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Reducing the Effects of Model Reduction on Stability Boundaries and Limit-Cycle Characteristics

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***Abstract*—In this note, the effects of model reduction on the stability boundaries of control systems with parameter variations, and the limit-cycle characteristics of nonlinear control systems are investigated. In order to reduce these effects, a method is used which can approximate the original transfer function at $s = 0$, $s = \infty$, and also match some selected points on the frequency response curve of the original transfer function. Examples are given, and comparisons to the methods given in the current literature are made**

I. INTRODUCTION

In the current literature, most of the methods for model reduction are based upon the assumption that the system has constant parameters and take care of the errors between the ste input (or frequency) responses of the reduced models and the original tı nsfer function [1]–[8]. Therefore, methods for reducing the effects of model reduction on either the stability boundaries of control systems with parameter variations [9], [10] or the limit-cycle characteristics of nonlinear control systems are still needed. For example, the Padé approximations [1]–[5] consider at $s = 0$; and some other methods consider approximations at both $s = 0$ and $s = \infty$ ($z = 1$ and $z = \infty$), or at $s = a (a = > 0)$ [6]–[8]. However, the response of the reduced model at intermediate frequency may not be matched well with that of the original system; thus the stability boundaries of the parameters or the limit-cycle characteristics of a closed-loop system with the reduced model may deviate considerably from those of the original system.

In the methods based on continued-fraction expansion, the high-order transfer function is expanded into the Cauer form [7], then the high-order terms are truncated, and finally the truncated continued fraction is converted into a rational low-order transfer function. In this note, the high-order terms are not truncated but equated to some rational transfer functions, the coefficients of the latter are obtained by matching the frequency responses of the reduced model with those of the original one at some intermediate frequencies. Thus, the reduced models obtained by the proposed method can approximate the original transfer function at $s = 0$, $s = \infty$, and also match some selected points on the frequency response curve of the original transfer function. It will be shown later in this note that one can take the matching points to be the phase-crossover, gain-crossover, limit-cycle, or some other frequencies [10]; then the effects of model reduction on the stability boundaries of control systems with parameter variations, and the limit-cycle characteristics of nonlinear control systems can be reduced.

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II. THE PROPOSED METHOD

Let the original transfer function and the reduced model be

$$G(s)=\frac{A_{21}+A_{22}s+\cdots+A_{2n}s^{n-1}}{A_{11}+A_{12}s+\cdots+A_{1,n+1}s^{n}} \tag{2.1}$$

and

$$R(s)=(\omega_1,\ \omega_2,\ \cdots,\ \omega_m)R[r-1,\ r]_j^i(s) = \frac{d_0+d_1s+\cdots+d_{r-1}s^{r-1}}{c_0+c_1s+\cdots+c_rs^r}, \tag{2.2}$$

respectively. In (2.2), r and $r - 1$ represent the numbers of poles and zeros of $R(s)$, respectively; i and j are the numbers of terms of the continued-fraction expansion of $G(s)$ about $s = 0$ and $s = \infty$, respectively; and $\omega_1, \omega_2, \cdots, \omega_m$ are the frequencies at which the frequency response of $G(s)$ are matched by $R(s)$. The procedure for finding a reduced model is as follows.

Step 1: Expand $G(s)$ about $s = 0$ with i (even number) terms, i.e.,

$$G(s)=\left[h_1+\left[\frac{h_2}{s}+\left[\cdots+\left[\frac{h_i}{s}+\frac{H_N(s)}{H_D(s)}\right]^{-1}\right]^{-1}\right]^{-1}\right]^{-1} \tag{2.3}$$

where

$$H_N(s)=A_{i+2,1}+A_{i+2,2}s+\cdots+A_{i+2,n-i/2}s^{n-1-i/2} \tag{2.4}$$

and

$$H_D(s)=A_{i+1,1}+A_{i+1,2}s+\cdots+A_{i+1,n+1-i/2}s^{n-i/2}. \tag{2.5}$$

Step 2: Reverse the polynomial sequences in (2.4) and (2.5), and continue to expand (2.3) about $s = \infty$ with j (even number) terms, then one has

$$\frac{H_N(s)}{H_D(s)}=\left[E_1s+\left[E_2+\left[\cdots+\left[E_j+\frac{F_N(s)}{F_D(s)}\right]^{-1}\right]^{-1}\right]^{-1}\right]^{-1} \tag{2.6}$$

where

$$F_N(s)=A_{i+j+2,n-(i+j)/2}s^{n-1-(i+j)/2}+\cdots+A_{i+j+2,2}s+A_{i+j+2,1} \tag{2.7}$$

and

$$F_D(s)=A_{i+j+1,n+1-(i+j)/2}s^{n-(i+j)/2}+\cdots+A_{i+j+1,2}s+A_{i+j+1,1}. \tag{2.8}$$

