國 立 交 通 大 學

應用數學系

博 士 論 文

圖形之路徑分割及其變型

Path Partition and Its Variations in Graphs

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圖形之路徑分割及其變型

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摘　要

假設 P 是一個圖形性質。圖形的 *P−分割*是將點集分割成互不相交的的集合，使得這些集合都誘導出滿足性質 P 的子圖。*P−分割數*是是圖中所有 P−分割的最小數，而 *P−分割問題*是找出 P−分割數的問題。同樣的，我們也可以定義所謂的 *P−覆蓋*和 *P−覆蓋數*，而他們和 P−分割、P−分割數只差在不要求集合要互不相交而已。

各式各樣的 P−分割和 P−覆蓋早已在文獻中被探討。比如，著色數是 P 為「沒有邊」這個性質的 P−分割數。由 Chartrand，Kronk 和 Wall [8]所定義的*點蔭度* $a(G)$，其 P 為「森林」這個性質。由 Harary [24]定義的*線性點蔭度* lva(G)，其 P 為「線性森林」這個性質。

這篇論文的目的是考慮 P 為「有一條漢米爾頓路徑」、「誘導路徑」或「原圖的同距路徑」性質。也就是說，此篇論文探討路徑分割問題、誘導路徑分割問題和同距路徑覆蓋問題。

就路徑分割問題而言，我們在塊形為完全圖、圈或完全二分圖的圖形上，給了一個 $O(|V|+|E|)$-時間的演算法。

就誘導路徑分割問題而言，我們在塊形為完全圖、圈或完全二分圖的圖形上，給了一個 $O(|V|+|E|)$-時間的演算法。我們也在補可約的圖形上給了一個多項式時間的演算法。

在同距路徑覆蓋問題上我們有三個結果。首先，我們決定了塊形圖形的同距路徑覆蓋數，並給了一個找出對應的路徑，時間為 $O(|V|+|E|)$的演算法。最後，我們算出完全 r 分圖、2 維和 3 維漢明圖的同距路徑覆蓋數。

Path Partition and Its Variations in Graphs

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Abstract

Suppose P is a graphical property. A *P-partition* of a graph $G = (V, E)$ is a partition of V into pairwise disjoint sets such that each set induces a subgraph satisfying property P. The *P-partition problem* is to find the *P-partition number* which is the minimum cardinality of a P-partition of a graph. We can define *P-cover* and *P-cover number* in a similar way, except now the subsets are not required to be disjoint.

Various P-partition and P-cover problems have been studied in the literature. For instance, the chromatic number is the P-partition number with the property P being "has no edges". For the *vertex-arboricity* $a(G)$ defined by Chartrand, Kronk and Wall [8], the property P is "induces a forest". For the *linear vertex arboricity* $\text{lva}(G)$ defined by Harary [24], the property P is "induces a linear forest".

The purpose of this thesis is to consider the problems in which property P is "containing a Hamiltonian path", "an induced path" or "an isometric path of the original graph". That is, we study the path-partition problem, the induced-path-partition problem and the isometric-path-cover problem.

For the path-partition problem, we give an $O(|V| + |E|)$-time algorithm for graphs whose blocks are complete graphs, cycles or complete bipartite graphs.

For the induced-path-partition problem, we present an $O(|V| + |E|)$-time algorithm for graphs whose blocks are complete graphs, cycles or complete bipartite graphs. We also give a polynomial-time algorithm for cographs.

We have three results for the isometric-path-cover problem. First, we determine isometric-path numbers of block graphs, and also give an $O(|V|+|E|)$-time algorithm for finding the corresponding paths. Second, we give isometric-path numbers of complete r-partite graphs and Hamming graphs of dimensions 2 and 3.

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Chapter 1

Introduction

A *path partition* of a graph is a collection of vertex-disjoint paths that cover all vertices of the graph. The *path-partition problem* is to find the *path-partition number* $p(G)$ of a graph G, which is the minimum cardinality of a path partition of G. The concept of path-partition number was introduced by Skupień [38], who studied the concept of *Hamiltonian shortage* of a graph G, written

$$S_H(G) = \min\{p : G \times K_p \text{ is Hamiltonian}\}.$$

He [38] proved that

$$S_H(G) = \begin{cases} p(G) - 1 = 0, & \text{if } G \text{ is Hamiltonian}, \\ p(G) + 1 = 2, & \text{if } G = K_1, \\ p(G) \geq 1, & \text{if } G \text{ is not Hamiltonian and } G \neq K_1. \end{cases}$$

He [38] also used an variation of Gallai-Milgram Theorem [20], saying $p(G) \leq \alpha(G)$ for any graph G, to prove that $S_H(G) \leq \alpha(G)$ for any graph G. Notice that G has a Hamiltonian path if and only if $p(G) = 1$.

The concept of path-partition number also has a close relationship with $L'(2,1)$-labeling number [7] describes as follows. An $L'(2,1)$-*labeling* of a graph G is a one to one function f from the vertex set $V(G)$ to the set of all nonnegative integers such that $|f(x) - f(y)| \geq 2$ if $d(x,y) = 1$ and $|f(x) - f(y)| \geq 1$ if $d(x,y) = 2$. The $L'(2,1)$-labeling number, denoted by $\lambda'(G)$, is the smallest number k such that G has a a $L'(2,1)$-labeling with $\max\{f(v) : v \in V(G)\} = k$. Thus, $p(G) = \lambda'(G^c) - |V(G)| + 2$, where G^c is the graph with vertex set $V(G)$ defined by $uv \in E(G^c)$ if and only if $uv \in E(G)$ in [7]. For more details about $L'(2,1)$-labeling, see [7].

We may extend the concept of path-partition number in a more general setting. Suppose P is a graphical property. A *P-partition* of a graph $G = (V, E)$ is a partition of V into pairwise disjoint sets such that each set induces a subgraph satisfying property P. The *P-partition problem* is to find the *P-partition number* which is the minimum cardinality of a P-partition of a graph. We can define *P-cover* and *P-cover number* in a similar way, except now the subsets are not required to be disjoint.

Various P-partition and P-cover problems have been studied in the literature. For instance, the chromatic number is the P-partition number with the property P being "has no edges". For the *vertex-arboricity* $a(G)$ defined by Chartrand, Kronk and Wall [8], the property P is "induces a forest". For the *linear vertex arboricity* $\text{lva}(G)$ defined by Harary [24], the property P is "induces a linear forest".

The purpose of this thesis is to consider the problems in which property P is "containing a Hamiltonian path", "an induced path" or "an isometric path of the original graph". That is, we study the path-partition problem, the induced-path-partition problem and the isometric-path-cover problem.

In this chapter, we first introduce some definitions needed in later chapters. Then, we describe motivations for studying the three problems mentioned above and give an overview of our results.

1.1 Basic definitions in graphs

A *graph* $G = (V, E)$ consists of a finite *vertex set* V and a finite *edge set* E, where each edge is an unordered pair $\{u, v\}$ of vertices called its *end-vertices*. For convenience, we write uv for an edge $\{u, v\}$. If $uv \in E$, then u and v are *adjacent*. The cardinality of V is called the *order* of G, and the cardinality of E the *size*. The *degree* of a vertex v in a graph G, written $d_G(v)$, is the number of edges containing v. The *maximum degree* is denoted by $\Delta(G)$; the *minimum degree* by $\delta(G)$. The *independence number* $\alpha(G)$ of G is the maximum size of a pairwise nonadjacent vertex set in G.

We illustrate a graph on paper by assigning a point to each vertex and drawing a curve for each edge between the points representing its end-vertices, sometimes omitting the names of the vertices or edges. Figure 1.1 is a graph with vertex set $V = \{a, b, c, d\}$, and edge set $E = \{\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}\}$.

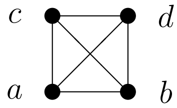

Figure 1.1: A graph $G = (V, E)$.

A *directed graph* or *digraph* $D = (V, E)$ consists of a *vertex set* V and an *edge set* E, where each edge is an ordered pair of vertices. We also write uv for the edge (u, v), with u being the *tail* and v being the *head*. We write $u \to v$ when $uv \in E$, meaning "there is an edge from u to v".

Let v be a vertex in a digraph. The *out-degree* $d^+_D(v)$ is the number of edges with tail v, and the *in-degree* $d^-_D(v)$ is the number of edges with head v. Figure 1.2 shows a digraph D with vertex set $V = \{a, b, c, d, c, f\}$ and edge set $E = \{(a, b), (b, c), (c, d), (d, e), (e, a), (f, a)\}$. Notice that $d^+_D(a) = 1$ and $d^-_D(a) = 2$.

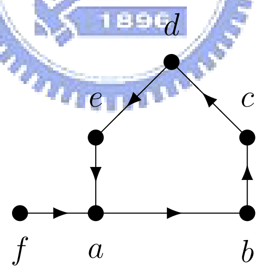

Figure 1.2: A digraph D.

A *subgraph* of a graph $G = (V, E)$ is a graph $H = (V', E')$ such that $V' \subseteq V$ and $E' \subseteq E$. For a subset $S \subseteq V$, the *subgraph induced by* S is the graph $H = (S, E')$ with $E' = \{xy \in E :\ x, y \in S\}$. For a subset $T \subseteq E$, the *subgraph induced by* T is the graph $H = (V', T)$ with $V' = \{x \in V :\ x \in e \text{ for some } e \in T\}$. Figure 1.3 is a subgraph of the graph in Figure 1.1.

A *path* is an ordered list of distinct vertices $(v_0, v_1, \ldots, v_n)$ such that $v_{i-1}v_i$ is

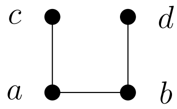

Figure 1.3: A subgraph of the graph in Figure 1.1.

an edge for $1 \leq i \leq n$. The first and last vertices of a path are its *end-vertices*; a u, v-path is a path with end-vertices u and v. If a graph G has a u, v-path, then the *distance* from u to v, written $d(u, v)$, is the least length of a u, v-path; if G has no such path, then $d(u, v) = \infty$. The *diameter* $\mathrm{diam}(G)$ of a graph G is the maximum distance between two vertices in G. An *induced path* is a path in which two vertices are adjacent only for those with consecutive indices. A *cycle* is an an ordered list of distinct vertices $(v_0, v_1, \ldots, v_n)$, except $v_0 = v_n$ such that all $v_{i-1}v_i$ for $1 \leq i \leq n$ are edges. A graph is called *Hamiltonian* if it has a cycle containing all vertices of the graph. A graph with n vertices that is a path or a cycle is denoted by P_n or C_n, respectively. A graph $G = (V, E)$ is *connected* if it has a u, v-path for each pair of vertices $u, v \in V$. The ordered list (c, a, b) of the graph in Figure 1.1 is a path, and (c, a, b, c) a cycle. The ordered list (a, b, c, d, e, a) of the graph in Figure 1.2 is a directed cycle.

A *complete graph* of order n, written K_n, is a graph in which every pair of vertices is an edge. Figure 1.1 is a complete graph of order 4. A *complete bipartite graph* is a graph whose vertex set is the union of the two disjoint sets and edge set consists of all pairs having a vertex from each of two disjoint sets covering the vertices. A *complete r-partite graph* is a graph whose vertex set can be partitioned into disjoint union of r nonempty parts, and two vertices are adjacent if and only if they are in different parts. We use $K_{n_1, n_2, \ldots, n_r}$ to denote the complete r-partite graph whose parts are of sizes $n_1, n_2, \ldots, n_r$, respectively. Figure 1.4 is the complete bipartite graph $K_{2,2}$.

The *union* of two graphs $G = (V, E)$ and $H = (V', E')$, written $G \cup H$ is the graph having vertex set $V \cup V'$ and edge set $E \cup E'$. To specify the *disjoint union* with $V \cap V' = \emptyset$, we write $G + H$. The *join* of G and H, written $G \times H$, is obtained

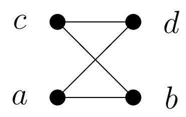

Figure 1.4: The complete bipartite graph $K_{2,2}$.

from $G + H$ by adding the edges $\{xy : x \in V \text{ and } y \in V'\}$. The *Cartesian product* of graphs G and H, written $G \Box H$, is the graph with vertex set $V \times V'$ specified by putting (u, u') adjacent to (v, v') if and only if (1) $u = v$ and $u'v' \in E'$, or (2) $u' = v'$ and $uv \in E$. *Complement reducible graphs* (also called *cographs*) are defined recursively by the following rules: (i) K_1 is a cograph; (ii) if G and H are cographs, then so are $G + H$ and $G \times H$; (iii) no other graphs are cographs. For more details on cographs, see [12, 13, 26]. Figure 1.5 is the Cartesian product of P_2 and P_2, and Figure 1.6 is a cograph since we can use the following construction.

First, let $G_1 = a$, $G_2 = b$, $G_3 = c$, $G_4 = d$ and $G_5 = e$ by rule (i). Second, we get $G_1 + G_2$ and $G_4 \times G_5$ by rule (ii). Third, we obtain $(G_1 + G_2) \times G_3$ by rule (ii). Finally, we get the required graph $((G_1 + G_2) \times G_3) \times (G_4 \times G_5)$ by rule (ii).

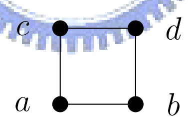

Figure 1.5: The Cartesian product $P_2 \Box P_2$ of P_2 and P_2 .

A *Hamming graph* is the Cartesian product of complete graphs, which is the graph $K_{n_1} \Box K_{n_2} \Box \ldots \Box K_{n_r} = (V, E)$ with vertex set

$$V = \{(x_1, x_2, \ldots, x_r) : 0 \leq x_i < n_i \text{ for } 1 \leq i \leq r\}$$

and edge set

$$E = \{(x_1, x_2, \ldots, x_r)(y_1, y_2, \ldots, y_r) : x_i = y_i \text{ for all } i \text{ except just one } x_j \neq y_j\}.$$

Figure 1.7 is the Hamming graph $K_2 \Box K_3$.

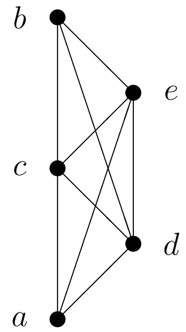

Figure 1.6: A cograph $K_{1,3}$.

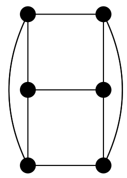

Figure 1.7: The Hamming graph $K_2 \square K_3$.

A *cut-vertex* of a graph is a vertex whose removal results in a graph having more components than the original graph. A *block* is a maximal connected subgraph without a cut-vertex. Notice that the intersection of two distinct blocks contains at most one vertex; and a vertex is a cut-vertex if and only if it is the intersection of two or more blocks. Consequently, a graph with one or more cut-vertices has at least two blocks. An *end block* is a block with exactly one cut-vertex. A graph is a *block graph* if it is the intersection graph of the family of blocks of some graph. Harary [23] proved that a graph is a block graph if and only if all its blocks are complete graphs. Figure 1.8 shows a block graph having two blocks $B_1 = (\{a, b, x\}, \{ab, ax, bx\})$ and $B_2 = (\{c, d, x\}, \{cd, cx, dx\})$, and a cut-vertex x.

1.2 Basic definitions in algorithms

In this section, we introduce some concepts on algorithms as some of our results are in terms of algorithm.

An *algorithm* is a finite sequence of deterministic computational steps that trans-

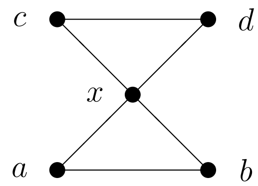

Figure 1.8: A graph $G = (V, E)$ having two blocks.

form the input into the output. The time needed for an algorithm, in worst case, expressed as a function of the size of the input of a problem is called the *time complexity* of the algorithm. The limiting behavior of the complexity as size increases is called the *asymptotic time complexity*. A function $f(n)$ is said to be $O(g(n))$ if there exists two positive constant c and n_0 such that $0 \leq f(n) \leq cg(n)$ for all $n \geq n_0$.

A *depth-first search*, as its name implies, is to search "deeper" in the graph whenever possible. In a depth-first search, we select and "visit" a starting vertex v. Then we select any edge vw incident to v, and visit w. In general, suppose x is the most recently visited vertex. The search is continued by selecting some unexplored edge xy. If y has been previously visited, we find another new edge incident to x. If y has not been previously visited, then we visit y and begin a new search starting at vertex y. After completing the search through all paths beginning at y, the search returns to x, the vertex from which y was first reached. The process of selecting unexplored edges incident to x is continued until the list of these edges is exhausted. The depth-first search can find all blocks of a graph G and spend $O(e)$ time if G has e edges.

A *nondeterministic algorithm* consists of two phases: a *guessing stage* and a *checking stage* which is a deterministic algorithm. Furthermore, it is assumed that a nondeterministic algorithm always makes a correct guessing. If the checking stage of a nondeterministic algorithm is of polynomial-time complexity, then this nondeterministic algorithm is called a *nondeterministic polynomial algorithm*. If a problem can be solved by a *nondeterministic polynomial algorithm*, this problem is called a nondeterministic polynomial ($\mathcal{NP}$ for short) problem. All of the problems which can be solved in polynomial time are called $\mathcal{P}$ problems. Cook [10] proved the following important theorem, we now call Cook's Theorem.

Theorem 1.1 [10] *$\mathcal{NP} = \mathcal{P}$ if and only if the Satisfiability Problem is a $\mathcal{P}$ problem.*

Let A_1 and A_2 be two problems. A_1 is *reducible* to A_2 if and only if A_1 can be solved in polynomial time by using a polynomial-time algorithm which solves A_2. A problem A is *$\mathcal{NP}$-complete* if A is in $\mathcal{NP}$ and every $\mathcal{NP}$ problem reduces to A. The Satisfiability Problem is $\mathcal{NP}$-complete according to Cook's Theorem.

For more details on the design and analysis of algorithms. see [1, 11].

1.3 Path partition

A *path partition* of a graph is a collection of vertex-disjoint paths that cover all vertices of the graph. The *path-partition problem* is to find the *path-partition number* $p(G)$ of a graph G, which is the minimum cardinality of a path partition of G. For the graph G in Figure 1.9, $p(G) = 1$.

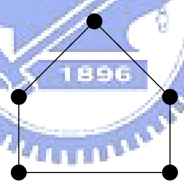

Figure 1.9: A graph G with $p(G) = 1$.

The concept of path-partition number was introduced by Skupień [38], who studied the concept of *Hamiltonian shortage* of a graph G, written

$$S_H(G) = \min\{p : G \times K_p \text{ is Hamiltonian}\}.$$

He [38] proved that

$$S_H(G) = \begin{cases} p(G) - 1 = 0, & \text{if } G \text{ is Hamiltonian,} \\ p(G) + 1 = 2, & \text{if } G = K_1, \\ p(G) \geq 1, & \text{if } G \text{ is not Hamiltonian and } G \neq K_1. \end{cases}$$

He [38] also used an variation of Gallai-Milgram Theorem [20], saying $p(G) \leq \alpha(G)$ for any graph G, to prove that $S_H(G) \leq \alpha(G)$ for any graph G. Notice that G has a Hamiltonian path if and only if $p(G) = 1$. Since the Hamiltonian path problem is $\mathcal{NP}$-complete for planar graphs [21], bipartite graphs [22], chordal graphs [22], chordal bipartite graphs [31] and strongly chordal graphs [31], so is the path-partition problem. On the other hand, the path-partition problem is polynomially solvable for trees [25, 38], interval graphs [4, 5, 14], circular-arc graphs [5, 14], cographs [7, 12, 30], cocomparability graphs [15], block graphs [39, 40, 41] and bipartite distance-hereditary graphs [43].

The concept of path-partition number also has a close relationship with $L'(2,1)$-labeling number [7] describes as follows. An $L'(2,1)$-*labeling* of a graph G is a one to one function f from the vertex set $V(G)$ to the set of all nonnegative integers such that $|f(x) - f(y)| \geq 2$ if $d(x,y) = 1$ and $|f(x) - f(y)| \geq 1$ if $d(x,y) = 2$. The $L'(2,1)$-labeling number, denoted by $\lambda'(G)$, is the smallest number k such that G has a a $L'(2,1)$-labeling with $\max\{f(v) : v \in V(G)\} = k$. Thus, $p(G) = \lambda'(G^c) - |V(G)| + 2$, where G^c is the graph with vertex set $V(G)$ defined by $uv \in E(G^c)$ if and only if $uv \in E(G)$ in [7]. For more details about $L'(2,1)$-labeling, see [7].

