國 立 交 通 大 學

應用數學系
數學建模與科學計算研究所

碩 士 論 文

產權市場短期套利的探討

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摘　要

我們給予了一個在任意短期時間內，以及任意產權市場會存在套利的充分條件。相較於 Banner 和 Fernholz 在 2008 年所提出的充分條件，我們給的條件更具一般性，然而，另一方面，我們給的條件也比較難去驗證。儘管如此，我們給出了一類的財務模型，它們可能不會滿足 Banner 和 Fernholz 所給出的充分條件，可是卻滿足我們這篇論文所給出的條件。

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Abstract

We provide a sufficient condition for the existence of relative arbitrage over arbitrarily short time horizon in equity markets. Compared with the sufficient condition given in Banner & Fernholz(2008), our condition is much general, but, on the other hand, it is more difficult to check. However, we give a family of abstract models which do not satisfy the criteria in Banner & Fernholz (2008) but satisfy the criteria here.

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Contents

1 Introduction

In a financial market, there are three major topics for study, that is arbitrage, pricing and hedging. Arbitrage, in some sense, is to make money without any cost. Simply speaking, if there are two objects in the market with the same price today, and assume we are very sure about that which one of them will be cheaper than the other some specific day in the future, then we can sell the will-be-cheaper one and buy another one today, also, buy the cheaper one and sell another in the future, in this way, we will earn the price difference between these two objects without any cost today. Obviously, many people are really interesting about finding arbitrages in equity markets. Therefore, we may ask that do there exist arbitrages in any equity market? If not, then under what kind of equity markets do arbitrages exist? If arbitrages really exist in some equity markets, then we want to know more about how long can we achieve these arbitrages? a month? or a year? and how can we make these arbitrages? that is, by what combination of stocks in equity markets?

Pricing is another topic about how to price a derivative financial product with some sort of function. For example, futures contracts, if I have a deal with you that I will give you 1000 ipad2s next month but you want to pay me right now for some reasons, then how much should I charge for those ipad2s? Or from your point of view, how much money should you pay for those ipad2s? Such questions are highly related to another topic — hedging. Since no matter much money I get from you today, I still have the risk to lose money next month. For a relative conservative person as me, we don't want to take a chance on it. Hence, we would like to ask that whether there some products which are highly correlated to ipad2 so that we can buy an appropriate combination of them to reduce the possible-loss. In other words, use the money I have today to make enough money at least to cover the possible-loss in the future. Furthermore, we may like to know that what are the best combination among all of them?

There are many theories for the three topics above, the most well-known one is Dynamic Asset Pricing Theory. In this theory, the market structure is analyzed under strong normative assumptions with respect to the behavior of market participants. This theory is based on a market model with existence of equivalent martingale measure(s) and the absence of arbitrage, so we can only use this theory to answer the topics about pricing and hedging. Black-Scholes Option Pricing Formula is one of the most famous results among Dynamic Asset Pricing Theory.

Another recently-developed theory is Stochastic Portfolio Theory. This is a theory constructed on a rather general setting. It uses the class of semi-martingales to describe the stocks in the real markets, and by making some descriptive assumptions to study arbitrage, pricing and hedging. The differences between Dynamic Asset Pricing Theory and Stochastic Portfolio Theory is that the assumptions made in Stochastic Portfolio Theory are usually some observable properties of the markets, while the assumptions in Dynamic Asset Pricing Theory are usually made for technical concerns. One of the most powerful tools of Stochastic Portfolio Theory is portfolio generating function, it is a tool developed by Fernholz, R.(1999). With this tool and another powerful tool in stochastic analysis — Itô's formula, we can construct a class of bounded portfolios whose returns can be decomposed into several components with proper characteristics. Also, we can use these portfolios to investigate the issue of arbitrage for time large enough, or even better, for arbitrarily time-horizon. Moreover, we can use such tools to understand that what kind of portfolios can achieve arbitrage. Recently, there are a few works which concerns about arbitrage, and the conditions from those works may be present in actual market and the corresponding portfolio can be implemented in practical. This theory has been the basis of successful equity investment strategies for a decade.