Step 3: Replace $F_N(s)/F_D(s)$ in (2.6) by a low-order model defined as

$$\frac{T_N(s)}{T_D(s)}=\frac{y_{m-1}s^{m-1}+y_{m-2}s^{m-2}+\cdots+y_1s+y_0}{x_ms^m+x_{m-1}s^{m-1}+\cdots+x_1s+x_0} \tag{2.9}$$

where

$$y_{m-1}=1 \tag{2.10}$$

and m is the number of points on the frequency response curve of $G(s)$ to be matched by $R(s)$. The $2m$ unknowns $y_{m-2}, y_{m-3}, \cdots, y_1, y_0, x_m, \cdots, x_1, x_0$ can be obtained by setting

$$\left.\frac{T_N(s)}{T_D(s)}\right|_{s=j\omega_k}=\left.\frac{F_N(s)}{F_D(s)}\right|_{s=j\omega_k}=r_k+jm_k \qquad k=1,\ 2,\ \cdots,\ m \tag{2.11}$$

where r_k and m_k are the real part and the imaginary part of $F_N(s)/F_D(s)$

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for $s = j\omega_k$, respectively. Equating the real part and the imaginary part in (2.11), one can obtain a set of simultaneously independent equations. Therefore, the $2m$ unknowns $y_{m-2}, y_{m-3}, \cdots, y_1, y_0, x_m, \cdots, x_1, x_0$ can be obtained by solving the $2m$ simultaneously independent equations.

Step 4: Replace $F_N(s)/F_D(s)$ in (2.6) by (2.9) and invert the continued-fraction; the reduced model defined in (2.2) is obtained, i.e.,

$$(\omega_1, \omega_2, \cdots, \omega_m)R[r-1, r]_j^i(s) = \left[h_1 + \left[\frac{h_2}{s} + \left[\cdots + \left[\frac{h_i}{s} + \left[E_1 s + \left[E_2 + \left[\cdots + \left[E_j + \frac{T_N(s)}{T_D(s)}\right]^{-1}\right]^{-1}\right]^{-1}\right]^{-1}\right]^{-1}\right]^{-1}\right]^{-1}\right]^{-1}. \tag{2.12}$$

The order of the denominator of the reduced model $R(s)$ is

$$r = m + (i+j)/2. \tag{2.13}$$

By use of (2.3) and (2.6), one has

$$G(s) = \left[h_1 + \left[\frac{h_2}{s} + \left[\cdots + \left[\frac{h_i}{s} + \left[E_1 s + \left[E_2 + \left[\cdots + \left[E_j + \frac{F_N(s)}{F_D(s)}\right]^{-1}\right]^{-1}\right]^{-1}\right]^{-1}\right]^{-1}\right]^{-1}\right]^{-1}\right]^{-1}. \tag{2.14}$$

From (2.11), (2.12), and (2.14) it can be seen that

$$G(s) = (\omega_1, \omega_2, \cdots, \omega_m)R[r-1, r]_j^i(s) \quad \text{for } s = j\omega_k \quad k = 1, 2, \cdots, m. \tag{2.15}$$

Equation (2.15) indicates that the frequency response of $G(s)$ and $R(s)$ are matched at $s = j\omega_k$, $k = 1, 2, \cdots, m$. Note that if both i and j are odd numbers, (2.3)–(2.6) may have some minor differences, but the procedure is the same.

In comparison to the original continued-fraction methods [1], [6], [7], [11] the advantage of the proposed method is due to the fact that the remainders $(F_N(s)/F_D(s))$ of the continued fraction, which are disregarded in the continued-fraction methods, are now used to match some desirable points on the frequency response curve of the original transfer function. Thus, the frequency response at low, intermediate, and high frequencies can be matched; and the effects of model reduction on the stability boundaries of control systems with parameter variations and the limit-cycle characteristics of nonlinear control systems can be reduced.