1.4 Induced-path partition

The concept of induced-path partition was considered by Chartrand *et al.* [9] as the P-partition with the property of being a path. More precisely, an *induced-path partition* of a graph is a collection of vertex-disjoint induced paths that cover all vertices of the graph. The *induced-path-partition problem* is to find the *induced-path number* $\rho(G)$ of a graph G, which is the minimum cardinality of an induced-path partition of G. For the graph G in of Figure 1.10, $\rho(G) = 2$.

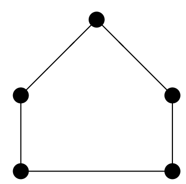

Figure 1.10: A graph G with $\rho(G) = 2$.

Chartrand *et al.* [9] gave the induced-path numbers of complete bipartite graphs, complete binary trees, 2-dimensional meshs, butterflies and general trees. Broere *et al.* [6] determined exact values for complete multipartite graphs. Chartrand *et al.* [9] conjectured that $\rho(Q_d) \leq d$ for the d-dimensional hypercube Q_d with $d \geq 2$. Alsardary [3] proved that $\rho(Q_d) \leq 16$. From an algorithmic point of view, Le *et al.* [27] proved that the induced-path-partition problem is $\mathcal{NP}$-complete for general graphs.

1.5 Isometric-path cover

An *isometric path* between two vertices in a graph G is a shortest path joining them. An *isometric-path cover* of a graph is a collection of isometric paths that cover all vertices of the graph. The *isometric-path-cover problem* is to find the *isometric-path number* $\text{ip}(G)$ of a graph G which is the minimum cardinality of an isometric-path cover. The concept of the isometric-path number has a close relationship with the game of cops and robbers described as follows.

The game is played by two players, the *cop* and the *robber*, on a graph. The two players move alternatively, starting with the cop. Each player's first move consists of choosing a vertex at which to start. At each subsequent move, a player may choose either to stay at the same vertex or to move to an adjacent vertex. The object for the cop is to catch the robber, and for the robber is to prevent this from happening. Nowakowski and Winkler [32] and Quilliot [37] independently proved that the cop wins if and only if the graph can be reduced to a single vertex by successively removing pitfalls, where a *pitfall* is a vertex whose close neighborhood is a subset of the close neighborhood of another vertex.

As not all graphs are cop-win graphs, Aigner and Fromme [2] introduced the concept of the *cop-number* of a general graph G, denoted by $c(G)$, which is the minimum number of cops needed to put into the graph in order to catch the robber. On the way to giving an upper bound for the cop-numbers of planar graphs, they showed that a single cop moving on an isometric path P guarantees that after a finite number of moves the robber will be immediately caught if he moves onto P.

Observing this fact, Fitzpatrick [16] then introduced the concept of isometric-path cover and pointed out that $c(G) \leq \mathrm{ip}(G)$. For the graph G of Figure 1.11, $\mathrm{ip}(G) = 2$.

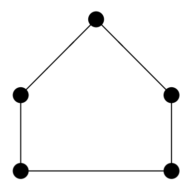

Figure 1.11: A graph G with $\mathrm{ip}(G) = 2$.

The isometric-path number of the Cartesian product $P_{n_1} \Box P_{n_2} \Box \ldots \Box P_{n_d}$ has been studied in the literature. Fitzpatrick [17] gave bounds for the case when $n_1 = n_2 = \ldots = n_d$. Fisher and Fitzpatrick [18] gave exact values for the case $d = 2$. Fitzpatrick *et al.* [19] gave a lower bound, which is in fact the exact value if $d + 1$ is a power of 2, for the case when $n_1 = n_2 = \ldots = n_d = 2$.

1.6 Overview of the thesis

In this thesis, we study path-partition numbers, induced-path numbers and isometric-path numbers. We give a brief overview of the thesis.

In Chapter 1, we introduce basic terminology in graphs and algorithms. We also describe motivations of the three problems studied in this thesis, namely the path-partition problem, the induced-path-partition problem and the isometric-path-cover problem.

Chapter 2 is devoted to the path-partition problem. This problem has been proved to be $\mathcal{NP}$-complete for many classes of graphs, while it is also polynomially solvable for some classes of graphs such as trees and block graphs. As these graphs all have tree structures, the purpose of this chapter is to use a unified method, called a *labeling algorithm*, to give an $O(|V| + |E|)$-time algorithm for the path-partition problem for graphs whose blocks are complete graphs, cycles or complete bipartite graphs.

Chapter 3 considers the induced-path-partition problem. Le *et al.* [27] used the fact that Not-All-Equal 3SAT is $\mathcal{NP}$-complete to prove that the induced-path-partition problem is $\mathcal{NP}$-complete for general graphs. The main purpose of this chapter is to present an $O(|V| + |E|)$-time algorithm for finding the induced-path numbers of graphs whose blocks are complete graphs, cycles or complete bipartite graphs. We also give a polynomial-time algorithm for finding the induced-path numbers of cographs.

In Chapter 4, we discuss the isometric-path-cover problem. This is a relatively new problem. Previous and our results on this problem are most non-algorithmic. We have three results for this problem. First, we determine isometric-path numbers of block graphs, and also give an $O(|V| + |E|)$-time algorithm for finding the corresponding paths. Second, we give isometric-path numbers of complete r-partite graphs and Hamming graphs of dimensions 2 and 3.

Chapter 5 makes a conclusion, in which we give some open problems on the path-partition problem, the induced-path-partition problem and the isometric-path-cover problem.

Chapter 2

Path Partition

2.1 Preliminary of path partition

Recall that a *path partition* of a graph is a collection of vertex-disjoint paths that cover all vertices of the graph. The *path-partition problem* is to find the *path-partition number* $p(G)$ of a graph G, which is the minimum cardinality of a path partition of G. Notice that G has a Hamiltonian path if and only if $p(G) = 1$. Since the Hamiltonian path problem is $\mathcal{NP}$-complete for planar graphs [21], bipartite graphs [22], chordal graphs [22], chordal bipartite graphs [31] and strongly chordal graphs [31], so is the path-partition problem. On the other hand, the path-partition problem is polynomially solvable for trees [25, 38], interval graphs [4, 5, 14], circular-arc graphs [5, 14], cographs [7, 12, 30], cocomparability graphs [15], block graphs [39, 40, 41] and bipartite distance-hereditary graphs [43].

The purpose of this chapter is to give a linear-time algorithm for the path-partition problem for graphs whose blocks are complete graphs, cycles or complete bipartite graphs. For technical reasons, we consider the following generalized problem, which is a *labeling approach* for the problem.

Suppose every vertex v in the graph G is associated with an integer $f(v) \in \{0, 1, 2, 3\}$. An *f-path partition* is a collection $\mathcal{P}$ of vertex-disjoint paths such that the following conditions hold.

(P1) Any vertex v with $f(v) \neq 3$ is in some path in $\mathcal{P}$.

(P2) If $f(v) = 0$, then v itself is a path in $\mathcal{P}$.

(P3) If $f(v) = 1$, then v is an end-vertex of some path in $\mathcal{P}$.

The *f-path-partition problem* is to determine the *f-path-partition number* $p_f(G)$ which is the minimum cardinality of an f-path partition of G. It is clear that $p(G) = p_f(G)$ when $f(v) = 2$ for all vertices v in G. Notice that as there may have some vertices of labels 3, an f-path partition is not necessary a path partition.

2.2 Path partition in graphs

The labeling approach used in this chapter starts from an end block. Suppose $B = (V, E)$ is an end block whose only cut-vertex is x. Let A be the graph $G - (V - \{x\})$. Notice that we can view G as the "composition" of A and B, *i.e.*, G is the union of A and B which meet at a common vertex x. The idea is to get the path-partition number of G from those of A and B.

In the lemmas and theorems of this chapter, we use the following notation. Suppose x is a specified vertex of a graph $H = (V, E)$ in which f is a vertex labeling. For $i = 0, 1, 2, 3$, we define the function $f_i : V \to \{0, 1, 2, 3\}$ by $f_i(y) = f(y)$ for all vertices y except $f_i(x) = i$.

Lemma 2.1 *Suppose x is a specified vertex in a graph H. Then the following statements hold.*

(1) $p_{f_3}(H) \le p_{f_2}(H) \le p_{f_1}(H) \le p_{f_0}(H)$.

(2) $p_{f_1}(H) \le p_{f_0}(H) \le p_{f_1}(H) + 1$.

(3) $p_{f_2}(H) \le p_{f_1}(H) \le p_{f_2}(H) + 1$.

(4) $p_{f_3}(H) = \min\{p_{f_2}(H), p_f(H - x)\} \le p_f(H - x) = p_{f_0}(H) - 1$.

(5) $p_f(H) \ge p_{f_1}(H) - 1$.

Proof. (1) The inequalities follow from that an f_i-path partition is an f_j-path partition whenever $i < j$.

(2) The second inequality follows from that replacing the path Px in an f_1-path partition by two paths P and x results in an f_0-path partition of H.

(3) The second inequality follows from that replacing the path PxQ in an f_2-path partition by two paths Px and Q results in an f_1-path partition of H.

(4) The first equality follows from that one is an f_3-path partition of H if and only if it is either an f_2-path partition of H or an f-path partition of $H-x$. The second equality follows from that $\mathcal{P}$ is an f_0-path partition of H if and only if it is the union of $\{x\}$ and an f-path partition of $H-x$.

(5) According to (1), (3) and (4), we have

$$p_f(H) \geq p_{f_3}(H) = \min\{p_{f_2}(H), p_f(H-x)\} \geq \min\{p_{f_1}(H)-1, p_{f_0}(H)-1\} = p_{f_1}(H)-1.$$

∎

Lemma 2.2 (1) $p_f(G) \leq \min\{p_f(A)+p_{f_0}(B)-1, p_{f_0}(A)+p_f(B)-1\}$.

(2) $p_{f_2}(G) \leq p_{f_1}(A)+p_{f_1}(B)-1$.

Proof. (1) Suppose $\mathcal{P}$ is an optimal f-path partition of A, and $\mathcal{Q}$ an f_0-path partition of B. Then $x \in \mathcal{Q}$ and so $(\mathcal{P} \cup \mathcal{Q}) - \{x\}$ is an f-path partition of G. This gives $p_f(G) \leq p_f(A)+p_{f_0}(B)-1$. Similarly, $p_f(G) \leq p_{f_0}(A)+p_f(B)-1$.

(2) The inequality follows from that if $\mathcal{P}$ (respectively, $\mathcal{Q}$) is an optimal f_1-path partition of A (respectively, B) in which $Px \in \mathcal{P}$ (respectively, $xQ \in \mathcal{Q}$) contains x, then $(\mathcal{P} \cup \mathcal{Q} \cup \{PxQ\}) - \{Px, xQ\}$ is an f_2-path partition of G. ∎

We now have the following theorem which is the key for the inductive step of our algorithm.

Theorem 2.3 *Suppose* $\alpha = p_{f_0}(B) - p_{f_1}(B)$ *and* $\beta = p_{f_1}(B) - p_{f_2}(B)$. (*Notice that* $\alpha, \beta \in \{0,1\}$.) *Then the following statements hold.*

(1) *If* $f(x) = 0$, *then* $p_f(G) = p_f(A) + p_f(B) - 1$.

(2) *If* $f(x) = 1$, *then* $p_f(G) = p_{f_{1-\alpha}}(A) + p_{f_\alpha}(B) - 1$.

(3) *If* $f(x) \geq 2$ *and* $\alpha = \beta = 0$*, then* $p_f(G) = p_f(A) + p_{f_0}(B) - 1$*.*

(4) *If* $f(x) \geq 2$ *and* $\alpha = 0$ *and* $\beta = 1$*, then* $p_f(G) = p_{f_3}(A) + p_f(B)$*.*

(5) *If* $f(x) \geq 2$ *and* $\alpha = 1$*, then* $p_f(G) = p_{f_{1-\beta}}(A) + p_{f_{1+\beta}}(B) - 1$*.*

Proof. Suppose $\mathcal{P}$ is an optimal f-path partition of G. Let P^* be the path in $\mathcal{P}$ that contains x. (It is possible that there is no such path when $f(x) = 3$.) There are three possibilities for P^*: (a) P^* does not exist or $P^* \subseteq A$; (b) $P^* \subseteq B$; (c) x is an internal vertex of P^*, say $P^* = P'xP''$, with $P'x \subseteq A$ and $xP'' \subseteq B$. (The latter is possible only when $f(x) \geq 2$.)

For the case when (a) holds, $\{P \in \mathcal{P} : P \subseteq A\}$ is an f-path partition of A and $\{P \in \mathcal{P} : P \subseteq B\} \cup \{x\}$ is an f_0-path partition of B. We then have the inequality in (a′). Similarly, we have (b′) and (c′) corresponding to (b) and (c).

(a′) $p_f(G) \geq p_f(A) + p_{f_0}(B) - 1$.

(b′) $p_f(G) \geq p_{f_0}(A) + p_f(B) - 1$. (We may replace $p_f(B)$ by $p_{f_2}(B)$ when $f(x) \geq 2$.)

(c′) $p_f(G) \geq p_{f_1}(A) + p_{f_1}(B) - 1$. (This is possible only when $f(x) \geq 2$.)

We are now ready to prove the theorem.

(1) Since $f(x) = 0$, we have $f = f_0$. According to Lemma 2.2 (1), $p_f(G) \leq p_f(A) + p_f(B) - 1$. On the other hand, (a′) and (b′) give $p_f(G) \geq p_f(A) + p_f(B) - 1$.

(2) Since $f(x) = 1$, we have $f = f_1$. Lemma 2.2 (1), together with (a′) and (b′), gives $p_f(G) = \min\{p_{f_1}(A) + p_{f_0}(B) - 1, p_{f_0}(A) + p_{f_1}(B) - 1\}$. If $\alpha = 0$, then

$$p_{f_0}(A) + p_{f_1}(B) - 1 \geq p_{f_1}(A) + (p_{f_0}(B) - \alpha) - 1 = p_{f_1}(A) + p_{f_0}(B) - 1;$$

and if $\alpha = 1$, then

$$p_{f_1}(A) + p_{f_0}(B) - 1 \geq (p_{f_0}(A) - 1) + (p_{f_1}(B) + \alpha) - 1 = p_{f_0}(A) + p_{f_1}(B) - 1.$$

Hence $p_f(G) = p_{f_{1-\alpha}}(A) + p_{f_\alpha}(B) - 1$.

(3) According to Lemma 2.2 (1), $p_f(G) \leq p_f(A) + p_{f_0}(B) - 1$. On the other hand, as $p_{f_0}(A) \geq p_{f_1}(A) \geq p_f(A)$ and $p_{f_0}(B) = p_{f_1}(B) = p_{f_2}(B)$, (a′)–(c′) give $p_f(G) \geq p_f(A) + p_{f_0}(B) - 1$.

(4) According to Lemma 2.1 (4) and $\alpha = 0$ and $\beta = 1$, we have

$$p_f(B-x) = p_{f_0}(B) - 1 = p_{f_1}(B) - 1 = p_{f_2}(B).$$

This, together with Lemma 2.1 (4), gives that the above value is also equal to $p_{f_3}(B)$ and so $p_f(B)$. Then, an optimal f_3-path partition $\mathcal{P}$ of A, together with an optimal p_f-path partition of $B-x$ (respectively, B) when x is (respectively, is not) in a path of $\mathcal{P}$, forms an f_2-path partition of G. Thus, $p_f(G) \leq p_{f_2}(G) \leq p_{f_3}(A) + p_f(B)$.

On the other hand, since $p_{f_1}(A) \geq p_f(A) \geq p_{f_3}(A)$ and $p_{f_0}(B)-1 = p_{f_1}(B)-1 = p_f(B)$, (a$'$) or (c$'$) implies $p_f(G) \geq p_{f_3}(A) + p_f(B)$. Also, as $p_{f_0}(A) - 1 \geq p_{f_3}(A)$ by Lemma 2.1 (4), (b$'$) implies $p_f(G) \geq p_{f_3}(A) + p_f(B)$.

(5) According to Lemma 2.1 (1) and Lemma 2.2, we have

$$p_f(G) \leq p_{f_2}(G) \leq \min\{p_{f_0}(A) + p_{f_2}(B) - 1, p_{f_1}(A) + p_{f_1}(B) - 1\}.$$

On the other hand, if (a$'$) holds, then by Lemma 2.1 (5) and that $p_{f_0}(B) = p_{f_1}(B)+1$,

$$p_f(G) \geq p_f(A) + p_{f_0}(B) - 1 \geq (p_{f_1}(A) - 1) + (p_{f_1}(B) + 1) - 1 = p_{f_1}(A) + p_{f_1}(B) - 1.$$

This, together with (b$'$) and (c$'$), gives

$$p_f(G) = \min\{p_{f_0}(A) + p_{f_2}(B) - 1, p_{f_1}(A) + p_{f_1}(B) - 1\}.$$

If $\beta = 0$, then

$$p_{f_0}(A) + p_{f_2}(B) - 1 \geq p_{f_1}(A) + (p_{f_1}(B) - \beta) - 1 = p_{f_1}(A) + p_{f_1}(B) - 1;$$

and if $\beta = 1$, then

$$p_{f_1}(A) + p_{f_1}(B) - 1 \geq (p_{f_0}(A) - 1) + (p_{f_2}(B) + \beta) - 1 = p_{f_0}(A) + p_{f_2}(B) - 1.$$

Hence $p_f(G) = p_{f_{1-\beta}}(A) + p_{f_{1+\beta}}(B) - 1.$ ∎

2.3 Path partitions for special blocks

Notice that the inductive theorem (Theorem 2.3) can be applied to solve the path-partition problem on graphs for which the problem can be solved on its blocks. In this section, we mainly consider the case when the blocks are complete graphs, cycles or complete bipartite graphs.

Now, we assume that $B = (V, E)$ is a graph in which each vertex v has a label $f(v) \in \{0, 1, 2, 3\}$. Recall that $f^{-1}(i)$ is the set of *pre-images* of i, *i.e.*,

$$f^{-1}(i) = \{v \in V : f(v) = i\}.$$

According to Lemma 2.1 (4), we have $p_f(B) = p_f(B - f^{-1}(0)) + |f^{-1}(0)|$. *Therefore, in this section we only consider the function f with $f^{-1}(0) = \emptyset$.*

We first consider the case when B is a complete graph.

Lemma 2.4 *Suppose B is a complete graph. If $f^{-1}(1) \neq \emptyset$ or $f^{-1}(2) = \emptyset$, then $p_f(B) = \lceil |f^{-1}(1)|/2 \rceil$ else $p_f(B) = 1$.*

Proof. It is clear that $p_f(B) \geq \lceil |f^{-1}(1)|/2 \rceil$. For the case when $f^{-1}(1) \neq \emptyset$ or $f^{-1}(2) = \emptyset$, we can pair the vertices in $f^{-1}(1)$ as end-vertices of paths to form an f-path partition; and so $p_f(B) \leq \lceil |f^{-1}(1)|/2 \rceil$. For the case when $f^{-1}(1) = \emptyset$ and $f^{-1}(2) \neq \emptyset$, it is clear that a Hamiltonian path forms an f-path partition; and so $p_f(B) = 1$. ∎

Next, consider the case when B is a path. This is useful as a subroutine for handling cycles.

Lemma 2.5 *Suppose B is a path.*

(1) *If x is an end-vertex of B with $f(x) = 3$, then $p_f(B) = p_f(B - x)$.*

(2) *If x is an end-vertex of B with $f(x) \in \{1, 2\}$ and another vertex y with $f(y) = 1$ such that no vertex between x and y has a label 1 (choose y the other end-vertex of B if there is no such vertex), then $\rho_f(B) = \rho_f(B') + 1$ where B' is the path obtained from B by deleting x, y and all vertices between them.*

Proof. (1) Since $f(x) = 3$, by Lemma 2.1 (4), $p_f(B) \leq p_f(B - x)$. As x is an end-vertex of B, $p_f(B) \geq p_f(B - x)$ follows from that deleting x from a path (if any) in an f-path partition of B results in an f-path partition of $B - x$.