The most significant concern of this paper is that under what conditions, we can have strong arbitrage over any time-horizon. In section 2, we proceed with an introduction to the standard equity market model that we use, and then we introduce the concepts of investment strategies, portfolios and arbitrage in an equity market. Also, we mention a few important quantities corresponding to portfolios which are related to existence of arbitrages. At the end of this section, we also give some examples which exist an arbitrage opportunity for time large enough. Then, we introduce a powerful tool for Stochastic Portfolio Theory—portfolio generating functions, in section 3. With portfolio generating function, we can derive the master formula, we also give a simple example to show how to use the master formula to achieve arbitrage. Moreover, we mention some properties of concave functions in the same section for the later use. Section 4 is the main work of this paper, we gives a sufficient condition for arbitrarily strong arbitrage, and weaken the sufficient condition in Banner and Fernholz (2008). As for section 5, we propose an abstract model which may satisfy the condition for arbitrage given in section 4 but may not satisfy the sufficient condition in Banner and Fernholz (2008) .

2 Preliminaries

2.1 The Model

On a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$, $\mathbb{F} = \{\mathcal{F}(t)\}_{0 \le t < \infty}$ with the usual conditions, that is, right-continuity and augmentation by $\mathbb{P}$-negligible sets. Also, we assume $\mathcal{F}(0) = \{\emptyset, \Omega\}$ modulo $\mathbb{P}$. Consider the following market model :

$$d \log X_i(t) = \gamma_i(t)dt + \sum_{k=1}^{n} \sigma_{ik}(t)dW_k(t) , \qquad i = 1, \cdots, n . \tag{1}$$

Here the vector-valued process $\gamma(t) = (\gamma_1(t), ..., \gamma_n(t))$ is the growth rates process for the stocks, and the matrix-valued process $\sigma(t) = [\sigma_{ik}(t)]_{1 \le i,k \le n}$ is called the volatility of the stocks in the market. The covariance process of the stocks in the market is the matrix-valued process $\alpha(t) = \sigma(t)\sigma'(t)$ with elements

$$\alpha_{ij}(t) = \sum_{k=1}^{n} \sigma_{ik}(t)\sigma_{jk}(t) = \frac{d}{dt} \langle \log X_i, \log X_j \rangle (t), 1 \le i, j \le n. \tag{2}$$

By Itô's formula, the market model can be formulated as

$$dX_i(t) = X_i(t) \left(\beta_i(t)dt + \sum_{k=1}^{n} \sigma_{ik}(t)dW_k(t) \right), \qquad i = 1, \cdots, n \tag{3}$$

where

$$\beta_i(t) = \gamma_i(t) + \frac{1}{2}\alpha_{ii}(t), \qquad \text{for all } t > 0 \tag{4}$$

is the mean rate of return for the stock i, for each $i = 1, \cdots, n$.

Remark 1. *(A)A special case of* (1) *is the so-called volatility-stabilized market given by*

$$d\log X_i(t) = \frac{\delta}{2\mu_i(t)}dt + \frac{1}{\sqrt{\mu_i(t)}}dW_i(t)\ , \qquad i = 1, \cdots, n \tag{5}$$

where $\delta \geq 0$ *is a constant and* $\mu_i(t) = X_i(t)/\sum_{j=1}^n X_j(t)$, $1 \leq i \leq n$.
This model captures the property in real markets that the smaller stocks are more likely to have greater growth rates and volatilities than the larger stocks. Therefore, it is not surprising that each stock in such market fluctuates heavily. See Fernholz and Karatzas (2005) for details.
(B)Another case of (1) *is called Atlas model given by*

$$d\log X_i(t) = ng1_{\{X_i(t)=X_{p_t(n)}(t)\}} + \sigma dW_i(t)\ , \qquad i = 1, \cdots, n \tag{6}$$

where $g > 0$ *and* σ *are constants. (Here* $p_k(t)$ *is the name of stock which rank* k*th position of all stocks at time* t*; If there are stocks with the same value, then sort them by their indexes.) This model captures the characteristic in the real market that smaller stocks should have higher growth rates, by assigning zero growth rate to all the stocks except the smallest one.(For details, see Banner, Fernholz and Karatzas(2005).)*

