-loop poles inside a circular region as shown in Fig. 1. Recall that given plant dynamics (1) and the performance index

$$J=\sum_{p=0}^{\infty}\left(\frac{1}{\alpha}\right)^{2p}\{X^T(p)QX(p)+U^T(p)RU(p)\} \tag{2}$$

where $Q = Q^T \geq 0$, and $R = R^T > 0$, it is well known [2] that the optimal control which minimizes (2) will have closed-loop poles inside a circle centered at the origin with radius $r = \alpha$.

Now it becomes clear that if all the poles of the feedback control that minimizes (2) are shifted by $-\beta$, then all the poles will be inside the circular region as shown in Fig. 1. That is, if

$$\bar{X}(p+1)=A_\beta\bar{X}(p)+B\bar{U}(p), \text{ where } A_\beta=A-\beta I \tag{3}$$

then the optimal control which minimizes

$$J=\sum_{p=0}^{\infty}\left(\frac{1}{\alpha}\right)^{2p}[\bar{X}^T(p)Q\bar{X}(p)+\bar{U}^T(p)R\bar{U}(p)] \tag{4}$$

will have all poles inside the circular region of Fig. 1. Thus, the problem boils down to finding a performance index so that the resultant optimal control subject to (1) is equivalent to the optimal control that minimizes (4) subject to (3). The following theorem states the result.

Theorem 1: The minimization problem with respect to (3) and the performance index (4) is equivalent to the minimization problem with respect to (1) and the performance index

$$J=\sum_{p=0}^{\infty}\left(\frac{1}{\alpha}\right)^{2p}\left\{\left[\sum_{i=0}^{p}d_{p,i}X(i)\right]^T Q\left[\sum_{i=0}^{p}d_{p,i}X(i)\right]+\left[\sum_{i=0}^{p}d_{p,i}U(i)\right]^T R\left[\sum_{i=0}^{p}d_{p,i}U(i)\right]\right\} \tag{5}$$

where

$$d_{p,i}=C_i^p(-\beta)^{p-i},\quad C_i^p=\frac{p!}{(p-i)!i!} \tag{6}$$

$Q = Q^T \geq 0$, $R = R^T > 0$, and $(A, B, Q^{1/2})$ is minimal, in the following sense.

i) The minimum value of (4) is the same as the minimum value of (5).

ii) If $U(p)$ is the optimal control for (1) and (5), $\bar{U}(p) = \sum_{j=0}^{p} d_{p,j}U(j)$ is the optimal control for (3) and (4) and conversely.

Note that Theorem 1 can be proven from the following lemma.

Lemma 1: The transformations

$$\bar{X}(p)=\sum_{j=0}^{p}C_j^p(-\beta)^{p-j}X(j) \tag{7}$$

and

$$\bar{U}(p)=\sum_{j=0}^{p}C_j^p(-\beta)^{p-j}U(j) \tag{8}$$

where

$$C_j^p=\frac{p!}{(p-j)!j!} \tag{9}$$

will transform the plant dynamics

$$X(p+1)=AX(p)+BU(p) \tag{10}$$

into

$$\bar{X}(p+1)=(A-\beta I)\bar{X}(p)+B\bar{U}(p)=A_\beta\bar{X}(p)+B\bar{U}(p). \tag{11}$$

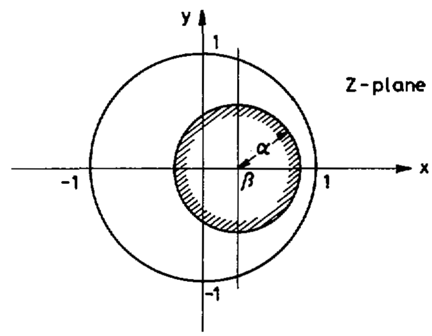

Fig. 1. The desired circular region for closed-loop poles. ($0 < \alpha \leq 1$, $\alpha + |\beta| \leq 1$).