(2) First, we claim that if $f(x) = 2$, then $\rho_f(B) = \rho_{f_1}(B)$. By Lemma 2.1 (1), $\rho_f(B) \leq \rho_{f_1}(B)$. Since x is an end-vertex of B and $f(x) = 2$, an f-path partition is in fact an f_1-path partition of B. Thus $\rho_f(B) \geq \rho_{f_1}(B)$. Now, we can assume that $f(x) = 1$.

Let P denotes the path from x to y in B. First, $\rho_f(B) \leq \rho_f(B') + 1$ follows from that an f-path partition of B', together with P, forms an f-path partition of B. On the other hand, suppose $\mathcal{P}$ is an optimal f-path partition of B. Since $f(x) = f(y) = 1$ and x is an end vertex of B, $\mathcal{P}$ has some $P' \subseteq P$ with $x \in P'$. Deleting all vertices of P from the paths in $\mathcal{P}$ results in an f-path partition of B' whose size is less than $|\mathcal{P}|$ by at least one. Thus, $\rho_f(B) - 1 \geq \rho_f(B')$. ∎

We then consider the case when B is a cycle.

Lemma 2.6 *Suppose B is a cycle.*

(1) *If $f^{-1}(2) = \emptyset$, then $p_f(B) = \lceil |f^{-1}(1)|/2 \rceil$.*

(2) *If P is a path from x to y in B such that $f^{-1}(1) \cap P = \{x, y\}$ and $f^{-1}(2) \cap P \neq \emptyset$, then $p_f(B) = p_f(B - P) + 1$.*

Proof. (1) It is clear that $p_f(B) \geq \lceil |f^{-1}(1)|/2 \rceil$. As $f^{-1}(2) = \emptyset$, we can pair the vertices in $f^{-1}(1)$ as end-vertices of paths to form an f-path partition; and so $p_f(B) \leq \lceil |f^{-1}(1)|/2 \rceil$.

(2) First, $p_f(B) \leq p_f(B - P) + 1$ follows from that an f-path partition of $B - P$ together with P forms an f-path partition of B. On the other hand, suppose $\mathcal{P}$ is an optimal f-path partition of B. Since $f^{-1}(1) \cap P = \{x, y\}$ and $f^{-1}(2) \cap P \neq \emptyset$, $\mathcal{P}$ must contain some $P' \subseteq P$ using x or y as one of its end-vertex. Deleting all vertices of P from the paths in $\mathcal{P}$ results in an f-path partition of $B - P$ whose size is less than $|\mathcal{P}|$ by at least one. Thus, $p_f(B) - 1 \geq p_f(B - P)$. ∎

Finally, we consider the case when B is a complete bipartite graph with $C \cup D$ as a bipartition of the vertex set. For $i \in \{0, 1, 2, 3\}$, let

$$C_i = \{u \in C : f(u) = i\} \text{ with } c_i = |C_i|;$$

$$D_i = \{v \in D : f(v) = i\} \text{ with } d_i = |D_i|.$$

We have the following lemmas.

Lemma 2.7 *If $c_1 = d_1 = 0$ and $c_2 \geq d_2$ and $x \in C_2$, then $p_f(B) = p_{f'}(B)$ where f' is the same as f except $f'(x) = 1$.*

Proof. $p_f(B) \leq p_{f'}(B)$ follows from the fact that any f'-path partition of B is an f-partition.

Suppose $\mathcal{P}$ is an optimal f-path partition of B. We may assume that $\mathcal{P}$ is chosen so that the paths in $\mathcal{P}$ cover as few vertices as possible. For the case when $\mathcal{P}$ has a path Py with $y \in C$, we may interchange y and x to assume that $Px \in \mathcal{P}$. In this case, $\mathcal{P}$ is an f'-path partition of B and so $p_{f'}(B) \leq p_f(B)$. So, now assume that all end-vertices of paths in $\mathcal{P}$ are in D. Then, these end-vertices are all in D_2 for otherwise we may delete those end-vertices in D_3 to get a new $\mathcal{P}$ which covers fewer vertices. We may further assume that paths in $\mathcal{P}$ cover no vertices in D_3, for otherwise we may interchange such a vertex with an end-vertex of a path in $\mathcal{P}$ and then delete it from the path. Thus each path of $\mathcal{P}$ uses vertices in $C_2 \cup C_3 \cup D_2$, and has end-vertices in D_2. These imply that $d_2 > c_2$, contradicting that $c_2 \geq d_2$. ∎

By symmetry, we may prove a similar theorem for the case when $d_1 = c_1 = 0$ and $d_2 \geq c_2$ and $d_2 \geq 1$.

Lemma 2.8 *Suppose $x \in C_1$. Also, either $d_2 \geq 1$ with $y \in D_2$, or else $c_1 > d_1$ and $d_2 = 0 < d_3$ with $y \in D_3$. Then $p_f(B) = p_{f'}(B - x)$, where f' is the same as f except $f'(y) = 1$.*

Proof. Suppose Py is in an optimal f'-path partition $\mathcal{P}$ of $B - x$. Then $(\mathcal{P} - \{Py\}) \cup \{Pyx\}$ is an f-path partition of B and so $p_f(B) \leq p_{f'}(B - x)$.

On the other hand, suppose Px is in an optimal f-path partition $\mathcal{P}$ of B. For the case when y is not covered by any path in $\mathcal{P}$, we have $y \in D_3$ and so $c_1 > d_1$ and $d_2 = 0$. Consequently, there is some $Qz \in \mathcal{P}$ with $z \in C_2 \cup C_3$ or $z \in D_3$. For the former case, we replace Qz by Qzy in $\mathcal{P}$; for the latter, we replace Qz by Qy. So, in any case we may assume that y is covered by some path RyS in $\mathcal{P}$. If $RyS = Px$, then again we may interchange y with the last vertex of P to assume that $RyS = Tyx$ in $\mathcal{P}$ for some T. If $RyS \neq Px$, then we may replace the two paths RyS and Px by Ryx and PS. So, in any case, we may assume that $\mathcal{P}$ has a path Uyx. Then, $(\mathcal{P} - \{Uyx\}) \cup \{Uy\}$ is an f'-path partition of $B - x$. Thus $p_{f'}(B - x) \leq p_f(B)$. ∎

By symmetry, we may prove a similar theorem for the case when $x \in D_1$; and either $c_2 \geq 1$ with $y \in C_2$, or else $d_1 > c_1$ and $c_2 = 0 < c_3$ with $y \in C_3$.

2.4 Algorithm for graphs with special blocks

We are ready to give a linear-time algorithm for the path-partition problem in graphs whose blocks are complete graphs, cycles or complete bipartite graphs. Notice that we may consider only connected graphs. We present five procedures. The first four are subroutines which calculate f-path-partition numbers of complete graphs, paths, cycles and complete bipartite graphs, respectively, by using Lemmas 2.4 to 2.8. The last one is the main routine for the problem.

First, Lemmas 2.1 (4) and 2.4 lead to the following subroutine for complete graphs.

Algorithm PCG. Find the f-path partition number $p_f(B)$ of a complete graph B.
Input. A complete graph B and a vertex labeling f.
Output. $p_f(B)$.
Method.

if $(f^{-1}(1) \neq \emptyset$ or $f^{-1}(2) = \emptyset)$
then $p_f(B) = |f^{-1}(0)| + \lceil |f^{-1}(1)|/2 \rceil$;
else $p_f(B) = |f^{-1}(0)| + 1$;
return $p_f(B)$.

Lemma 2.5 leads to the following subroutine for paths, which is useful for the cycle subroutine.

Algorithm PP. Find the f-path partition number $p_f(B)$ of the path B.

Input. A path B and a vertex labeling f with $f^{-1}(0) = \emptyset$.

Output. $p_f(B)$.

Method.

$p_f(B) \leftarrow 0$;

$B' \leftarrow B$;

while $(B' \neq \emptyset)$ **do**

choose an end-vertex x of B';

if $(f(x) = 3)$ **then** $B' \leftarrow B' - x$ **else**

choose a vertex y nearest to x with $f(y) = 1$

(let y be the other end-vertex if there is no such vertex);

$p_f(B) \leftarrow p_f(B) + 1$;

$B' \leftarrow B'-$ all vertices between (and including) x and y;

end else;

end while;

return $p_f(B)$.

Lemmas 2.1 (4) and 2.6 lead to the following subroutine for cycles.

Algorithm PC. Find the f-path partition number $p_f(B)$ of a cycle B.

Input. A cycle B and a vertex labeling f.

Output. $p_f(B)$.

Method.

> **if** $(f^{-1}(0) = \emptyset$ **and** $f^{-1}(2) = \emptyset)$
>
> **then** $p_f(B) \leftarrow \lceil f^{-1}(1)/2 \rceil$;
>
> **else if** $(f^{-1}(0) = \emptyset$ and $f^{-1}(2) \neq \emptyset$ and $|f^{-1}(1)| \leq 1)$ **then**
>
> > $p_f(B) \leftarrow 1$;
>
> **else if** $(f^{-1}(0) = \emptyset$ and $f^{-1}(2) \neq \emptyset$ and $|f^{-1}(1)| \geq 2)$ **then**
>
> > choose a path P from x to y such that
> >
> > > $f^{-1}(1) \cap P = \{x, y\}$ and $f^{-1}(2) \cap P \neq \emptyset$;
> >
> > $p_f(B) \leftarrow p_f(B - P) + 1$ by calling **PP**$(B - P)$;
>
> **else** // now $f^{-1}(0) \neq \emptyset$ //
>
> > let $B - f^{-1}(0)$ be the disjoint union of paths $P_1, P_2, \ldots, P_k$;
> >
> > $p_f(B) \leftarrow |f^{-1}(0)|$;
> >
> > **for** $i = 1$ **to** k **do** $p_f(B) \leftarrow p_f(B) + p_f(P_i)$ by calling **PP**(P_i);
>
> **end else**;
>
> **return** $p_f(B)$.

Lemmas 2.1 (4), 2.7 and 2.8 lead to the following subroutine for complete bipartite graphs. In the subroutine, we inductively reduce the size of $C \cup D$. Besides the reduction of C_0 and D_0 in the second line, we consider 9 cases. The first case is for $C = \emptyset$ or $D = \emptyset$. The next 5 cases are for $c_1 \geq 1$ or $d_1 \geq 1$. In particular, the case of $c_1 \geq 1$ is covered by cases 2 and 3, except when $d_2 = 0$ and ($c_1 \leq d_1$ or $d_3 = 0$). The case of $d_1 \geq 1$ is covered by cases 4 and 5, except when $c_2 = 0$ and ($d_1 \leq c_1$ or $c_3 = 0$). The exceptions are then covered by case 6. Finally, the last 3 cases are for $c_1 = d_1 = 0$.

Algorithm PCB. Find the f-path partition number $p_f(B)$ of a complete bipartite graph B.

Input. A complete bipartite graph B with a bipartition $C \cup D$ of vertices and a vertex labeling f.

Output. $p_f(B)$.

Method.

$c_i \leftarrow |f^{-1}(i) \cap C|$ and $d_i \leftarrow |f^{-1}(i) \cap D|$ for $0 \le i \le 3$;

$p_f(B) \leftarrow c_0 + d_0$;

while (true) **do**

  if ($c_1 = c_2 = c_3 = 0$ or $d_1 = d_2 = d_3 = 0$) **then**

    $p_f(B) \leftarrow p_f(B) + c_1 + c_2 + d_1 + d_2$; **return** $p_f(B)$;

  else if ($c_1 \ge 1$ and $d_2 \ge 1$) **then** // use Lemma 2.8 //

    $c_1 \leftarrow c_1 - 1$; $d_2 \leftarrow d_2 - 1$; $d_1 \leftarrow d_1 + 1$;

  else if ($c_1 \ge 1$ and $c_1 > d_1$ and $d_2 = 0 < d_3$) **then** // use Lemma 2.8 //

    $c_1 \leftarrow c_1 - 1$; $d_3 \leftarrow d_3 - 1$; $d_1 \leftarrow d_1 + 1$;

  else if ($d_1 \ge 1$ and $c_2 \ge 1$) **then** // use the remark after Lemma 2.8 //

    $d_1 \leftarrow d_1 - 1$; $c_2 \leftarrow c_2 - 1$; $c_1 \leftarrow c_1 + 1$;

  else if ($d_1 \ge 1$ and $d_1 > c_1$ and $c_2 = 0 < c_3$) **then** // remark after Lemma 2.8 //

    $d_1 \leftarrow d_1 - 1$; $c_3 \leftarrow c_3 - 1$; $c_1 \leftarrow c_1 + 1$;

  else if ($c_2 = d_2 = 0$ and ($c_1 = d_1 \ge 1$ or $c_1 > d_1 \ge 1$ with $d_3 = 0$

      or $d_1 > c_1 \ge 1$ with $c_3 = 0$)) **then**

    $p_f(B) \leftarrow p_f(B) + \max\{c_1, d_1\}$; **return** $p_f(B)$;

  else // by now $c_1 = d_1 = 0$ // **if** ($c_2 = d_2 = 0$) **then**

    return $p_f(B)$;

  else if ($c_2 \ge d_2$) **then** // use Lemma 2.7 //

    $c_1 \leftarrow 1$; $c_2 \leftarrow c_2 - 1$;

  else if ($c_2 < d_2$) **then** // use the remark after Lemma 2.7 //

    $d_1 \leftarrow 1$; $d_2 \leftarrow d_2 - 1$;

end while.

Finally, Theorem 2.3 and the subroutines above lead to the main algorithm.

Algorithm PG. Find the path-partition number $p_f(G)$ of the connected graph G whose blocks are complete graphs, cycles or complete bipartite graphs.

Input. A graph G and a vertex labeling f.

Output. $p_f(G)$.

Method.

$p_f(G) \leftarrow 0$; $G' \leftarrow G$;

while $(G' \neq \emptyset)$ **do**

choose a block B of G' with only one cut-vertex x or with no cut-vertex;

if (B is a complete graph) **then**

find $p_{f_i}(B)$ by calling **PCG**(B, f_i) for $0 \leq i \leq 3$;

if (B is a cycle) **then**

find $p_{f_i}(B)$ by calling **PC**(B, f_i) for $0 \leq i \leq 3$;

if (B is a complete bipartite graph) **then**

find $p_{f_i}(B)$ by calling **PCB**(B, f_i) for $0 \leq i \leq 3$;

$\alpha := p_{f_0}(B) - p_{f_1}(B)$; $\beta := p_{f_1}(B) - p_{f_2}(B)$;

if $(f(x) = 0)$ **then** $p_f(G) \leftarrow p_f(G) + p_f(B) - 1$;

else if $(f(x) = 1)$ **then**

$p_f(G) \leftarrow p_f(G) + p_{f_\alpha}(B) - 1$; $f(x) \leftarrow 1 - \alpha$;

else // by now $f(x) = 2$ or 3 //

case 1: $\alpha = \beta = 0$

$p_f(G) \leftarrow p_f(G) + p_{f_0}(B) - 1$;

case 2: $\alpha = 0$ and $\beta = 1$

$p_f(G) \leftarrow p_f(G) + p_f(B)$; $f(x) \leftarrow 3$;

case 3: $\alpha = 1$

$p_f(G) \leftarrow p_f(G) + p_{f_{1+\beta}}(B) - 1$; $f(x) \leftarrow 1 - \beta$;

$G' := G' - (B - \{x\})$;

end while;

output $p_f(G)$.

Theorem 2.9 *Algorithm* **PG** *computes the f-path partition number of a connected graph whose blocks are complete graphs, cycles or complete bipartite graphs in linear time.*

Proof. The correctness of the algorithm follows from Lemma 2.1 (4) and Lemmas 2.4 to 2.8. The algorithm takes only linear time since the depth-first search can be used to find blocks one by one in linear time, and each subroutine requires only $O(|B|)$ operations. ∎

We close this section by giving an example that demonstrates the algorithm.

Example 2.1 Consider the graph G_1 of 12 vertices and 5 blocks in Figure 2.1. Notice that its blocks are three complete graphs, a cycle and a complete bipartite graph.

1. We begin with the assignment $f(v) = 2$ for every vertex v. Set $p_f(G) = 0$.

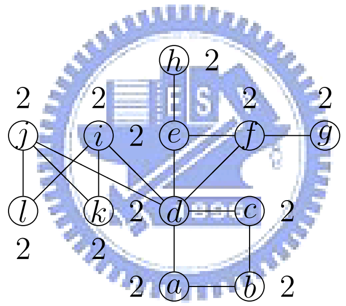

Figure 2.1: Graph G_1 of 12 vertices and 5 blocks.

2. Choose the block $B_1 = \{f, g\}$, which is a complete graph, with the only cut-vertex f in G_1. Call the subroutine **PCG**. Thus, $\alpha = 2-1 = 1$ and $\beta = 1-1 = 0$. Then, $p_f(G) = 0$ and $f(f) = 1$ (with a path fg results). Delete $B_1 - \{f\}$ from G_1 to get the graph G_2 in Figure 2.2.

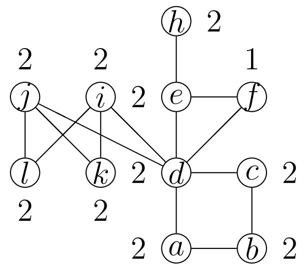

Figure 2.2: Graph G_2 results from G_1 by deleting $\{g\}$.

3. Choose the block $B_2 = \{e, h\}$, which is a complete graph, with the only cut-vertex e in G_2. Call the subroutine **PCG**. Thus, $\alpha = 2-1 = 1$ and $\beta = 1-1 = 0$. Then, $p_f(G) = 0$ and $f(e) = 1$ (with a path eh results). Delete $B_2 - \{e\}$ from G_2 to get the graph G_3 in Figure 2.3.

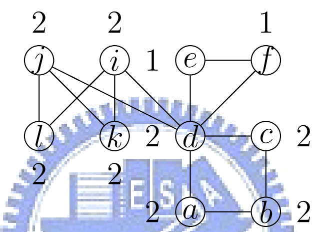

Figure 2.3: Graph G_3 results from G_2 by deleting $\{h\}$.

4. Choose the block $B_3 = \{d, e, f\}$, which is a complete graph, with the only cut-vertex d. Call the subroutine **PCG**. Thus, $\alpha = 2 - 2 = 0$ and $\beta = 2 - 1 = 1$. Then, $p_f(G) = 1$ and $f(d) = 3$ (with the path $P_1 = gfeh$ or $gfdeh$ results). Delete $B_3 - \{d\}$ to get the graph G_4 from G_3 in Figure 2.4.

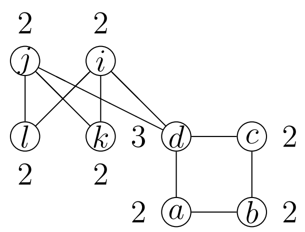

Figure 2.4: Graph G_4 results from G_3 by deleting $\{e, f\}$.

5. Choose the block $B_4 = \{a, b, c, d\}$, which is a cycle, with the only cut-vertex

d. Call the subroutine **PC**. Thus, $\alpha = 2 - 1 = 1$ and $\beta = 1 - 1 = 0$. Then, $p_f(G) = 1$, $P_1 = gfeh$ and $f(d) = 1$ (with a path $dcba$ results). Delete $B_4 - \{d\}$ from G_4 to get the graph G_5 from G_4 in Figure 2.5.

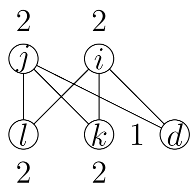

Figure 2.5: Graph G_5 results from G_4 by deleting $\{a, b, c\}$.

6. Choose the final block $B_5 = \{d, i, j, k, l\}$, which is a complete bipartite graph. Call the subroutine **PCB**. Set $c_2 = d_2 = 2$, $d_1 = 1$ and $c_0 = d_0 = c_1 = c_3 = d_3 = 0$. Since $d_1 = 1$ and $c_2 = 2 \geq 1$, by the remark after Lemma 2.8, we have $d_1 = 0$, $c_2 = 2 - 1 = 1$ and $c_1 = 1$. That is, we delete d from G_5 to get a new label 1 at vertex i. Then, we get the graph G_6 in Figure 2.6.

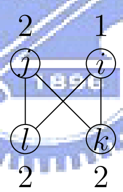

Figure 2.6: Graph G_6 results from G_5 by deleting d and set $f(i) = 1$.

7. Since $c_1 = 1$ and $d_2 = 2$, by Lemma 2.8, we have $c_1 = 0$, $d_2 = 1$ and $d_1 = 1$. That is, we delete i from G_6 to get a new label 1 at vertex k. Then, we get the graph G_7 in Figure 2.7.