Marshall [12] proposed a method of model reduction by retaining the dominant poles and zeros of the original high-order transfer function first, and then cascading an adjustable time-delay and/or a lead-lag compensator such that the frequency response of the system with the reduced model matches that of the original system at the phase-crossover or some other frequencies. If there are no dominant poles or zeros, this method cannot be applied. However, the method proposed in this note is straightforward and clear, i.e., one can make the frequency response of the reduced model at $s = 0$ (or $s = \infty$) match much better with that of the original high-order transfer function just by choosing larger i (or j) defined in (2.2). In addition, the number of points and frequencies to be matched can be arbitrarily chosen following the constraint defined by (2.13).

III. Applications of the Proposed Method

The main purpose of this section is to apply the proposed method to show that, when reduced models are used, the effects of model reduction on the stability boundaries of control systems with parameter variations and the limit-cycle characterstics of nonlinear control systems can be reduced.

Example 1: Consider the system shown in Fig. 1 where $r = 0.3$, the stability boundaries of the parameters α and β are to be determined. The nonlinear elements N_1 and N_2 are neglected for this example. The transfer

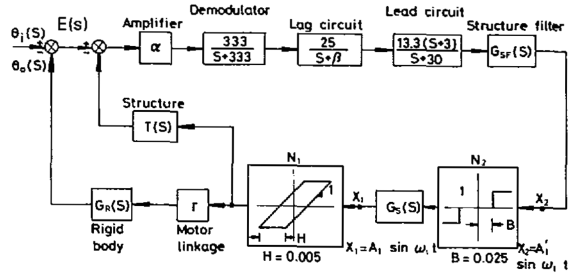

Fig. 1. Block diagram of a missile control system.

functions are defined as follows [9]:

$$G_R(s) = \frac{7.21}{(s+1.6)(s-1.48)} \tag{3.1}$$

$$G_S(s) = \frac{2750}{s^2 + 42.2s + 2750} \tag{3.2}$$

$$G_{SF}(s) = \frac{(s^2 + 70s + 4000)(s^2 + 22s + 12800)}{(s^2 + 30s + 5810)(s^2 + 30s + 12800)} \tag{3.3}$$

and the structure $T(s)$ is defined as

$$T(s) = [0.686(s+53)(s-53)(s^2 - 152.2s + 14500)(s^2 + 153.8s + 14500)] / [(s^2 + s + 605)(s^2 + 45.5s + 2660)(s^2 + 2.51s + 3900) \cdot (s^2 + 3.99s + 22980)]. \tag{3.4}$$

The proposed method is applied to reduce $T(s)$ with frequency responses at $\omega_{1p} = 21.021$ rad/s and $\omega_l = 54.844$ rad/s to be matched by the reduced model, where ω_{1p} is the phase-crossover frequency of the open-loop transfer function $\theta_0(s)/E(s)$ when the nonlinear elements N_1 and N_2 are neglected and the parameters α, β, and r are assumed to be 15, 100, and 0.3, respectively; and ω_l is the frequency of the limit cycle for the parameters α, β, and r assumed to be 18, 90, and 0.3 in Example 2, respectively. The result is

$$W_1(s) = (\omega_l, \omega_{1p})R[3, 5]_2^4(s) = [0.686(s+24.6442)(s-40.9717)(s+219.7768)]/[(s+28.5945)(s^2 + 1.2351s + 593.7862)(s^2 + 10.0018s + 3191.7797)]. \tag{3.5}$$

By use of other methods, the reduced models of $T(s)$ are found as follows.

1) Continued-Fraction Method [1]:

$$C_1(s) = R[4, 5]_0^{10}(s) = [-2.44 \times 10^{-2}(s - 61.3836)(s - 67.5164)(s^2 + 81.5000s + 1748.6381)]/[(s + 41.3000)(s^2 + 0.9700s + 605.9021)(s^2 + 25.4000s + 2447.8033)]. \tag{3.6}$$

2) Stability-Equation Method and Padé Approximation Method [13]:

$$B_1(s) = [-2.3837 \times 10^{-2}(s + 38.1380)(s + 130.4869)(s^2 - 101.2398s + 2997.6391)]/[(s + 72.5638)(s^2 + 0.91s + 602.1438)(s^2 + 12.4262s + 2902.0190)]. \tag{3.7}$$

3) Modified Padé Approximation Method [6]:

$$M_1(s) = R[3, 5]_6^4(s) = [0.686(s - 111.3723)(s^2 + 321.1828s + 38950.873)] / [(s^2 + 1.2982s + 401.8666)(s^2 + 152.1748s + 28159.032) \cdot (s + 107.7370)]. \tag{3.8}$$