Proof: From (7)–(9), it follows that $\bar{X}(0) = X(0)$ and $\bar{U}(0) = U(0)$. From (7),

$$\bar{X}(1)=X(1)-\beta X(0)=AX(0)-\beta X(0)+BU(0)=(A-\beta I)\bar{X}(0)+B\bar{U}(0).$$

Thus, (11) is true for $P = 0$. In the following, we shall prove that (11) holds for $P = K$ ($K = 1, 2, \cdots$).

Substitute $P = K$ into (11) yielding

$$\bar{X}(K+1)=(A-\beta I)\bar{X}(K)+B\bar{U}(K). \tag{12}$$

The left-hand side of (12), after using the relation of (7), can be written as

$$\bar{X}(K+1)=\sum_{j=0}^{K+1}C_j^{K+1}(-\beta)^{K+1-j}X(j)=X(K+1)+\sum_{j=1}^{K}C_j^{K+1}(-\beta)^{K+1-j}X(j)+(-\beta)^{K+1}X(0). \tag{13}$$

Recall that $C_j^{K+1} = C_j^K + C_{j-1}^K$. Therefore, (13) can be rewritten as

$$\bar{X}(K+1)=AX(K)+BU(K)+\sum_{j=1}^{K}C_{j-1}^K(-\beta)^{K+1-j}X(j)+\sum_{j=1}^{K}C_j^K(-\beta)^{K+1-j}X(j)-\beta C_0^K(-\beta)^K X(0). \tag{15}$$

Note that the term

$$\sum_{j=1}^{K}C_j^K(-\beta)^{K+1-j}X(j)-\beta C_0^K(-\beta)^K X(0)=(-\beta)\sum_{j=0}^{K}C_j^K(-\beta)^{K-j}X(j)=-\beta\bar{X}(K). \tag{16}$$

Hence, (15) can be reexpressed as

$$\bar{X}(K+1)=AC_K^KX(K)+BC_K^KU(K)+\sum_{j=1}^{K}C_{j-1}^K(-\beta)^{K+1-j}\cdot(AX(j-1)+BU(j-1))-\beta\bar{X}(K)=AC_K^KX(K)+BC_K^KU(K)+\sum_{j=0}^{K-1}C_j^K(-\beta)^{K-j}(AX(j)+BU(j))-\beta\bar{X}(K) \tag{17}$$

which can be further simplified to

$$\bar{X}(K+1)=A\sum_{j=0}^{K}C_j^K(-\beta)^{K-j}X(j)+B\sum_{j=0}^{K}C_j^K(-\beta)^{K-j}U(j)-\beta\bar{X}(K). \tag{18}$$

Hence,

$$\bar{X}(K+1)=(A-\beta I)\bar{X}(K)+B\bar{U}(K). \quad (19)$$

Q.E.D.

Note also that the minimization problem with dynamics (3) and performance index (4) can be further reduced to a linear quadratic regulator problem. Indeed, if we let

$$\hat{X}(p)=\left(\frac{1}{\alpha}\right)^{p}\bar{X}(p),\ \hat{U}(p)=\left(\frac{1}{\alpha}\right)^{p}\bar{U}(p) \quad (20)$$

$$\hat{A}=\left(\frac{1}{\alpha}\right)A_{\beta},\ \hat{B}=\left(\frac{1}{\alpha}\right)B \quad (21)$$

then dynamical equation (3) becomes

$$\hat{X}(p+1)=\hat{A}\hat{X}(p)+\hat{B}\hat{U}(p). \quad (22)$$

The performance index (4) can be rewritten as

$$J=\sum_{p=0}^{\infty}[\hat{X}(p)Q\hat{X}(p)+\hat{U}(p)R\hat{U}(p)]. \quad (23)$$

Thus, the minimization problem with respect to dynamics (3) and the performance index (4) is equivalent to the minimization problem with respect to dynamics (22) and performance index (23) in the following sense.

a) The minimum value of (4) is the same as the minimum value of (23).

b) IF $\hat{U}(p) = G(\hat{X}(p))$ is the optimal control for (22) and (23), $\bar{U}(p) = (1/\alpha)^{-p}G((1/\alpha)^{p}\bar{X}(p))$ is the optimal control for (3) and (4) and conversely.