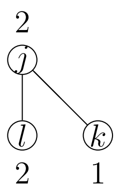

Figure 2.7: Graph G_7 results from G_6 by deleting i and set $f(k) = 1$.

8. Since $d_1 = 1$ and $c_2 = 1$, by the remark after Lemma 2.8, we have $d_1 = 0$, $c_2 = 0$ and $c_1 = 1$. That is, we delete k from G_7 to get a new label 1 at vertex j. Then, we get the graph G_8 in Figure 2.8.

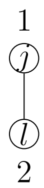

Figure 2.8: Graph G_8 results from G_7 by deleting k and set $f(j) = 1$.

9. Since $c_1 = 1$ and $d_2 = 1$, by Lemma 2.8, we have $c_1 = 0$, $d_2 = 0$ and $d_1 = 1$. That is, we delete j from G_8 to get a new label 1 at vertex l. Then, we get the graph G_9 in Figure 2.9.

Figure 2.9: Graph G_9 results from G_8 by deleting j and set $f(l) = 1$.

10. Since $c_1 = c_2 = c_3 = 0$, we have $p_f(B_5) = d_1 + d_2 + d_3 = 1$ (a path $abcdijkl$ results). Hence, $p_f(G) = 1 + p_f(B_5) = 2$ and an optimal path partition $\mathcal{P} = \{gfeh, abcdikjl\}$.

Chapter 3

Induced-path Partition

3.1 Preliminary of induced-path partition

Recall that an *induced path* is a path in which two vertices are adjacent only for those with consecutive indices. An *induced-path partition* of a graph is a collection of vertex-disjoint induced paths that cover all vertices of the graph. The *induced-path-partition problem* is to find the *induced-path number* $\rho(G)$ of a graph G, which is the minimum cardinality of an induced-path partition of G.

The concept of induced-path number was introduced by Chartrand *et al.* [9], who gave the induced-path numbers of complete bipartite graphs, complete binary trees, 2-dimensional meshs, butterflies and general trees. Broere *et al.* [6] determined exact values for complete multipartite graphs. Chartrand *et al.* [9] conjectured that $\rho(Q_d) \leq d$ for the d-dimensional hypercube Q_d with $d \geq 2$. Alsardary [3] proved that $\rho(Q_d) \leq 16$. From an algorithmic point of view, Le *et al.* [27] proved that the induced path partition problem is $\mathcal{NP}$-complete for general graphs.

The purpose of this chapter is to give a linear-time algorithm for the induced-path numbers of graphs whose blocks are complete graphs, cycles or complete bipartite graphs and a polynomial-time algorithm for cographs.

3.2 Induced-path partition in graphs with special blocks

In this section, we shall present a linear-time algorithm for the induced-path numbers for graphs whose blocks are complete graphs, cycles or complete bipartite graphs.

We use the same approach as in above chapter on this problem. Since the structure is the same as the above chapter, the results of the Subsection 3.2.1 is similar to those in Section 2.2. For completeness, we still present the results in detail.

For technical reasons, we consider the following generalized problem, which is a *labeling approach* for the problem.

Suppose every vertex v in the graph G is associated with an integer $f(v) \in \{0, 1, 2, 3\}$. An *f-induced-path partition* is a collection $\mathcal{P}$ of vertex-disjoint induced paths such that the following conditions hold.

(P1) Any vertex v with $f(v) \neq 3$ is in some induced path in $\mathcal{P}$.

(P2) If $f(v) = 0$, then v itself is an induced path in $\mathcal{P}$.

(P3) If $f(v) = 1$, then v is an end-vertex of some induced path in $\mathcal{P}$.

The *f-induced-path-partition problem* is to determine the *f-induced-path number* $\rho_f(G)$ which is the minimum cardinality of an f-induced-path partition of G. It is clear that $\rho(G) = \rho_f(G)$ when $f(v) = 2$ for all vertices v in G. Notice that as there may have some vertices of labels 3, an f-induced-path partition is not necessary a induced-path partition.

3.2.1 Inductive theorem

The labeling approach used in this subsection starts from an end block. Suppose $B = (V, E)$ is an end block whose only cut-vertex is x. Let A be the graph $G - (V - \{x\})$. Notice that we can view G as the "composition" of A and B, *i.e.*, G is the union of A and B which meet at a common vertex x. The idea is to get the induced-path number of G from those of A and B.

In the lemmas and theorems of this subsection, we use the following notation. Suppose x is a specified vertex of a graph $H = (V, E)$ in which f is a vertex labeling. For $i = 0, 1, 2, 3$, we define the function $f_i : V \to \{0, 1, 2, 3\}$ by $f_i(y) = f(y)$ for all vertices y except $f_i(x) = i$.

Lemma 3.1 *Suppose x is a specified vertex in a graph H. Then the following statements hold.*

(1) $\rho_{f_3}(H) \le \rho_{f_2}(H) \le \rho_{f_1}(H) \le \rho_{f_0}(H)$.

(2) $\rho_{f_1}(H) \le \rho_{f_0}(H) \le \rho_{f_1}(H) + 1$.

(3) $\rho_{f_2}(H) \le \rho_{f_1}(H) \le \rho_{f_2}(H) + 1$.

(4) $\rho_{f_3}(H) = \min\{\rho_{f_2}(H), \rho_f(H - x)\} \le \rho_f(H - x) = \rho_{f_0}(H) - 1$.

(5) $\rho_f(H) \ge \rho_{f_1}(H) - 1$.

Proof. (1) The inequalities follow from that an f_i-induced-path partition is an f_j-induced-path partition whenever $i < j$.

(2) The second inequality follows from that replacing the induced path Px in an f_1-induced-path partition by two induced paths P and x results in an f_0-induced-path partition of H.

(3) The second inequality follows from that replacing the induced path PxQ in an f_2-induced-path partition by two induced paths Px and Q results in an f_1-induced-path partition of H.

(4) The first equality follows from that one is an f_3-induced-path partition of H if and only if it is either an f_2-induced-path partition of H or an f-induced-path partition of $H - x$. The second equality follows from that $\mathcal{P}$ is an f_0-induced-path partition of H if and only if it is the union of $\{x\}$ and an f-induced-path partition of $H - x$.

(5) According to (1), (3) and (4), we have

$$\rho_f(H) \ge \rho_{f_3}(H) = \min\{\rho_{f_2}(H), \rho_f(H-x)\} \ge \min\{\rho_{f_1}(H)-1, \rho_{f_0}(H)-1\} = \rho_{f_1}(H)-1.$$

∎

Lemma 3.2 (1) $\rho_f(G) \le \min\{\rho_f(A) + \rho_{f_0}(B) - 1, \rho_{f_0}(A) + \rho_f(B) - 1\}$.

(2) $\rho_{f_2}(G) \le \rho_{f_1}(A) + \rho_{f_1}(B) - 1$.

Proof. (1) Suppose $\mathcal{P}$ is an optimal f-induced-path partition of A, and $\mathcal{Q}$ an f_0-induced-path partition of B. Then $x \in \mathcal{Q}$ and so $(\mathcal{P} \cup \mathcal{Q}) - \{x\}$ is an f-induced-path partition of G. This gives $\rho_f(G) \le \rho_f(A) + \rho_{f_0}(B) - 1$. Similarly, $\rho_f(G) \le \rho_{f_0}(A) + \rho_f(B) - 1$.

(2) The inequality follows from that if $\mathcal{P}$ (respectively, $\mathcal{Q}$) is an optimal f_1-induced-path partition of A (respectively, B) in which $Px \in \mathcal{P}$ (respectively, $xQ \in \mathcal{Q}$) contains x, then $(\mathcal{P} \cup \mathcal{Q} \cup \{PxQ\}) - \{Px, xQ\}$ is an f_2-induced-path partition of G.

∎

We now have the following theorem which is key for the inductive step of our algorithm.

Theorem 3.3 *Suppose* $\alpha = \rho_{f_0}(B) - \rho_{f_1}(B)$ *and* $\beta = \rho_{f_1}(B) - \rho_{f_2}(B)$. *(Notice that* $\alpha, \beta \in \{0, 1\}$.*) Then the following statements hold.*

(1) *If* $f(x) = 0$, *then* $\rho_f(G) = \rho_f(A) + \rho_f(B) - 1$.

(2) *If* $f(x) = 1$, *then* $\rho_f(G) = \rho_{f_{1-\alpha}}(A) + \rho_{f_\alpha}(B) - 1$.

(3) *If* $f(x) \ge 2$ *and* $\alpha = \beta = 0$, *then* $\rho_f(G) = \rho_f(A) + \rho_{f_0}(B) - 1$.

(4) *If* $f(x) \ge 2$ *and* $\alpha = 0$ *and* $\beta = 1$, *then* $\rho_f(G) = \rho_{f_3}(A) + \rho_f(B)$.

(5) *If* $f(x) \ge 2$ *and* $\alpha = 1$, *then* $\rho_f(G) = \rho_{f_{1-\beta}}(A) + \rho_{f_{1+\beta}}(B) - 1$.

Proof. Suppose $\mathcal{P}$ is an optimal f-induced-path partition of G. Let P^* be the induced path in $\mathcal{P}$ that contains x. (It is possible that there is no such induced path when $f(x) = 3$.) There are three possibilities for P^*: (a) P^* does not exist or $P^* \subseteq A$; (b) $P^* \subseteq B$; (c) x is an internal vertex of P^*, say $P^* = P'xP''$, with $P'x \subseteq A$ and $xP'' \subseteq B$. (This is possible only when $f(x) \ge 2$.)

For the case when (a) holds, $\{P \in \mathcal{P} : P \subseteq A\}$ is an f-induced-path partition of A and $\{P \in \mathcal{P} : P \subseteq B\} \cup \{x\}$ is an f_0-induced-path partition of B. We then have the inequality in (a′). Similarly, we have (b′) and (c′) corresponding to (b) and (c).

(a′) $\rho_f(G) \geq \rho_f(A) + \rho_{f_0}(B) - 1$.

(b′) $\rho_f(G) \geq \rho_{f_0}(A) + \rho_f(B) - 1$. (We may replace $\rho_f(B)$ by $\rho_{f_2}(B)$ when $f(x) \geq 2$.)

(c′) $\rho_f(G) \geq \rho_{f_1}(A) + \rho_{f_1}(B) - 1$. (This is possible only when $f(x) \geq 2$.)

We are now ready to prove the theorem.

(1) Since $f(x) = 0$, we have $f = f_0$. According to Lemma 3.2 (1), $\rho_f(G) \leq \rho_f(A) + \rho_f(B) - 1$. On the other hand, (a′) and (b′) give $\rho_f(G) \geq \rho_f(A) + \rho_f(B) - 1$.

(2) Since $f(x) = 1$, we have $f = f_1$. Lemma 3.2 (1), together with (a′) and (b′), gives $\rho_f(G) = \min\{\rho_{f_1}(A) + \rho_{f_0}(B) - 1, \rho_{f_0}(A) + \rho_{f_1}(B) - 1\}$. If $\alpha = 0$, then

$$\rho_{f_0}(A) + \rho_{f_1}(B) - 1 \geq \rho_{f_1}(A) + (\rho_{f_0}(B) - \alpha) - 1 = \rho_{f_1}(A) + \rho_{f_0}(B) - 1;$$

and if $\alpha = 1$, then

$$\rho_{f_1}(A) + \rho_{f_0}(B) - 1 \geq (\rho_{f_0}(A) - 1) + (\rho_{f_1}(B) + \alpha) - 1 = \rho_{f_0}(A) + \rho_{f_1}(B) - 1.$$

Hence $\rho_f(G) = \rho_{f_{1-\alpha}}(A) + \rho_{f_\alpha}(B) - 1$.

(3) According to Lemma 3.2 (1), $\rho_f(G) \leq \rho_f(A) + \rho_{f_0}(B) - 1$. On the other hand, as $\rho_{f_0}(A) \geq \rho_{f_1}(A) \geq \rho_f(A)$ and $\rho_{f_0}(B) = \rho_{f_1}(B) = \rho_{f_2}(B)$, (a′)–(c′) give $\rho_f(G) \geq \rho_f(A) + \rho_{f_0}(B) - 1$.

(4) According to Lemma 3.1 (4) and $\alpha = 0$ and $\beta = 1$, we have

$$\rho_f(B - x) = \rho_{f_0}(B) - 1 = \rho_{f_1}(B) - 1 = \rho_{f_2}(B).$$

This, together with Lemma 3.1 (4), gives that the above value is also equal to $\rho_{f_3}(B)$ and so $\rho_f(B)$. Then, an optimal f_3-induced-path partition $\mathcal{P}$ of A, together with an optimal f-induced-path partition of $B - x$ (respectively, B) when x is (respectively, is not) in an induced path of $\mathcal{P}$, forms an f_2-induced-path partition of G. Thus, $\rho_f(G) \leq \rho_{f_2}(G) \leq \rho_{f_3}(A) + \rho_f(B)$.

On the other hand, since $\rho_{f_1}(A) \geq \rho_f(A) \geq \rho_{f_3}(A)$ and $\rho_{f_0}(B) - 1 = \rho_{f_1}(B) - 1 = \rho_f(B)$, (a′) or (c′) implies $\rho_f(G) \geq \rho_{f_3}(A) + \rho_f(B)$. Also, as $\rho_{f_0}(A) - 1 \geq \rho_{f_3}(A)$ by Lemma 3.1 (4), (b′) implies $\rho_f(G) \geq \rho_{f_3}(A) + \rho_f(B)$.

(5) According to Lemma 3.1 (1) and Lemma 3.2, we have

$$\rho_f(G) \leq \rho_{f_2}(G) \leq \min\{\rho_{f_0}(A) + \rho_{f_2}(B) - 1, \rho_{f_1}(A) + \rho_{f_1}(B) - 1\}.$$

On the other hand, if (a′) holds, then by Lemma 3.1 (5) and that $\rho_{f_0}(B) = \rho_{f_1}(B) + 1$,

$$\rho_f(G) \geq \rho_f(A) + \rho_{f_0}(B) - 1 \geq (\rho_{f_1}(A) - 1) + (\rho_{f_1}(B) + 1) - 1 = \rho_{f_1}(A) + \rho_{f_1}(B) - 1.$$

This, together with (b′) and (c′), gives

$$\rho_f(G) = \min\{\rho_{f_0}(A) + \rho_{f_2}(B) - 1, \rho_{f_1}(A) + \rho_{f_1}(B) - 1\}.$$

If $\beta = 0$, then

$$\rho_{f_0}(A) + \rho_{f_2}(B) - 1 \geq \rho_{f_1}(A) + (\rho_{f_1}(B) - \beta) - 1 = \rho_{f_1}(A) + \rho_{f_1}(B) - 1;$$

and if $\beta = 1$, then

$$\rho_{f_1}(A) + \rho_{f_1}(B) - 1 \geq (\rho_{f_0}(A) - 1) + (\rho_{f_2}(B) + \beta) - 1 = \rho_{f_0}(A) + \rho_{f_2}(B) - 1.$$

Hence $\rho_f(G) = \rho_{f_{1-\beta}}(A) + \rho_{f_{1+\beta}}(B) - 1$. ∎

Before we use the theorems of this subsection to design an efficient algorithm, let us use them to give an alternative proof for a result on trees.

Let T be a tree. For a vertex v of T with $d_T(v) \geq 3$, the excess degree $\varepsilon(v)$ of v is equal to $d_T(v) - 2$. A *penultimate vertex* is a vertex that is not a leaf and all of whose neighbors are leaves, with the possible exception of one.

Corollary 3.4 [9] *Let T be a tree, and let H be the forest induced by the vertices of T having degree 3 or more. Let H' be a spanning sub-forest of H of maximum size that $d_{H'}(v) \leq \varepsilon(v)$ for every vertex v of H. Then,*

$$\rho(T) = 1 + |E(H')| + \sum_{v \in V(H)} [\varepsilon(v) - d_{H'}(v)].$$

Proof. The corollary is clear when the tree has just one vertex. Suppose now T has at least two vertices. Choose a penultimate vertex x whose with leaf-neighbors x_1, x_2,..., x_r. Let $T' = T - \{x, x_1, x_2, \ldots, x_r\}$. By Theorem 3.3 (5), (2) and (1) and the induction hypothesis,

$$\rho(T) = \rho(T') + r - 1 = 1 + |E(H'_{T'})| + \sum_{v \in V(H_{T'})} (\varepsilon(v) - d_{H'_{T'}}(v)) + r - 1,$$

where $H_{T'}$ is the forest induced by the vertices of T' having degree 3 or more, and $H'_{T'}$ is a spanning sub-forest of H of maximum size such that $d_{H'}(v) \leq \varepsilon(v)$ for every vertex v of $H_{T'}$. Since $\varepsilon(x) = r - 1$, $d_{H'}(x) = 1$ and $|E(H')| = |E(H'_{T'})| + 1$, the corollary then follows. ∎

3.2.2 Induced-path partitions for special blocks

Besides the inductive theorem (Theorem 3.3) we also need to establish formula for the induced-path numbers of special graphs including complete graphs, cycles or complete bipartite graphs. Here we assume that B is a graph in which each vertex v has a label $f(v) \in \{0, 1, 2, 3\}$. Recall that $f^{-1}(i)$ is the set of *pre-images* of i, *i.e.,*

$$f^{-1}(i) = \{v \in V(B) : f(v) = i\}.$$

Also, $f^{-1}(I) = \cup_{i \in I} f^{-1}(i)$ for any $I \subseteq \{0, 1, 2, 3\}$. According to Lemma 3.1 (4), $\rho_f(B) = \rho_f(B - f^{-1}(0)) + |f^{-1}(0)|$. *Therefore, in this section we only consider the function f with $f^{-1}(0) = \emptyset$.*

We first consider the case when B is a complete graph.

Lemma 3.5 *If B is a complete graph, then $\rho_f(B) = \lceil |f^{-1}(\{1, 2\})|/2 \rceil$.*

Proof. The equality holds since an induced path of a complete graph is a 2-path or a 1-path. ∎

Next, we consider the case when B is a path. This is useful as a subroutine for handling cycles.

Lemma 3.6 *Suppose B is a path.*

(1) *If x is an end-vertex of B with $f(x) = 3$, then $\rho_f(B) = \rho_f(B - x)$.*

(2) *If x is an end-vertex of B with $f(x) \in \{1, 2\}$ and another vertex y with $f(y) = 1$ such that no vertex between x and y has a label 1 (choose y the other end-vertex of B if there is no such vertex), then $\rho_f(B) = \rho_f(B') + 1$ where B' is the path obtained from B by deleting x, y and all vertices between them.*

Proof. (1) Since $f(x) = 3$, by Lemma 3.1 (4), $\rho_f(B) \le \rho_f(B - x)$. As x is an end-vertex of B, $\rho_f(B) \ge \rho_f(B - x)$ follows from that deleting x from an induced path (if any) in an f-induced-path partition of B results in an f-induced-path partition of $B - x$.

(2) First, we claim that if $f(x) = 2$, then $\rho_f(B) = \rho_{f_1}(B)$. By Lemma 3.1 (1), $\rho_f(B) \le \rho_{f_1}(B)$. Since x is an end-vertex of B and $f(x) = 2$, an f-induced-path partition is in fact an f_1-induced-path partition of B. Thus $\rho_f(B) \ge \rho_{f_1}(B)$. Now, we can assume that $f(x) = 1$.

Let P denotes the path from x to y in B. First, $\rho_f(B) \le \rho_f(B') + 1$ follows from that an f-induced-path partition of B', together with P, forms an f-induced-path partition of B. On the other hand, suppose $\mathcal{P}$ is an optimal f-induced-path partition of B. Since $f(x) = f(y) = 1$ and x is an end-vertex of B, $\mathcal{P}$ has some $P' \subseteq P$ with $x \in P'$. Deleting all vertices of P from the paths in $\mathcal{P}$ results in an f-induced-path partition of B' whose size is less than $|\mathcal{P}|$ by at least one. Thus, $\rho_f(B) - 1 \ge \rho_f(B')$. ∎

We now consider the case when B is a cycle.