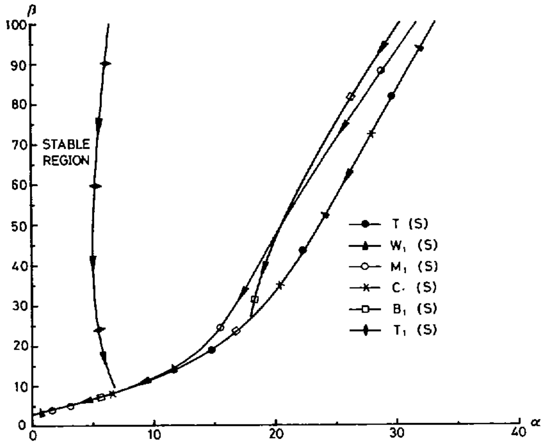

Fig. 2. Stability boundaries of the system with $T(s)$ and its reduced models.

TABLE I
LIMIT-CYCLE CHARACTERISTICS FOR THE SYSTEM WITH $T(s)$ AND ITS REDUCED MODELS

Models	Amplitudes		Frequency
	A_1'	A_1	ω_1 (rad/s)
Original $T(S)$	0.6841	1.50256	54.844
$W_1(S)$	0.6841	1.50256	54.844
$C_1(S)$	0.401	1.64019	50.191
$B_1(S)$	0.813	1.60078	51.782
$M_1(S)$	0.7775	1.38277	18.513
$T_1(S)$	3.777	1.26043	61.32

4) Routh Stability Array Method [14]:

$$T_1(s)=[-1.5899\times10^{-2}(s+53)(s+150.2440)(s-53)\cdot(s-157.0873)]/[(s+161.1435)(s^2+0.7348s+611.5744)\cdot(s^2+2.0104s+3807.4562)]. \quad (3.9)$$

By use of the parameter space method [9], the stability boundaries of the original closed-loop system and the system with reduced models are plotted in the α versus β plane as shown in Fig. 2, where the stable regions are to the right-hand side of each of the boundaries along the arrow direction. It can be seen that the stable regions are reduced when $B_1(s)$, $M_1(s)$, and $T_1(s)$ are used. In other words, due to the effects of parameter variations, a stable system may become unstable if an improper reduced model is used. However, when the reduced model $W_1(s)$ is used, the stability boundary is very close to that of the original system. This is due to the fact that the frequency response of $T(s)$ at $s = 0$, $s = \infty$, and two intermediate frequencies are matched by the reduced model $W_1(s)$.

Example 2: Consider the system shown in Fig. 1. Let the parameters α, β, and r be 18, 90, and 0.3, respectively. By use of the describing function method [9] and the parameter space method, a limit cycle can be found. The amplitude and frequency of the limit cycle are at

$$A_1=1.50256 \quad (3.10)$$

$$A_1'=0.6841 \quad (3.11)$$

and

$$\omega_1=54.844 \text{ rad/s} \quad (3.12)$$

respectively, where A_1 and A_1' are the amplitudes of the sinusoidal waves at the input terminals of N_1 and N_2, respectively. Replacing $T(s)$ by models as defined from (3.6) to (3.9), the limit-cycle characteristics are obtained as shown in Table I. It can be seen that the limit-cycle characteristics are changed when the reduced models are used. In other words, the result of limit cycle analysis may be incorrect if an improper reduced model is used. Note that both the amplitudes and the frequency of the limit cycle can be preserved by the system with the proposed model $W_1(s)$ as also shown in Table I. This is due to the fact that the frequency response of $W_1(s)$ at the limit-cycle frequency is matched with that of $T(s)$.

CONCLUSIONS

The effects of model reduction on the stability boundaries of control systems with parameter variations and the limit-cycle characteristics of nonlinear control systems have been investigated. Applications of the proposed method to reduce these effects have been presented. In comparison to the results obtained by the methods given in the current literature, the proposed method can, in general, give better results.

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Path Controllability of Linear Input-Output Systems

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Abstract—**The varied definitions used in four studies of path controllability of linear input-output systems are given, their similarities and differences examined, and some of their corresponding results compared and discussed.**

INTRODUCTION

It is often of interest, in studying input-output systems, to determine whether the output of the system can be made to follow a preassigned

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