Therefore, the minimization problem with plant dynamics (1) and the performance index (5) is reduced to a standard LQ optimization problem with plant dynamics (22) and performance index (23). The optimal control law that minimizes (23) subject to constraint (22) is

$$\hat{U}_{\text{opt}}(p)=-G_f\hat{X}(p) \quad (24)$$

where

$$G_f=[R+\hat{B}^T S\hat{B}]^{-1}[\hat{B}^T S\hat{A}] \quad (25)$$

and S is the unique symmetric positive definite solution of the discrete Ricatti equation

$$S=\hat{A}^T S\hat{A}+Q-[\hat{B}^T S\hat{A}]^T[R+\hat{B}^T S\hat{B}]^{-1}[\hat{B}^T S\hat{A}]. \quad (26)$$

Moreover, if $[\hat{A}, \hat{B}]$ is either completely controllable or stabilizable, and if $[\hat{A}, D]$ is completely observable, where D is any $n \times n$ matrix such that $DD^T = Q$, then the feedback system is asymptotically stable. The closed-loop system is

$$\hat{X}(p+1)=(\hat{A}-\hat{B}G_f)\hat{X}(p). \quad (27)$$

Since the poles of this system being given by the eigenvalues of $(\hat{A} - \hat{B}G_f)$ have an eigenvalue less than 1, it follows that the eigenvalues of $\bar{X}(p+1) = (A_\beta - BG_f)\bar{X}(p)$ are less than α. Hence, the eigenvalues of

$$X(p+1)=(A-BG_f)X(p)$$

are all inside a circle centered at $(\beta, 0)$ with radius α. Thus, the optimal control which minimizes (5) subject to plant dynamics (1) ensures that all the closed-loop poles are inside a circle centered at $(\beta, 0)$ with radius α. Notice that the optimization problem with dynamics (1) and performance index (5) can be solved in the following way.

First, solve the optimal control problem with dynamics (22) and performance index (23) to obtain $\hat{U}(p)$, and then from (20) to obtain $\bar{U}(p) = (1/\alpha)^{-p}\hat{U}(p)$. Now the problem remains to obtain $U(p)$ from $\bar{U}(p)$. From (19), it is easy to show that

$$\begin{aligned} U(0) &= \bar{U}(0)\\ U(1) &= \bar{U}(1)+\beta\bar{U}(0)\\ U(2) &= \bar{U}(2)+2\beta\bar{U}(1)+\beta^2\bar{U}(0).\\ &\vdots \end{aligned} \quad (28)$$

In fact, the general expression for $U(p)$ is

$$U(p)=\sum_{k=0}^{p}C_k^p\beta^k\bar{U}(p-K). \quad (29)$$

From (29), the optimal control that minimizes (5) is achieved.

III. Nonlinear Perturbations

In this section, we will study the robustness of a discrete-time linear quadratic state feedback (LQSF) design in the presence of some nonlinear perturbations.

Consider a discrete-time system with dynamics

$$X(p+1)=AX(p)+BU(p)+F\{X(p),\ U(p)\} \quad (30)$$

where F is a nonlinear vector function. A difference equation of this form may be considered as a linearization of a general nonlinear equation of the form $X(p+1) = G\{X(p), U(p)\}$, A and B denoting the Jacobian of G with respect to the state vector $X(p)$ and the control vector $U(p)$, respectively, and F denoting higher order terms.