Lemma 3.7 *Suppose B is a cycle.*

(1) *If $f^{-1}(\{1, 2\}) = \emptyset$, then $\rho_f(B) = 0$.*

(2) *When $f^{-1}(1) = \emptyset$ and $f^{-1}(2) \ne \emptyset$, if there exists a vertex with label 3, then $\rho_f(B) = 1$ else $\rho_f(B) = 2$.*

(3) *When* $f^{-1}(1) = \{x\}$, *if* x *has at least one neighbor labeled with* 3, *then* $\rho_f(B) = 1$ *else* $\rho_f(B) = 2$.

(4) *If* $|f^{-1}(1)| \geq 2$ *and* $f^{-1}(2) = \emptyset$, *then* $\rho_f(B) = \lceil |f^{-1}(1)|/2 \rceil$.

(5) *When* $f^{-1}(1)$ *contains exactly two vertices which are adjacent and* $f^{-1}(2) \neq \emptyset$, *then* $\rho_f(B) = 2$.

(6) *If* P *is an induced path from* x *to* y *in* B *such that* $f^{-1}(1) \cap P = \{x, y\}$ *and* $f^{-1}(2) \cap P \neq \emptyset$, *then* $\rho_f(B) = \rho_f(B - P) + 1$.

Proof. The proof from (1) to (5) are obvious.

(5) First, $\rho_f(B) \leq \rho_f(B - P) + 1$ follows from that an f-induced-path partition of $B - P$ together with P forms an f-induced-path partition of B. On the other hand, suppose $\mathcal{P}$ is an optimal f-induced-path partition of B. Since $f^{-1}(1) \cap P = \{x, y\}$ and $f^{-1}(2) \cap P \neq \emptyset$, $\mathcal{P}$ must contain some $P' \subseteq P$. Deleting all vertices of P from the paths in $\mathcal{P}$ results in an f-induced-path partition of $B - P$ whose size is less than $|\mathcal{P}|$ by at least one. Thus, $\rho_f(B) - 1 \geq \rho_f(B - P)$. ∎

We now consider the case when B is a complete bipartite graph with $C \cup D$ as a bipartition of the vertex set. For $i = 0, 1, 2, 3$, let

$$C_i = \{x \in C : f(x) = i\} \text{ and } c_i = |C_i|;$$

$$D_i = \{y \in D : f(y) = i\} \text{ and } d_i = |D_i|.$$

Notice that an induced path of a complete bipartite graph has at most 3 vertices. We then have the following lemmas.

Lemma 3.8 *Suppose* $c_1 \geq 2$ *and* $d_2 \geq 1$. *If* $x, z \in C_1$ *and* $y \in D_2$, *then* $\rho_f(B) = \rho_f(B - \{x, y, z\}) + 1$.

Proof. First, $\rho_f(B) \leq \rho_f(B - \{x, y, z\}) + 1$ since xyz is an induced path. On the other hand, suppose $\mathcal{P}$ is an optimal f-induced-path partition of B. We claim that there exists a path xyT in $\mathcal{P}$. Otherwise, suppose xP and QyR are in $\mathcal{P}$ with $|R| \leq 1$.

When $P = y'S$, we may replace $xP = xy'S$ by xyS and QyR by $Qy'R$; when $P = \emptyset$, we may replace $xP = x$ by xyR and QyR by Q. Next we claim that $T = z$. Otherwise, suppose Sz is in $\mathcal{P}$. In this case we may replace xyT by xyz and Sz by ST. Therefore, we may assume that $\mathcal{P}$ contains xyz, and so $\rho_f(B) - 1 \geq \rho_f(B - \{x, y, z\})$. ∎

By symmetry, we may prove a similar lemma for the case when $d_1 \geq 2$ and $c_2 \geq 1$.

Lemma 3.9 *Suppose* $2c_2 > d_1 + d_2$. *If* $x \in C_2$, *then* $\rho_f(B) = \rho_{f'}(B)$ *where* f' *is the same as* f *except* $f'(x) = 1$.

Proof. First, $\rho_f(B) \leq \rho_{f'}(B)$ since an f'-induced-path partition of B is an f-induced-path partition of B. On the other hand, suppose $\mathcal{P}$ is an optimal f-induced-path partition of B. If every vertex in C_2 is an internal vertex of some induced path in $\mathcal{P}$, then the two end-vertices of this induced path are in $D_1 \cup D_2$, and so $2c_2 \leq d_1 + d_2$ which is impossible. Hence, we may assume that x is the end-vertex of an induced path in $\mathcal{P}$. This gives $\rho_f(B) \geq \rho_{f'}(B)$. ∎

By symmetry, we may prove a similar lemma for the case when $2d_2 > c_1 + c_2$.

We may repeatedly apply Lemmas 3.8 and 3.9 and the remarks after them until the following conditions hold:

$$(d_1 \leq 1 \text{ or } c_2 = 0), \quad (c_1 \leq 1 \text{ or } d_2 = 0), \quad 2c_2 \leq d_1 + d_2, \quad 2d_2 \leq c_1 + c_2.$$

Notice that it is impossible that $c_2 = 0 < d_2$, for otherwise the second condition gives $c_1 \leq 1$ while the forth gives $2 \leq 2d_2 \leq c_1 \leq 1$, a contradiction. So, either $c_2 = d_2 = 0$ or both c_2 and d_2 are nonzero. The latter case implies $c_1 = c_2 = d_1 = d_2 = 1$, in which case $\rho(B) = 2$.

Lemma 3.10 *Suppose* $c_2 = d_2 = 0$, $c_1 \geq 1$ *and* $d_1 \geq 1$. *If* $x \in C_1$ *and* $y \in D_1$, *then* $\rho_f(B) = \rho_f(B - \{x, y\}) + 1$.

Proof. First, $\rho_f(B) \leq \rho_f(B - \{x, y\}) + 1$ since xy is an induced path. On the other hand, suppose $\mathcal{P}$ is an optimal f-induced-path partition of B. If xy is not in $\mathcal{P}$, then

$\mathcal{P}$ contains xP and yQ. For the case when $P = \emptyset$, we may replace $xP = x$ by xy and yQ by Q. For the case when $P = y'$, we may replace $xP = xy'$ by xy and yQ by $y'Q$. So, we may assume that $xP = xy'z$. By symmetry, we may also assume that $yQ = yz'x'$. As $c_2 = d_2 = 0$, it is the case that $y' \in D_3$ and $z' \in C_3$. Then we may replace $xy'z$ by xy and $yz'x'$ by $x'z$. Therefore, we may assume that xy is in $\mathcal{P}$ and so $\rho_f(B) - 1 \geq \rho_f(B - \{x, y\})$. ∎

Lemma 3.11 *Suppose* $d_1 = c_2 = d_2 = 0$, $c_1 \geq 2$ *and* $d_3 \geq 1$. *If* $x, z \in C_1$ *and* $y \in D_3$, *then* $\rho_f(B) = \rho_f(B - \{x, y, z\}) + 1$.

Proof. First, $\rho_f(B) \leq \rho_f(B - \{x, y, z\})$ since xyz is an induced path. On the other hand, suppose $\mathcal{P}$ is an optimal f-induced path of B. By the condition $d_1 = c_2 = d_2 = 0$, it is easy to see that we may assume that xyz is an induced path in $\mathcal{P}$. Hence, $\rho_f(B) - 1 \geq \rho_f(B - \{x, y, z\})$. ∎

By symmetry, we may prove a similar lemma for the case when $c_1 = c_2 = d_2 = 0$, $d_1 \geq 2$ and $c_3 \geq 1$.

3.2.3 Algorithm for graphs with special blocks

We are ready to give a linear-time algorithm for the induced-path number of graphs whose blocks are complete graphs, cycles or complete bipartite graphs. Notice that we may consider only connected graphs. We present five procedures. The first four are subroutines which calculate f-induced-path numbers of complete graphs, paths, cycles and complete bipartite graphs, respectively, by using Lemmas 3.5 to 3.11. The last one is the main routine for the problem.

Lemmas 3.1 (4) and 3.5 lead to the following subroutine for complete graphs.

Algorithm IPCG. Find the f-induced-path number $\rho_f(B)$ of a complete graph B.

Input. A complete graph B and a vertex labeling f.

Output. $\rho_f(B)$.

Method.

$\rho_f(B) = |f^{-1}(0)| + \lceil |f^{-1}(\{1,2\})|/2 \rceil$;

return $\rho_f(B)$.

Lemma 3.6 leads to the following subroutine for paths, which is used in the cycle subroutine.

Algorithm IPP. Find the f-induced-path number $\rho_f(B)$ of a path B.

Input. A path B and a vertex labeling f with $f^{-1}(0) = \emptyset$.

Output. $\rho_f(B)$.

Method.

$\rho_f(B) \leftarrow 0$;

$B' \leftarrow B$;

while $(B' \neq \emptyset)$ **do**

choose an end-vertex x of B';

if $(f(x) = 3)$ **then** $B' \leftarrow B' - x$ **else**

choose a vertex y nearest to x with $f(y) = 1$

(let y be the other end-vertex if there is no such vertex);

$\rho_f(B) \leftarrow \rho_f(B) + 1$;

$B' \leftarrow B' -$ all vertices between (and including) x and y;

end else;

end while;

return $\rho_f(B)$.

Lemmas 3.1 (4) and 3.7 lead to the following subroutine for cycles.

Algorithm IPC. Find the f-induced-path number $\rho_f(B)$ of a cycle B.

Input. A cycle B and a vertex labeling f.

Output. $\rho_f(B)$.

Method.

if $(f^{-1}(\{0,1,2\}) = \emptyset)$ **then** $\rho_f(B) \leftarrow 0$;

else if $(f^{-1}(\{0,1\}) = \emptyset \neq f^{-1}(2))$ **then**

  if there exists a vertex with label 3 **then** $\rho_f(B) \leftarrow 1$ **else** $\rho_f(B) \leftarrow 2$;

else if $(f^{-1}(0) = \emptyset$ and $f^{-1}(1) = \{x\})$ **then**

  if x has a neighbor labeled with 3 **then** $\rho_f(B) \leftarrow 1$ **else** $\rho_f(B) \leftarrow 2$;

else if $(f^{-1}(0) = \emptyset$ and $|f^{-1}(1)| \geq 2$ and $f^{-1}(2) = \emptyset)$ **then**

  $\rho_f(B) \leftarrow \lceil |f^{-1}(1)|/2 \rceil$;

else if $(f^{-1}(0) = \emptyset$ and $|f^{-1}(1)| \geq 2$ and $f^{-1}(2) \neq \emptyset)$ **then**

  if ($f^{-1}(1)$ contains exactly two vertices which are adjacent) **then** $\rho_f(B) \leftarrow 2$;

  else choose an x-y induced path P with $f^{-1}(1) \cap P = \{x, y\}$ and $f^{-1}(2) \cap P \neq \emptyset$;

    $p_f(B) \leftarrow p_f(B - P) + 1$ by calling **PP**$(B - P)$;

else // now $f^{-1}(0) \neq \emptyset$ //

  let $B - f^{-1}(0)$ be the disjoint union of paths $P_1, P_2, \ldots, P_k$;

  $\rho_f(B) \leftarrow |f^{-1}(0)|$;

  for $i = 1$ **to** k **do** $\rho_f(B) \leftarrow \rho_f(B) + \rho_f(P_i)$ by calling **PP**(P_i);

end else;

return $\rho_f(B)$.

Lemma 3.1 (4) and Lemmas 3.8 to 3.11 lead to the following subroutine for complete bipartite graphs.

Algorithm IPCB. Find the f-induced-path number $\rho_f(B)$ of a complete bipartite graph B.

Input. A complete bipartite graph B with a bipartition $C \cup D$ of vertices and a vertex labeling f.

Output. $\rho_f(B)$.

Method.

$c_i \leftarrow |f^{-1}(i) \cap C|$ and $d_i \leftarrow |f^{-1}(i) \cap D|$ for $0 \le i \le 3$; $\rho_f(B) \leftarrow c_0 + d_0$;

while (true) **do**

    if ($c_1 \ge 2$ and $d_2 \ge 1$) **then** // use Lemma 3.8 //

        $c_1 \leftarrow c_1 - 2$; $d_2 \leftarrow d_2 - 1$; $\rho_f(B) \leftarrow \rho_f(B) + 1$;

    else if ($d_1 \ge 2$ and $c_2 \ge 1$) **then** // remark after Lemma 3.8 //

        $d_1 \leftarrow d_1 - 2$; $c_2 \leftarrow c_2 - 1$; $\rho_f(B) \leftarrow \rho_f(B) + 1$;

    else if ($2c_2 > d_1 + d_2$) **then** // use Lemma 3.9 //

        $c_2 \leftarrow c_2 - 1$; $c_1 \leftarrow c_1 + 1$;

    else if ($2d_2 > c_1 + c_2$) **then** // remark after Lemma 3.9 //

        $d_2 \leftarrow d_2 - 1$; $d_1 \leftarrow d_1 + 1$;

    else if ($c_1 = c_2 = d_1 = d_2 = 1$) **then**

        $\rho_f(B) \leftarrow \rho_f(B) + 2$; **return** $\rho_f(B)$;

    else if ($c_2 = d_2 = 0$ and $c_1 \ge 1$ and $d_1 \ge 1$) **then** // use Lemma 3.10 //

        $c_1 \leftarrow c_1 - 1$; $d_1 \leftarrow d_1 - 1$; $\rho_f(B) \leftarrow \rho_f(B) + 1$;

    else if ($d_1 = c_2 = d_2 = 0$ and $c_1 \ge 2$ and $d_3 \ge 1$) **then** // use Lemma 3.11 //

        $c_1 \leftarrow c_1 - 2$; $d_3 \leftarrow d_3 - 1$; $\rho(B) \leftarrow \rho(B) + 1$;

    else if ($c_1 = c_2 = d_2 = 0$ and $d_1 \ge 2$ and $c_3 \ge 1$) **then** // remark after Lemma 3.11 //

        $d_1 \leftarrow d_1 - 2$; $c_3 \leftarrow c_3 - 1$; $\rho(B) \leftarrow \rho(B) + 1$;

    else // now $c_2 = d_2 = 0$ with ($c_1 + d_1 \le 1$ or $c_1 = c_3 = 0$ or $d_1 = d_3 = 0$) //

        $\rho_f(B) \leftarrow \rho_f(B) + c_1 + c_2 + d_1 + d_2$; **return** $\rho_f(B)$;

end while.

Finally, Theorem 3.3 leads to the following main algorithm.

Algorithm IPG. Find the f-induced-path number $\rho_f(G)$ of the connected graph G whose blocks are complete graphs, cycles or complete bipartite graphs.

Input. A graph G and a vertex labeling f.

Output. $\rho_f(G)$.

Method.

$\rho_f(G) \leftarrow 0$;
while $(G \neq \emptyset)$ **do**
  choose a block B with cut-vertex x or with no cut-vertex;
  if (B is a complete graph) **then**
    find $\rho_{f_i}(B)$ by calling **IPCG**(B, f_i) for $0 \le i \le 3$;
  else if (B is a cycle) **then**
    find $\rho_{f_i}(B)$ by calling **IPC**(B, f_i) for $0 \le i \le 3$;
  else if (B is a complete bipartite graph) **then**
    find $\rho_{f_i}(B)$ by calling **IPCB**(B, f_i) for $0 \le i \le 3$;
  $\alpha := \rho_{f_0}(B) - \rho_{f_1}(B)$; $\beta := \rho_{f_1}(B) - \rho_{f_2}(B)$;
  if $(f(x) = 0)$ **then** $\rho_f(G) \leftarrow \rho_f(G) + \rho_f(B) - 1$;
  else if $(f(x) = 1)$ **then**
    $\rho_f(G) \leftarrow \rho_f(G) + \rho_{f_\alpha}(B) - 1$; $f(x) \leftarrow 1 - \alpha$;
  else // by now $f(x) = 2$ or 3 //
    case 1: $\alpha = \beta = 0$
      $\rho_f(G) \leftarrow \rho_f(G) + \rho_{f_0}(B) - 1$;
    case 2: $\alpha = 0$ and $\beta = 1$
      $\rho_f(G) \leftarrow \rho_f(G) + \rho_f(B)$; $f(x) \leftarrow 3$;
    case 3: $\alpha = 1$
      $\rho_f(G) \leftarrow \rho_f(G) + \rho_{f_{1+\beta}}(B) - 1$; $f(x) \leftarrow 1 - \beta$;
  $G := G - (B - \{x\})$;
end while;
output $\rho_f(G)$.

Theorem 3.12 *Algorithm* **IPG** *computes the induced-path number of a connected graph whose blocks are complete graphs, cycles or complete bipartite graphs in linear time.*

Proof. The correctness of the algorithm follows from Theorem 3.3, Lemma 3.1 (4) and Lemmas 3.5 to 3.11. The algorithm takes only linear time since the depth-first search can be used to find blocks one by one in linear time, and each subroutine requires only $O(|B|)$ operations. ∎

We now give an example to demonstrate the algorithm.

Example 3.1 Consider the graph G_1 of 9 vertices and 3 blocks in Figure 3.1. Notice that its blocks are a complete graph, a cycle and a complete bipartite graph.

1. We begin with the assignment $f(v) = 2$ for every vertex v. Set $\rho_f(G_1) = 0$.

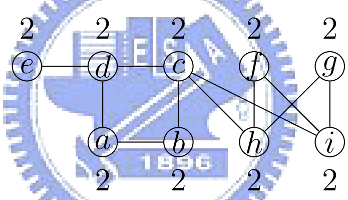

Figure 3.1: Graph G_1 of 9 vertices and 3 blocks.

2. Choose the block $B_1 = \{d, e\}$, which is a complete graph, with the only cut-vertex d in G_1. Call the subroutine **IPCG**. Thus, $\alpha = 2 - 1 = 1$ and $\beta = 1 - 1 = 0$. Then, $\rho_f(G) = 0 + 1 - 1 = 0$ and relabel $f(d) = 1$ (with an induced path de results). Delete $B_1 - \{d\}$ from G_1 to get the graph G_2 in Figure 3.2.

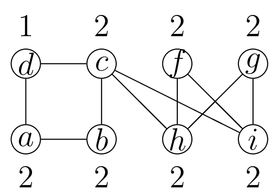

Figure 3.2: Graph G_2 results from G_1 by deleting $\{e\}$.

3. Choose the block $B_2 = \{a, b, c, d\}$, which is a cycle, with the only cut-vertex c in G_2. Call the subroutine **IPC**. Thus, $\alpha = 2 - 2 = 0$ and $\beta = 2 - 2 = 0$. Then, $\rho_f(G) = 0 + 2 - 1 = 1$ (with an induced path $edab$ results). Delete $B_2 - \{c\}$ from G_2 to get the graph G_3 in Figure 3.3.

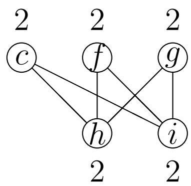

Figure 3.3: Graph G_3 results from G_2 by deleting $\{a, b, d\}$.

4. Choose the final block $B_3 = \{c, f, g, h, i\}$, which is a complete bipartite graph. Call the subroutine **IPCB**. Notice that $c_2 = 3$, $d_2 = 2$ and $c_0 = c_1 = c_3 = d_0 = d_1 = d_3 = 0$. Since $2c_2 > d_1 + d_2$, by using Lemma 3.9, we get a new label 1 at vertex c as in Figure 3.4.

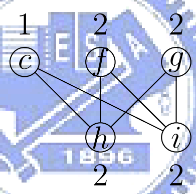

Figure 3.4: Graph G_3 with a new label at vertex c.

5. Now, $c_1 = 1$, $c_2 = d_2 = 2$ and $c_0 = c_3 = d_0 = d_3 = 0$. Since $2c_2 > d_1 + d_2$, again by Lemma 3.9, we relabel vertex f by 1 as in Figure 3.5.

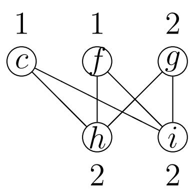

Figure 3.5: Graph G_3 with a new label at vertex f.

6. Now $c_2 = 1$, $c_1 = d_2 = 2$ and $c_0 = c_3 = d_1 = d_3 = 0$. Since $c_1 \geq 2$ and $d_2 \geq 1$, by Lemma 3.8, we have $\rho_f(B_3) = 1 + \rho_f(B_3 - \{c, h, f\})$ (with a path chf results).

Continue this process to calculate $\rho_f(B_3 - \{c, h, f\})$, we get $\rho_f(B_3) = 2$ (with an induced path gi results). Hence, $p_f(G) = 1 + p_f(B_3) = 3$, and an optimal induced-path partition is $\mathcal{P} = \{edab, chf, gi\}$.

3.3 Induced-path partition in cographs

This section gives a polynomial-time algorithm for the induced-path number of cographs. Recall that *cographs* are defined by the following rules:

(i) K_1 is a cograph;

(ii) if G and H are cographs, then so are $G + H$ and $G \times H$;

(iii) no other graphs are cographs.

For more details on cographs, see [12, 13, 26].

For technical reasons, we consider the following generalized definition. Let $\rho(G, t, p)$ be the minimum among all induced-path numbers of all graphs $G(t, p)$ obtained from G by removing t vertices and p pairs of nonadjacent vertices. It is clear that $\rho(G) = \rho(G, 0, 0)$.

In the following lemma, suppose $G = (V, E)$ and $H = (V', E')$.

Lemma 3.13 *For $t + 2p \le |V| + |V'|$, we have*

$$\rho(G + H, t, p) = \min_C \{\rho(G, t_1 + a, p_1) + \rho(H, t_2 + a, p_2 - a)\},$$

where

$$C = \{(t_1, t_2, p_1, p_2, a) : \begin{array}{l} t = t_1 + t_2,\ p = p_1 + p_2, \\ t_1 + a + 2p_1 \le |V|, \\ t_2 + a + 2(p_2 - a) \le |V'|, \\ p_2 \ge a \ge 0, \\ t_1 \ge 0,\ t_2 \ge 0,\ p_1 \ge 0,\ p_2 \ge 0,\ a \ge 0\}. \end{array}$$

Proof. Suppose $\mathcal{P}$ is an optimal induced-path partition of $(G + H)(t, p)$. Then, $\rho(G + H, t, p) \ge \rho(G, t_1 + a, p_1) + \rho(H, t_2 + a, p_2 - a)$ for some t_1 and a. Thus,

$$\rho(G + H, t, p) \ge \min_C \{\rho(G, t_1 + a, p_1) + \rho(H, t_2 + a, p_2 - a)\},$$

where

$$
\begin{aligned}
C = \{(t_1, t_2, p_1, p_2, a) :\ & t = t_1 + t_2,\ \ p = p_1 + p_2, \\
& t_1 + a + 2p_1 \le |V)|, \\
& t_2 + a + 2(p_2 - a) \le |V'|, \\
& p_2 \ge a \ge 0, \\
& t_1 \ge 0,\ \ t_2 \ge 0,\ \ p_1 \ge 0,\ \ p_2 \ge 0,\ \ a \ge 0\}.
\end{aligned}
$$

On the other hand, suppose $\mathcal{Q}$ (respectively, $\mathcal{R}$) is an optimal induced-path partition of $G(t_1+a, p_1)$ (respectively, $H(t_2+a, p_2-a)$). Then $\mathcal{Q} \cup \mathcal{R}$ is an induced-path partition of $(G+H)(t,p)$. Thus,

$$\rho(G+H, t, p) \le \min_{C}\{\rho(G, t_1+a, p_1) + \rho(H, t_2+a, p_2-a)\},$$

where

$$
\begin{aligned}
C = \{(t_1, t_2, p_1, p_2, a) :\ & t = t_1 + t_2,\ \ p = p_1 + p_2, \\
& t_1 + a + 2p_1 \le |V|, \\
& t_2 + a + 2(p_2 - a) \le |V|, \\
& p_2 \ge a \ge 0,\ \ t_1 \ge 0,\ \ t_2 \ge 0,\ \ p_1 \ge 0,\ \ p_2 \ge 0,\ \ a \ge 0\}.
\end{aligned}
$$

Hence

$$\rho(G+H, t, p) = \min_{C}\{\rho(G, t_1+a, p_1) + \rho(H, t_2+a, p_2-a)\},$$

where

$$
\begin{aligned}
C = \{(t_1, t_2, p_1, p_2, a) :\ & t = t_1 + t_2,\ \ p = p_1 + p_2, \\
& t_1 + a + 2p_1 \le |V|, \\
& t_2 + a + 2(p_2 - a) \le |V'|, \\
& p_2 \ge a \ge 0,\ \ t_1 \ge 0,\ \ t_2 \ge 0,\ \ p_1 \ge 0,\ \ p_2 \ge 0,\ \ a \ge 0\}. \qquad \blacksquare
\end{aligned}
$$

Lemma 3.14 *For $t + 2p \le |V| + |V'|$, we have*

$$\rho(G \times H, t, p) = \min_{D}\{\rho(G, t_1+a+c, p_1+b) + \rho(H, t_2+b+c, p_2+a) + a + b + c\},$$

where

$$
\begin{aligned}
D = \{(t_1, t_2, p_1, p_2, a, b, c) :\ & t = t_1 + t_2,\ \ p = p_1 + p_2, \\
& a + 2b + c + t_1 + 2p_1 \le |V|, \\
& 2a + b + c + t_2 + 2p_2 \le |V'|, \\
& t_1 \ge 0,\ \ t_2 \ge 0,\ \ p_1 \ge 0,\ \ p_2 \ge 0, \\
& a \ge 0,\ \ b \ge 0,\ \ c \in \{0, 1\}\}.
\end{aligned}
$$

Proof. Suppose $\mathcal{P}$ is an optimal induced-path partition of $(G \times H)(t,p)$, $\mathcal{P}$ has a (respectively, b) P_3 whose internal vertex is in $G(t_1+a+c, p_1+b)$ (respectively, $H(t_2+b+c, p_2+a)$), and c edges whose end-vertices are in the different parts. If $c \geq 2$ and at least two vertices in the same part in c edges are nonadjacent, then we can interchange two edges with a P_3 and a vertex. If there exists two edges in c edges whose end-vertices in the same part (also the other part) are adjacent, then we can interchange these two edges with two other edges whose end-vertices are in the same part. Thus,

$$\rho(G \times H, t, p) \geq \min_{D}\{\rho(G, t_1+a+c, p_1+b) + \rho(H, t_2+b+c, p_2+a) + a + b + c\},$$

where

$$D = \{(t_1, t_2, p_1, p_2, a, b, c) : \begin{array}{l} t = t_1 + t_2,\ p = p_1 + p_2, \\ a + 2b + c + t_1 + 2p_1 \leq |V|, \\ 2a + b + c + t_2 + 2p_2 \leq |V'|, \\ t_1 \geq 0,\ t_2 \geq 0,\ p_1 \geq 0,\ p_2 \geq 0, \\ a \geq 0,\ b \geq 0,\ c \in \{0,1\}\}. \end{array}$$

On the other hand, suppose $\mathcal{Q}$ (respectively, $\mathcal{R}$) is an optimal induced-path partition of the graph $G(t_1+a+c, p_1+b)$ (respectively, $H(t_2+b+c, p_2+a)$), and we have the set S containing $(a+b)$ P_3 and c edges. So $\mathcal{Q} \cup \mathcal{R} \cup S$ is an induced-path partition of a graph $G \times H(t,p)$. Thus,

$$\rho(G \times H, t, p) \leq \min_{D}\{\rho(G, t_1+a+c, p_1+b) + \rho(H, t_2+b+c, p_2+a) + a + b + c\},$$

where

$$D = \{(t_1, t_2, p_1, p_2, a, b, c) : \begin{array}{l} t = t_1 + t_2,\ p = p_1 + p_2, \\ a + 2b + c + t_1 + 2p_1 \leq |V|, \\ 2a + b + c + t_2 + 2p_2 \leq |V'|, \\ t_1 \geq 0,\ t_2 \geq 0,\ p_1 \geq 0,\ p_2 \geq 0, \\ a \geq 0,\ b \geq 0,\ c \in \{0,1\}\}. \end{array}$$

And so,

$$\rho(G \times H, t, p) = \min_{D}\{\rho(G, t_1+a+c, p_1+b) + \rho(H, t_2+b+c, p_2+a) + a + b + c\},$$

where

$$D = \{(t_1, t_2, p_1, p_2, a, b, c) : \begin{array}{l} t = t_1 + t_2,\ p = p_1 + p_2, \\ a + 2b + c + t_1 + 2p_1 \le |V|, \\ 2a + b + c + t_2 + 2p_2 \le |V'|, \\ t_1 \ge 0,\ t_2 \ge 0,\ p_1 \ge 0,\ p_2 \ge 0, \\ a \ge 0,\ b \ge 0,\ c \in \{0, 1\}\}. \end{array}$$

∎

Theorem 3.15 *There is a polynomial-time algorithm for computing the induced-path number of a cograph.*

Proof. At any iteration, Lemmas 3.13 uses polynomial time and Lemma 3.14 uses polynomial time. And by the definition of cographs, the theorem holds. ∎

Chapter 4

Isometric-path Cover

4.1 Preliminary of isometric-path cover

Recall that an *isometric path* between two vertices in a graph G is a shortest path joining them. An *isometric-path cover* of a graph is a collection of isometric paths that cover all vertices of the graph. The *isometric-path-cover problem* is to find the *isometric-path number* $\text{ip}(G)$ of a graph G which is the minimum cardinality of an isometric-path cover.

The isometric-path number of the Cartesian product $P_{n_1} \square P_{n_2} \square \ldots \square P_{n_d}$ has been studied extensively in the literature. Fitzpatrick [17] gave bounds for the case when $n_1 = n_2 = \ldots = n_d$. Fisher and Fitzpatrick [18] gave exact values for the case $d = 2$. Fitzpatrick *et al.* [19] gave a lower bound, which is in fact the exact value if $d+1$ is a power of 2, for the case when $n_1 = n_2 = \ldots = n_d = 2$.

The purpose of this chapter is to give a linear-time algorithm for the isometric-path-cover problem in block graphs. We also determine isometric-path numbers of complete r-partite graphs and Hamming graphs of dimensions 2 and 3.

4.2 Isometric-path cover in block graphs

The purpose of this section is to give isometric-path numbers of block graphs. We also give a linear-time algorithm to find the corresponding paths. For technical reasons, we consider a slightly more general problem as follows. Suppose every vertex v in the graph G is associated with a non-negative integer $f(v)$. We call such function f a *vertex labeling* of G. An *f-isometric-path cover* of G is a family $\mathcal{C}$ of isometric paths such that the following conditions hold.

(C1) If $f(v) = 0$, then v is in an isometric path in $\mathcal{C}$.

(C2) If $f(v) \geq 1$, then v is an end-vertex of at least $f(v)$ isometric paths in $\mathcal{C}$, while the counting is twice if v itself is a path in $\mathcal{C}$.

The *f-isometric-path number* of G, denoted by $\text{ip}_f(G)$, is the minimum cardinality of an f-isometric-path cover of G. It is clear that when $f(v) = 0$ for all vertices v in G, we have $\text{ip}(G) = \text{ip}_f(G)$. The attempt of this section is to determine the f-isometric-path number of a block graph.

4.2.1 Formula for block graphs

In this subsection, we determine the f-isometric-path numbers for block graphs G. Without loss of generality, we may assume that G is connected.

First, a useful lemma.

Lemma 4.1 *Suppose x is a non-cut-vertex of a block graph G with a vertex labeling f. If vertex labeling f' is the same as f except that $f'(x) = \max\{1, f(x)\}$, then $\text{ip}_f(G) = \text{ip}_{f'}(G)$.*

Proof. Notice that an internal vertex of an isometric path in a block graph is a cut-vertex. Since x is not a cut-vertex, x must be an end-vertex of any isometric path. It follows that a collection $\mathcal{C}$ is an f-isometric-path cover if and only if it is an f'-isometric-path cover. The lemma then follows. ∎

Now, we may assume that $f(v) \geq 1$ for all non-cut-vertices v of G, and call such a vertex labeling *regular*. We have the following theorem for the inductive step.

Theorem 4.2 *Suppose G is a block graph with a regular labeling f, and x is a non-cut-vertex in a block B with exactly one cut-vertex y or with no cut-vertex in which case let y be any vertex of $B - \{x\}$. When $f(x) = 1$, let $G' = G - x$ with a regular* *vertex labeling* f' *which is the same as* f *except* $f'(y) = f(y)+1$. *When* $f(x) \geq 2$, *let* $G' = G$ *with a regular vertex labeling* f' *which is the same as* f *except* $f'(x) = f(x)-1$ *and* $f'(y) = f(y)+1$. *Then* $\mathrm{ip}_f(G) = \mathrm{ip}_{f'}(G')$.

Proof. We first prove that $\mathrm{ip}_f(G) \geq \mathrm{ip}_{f'}(G')$. Suppose $\mathcal{C}$ is an optimal f-isometric-path cover of G. Choose an isometric path P in $\mathcal{C}$ having x as an end-vertex. We consider four cases.

Case 1.1. $P = x$ and $f(x) = 1$ (*i.e.*, $G' = G - x$).

In this case, $\mathcal{C}' = (\mathcal{C} - \{P\}) \cup \{y\}$ is an f'-isometric-path cover of G'. Hence, $\mathrm{ip}_f(G) = |\mathcal{C}| \geq |\mathcal{C}'| \geq \mathrm{ip}_{f'}(G')$.

Case 1.2. $P = x$ and $f(x) \geq 2$ (*i.e.*, $G' = G$).

In this case, $\mathcal{C}' = (\mathcal{C} - \{P\}) \cup \{xy\}$ is an f'-isometric-path cover of G'. Hence, $\mathrm{ip}_f(G) = |\mathcal{C}| \geq |\mathcal{C}'| \geq \mathrm{ip}_{f'}(G')$.

Case 1.3. $P = xz$ for some vertex z in $B - \{x, y\}$.

In this case, $\mathcal{C}' = (\mathcal{C} - \{P\}) \cup \{yz\}$ is an f'-isometric-path cover of G'. Hence, $\mathrm{ip}_f(G) = |\mathcal{C}| \geq |\mathcal{C}'| \geq \mathrm{ip}_{f'}(G')$.

Case 1.4. $P = xyQ$, where Q contains no vertices in B.

In this case, $\mathcal{C}' = (\mathcal{C} - \{P\}) \cup \{yQ\}$ is an f'-isometric-path cover of G'. Hence, $\mathrm{ip}_f(G) = |\mathcal{C}| \geq |\mathcal{C}'| \geq \mathrm{ip}_{f'}(G')$.

Next, we prove that $\mathrm{ip}_f(G) \leq \mathrm{ip}_{f'}(G')$. Suppose $\mathcal{C}'$ is an optimal f'-isometric-path cover of G'. Choose a path P' in $\mathcal{C}'$ having y as an end-vertex. We consider three cases.

Case 2.1. $P' = yx$.

In this case, $G' = G$ and $\mathcal{C} = (\mathcal{C}' - \{P'\}) \cup \{x\}$ is an f-isometric-path cover of G. Hence, $\mathrm{ip}_f(G) \leq |\mathcal{C}| \leq |\mathcal{C}'| = \mathrm{ip}_{f'}(G')$.

Case 2.2. $P' = yz$ for some z in $B - \{x, y\}$.

In this case, $\mathcal{C} = (\mathcal{C}' - \{P'\}) \cup \{xz\}$ is an f-isometric-path cover of G. Hence, $\mathrm{ip}_f(G) \leq |\mathcal{C}| \leq |\mathcal{C}'| = \mathrm{ip}_{f'}(G')$.

Case 2.3. $P' = yQ$, where Q contains no vertex in B.

In this case, $\mathcal{C} = (\mathcal{C}' - \{P'\}) \cup \{xyQ\}$ is an f-isometric-path cover of G. Hence,

$\text{ip}_f(G) \le |\mathcal{C}| \le |\mathcal{C}'| = \text{ip}_{f'}(G')$. ∎

Consequently, we have the following result for f-isometric-path numbers of connected block graphs.

Theorem 4.3 *If $G = (V, E)$ is a connected block graph with a regular vertex labeling f, then $\text{ip}_f(G) = \lceil \frac{s(G)}{2} \rceil$, where $s(G) = \sum_{v \in V} f(v)$.*

Proof. The theorem is obvious when G has only one vertex. For the case when G has more than one vertex, we apply Theorem 4.2 repeatedly until the graph becomes trivial. Notice that the $s(G') = s(G)$ when Theorem 4.2 is applied. ∎

For the isometric-path-cover problem, we have

Corollary 4.4 *If G is a connected block graph, then $\text{ip}(G) = \lceil \frac{\text{nc}(G)}{2} \rceil$, where $\text{nc}(G)$ is the number of non-cut-vertices of G.*

Proof. The corollary follows from Theorem 4.3 and the fact that $\text{ip}(G) = \text{ip}_f(G)$ for the regular vertex labeling f with $f(v) = 1$ if v is a non-cut-vertex and $f(v) = 0$ otherwise. ∎

Corollary 4.5 [18] *If T is any tree then $\text{ip}(T) = \lceil \frac{\ell(T)}{2} \rceil$, where $\ell(T)$ is the number of leaves in T.*

4.2.2 Algorithm for block graphs

Based on Theorem 4.2, we are able to design an algorithm for the isometric-path-cover problem in block graphs. Notice that we may only consider connected block graphs with regular vertex labelings. To speed up the algorithm, we may modify Theorem 4.2 a little bit so that each time a non-cut-vertex is handled.

Theorem 4.6 *Suppose G is a block graph with a regular labeling f, and x is a non-cut-vertex in a block B with exactly one cut-vertex y or with no cut-vertex in which let y be any vertex in $B - \{x\}$. Let $G' = G - x$ with a regular vertex labeling f' which is the same as f except $f'(y) = f(y) + f(x)$. Then $\text{ip}_f(G) = \text{ip}_{f'}(G')$.*

Proof. The theorem follows from repeatedly applying Theorem 4.2. ∎

Now, we are ready to give the algorithm.

Algorithm IP Find the f-isometric-path number $\text{ip}_f(G)$ of a connected block graph.

Input. A connected block graph G and a regular vertex labeling f.

Output. An optimal f-isometric-path cover $\mathcal{C}$ of G and $\text{ip}_f(G)$.

Method.

construct a stack S which is empty at the beginning;
let $G' \leftarrow G$;
while (G' has more than one vertex) **do**
    choose a block B with exactly one cut-vertex y
    or with no cut-vertex in which case choose any $y \in B$;
    for (all vertices x in $B - \{y\}$) **do**
        $f(y) \leftarrow f(y) + f(x)$;
        push $(x, y, f(x))$ into S;
        $G' \leftarrow G' - x$;
    end for;
end while;
$\text{ip}_f(G) \leftarrow \lceil f(r)/2 \rceil$, where r is the only vertex of G';
let $\mathcal{C}$ be the family of isometric paths containing $\text{ip}(G)$ copies of the path r;
while (S is not empty) **do**
    pop (x, y, i) from S;
    choose i copies of path P in $\mathcal{C}$ using y as an end-vertex;
    if ($P = yx$) **then**
        replace the i copies of P by i copies of x in $\mathcal{C}$;
    if ($P = yz$ for some vertex z in the block of G containing x) **then**
        replace the i copies of P by i copies of xz in $\mathcal{C}$;
    if ($P = yQ$ where Q has no vertices in the block of G containing x) **then**
        replace the i copies of P by the i copies of xyQ in $\mathcal{C}$;
end while.

Algorithm **IP** can be implemented in linear time to the number of vertices and edges.

We close this section by giving an example that demonstrates the algorithm

Example 4.1 Consider the graph G_1 of 5 vertices and 2 blocks in Figure 4.1. Notice that its blocks are two complete graphs.

1. Give a regular vertex labeling f such that $f(c) = 0$, and $f(v) = 1$ for $v \neq c$ of G_1 in Figure 4.1.

2. Construct an empty stack S in Figure 4.1.

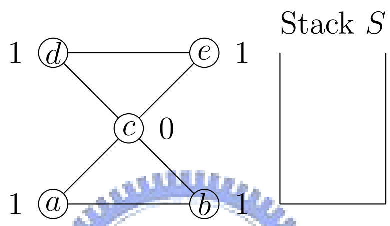

Figure 4.1: Graph G_1 of 5 vertices, and an empty stack S.

3. Choose the block $B_1 = \{a, b, c\}$, which is a complete graph, with the only cut-vertex c, and another vertex a. Thus, $f(c) = 0 + f(a) = 1$. Then, push $(a, c, 1)$ into S, and delete a from G_1 to get the graph G_2 in Figure 4.2.