Since the exact expression of the nonlinear function F is not usually available but only some bound on this function may be evaluated by a designer, we shall study the robustness of an LQSF design for the linear model

$$X(p+1)=AX(p)+BU(p) \quad (31)$$

in the presence of some nonlinear perturbation $F[X(p), U(p)]$. The pair (A, B) is assumed to be controllable. The performance index to be minimized is

$$J=\sum_{p=0}^{\infty}\left(\frac{1}{\alpha}\right)^{2p}\left\{\left[\sum_{i=0}^{p}d_{p,i}X(i)\right]^T Q\left[\sum_{i=0}^{p}d_{p,i}X(i)\right]+\left[\sum_{i=0}^{p}d_{p,i}U(i)\right]^T R\left[\sum_{i=0}^{p}d_{p,i}U(i)\right]\right\} \quad (32)$$

where $Q = Q^T \geq 0$, $R = R^T > 0$ and $\alpha > 0$.

The optimal control that minimizes (32) will ensure all closed-loop poles inside the circular region as shown in Fig. 1.

Recall that the minimization problem with respect to dynamics (31) and performance index (32) is equivalent to the minimization problem with respect to dynamics (3) and performance index (4). Therefore, the problem considered in this section may be restated as follows.

Given a plant dynamics

$$\bar{X}(p+1)=A_\beta\bar{X}(p)+B\bar{U}(p)+F[\bar{X}(p),\ \bar{U}(p)]. \quad (33)$$

We shall study the robustness of an LQSF design for the linear model

$$\bar{X}(p+1)=A_\beta\bar{X}(p)+B\bar{U}(p) \quad (34)$$

in the presence of some nonlinear perturbation $F[\bar{X}(p), \bar{U}(p)]$. The performance index to be minimized is

$$J=\sum_{p=0}^{\infty}\left(\frac{1}{\alpha}\right)^{2p}[\bar{X}^T(p)Q\bar{X}(p)+\bar{U}^T(p)R\bar{U}(p)]. \quad (35)$$

The optimization of performance index (35) with the system model (34) yields a state feedback control

$$\bar{U}(p) = -[R+\hat{B}^T S\hat{B}]^{-1}\hat{B}^T S\hat{A}\bar{X}(p) \tag{36}$$

where $\hat{B} = (1/\alpha)B$, $\hat{A} = (1/\alpha)A_\beta$, and S is the solution of (26). The resulting closed-loop system is given by

$$\bar{X}(p+1) = (A_\beta - B(R+\hat{B}^T S\hat{B})^{-1}\hat{B}^T S\hat{A})\bar{X}(P) + F[\bar{X}(P)]. \tag{37}$$

The problem which we will investigate in this section is to determine the bound on perturbation F which preserves the stability of (37).

Let a Lyapunov function be defined as

$$V(p) \triangleq \bar{X}^T(p) S\bar{X}(p) \tag{38}$$

where S is the solution of (26). Since S is positive definite, $V(p) > 0$, for all nonzero $\bar{X}(p)$, and $V(p) \to \infty$ as $\|\bar{X}\| \to \infty$. Here, and in the sequel,

$\|W\|_E$ denotes the Euclidean norm of a vector W (matrix W),

and

$$\|W\|_E = \left(\sum_{i,j} |W_{ij}|^2\right)^{1/2}.$$

Note that $\Delta V(p) < 0$ is required for the stability of the closed-loop system of (38). Now consider

$$\Delta V(p) = V(p+1) - V(p) = \bar{X}^T(p+1)S\bar{X}(p+1) - \bar{X}^T(p)S\bar{X}(p). \tag{39}$$

By simple manipulations, it yields

$$\begin{aligned}\Delta V(p) = \bar{X}^T(p)\{&A_\beta^T SA_\beta - 2(B^T SA_\beta)^T(R+\hat{B}^T S\hat{B})^{-1}(\hat{B}^T S\hat{A})\\ &+(\hat{B}^T S\hat{A})^T(R+\hat{B}^T S\hat{B})^{-1}B^T SB(R+\hat{B}^T S\hat{B})^{-1}(\hat{B}^T S\hat{A})\\ &-S\}\bar{X}(p) + 2F^T(X(p))S\{A_\beta - B(R+\hat{B}^T S\hat{B})^{-1}\\ &\cdot(\hat{B}^T S\hat{A})\}\bar{X}(p) + F^T(\bar{X}(p))SF(\bar{X}(p)).\end{aligned} \tag{40}$$

The bounds on the nonlinear perturbation $F(\bar{X}(p))$ for the stability of (37) can be summarized as the following theorem.