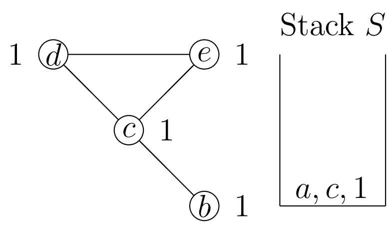

Figure 4.2: Graph G_2 results from G_1 by deleting a. Update stack S.

4. Choose the vertex b. Thus, $f(c) = 1 + f(b) = 2$. Then, push $(b, c, 1)$ into S, and delete b from G_2 to get the graph G_3 in Figure 4.3.

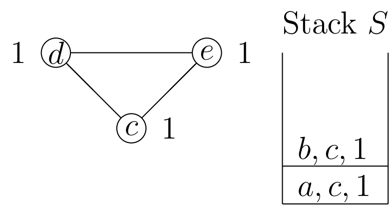

Figure 4.3: Graph G_3 results from G_2 by deleting b. Update stack S.

5. Choose the final block $B_2 = \{c, d, e\}$ and the vertex c. For all vertices of $V(B_2) - \{c\}$, continue this process. Thus, $f(c) = 4$. Then, $\mathrm{ip}(G) = 2$ and the isometric-path cover is $\mathcal{P} = \{c, c\}$. Hence, we obtain the graph G_4 in Figure 4.4.

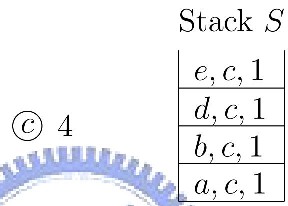

Figure 4.4: Graph G_4 results from G_3 by deleting d and e. Update stack S.

6. Pop $(e, c, 1)$ from stack S to update as in Figure 4.5. Thus, we get $\mathcal{P} = \{ce, c\}$.

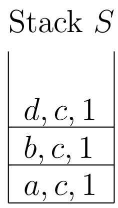

Figure 4.5: Update stack S by poping $(e, c, 1)$.

7. Also, pop $(d, c, 1)$ from S, and we get $\mathcal{P} = \{ce, cd\}$. Continue this Process. Pop $(b, c, 1)$ from S to obtain $\mathcal{P} = \{ceb, cd\}$. Finally, pop $(a, c, 1)$ form S. Hence, $\mathcal{P} = \{ceb, cda\}$.

4.3 Isometric-path cover in complete r-partite graphs

In this section we determine isometric-path numbers of all complete r-partite graphs.

Suppose G is the complete r-partite graph $K_{n_1,n_2,\ldots,n_r}$ of n vertices, where $r \geq 2$, $n_1 \geq n_2 \geq \ldots \geq n_r$ and $n = n_1 + n_2 + \ldots + n_r$. Let G has α parts of odd sizes. We notice that every isometric path in G has at most 3 vertices. Consequently,

$$\mathrm{ip}(G) \geq \left\lceil \frac{n}{3} \right\rceil .$$

Also, for any path of 3 vertices in an isometric-path cover $\mathcal{C}$, two end-vertices of the path is in a part of G and the center vertex in another part. In case when two paths of 3 vertices in $\mathcal{C}$ have a common end-vertex, we may replace one by a path of 2 vertices. And, a path of 1 vertex can be replaced by a path of 2 vertices. So, without loss of generality, *we may only consider isometric-path covers in which every path is of* 2 *or* 3 *vertices, and two* 3*-vertices paths have different end-vertices.*

Lemma 4.7 *If* $3n_1 > 2n$, *then* $\mathrm{ip}(G) = \lceil \frac{n_1}{2} \rceil$.

Proof. First, $\mathrm{ip}(G) \geq \lceil \frac{n_1}{2} \rceil$ since every isometric path contains at most two vertices in the first part.

On the other hand, we use an induction on $n - n_1$ to prove that $\mathrm{ip}(G) \leq \lceil \frac{n_1}{2} \rceil$. When $n - n_1 = 1$, we have $G = K_{n-1,1}$. In this case, it is clear that $\mathrm{ip}(G) \leq \lceil \frac{n_1}{2} \rceil$. Suppose $n - n_1 \geq 2$ and the claim holds for $n' - n'_1 < n - n_1$. Then we remove two vertices from the first part and one vertex from the second part to form an isometric 3-path P. Since $3n_1 > 2n$, we have $n_1 - 2 > 2(n - n_1 - 1) > 0$ and so $n_1 - 2 > n_2$. Then, the remaining graph G' has $r' \geq 2$, $n'_1 = n_1 - 2$ and $n' = n - 3$. It then still satisfies $3n'_1 > 2n'$. As $n' - n'_1 = n - n_1 - 1$, by the induction hypothesis, $\mathrm{ip}(G') \leq \lceil \frac{n'_1}{2} \rceil$ and so $\mathrm{ip}(G) \leq \lceil \frac{n'_1}{2} \rceil + 1 = \lceil \frac{n_1}{2} \rceil$. ∎

Lemma 4.8 *If* $3\alpha > n$, *then* $\mathrm{ip}(G) = \lceil \frac{n+\alpha}{4} \rceil$.

Proof. Suppose $\mathcal{C}$ is an optimum isometric-path cover with p_2 paths of 2 vertices and p_3 paths of 3 vertices. Then

$$2p_2 + 3p_3 \geq n.$$

Notice that there are at most $n - \alpha$ vertices in G can be paired up as the end-vertices of the 3-paths in $\mathcal{P}$. Hence $p_3 \leq \frac{n-\alpha}{2}$ and so

$$2p_2 + 2p_3 \geq n - \frac{n-\alpha}{2} = \frac{n+\alpha}{2} \quad \text{or} \quad \mathrm{ip}(G) = p_2 + p_3 \geq \left\lceil \frac{n+\alpha}{4} \right\rceil.$$

On the other hand, we use an induction on $n - \alpha$ to prove that $\mathrm{ip}(G) \leq \lceil \frac{n+\alpha}{4} \rceil$. When $n - \alpha \leq 1$, we have $n = \alpha$ and G is the complete graph of order n. So, $\mathrm{ip}(G) = \lceil \frac{n}{2} \rceil = \lceil \frac{n+\alpha}{4} \rceil$. Suppose $n - \alpha \geq 2$ and the claim holds for $n' - \alpha' < n - \alpha$. In this case, $3\alpha > n \geq \alpha + 2$ which implies $\alpha > 1$ and $n > 3$. Then we may remove two vertices from the first part of and one vertex form an odd part other than the first part to form an isometric 3-path P of G. The remaining graph G' has $n' = n - 3$ and $\alpha' = \alpha - 1$. It then satisfies $3\alpha' > n'$. Notice that $r' \geq 2$ unless $G = K_{2,1,1}$ in which $n = 4$ and $\alpha = 2$ imply $\mathrm{ip}(G) = 2 = \lceil \frac{n+\alpha}{4} \rceil$. By the induction hypothesis, $\mathrm{ip}(G') \leq \lceil \frac{n'+\alpha'}{4} \rceil$ and so $\mathrm{ip}(G) \leq \lceil \frac{n'+\alpha'}{4} \rceil + 1 = \lceil \frac{n+\alpha}{4} \rceil$. ∎

Lemma 4.9 *If* $3n_1 \leq 2n$ *and* $3\alpha \leq n$, *then* $\mathrm{ip}(G) = \lceil \frac{n}{3} \rceil$.

Proof. Since every isometric path in G has at most 3 vertices, $\mathrm{ip}(G) \geq \lceil \frac{n}{3} \rceil$.

On the other hand, we use an induction on n to prove that $\mathrm{ip}(G) \leq \lceil \frac{n}{3} \rceil$. When $n \leq 8$, by the assumptions that $3n_1 \leq 2n$ and $3\alpha \leq n$ we have $G \in \{K_{2,1}, K_{2,2}, K_{3,2}, K_{2,2,1}, K_{4,2}, K_{4,1,1}, K_{3,3}, K_{3,2,1}, K_{2,2,2}, K_{2,2,1,1}, K_{4,3}, K_{4,2,1}, K_{3,2,2}, K_{2,2,2,1}, K_{5,3}, K_{5,2,1}, K_{4,4}, K_{4,3,1}, K_{4,2,2}, K_{4,2,1,1}, K_{3,3,2}, K_{3,2,2,1}, K_{2,2,2,2}, K_{2,2,2,1,1}\}$. It is straightforward to check that $\mathrm{ip}(G) \leq \lceil \frac{n}{3} \rceil$.

Suppose $n \geq 9$ and the claim holds for $n' < n$. We remove two vertices from the first part and one vertex from the jth part to form an isometric 3-path P for

G, where j is the largest index such that $j \geq 2$ and n_j is odd (when n_i are even for all $i \geq 2$, we choose $j = r$). Then, the remaining subgraph G' has $n' = n - 3$ and $\alpha' = \alpha - 1$ or $\alpha' \leq 2$. Therefore, $3\alpha \leq n$ and $n \geq 9$ imply that $3\alpha' \leq n'$ in any case. We shall prove that $3n'_1 \leq 2n'$ according to the following cases.

Case 1. $n_1 \geq n_2 + 2$.

In this case, $n_1 - 2 \geq n_2 \geq n_i$ for all $i \geq 2$ and so $n'_1 = n_1 - 2$. Therefore, $3n'_1 = 3(n_1 - 2) \leq 2(n - 3) = 2n'$.

Case 2. $n_1 \leq n_2 + 1$ and $n_2 \leq 4$.

In this case, $n'_1 \leq n_2 \leq 4$ and $n' \geq 6$. Then, $3n'_1 \leq 12 \leq 2n'$.

Case 3. $n_1 \leq n_2 + 1$ and $n_2 \geq 5$ and $r = 2$.

In this case, $n'_1 \leq n_2 - 1$ and $n' = n - 3 = n_1 + n_2 - 3 \geq 2n_2 - 3$. Then, $3n'_1 \leq 3n_2 - 3 \leq 4n_2 - 8 < 2n'$.

Case 4. $n_1 \leq n_2 + 1$ and $n_2 \geq 5$ and $r \geq 3$.

In this case, $n'_1 \leq n_2$ and $n' = n - 3 \geq n_1 + n_2 + 1 - 3 \geq 2n_2 - 2$. Then, $3n'_1 \leq 3n_2 \leq 4n_2 - 5 < 2n'$. ∎

According to Lemma 4.7, 4.8 and 4.9, we have the following theorem.

Theorem 4.10 *Suppose G is the complete r-partite graph $K_{n_1,n_2,\ldots,n_r}$ of n vertices with $r \geq 2$, $n_1 \geq n_2 \geq \ldots \geq n_r$ and $n = n_1 + n_2 + \ldots + n_r$. If there are exactly α indices i with n_i odd, then*

$$\mathrm{ip}(G) = \begin{cases} \lceil \frac{n_1}{2} \rceil, & \text{if } 3n_1 > 2n; \\ \lceil \frac{n+\alpha}{4} \rceil, & \text{if } 3\alpha > n; \\ \lceil \frac{n}{3} \rceil, & \text{if } 3\alpha \leq n \text{ and } 3n_1 \leq 2n. \end{cases}$$

In the proofs of the lemmas above, the essential points for the arguments is the fact that each partite set of the complete r-partite graph is trivial. If we add some edges into the graph but still keep that each partite set can be partitioned into $\lfloor \frac{n_i}{2} \rfloor$ pairs of two nonadjacent vertices and $n_i - 2\lfloor \frac{n_i}{2} \rfloor$ vertex, then the same result still holds.

Corollary 4.11 *Suppose G is the graph obtained from the complete r-partite graph $K_{n_1,n_2,\ldots,n_r}$ of n vertices by adding edges such that each i-th part can be partitioned into $\lfloor \frac{n_i}{2} \rfloor$ pairs of two nonadjacent vertices and $n_i - 2\lfloor \frac{n_i}{2} \rfloor$ vertex, where $r \geq 2$, $n_1 \geq n_2 \geq \ldots \geq n_r$ and $n = n_1 + n_2 + \ldots + n_r$. If there are exactly α indices i with n_i odd, then*

$$\mathrm{ip}(G) = \begin{cases} \lceil \frac{n_1}{2} \rceil, & \textit{if } 3n_1 > 2n; \\ \lceil \frac{n+\alpha}{4} \rceil, & \textit{if } 3\alpha > n; \\ \lceil \frac{n}{3} \rceil, & \textit{if } 3\alpha \leq n \textit{ and } 3n_1 \leq 2n. \end{cases}$$

4.4 Isometric-path cover in Hamming graphs

In this section we determine isometric-path numbers of Cartesian products of 2 and 3 complete graphs. Recall that a *Hamming graph* is the Cartesian product of complete graphs, which is the graph $K_{n_1} \Box K_{n_2} \Box \ldots \Box K_{n_r} = (V, E)$ with vertex set

$$V = \{(x_1, x_2, \ldots, x_r) : 0 \leq x_i < n_i \text{ for } 1 \leq i \leq r\}$$

and edge set

$$E = \{(x_1, x_2, \ldots, x_r)(y_1, y_2, \ldots, y_r) : x_i = y_i \text{ for all } i \text{ except just one } x_j \neq y_j\}.$$

Suppose G is the Hamming graph $K_{n_1} \Box K_{n_2} \Box \ldots \Box K_{n_r}$ of n vertices, where $n = n_1 n_2 \ldots n_r$ and $n_i \geq 2$ for $1 \leq i \leq r$. We notice that every isometric path in G has at most $r + 1$ vertices. Consequently,

$$\mathrm{ip}(G) \geq \left\lceil \frac{n}{r+1} \right\rceil.$$

We first consider the case when $r = 2$

Theorem 4.12 *If $n_1 \geq 2$ and $n_2 \geq 2$, then* $\mathrm{ip}(K_{n_1} \Box K_{n_2}) = \lceil \frac{n_1 n_2}{3} \rceil$.

Proof. We only need to prove that $\mathrm{ip}(K_{n_1} \Box K_{n_2}) \leq \lceil \frac{n_1 n_2}{3} \rceil$. We shall prove this assertion by induction on $n_1 + n_2$. For the case when $n_1 + n_2 \leq 6$, the isometric-path covers

$$\begin{aligned}
\mathcal{C}_{2,2} &= \{(0,0)(0,1),(1,0)(1,1)\}, \\
\mathcal{C}_{2,3} &= \{(0,0)(0,1)(1,1),(0,2)(1,2)(1,0)\}, \\
\mathcal{C}_{2,4} &= \{(0,0)(0,1)(1,1),(0,2)(1,2)(1,0),(0,3)(1,3)\} \text{ and} \\
\mathcal{C}_{3,3} &= \{(0,0)(2,0)(2,2),(0,1)(0,2)(1,2),(1,0)(1,1)(2,1)\}
\end{aligned}$$

for $K_2\Box K_2$, $K_2\Box K_3$, $K_2\Box K_4$ and $K_3\Box K_3$ respectively, gives the assertion.

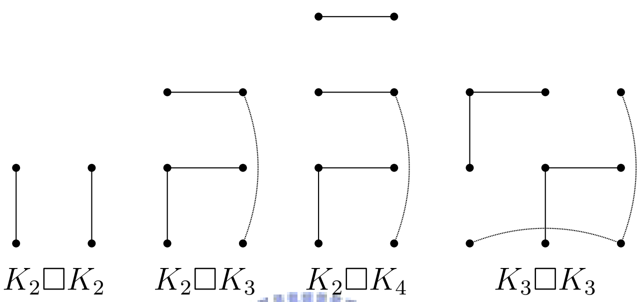

Figure 4.6: Isometric-path covers of $K_2\Box K_i$ for $2 \le i \le 4$, and $K_3\Box K_3$.

Suppose $n_1 + n_2 \ge 7$ and the assertion holds for $n_1' + n_2' < n_1 + n_2$. For the case when all $n_i \le 4$, without loss of generality we may assume that $n_1 = 4$ and $3 \le n_2 \le 4$. As we can partition the vertex set of $K_{n_1}\Box K_{n_2}$ into the vertex sets of two copies of distance invariant induced subgraphs $K_2\Box K_{n_2}$,

$$\mathrm{ip}(K_{n_1}\Box K_{n_2}) \le 2\mathrm{ip}(K_2\Box K_{n_2}) \le 2\left\lceil \frac{2n_2}{3} \right\rceil = \left\lceil \frac{n_1 n_2}{3} \right\rceil.$$

For the case when there is at least one $n_i \ge 5$, say $n_1 \ge 5$, again we can partition the vertex set of $K_{n_1}\Box K_{n_2}$ into the vertex sets of two distance invariant induced subgraphs $K_3\Box K_{n_2}$ and $K_{n_1-3}\Box K_{n_2}$. Then,

$$\begin{aligned}
\mathrm{ip}(K_{n_1}\Box K_{n_2}) &\le \mathrm{ip}(K_3\Box K_{n_2}) + \mathrm{ip}(K_{n_1-3}\Box K_{n_2}) \\
&\le \left\lceil \tfrac{3n_2}{3} \right\rceil + \left\lceil \tfrac{(n_1-3)n_2}{3} \right\rceil = \left\lceil \tfrac{n_1 n_2}{3} \right\rceil.
\end{aligned}$$

∎

Lemma 4.13 *If n_1, n_2 and n_3 are positive even integers, then*

$$\mathrm{ip}(K_{n_1}\Box K_{n_2}\Box K_{n_3}) = \frac{n_1 n_2 n_3}{4}.$$

Proof. We only need to prove that $\text{ip}(K_{n_1} \Box K_{n_2} \Box K_{n_3}) \le \frac{n_1 n_2 n_3}{4}$. First, the isometric-path cover $\mathcal{C}_{2,2,2} = \{(0,0,0)(0,0,1)(0,1,1)(1,1,1), (1,0,1)(1,0,0)(1,1,0)(0,1,0)\}$ for $K_2 \Box K_2 \Box K_2$ proves the assertion for the case when $n_1 = n_2 = n_3 = 2$.

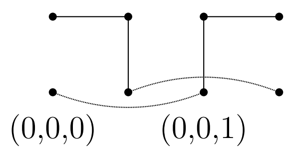

Figure 4.7: An isometric-path cover of $K_2 \Box K_2 \Box K_2$.

For the general case, as the vertex set of $K_{n_1} \Box K_{n_2} \Box K_{n_3}$ can be partitioned into the vertex sets of $\frac{n_1 n_2 n_3}{8}$ copies of distance invariant induced subgraphs $K_2 \Box K_2 \Box K_2$,

$$\text{ip}(K_{n_1} \Box K_{n_2} \Box K_{n_3}) \le \left(\frac{n_1 n_2 n_3}{8}\right) \text{ip}(K_2 \Box K_2 \Box K_2) \le \frac{n_1 n_2 n_3}{4}. \quad \blacksquare$$

Lemma 4.14 *If $n_3 \ge 3$ is odd, then* $\text{ip}(K_2 \Box K_2 \Box K_{n_3}) = n_3 + 1$.

Proof. First, we claim that $\text{ip}(K_2 \Box K_2 \Box K_{n_3}) \ge n_3 + 1$. Suppose to the contrary that the graph can be covered by n_3 isometric paths

$$P_i : (x_{i1}, x_{i2}, x_{i3})(y_{i1}, y_{i2}, y_{i3})(z_{i1}, z_{i2}, z_{i3})(w_{i1}, w_{i2}, w_{i3}),$$

$i = 1, 2, \ldots, n_3$. These paths are in fact vertex-disjoint paths of 4 vertices, each contains exactly one type-j edge for $j = 1, 2, 3$, where an edge $(x_1, x_2, x_3)(y_1, y_2, y_3)$ is type-j if $x_j \ne y_j$. For each P_i we then have $x_{i1} = 1 - w_{i1}$ and $x_{i2} = 1 - w_{i2}$, which imply that $x_{i1} + x_{i2}$ has the same parity with $w_{i1} + w_{i2}$. We call the path P_i *even* or *odd* when $x_{i1} + x_{i2}$ is even or odd, respectively. Also, as P_i has just one type-3 edge, by symmetry, we may assume either $x_{i3} \ne y_{i3} = z_{i3} = w_{i3}$ or $x_{i3} = y_{i3} \ne z_{i3} = w_{i3}$, for which we call P_i type 1-3 or type 2-2 respectively. For a type 2-2 path P_i we may further assume that $x_{i1} \ne y_{i1} = z_{i1} = w_{i1}$.

For $0 \le x_3 < n_3$, the x_3-*square* is the set

$$S(x_3) = \{(0,0,x_3), (0,1,x_3), (1,0,x_3), (1,1,x_3)\}.$$

Notice that a type 1-3 path P_i contains 1 vertex in $S(x_{i3})$ and 3 vertices in $S(w_{i3})$, while a type 2-2 path P_i contains 2 vertices in $S(x_{i3})$ and 2 vertices in $S(w_{i3})$. We call a type 1-3 path P_i is *adjacent to* another type 1-3 path P_j if the last 3 vertices of P_i and the first vertex of P_j form a square. This defines a digraph D whose vertices are all type 1-3 paths, in which each vertex has out-degree one and in-degree at most one. In fact, each vertex then has in-degree one. In other words, the "adjacent to" is a bijection. Consequently, vertices of all type 1-3 paths together form p squares; and so vertices of all type 2-2 paths form the other $n_3 - p$ squares.

Since $x_{i1} \neq y_{i1} = z_{i1} = w_{i1}$ for a type 2-2 path P_i, the first two vertices of a type 2-2 path together with the first two vertices of another type 2-2 path form a square. This shows that there is an even number of type 2-2 paths. Therefore, there is an odd number of type 1-3 paths.

On the other hand, in a type 1-3 path P_i we have $x_{i_1} + x_{i_2} = y_{i_1} + y_{i_2}$ has the different parity with $z_{i_1} + z_{i_2}$, and the same parity with $w_{i_1} + w_{i_2}$. So it is adjacent to a type 1-3 path whose parity is the same as $z_{i_1} + z_{i_2}$. That is, a type 1-3 path is adjacent to a type 1-3 path of different parity. Therefore, the digraph D is the union of some even directed cycle. This is a contradiction to the fact that there is an odd number of type 1-3 paths.

The arguments above prove that $\mathrm{ip}(K_2 \square K_2 \square K_{n_3}) \geq n_3 + 1$. On the other hand, since the vertex set of $K_2 \square K_2 \square K_{n_3}$ is the union of the vertex sets of $(n_3 + 1)/2$ copies of $K_2 \square K_2 \square K_2$, by the cover $\mathcal{C}_{2,2,2}$ in the proof of Lemma 4.13, we have $\mathrm{ip}(K_2 \square K_2 \square K_{n_3}) \leq n_3 + 1$. ∎

Theorem 4.15 *If all $n_i \geq 2$, then $\mathrm{ip}(K_{n_1} \square K_{n_2} \square K_{n_3}) = \lceil \frac{n_1 n_2 n_3}{4} \rceil$ except for the case when two n_i are 2 and the third is odd. In the exceptional case, $\mathrm{ip}(K_{n_1} \square K_{n_2} \square K_{n_3}) = \frac{n_1 n_2 n_3}{4} + 1$.*

Proof. The exceptional case holds according to Lemma 4.14.

For the main case, by Lemma 4.13, we may assume that at least one n_i is odd. Again, we only need to prove that $\mathrm{ip}(K_{n_1} \square K_{n_2} \square K_{n_3}) \leq \lceil \frac{n_1 n_2 n_3}{4} \rceil$. We shall prove the assertion by induction on $\sum_{i=1}^{3} n_i$. For the case when $\sum_{i=1}^{3} n_i \leq 10$, the following isometric-path covers for $K_2 \square K_3 \square K_3$, $K_2 \square K_3 \square K_4$, $K_2 \square K_3 \square K_5$, $K_3 \square K_3 \square K_3$ and $K_3 \square K_3 \square K_4$, respectively, prove the assertion:

$$\begin{aligned}\mathcal{C}_{2,3,3} = \{&(0,1,1)(0,1,0)(0,0,0)(1,0,0),\ (0,2,2)(0,2,0)(1,2,0)(1,1,0),\\ &(0,2,1)(1,2,1)(1,1,1),\ (0,0,2)(0,1,2)(1,1,2),\\ &(0,0,1)(1,0,1)(1,0,2)(1,2,2)\};\end{aligned}$$

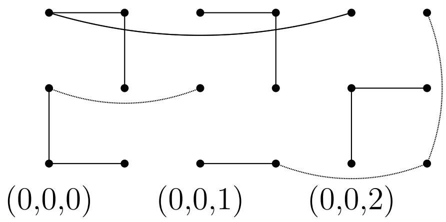

Figure 4.8: An isometric-path cover of $K_2 \square K_3 \square K_3$.

$$\left(\text{Let } \mathcal{C}^*_{2,3,3} = \begin{aligned}&\mathcal{C}_{2,3,3} \backslash \{(0,2,1)(1,2,1)(1,1,1),(0,0,2)(0,1,2)(1,1,2)\} \cup \\ &\{(0,2,1)(1,2,1)(1,1,1)(1,1,3),(0,0,2)(0,1,2)(1,1,2)(1,1,4)\}.\end{aligned} \right)$$

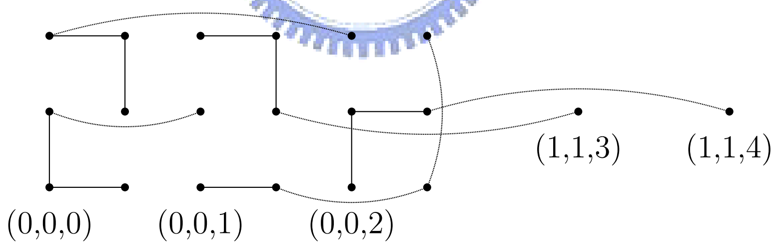

Figure 4.9: Another isometric-path cover of $K_2 \square K_3 \square K_3$.

$$\begin{aligned}\mathcal{C}_{2,3,4} = \{&(0,1,1)(0,1,0)(0,0,0)(1,0,0),\ (0,2,1)(0,2,0)(1,2,0)(1,1,0),\\ &(0,2,3)(0,2,2)(1,2,2)(1,1,2),\ (0,1,3)(0,1,2)(0,0,2)(1,0,2),\\ &(0,0,1)(1,0,1)(1,1,1)(1,1,3),\ (1,2,1)(1,2,3)(1,0,3)(0,0,3)\};\end{aligned}$$

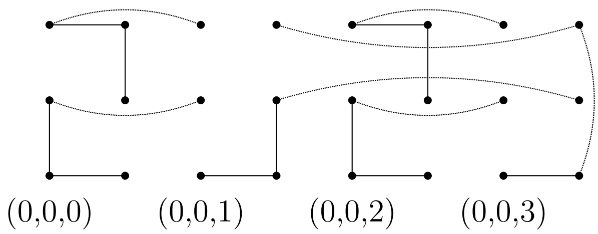

Figure 4.10: An isometric-path cover of $K_2\Box K_3\Box K_4$.

$$\begin{aligned}\mathcal{C}_{2,3,5} &= \mathcal{C}^*_{2,3,3}\cup\{(0,1,4)(0,1,3)(0,2,3)(1,2,3),\ (0,0,3)(0,0,4)(0,2,4)(1,2,4),\\ &\qquad (1,0,3)(1,0,4)\};\end{aligned}$$

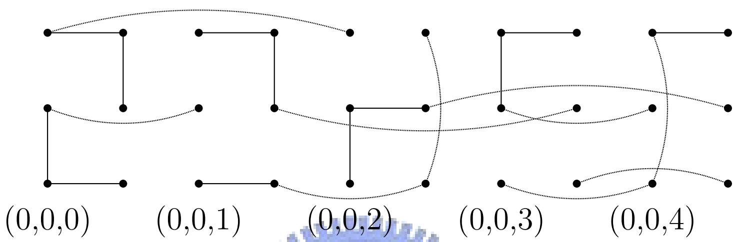

Figure 4.11: An isometric-path cover of $K_2\Box K_3\Box K_5$.

$$\begin{aligned}\mathcal{C}_{3,3,3} &= \{(0,0,0)(0,2,0)(1,2,0)(1,2,1),\ (1,1,0)(2,1,0)(2,2,0)(2,2,1),\\ &\quad (0,2,1)(0,1,1)(1,1,1)(1,1,2),\ (1,0,1)(2,0,1)(2,1,1)(2,1,2),\\ &\quad (0,1,0)(0,1,2)(0,2,2)(1,2,2),\ (0,0,1)(0,0,2)(2,0,2)(2,2,2),\\ &\quad (1,0,2)(1,0,0)(2,0,0)\};\end{aligned}$$

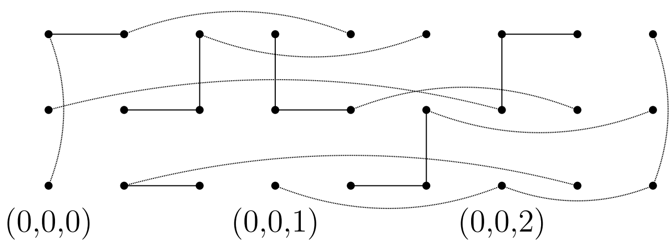

Figure 4.12: An isometric-path cover of $K_3\Box K_3\Box K_3$.

$$\begin{aligned}\mathcal{C}_{3,3,4} &= \{(0,0,0)(0,2,0)(1,2,0)(1,2,1),\ (1,1,0)(2,1,0)(2,2,0)(2,2,1),\\ &\quad (0,2,1)(0,1,1)(1,1,1)(1,1,2),\ (1,0,1)(2,0,1)(2,1,1)(2,1,2),\\ &\quad (0,1,0)(0,1,2)(0,2,2)(1,2,2),\ (0,0,2)(2,0,2)(2,2,2)(2,2,3),\\ &\quad (0,1,3)(1,1,3)(1,0,3)(1,0,2),\ (1,0,0)(2,0,0)(2,0,3)(2,1,3),\\ &\quad (0,0,1)(0,0,3)(0,2,3)(1,2,3)\}.\end{aligned}$$

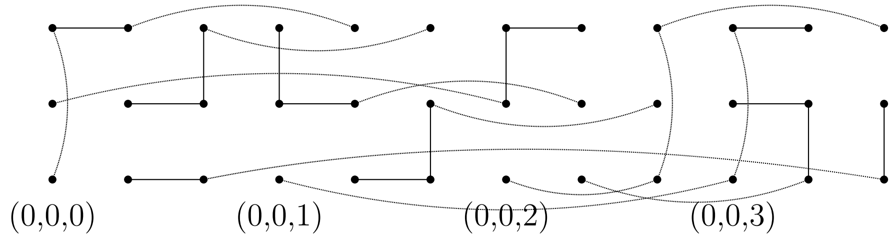

Figure 4.13: An isometric-path cover of $K_3 \Box K_3 \Box K_4$.

Suppose $\sum_{i=1}^{3} n_i \geq 11$ and the assertion holds for $\sum_{i=1}^{3} n_i' < \sum_{i=1}^{3} n_i$. We shall consider the following cases.

For the case when there is some i, say $i = 3$, such that $n_3 \geq 7$ or $n_3 = 6$ with all $n_j \geq 3$, we have $\text{ip}(K_{n_1} \Box K_{n_2} \Box K_{n_3}) \leq \text{ip}(K_{n_1} \Box K_{n_2} \Box K_4) + \text{ip}(K_{n_1} \Box K_{n_2} \Box K_{n_3-4}) \leq \lceil \frac{n_1 n_2 4}{4} \rceil + \lceil \frac{n_1 n_2 (n_3-4)}{4} \rceil = \lceil \frac{n_1 n_2 n_3}{4} \rceil$.

For the case when some n_i, say n_3, is equal to 4, we may assume $n_1 \geq n_2$ and so $n_1 \geq 4$. Then $\text{ip}(K_{n_1} \Box K_{n_2} \Box K_4) \leq \text{ip}(K_2 \Box K_{n_2} \Box K_4) + \text{ip}(K_{n_1-2} \Box K_{n_2} \Box K_4) = \lceil \frac{2 n_2 4}{4} \rceil + \lceil \frac{(n_1-2) n_2 4}{4} \rceil = \lceil \frac{n_1 n_2 n_3}{4} \rceil$.

There are 6 remaining cases. The following isometric-path covers prove the assertion for $K_2 \Box K_3 \Box K_6$, $K_2 \Box K_5 \Box K_5$ and $K_3 \Box K_5 \Box K_5$, respectively:

$$\begin{aligned} \mathcal{C}_{2,3,6} \; = \; & \mathcal{C}^*_{2,3,3} \cup \{(0,0,4)(0,0,3)(1,0,3)(1,2,3),\ (0,1,3)(0,1,4)(0,2,4)(1,2,4), \\ & (0,2,3)(0,2,5)(1,2,5)(1,1,5),\ (0,1,5)(0,0,5)(1,0,5)(1,0,4)\}; \end{aligned}$$

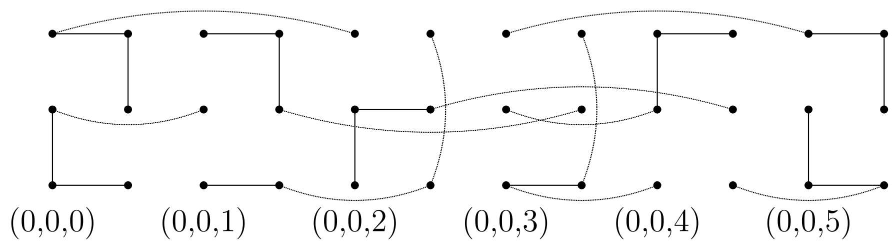

Figure 4.14: An isometric-path cover of $K_2 \Box K_3 \Box K_6$.

$$\begin{aligned} \mathcal{C}_{2,5,5} \; = \; & \mathcal{C}_{2,3,5} \backslash \{(1,0,3)(1,0,4)\} \cup \\ & \{(0,4,1)(0,4,0)(0,3,0)(1,3,0),\ (1,4,0)(1,4,1)(1,3,1)(0,3,1), \\ & (0,4,3)(0,4,2)(0,3,2)(1,3,2),\ (1,4,2)(1,4,3)(1,3,3)(0,3,3), \\ & (1,0,3)(1,0,4)(1,4,4),\ (0,4,4)(0,3,4)(1,3,4)\}; \end{aligned}$$

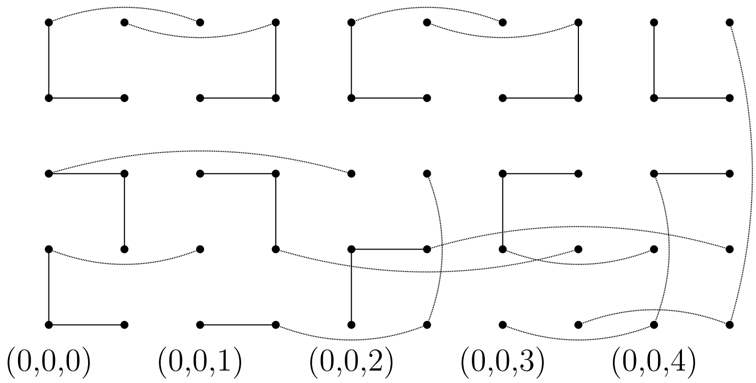

Figure 4.15: An isometric-path cover of $K_2\square K_5\square K_5$.

$$\begin{aligned}\mathcal{C}_{3,5,5} \;=\; & \mathcal{C}_{2,3,5}\backslash\{(1,0,3)(1,0,4)\}\cup \\ & \{(0,4,0)(2,4,0)(2,0,0)(2,0,1),\ (0,3,0)(2,3,0)(2,1,0)(2,1,1), \\ & \ (0,4,1)(0,3,1)(1,3,1)(1,3,0),\ (1,4,0)(1,4,1)(2,4,1)(2,2,1), \\ & \ (1,0,3)(2,0,3)(2,2,3)(2,2,0),\ (1,0,4)(2,0,4)(2,3,4)(2,3,1), \\ & \ (0,3,2)(2,3,2)(2,1,2)(2,1,3),\ (0,4,4)(0,4,2)(2,4,2)(2,0,2), \\ & \ (0,4,3)(1,4,3)(1,3,3)(1,3,2),\ (0,3,3)(2,3,3)(2,4,3)(2,4,4), \\ & \ (0,3,4)(1,3,4)(1,4,4)(1,4,2),\ (2,2,2)(2,2,4)(2,1,4)\}.\end{aligned}$$

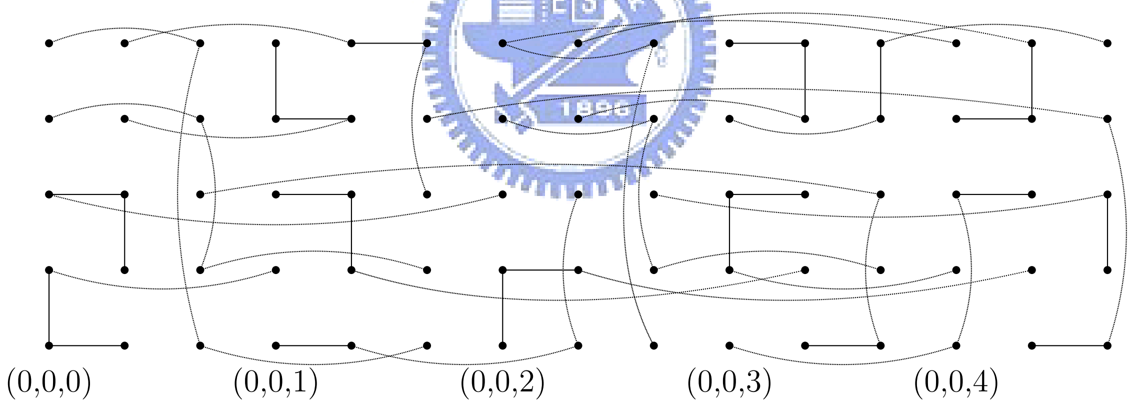

Figure 4.16: An isometric-path cover of $K_3\square K_5\square K_5$.

The other 3 cases follows from the following inequalities:

$$\mathrm{ip}(K_2\square K_5\square K_6) \leq \mathrm{ip}(K_2\square K_3\square K_6) + \mathrm{ip}(K_2\square K_2\square K_6) \leq 9 + 6 = 15,$$

$$\mathrm{ip}(K_3\square K_3\square K_5) \leq \mathrm{ip}(K_3\square K_3\square K_2) + \mathrm{ip}(K_3\square K_3\square K_3) \leq 5 + 7 = 12,$$

$$\mathrm{ip}(K_5\square K_5\square K_5) \leq \mathrm{ip}(K_5\square K_5\square K_3)+\mathrm{ip}(K_5\square K_5\square K_2) \leq 19+13 = 32. \qquad \blacksquare$$

Chapter 5

Conclusion

This thesis studies three problems on vertex partition/cover: the path-partition problem, the induced-path-partition problem and the isometric-path-cover problem. Many of our results are solved from algorithmic points of view.

For the path-partition problem, we give an $O(|V| + |E|)$-time algorithm for graphs whose blocks are complete graphs, cycles or complete bipartite graphs.

For the induced-path-partition problem, we present an $O(|V| + |E|)$-time algorithm for graphs whose blocks are complete graphs, cycles or complete bipartite graphs. We also give a polynomial-time algorithm for cographs.

We have three results for the isometric-path-cover problem. First, we determines isometric-path numbers of block graphs, and also give an $O(|V|+|E|)$-time algorithm for finding the corresponding paths. Second, we determine isometric-path numbers of complete r-partite graphs and Hamming graphs of dimensions 2 and 3.

Although some results of the above three problems are obtained, there are still many questions remain open. We describe below some of them that we concern most.

In Chapter 2, we use the tree structure to obtain an algorithm for the path-partition problem on graphs whose blocks are handleable. A nature question is that can we extend our result to graphs with small separator structure.

For the induced-path numbers, Alsardary [3] gave an upper bound on hypercubes. It is our hope to determine the exact values of them. It is also interesting to characterize graphs whose path-partition numbers are equal to induced-path numbers.

For the isometric-path-cover problem, a first question that we can not answer is that whether the isometric-path-cover problem is $\mathcal{NP}$-complete or not. We are also interested in finding an efficient algorithm on threshold graphs. Fitzpatrick *et al.* [19] gave an upper bound of isometric-path numbers on hypercubes. Can we find the exact values of them? We also study the isometric-path numbers on d-dimensional Hamming graphs for $d = 2$ and 3. Can we determine the isometric-path numbers for Hamming graphs with a general dimension d? It is also interesting to characterize graphs whose cop-numbers are equal to isometric-path numbers. Finally, it is our hope to study approximation algorithms for the above problem.

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