Theorem 2: Let

$$H \triangleq S\{A_\beta - B(R+\hat{B}^T S\hat{B})^{-1}(\hat{B}^T S\hat{A})\} \tag{41}$$

$$\begin{aligned}D &\triangleq (\hat{B}^T S\hat{A})^T(R+\hat{B}^T S\hat{B})^{-1}(\hat{B}^T S\hat{A}^T) + Q - (\hat{B}^T S\hat{A})^T\\ &\quad\cdot(R+\hat{B}^T S\hat{B})^{-1}\hat{B}^T S\hat{B}(R+\hat{B}^T S\hat{B})^{-1}(\hat{B}^T S\hat{A})\\ &\triangleq Q + G_f^T RG_f\end{aligned} \tag{42}$$

and let max $\lambda(W)$ and min $\lambda(W)$ denote the maximum and the minimum eigenvalue of a matrix W, respectively. Then if the nonlinear vector function $F(\bar{X}(p))$ satisfies the condition

$$\frac{\|F(\bar{X}(p))\|_E}{\|\bar{X}(p)\|_E} \le \frac{\alpha^2 \min\lambda(D)}{2\max|\lambda(H)|} + \frac{(1-\alpha^2)\min\lambda(S)}{2\max|\lambda(H)|} \tag{43}$$

for arbitrary nonzero $n \times 1$ state vector $\bar{X}(p)$, the closed-loop system (37) is asymptotically stable.

Proof: Since $D = Q + G_f^T RG_f$, D is positive definite.

Substitution of (26) into (40) yields

$$\Delta V(p) = -\alpha^2\bar{X}^T(p)D\bar{X}(p) - (1-\alpha^2)\bar{X}^T(p)S\bar{X}(p) + 2F^T(\bar{X}(p))H\bar{X}(p). \tag{44}$$

Since

$$\begin{aligned}F^T(\bar{X}(p))H\bar{X}(p) &\le \|F(\bar{X}(p))\|_E\|H\|_E\|\bar{X}(p)\|_E\\ &\le \left\{\frac{\alpha^2\min\lambda(D)}{2\max|\lambda(H)|} + \frac{(1-\alpha^2)\min\lambda(S)}{2\max|\lambda(H)|}\right\}\max|\lambda(H)|\|\bar{X}(p)\|_E^2\\ &= \frac{1}{2}\{\alpha^2\min\lambda(D) + (1-\alpha^2)\min\lambda(S)\}\|\bar{X}(p)\|_E^2\end{aligned} \tag{45}$$

where relation (43) has been used to obtain (45). Therefore,

$$\Delta V(p) = -\bar{X}^T(p)\{\alpha^2[D - \min\lambda(D)I_n] + (1-\alpha^2)[S - \min\lambda(S)I_n]\}\bar{X}(p). \tag{46}$$

It is easy to see that for $\alpha < 1$, $\Delta V(p) < 0$ for arbitrary nonzero $\bar{X}(p)$ and hence (37) is asymptotically stable.

Q.E.D.

Note that Theorem 2 reveals that for any weighting matrix $Q = DD^T \ge 0$ such that (A_β, D) is observable, and any weighting matrix $R = R^T > 0$, the optimal control law that minimizes (35) subject to the constraint of (34) will always have the closed-loop poles all inside a circular region as specified as long as the nonlinear function F satisfies (43).

IV. Illustrative Example

Consider a discrete-time controllable system

$$X(p+1) = \begin{bmatrix}1 & 0.1\\ 0 & 1\end{bmatrix}X(p) + \begin{bmatrix}0.005\\ 0.1\end{bmatrix}U(p). \tag{47}$$

In order to formulate an optimal control problem that will have all the closed-loop poles inside a circle centered at (0.5, 0) with radius $\alpha = 0.5$, we redefine a plant dynamics given by

$$\bar{X}(p+1) = A_\beta\bar{X}(p) + B\bar{U}(p) \tag{48}$$

where

$$A_\beta = \begin{bmatrix}0.5 & 0.1\\ 0 & 0.5\end{bmatrix},\quad B = \begin{bmatrix}0.005\\ 0.1\end{bmatrix} \tag{49}$$

and the performance index

$$J = \sum_{p=0}^{\infty}[\bar{X}^T(p)Q\bar{X}(p) + \bar{U}^T(p)R\bar{U}(p)]2^{2p}. \tag{50}$$

It is clear that for any Q satisfying $Q = DD^T \ge 0$ such that (A_β, D) is observable, and any $R = R^T > 0$, the optimal control that minimizes (50) with plant dynamics (49) will have its closed-loop poles inside a circle centered at (0.5, 0) with radius $\alpha = 0.5$.

For simplicity, let $Q = [\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}]$, $R = 1$. Then the steady-state solution of the discrete Riccati equation is

$$S = \begin{bmatrix}7.9176 & 5.4771\\ 5.4771 & 7.4556\end{bmatrix} \tag{51}$$

and the regulator gain is

$$G_f = [0.5061 \quad 0.7673]. \tag{52}$$

It is easy to verify that the closed-loop poles are at

$$0.9335 \pm j0.0662$$

which are located inside the specified circular region.

Let the plant include a nonlinear function $F(\bar{X}(p))$, as described by the following equation:

$$\bar{X}(p+1) = \begin{bmatrix}0.5 & 0.1\\ 0 & 0.5\end{bmatrix}\bar{X}(p) + \begin{bmatrix}0.005\\ 0.1\end{bmatrix}\bar{U}(p) + F(\bar{X}(p)). \tag{53}$$

Since

$$S = \begin{bmatrix}7.9176 & 5.4771\\ 5.4771 & 7.4556\end{bmatrix},\quad \lambda(S) = 2.2046;\ 13.1686$$

$$H = \begin{bmatrix}7.6207 & 5.8187\\ 5.0861 & 7.4107\end{bmatrix},\quad \lambda(H) = 12.9568;\ 2.0746$$

$$D = \begin{bmatrix}1.2561 & 0.3883\\ 0.3883 & 0.5887\end{bmatrix},\quad \lambda(D) = 0.4104;\ 1.4344$$

if the nonlinear function $F(\bar{X}(p))$ satisfies the bound

$$\frac{\|F(\bar{X}(p))\|_E}{\|\bar{X}\|_E} \leq 0.0678$$

then the closed-loop system is stable.

V. Conclusions

A discrete-time optimal control that guarantees that all the closed-loop poles will be inside a circular region has been formulated. The robustness properties of the exposed discrete-time quadratic regulator have been investigated. Results have been generated which quantitatively characterize the bounds of the nonlinear perturbations so that the resultant closed-loop system is stable. A related topic concerning how to synthesize a feedback law with a prescribed robustness sector is under investigation.

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Model Reduction of Digital Systems Using Discrete Walsh Series

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Abstract—**This study discusses the application of discrete Walsh series expansion to reduce the order of a linear time-invariant digital system described by z-transfer function. The approach is based on matching the discrete Walsh spectra to determine both the coefficients of the denominator and numerator of the reduced model. The proposed method is simple for computation, can preserve the dynamic characteristic of the original model satisfactorily, and guarantees to have the same zero initial response as the original system.**

I. Introduction

It is often desirable and sometimes necessary to reduce the order of a linear dynamic system in the analysis and design of complex systems. The main objective of model order reduction is to provide a simplified model which is computationally simpler to handle than the original high-order system. In order to facilitate digital image processing, the discrete Walsh series was developed by Kak [1] to manipulate the integral transform characterization of patterns of finite binary sequences. The order of the discrete Walsh spectra is a permutation of the continuous ones. Recently, Horng and Ho [2], [3] use the discrete Walsh series to deal with the analysis, identification, and optimal control of linear digital systems. Chou and Horng [4] introduce simple methods for finding three operational matrices to facilitate the study of control systems using discrete Walsh series. In this study, a new approach is presented for the model reduction of a discrete system described by a z-transfer function.

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First, the output data sequences of the original and reduced models with respect to unit-step input data sequence are transferred into discrete Walsh spectra. Then, by matching the two spectra, the parameters of the reduced model can thus be determined.

II. Discrete Walsh Series

Kak [1] defined $\phi_i(k)$ as the ith discrete Walsh series of k. The series is defined on $N = 2^m$ points; m is an integer, and i and k are less than N. Let

$$(i)_{\text{decimal}} = (i_{m-1}i_{m-2} \cdots i_0)_{\text{binary}} \tag{1}$$

$$(k)_{\text{decimal}} = (k_{m-1}k_{m-2} \cdots k_0)_{\text{binary}} \tag{2}$$

then the set of discrete Walsh series are defined by

$$\phi_i(k) = (-1)^{[\Sigma_{j=0}^{m-1} g_j(i)k_j]}, \qquad i = 0, 1, 2, \cdots, N-1 \tag{3}$$

where

$$\begin{aligned} g_0(i) &= i_{m-1} \\ g_1(i) &= i_{m-1} + i_{m-2} \\ g_2(i) &= i_{m-2} + i_{m-3} \\ &\cdots \quad \cdots \quad \cdots \\ g_{m-1}(i) &= i_1 + i_0. \end{aligned} \tag{4}$$

For example, to obtain $\phi_3(k)$ for $N = 2^4$, we first express $(3)_{\text{decimal}}$ in its binary representation using (1) as

$$(3)_{\text{decimal}} = (0\ 0\ 1\ 1)_{\text{binary}}$$

and

$$\begin{aligned} g_0(3) &= i_3 = 0 \\ g_1(3) &= i_3 + i_2 = 0 \\ g_2(3) &= i_2 + i_1 = 1 \\ g_3(3) &= i_1 + i_0 = 2. \end{aligned}$$

Using (3) we now obtain

$$\phi_3(k) = (-1)^{[g_0(3)k_0 + g_1(3)k_1 + g_2(3)k_2 + g_3(3)k_3]}.$$

That is,

$$\phi_3(k) = [1\ 1\ 1\ 1\ -1\ -1\ -1\ -1\ 1\ 1\ 1\ 1\ -1\ -1\ -1\ -1].$$

The discrete polynomials $\{\phi_i(k),\ i = 0, 1, 2, \cdots, N-1\}$ form a complete set and satisfy the orthogonal property

$$\sum_{k=0}^{N-1} \phi_i(k)\phi_j(k) = N\delta_{ij}; \qquad i, j = 0, 1, 2, \cdots, N-1 \tag{5}$$

where δ_{ij} is the Kronecker delta.

Let $f(k)$, $k = 0, 1, 2, \cdots, N-1$, be an arbitrary bounded signal sequence and can be expanded in terms of the discrete Walsh series as

$$f(k) = \sum_{i=0}^{N-1} f_i\phi_i(k) = F^T\phi(k) \tag{6}$$

where the superscript T means transpose, F is the discrete Walsh coefficient vector, and $\phi(k)$ is the discrete Walsh vector. These two vectors are defined as

$$F = [f_0, f_1, \cdots, f_{N-1}]^T \tag{7}$$