2.2 Investment Strategies and Portfolios

Now, consider a model with a money-market $dB(t) = B(t)r(t)dt$, $B(0) = 1$, and consider a small investor whose actions in the market cannot affect market prices. This investor decides, at each time t, that proportion $\pi_i(t)$ of current wealth $Z(t)$ to invest in the ith stock, $i = 1, \cdots, n$; the proportion $\pi_0(t) := 1 - \sum_{i=1}^n \pi_i(t)$ gets invested in the money market. Thus, given a strategy $\pi(\cdot)$ and initial capital $z \in (0, \infty)$, the corresponding wealth process $Z_{z,\pi}(\cdot)$ for this strategy satisfies :

$$\begin{cases} \dfrac{dZ_{z,\pi}(t)}{Z_{z,\pi}(t)} = \displaystyle\sum_{i=1}^n \pi_i(t)\frac{dX_i(t)}{X_i(t)} + \pi_0(t)\frac{dB(t)}{B(t)} = \pi'(t)[(\beta(t) + r(t))dt + \sigma(t)dW(t)]\ . \\ \\ Z_{z,\pi}(0) = z \end{cases} \tag{7}$$

We shall say $\pi(\cdot)$ is an investment strategy, and write $\pi(\cdot) \in \mathcal{H}$, if $\pi : [0, \infty) \times \Omega \to \mathbb{R}^n$ is a $\mathbb{F}$-progressively measurable process which satisfies for each $T \in (0, \infty)$

$$\int_0^T \left(|\pi'(t)(\beta(t) + r(t))| + \pi'(t)\alpha(t)\pi(t)\right) dt < \infty\ , \qquad \text{a.s.} \tag{8}$$

An investment strategy $\pi(\cdot) \in \mathcal{H}$ with $\sum_{i=1}^n \pi_i(t, \omega) = 1$ for all $(t, \omega) \in [0, \infty) \times \Omega$ will be called a portfolio. A portfolio never invests in the money market and never borrows from it. And we shall say a portfolio $\pi(\cdot)$ is bounded if there exist a constant $M > 0$ such that $||\pi(t)|| \leq M$, for all $(t, \omega) \in [0, \infty) \times \Omega$. We shall call a portfolio long-only, if it never sells any stock short. Clearly, a long-only portfolio is also bounded.

Remark 2. *We will write* $Z_\pi(t)$ *for* $Z_{z,\pi}(t)$*, if* $Z_{z,\pi}(0) = 1$.

Remark 3. *An important long-only portfolio is the market portfolio; this invests in all stocks in proportion to their relative weights,*

$$\mu_i(t) = \frac{X_i(t)}{X(t)}, \qquad 1 \leq i \leq n, \tag{9}$$

where $X(t) := X_1(t) + \cdots + X_n(t)$*.Clearly, we have* $Z_{z,\mu}(\cdot) = zX(\cdot)/X(0)$ *and the resulting vector process* $\mu(\cdot) = (\mu_1(\cdot), \cdots, \mu_n(\cdot))$ *of market weights takes values in the positive simplex*

$$\Delta^n = \{x \in \mathbb{R}^n : x_1 + \cdots + x_n = 1; 0 < x_i < 1, i = 1 \cdots, n\}.$$

For a portfolio $\pi(\cdot)$ with initial capital $z > 0$, since a portfolio never invests in the money market and never borrows from it(i.e. $\pi_0(\cdot) \equiv 0$), we can find by (7) that the corresponding wealth process for this portfolio is the solution of the following SDE :

$$\begin{cases} \dfrac{dZ_{z,\pi}(t)}{Z_{z,\pi}(t)} = \displaystyle\sum_{i=1}^{n} \pi_i(t) \frac{dX_i(t)}{X_i(t)} = \pi'(t)[\beta(t)dt + \sigma(t)dW(t)] \ . \\ \\ Z_{z,\pi}(0) = z \end{cases} \tag{10}$$

or, equivalently,

$$Z_{z,\pi}(t) = z \exp\left\{ \int_0^t \left(\pi'(s)\beta(s) - \frac{1}{2}\pi'(s)\alpha(s)\pi(s) \right) ds + \int_0^t \pi'(s)\sigma(s)dW(s) \right\}. \tag{11}$$

By analogy with (1) we can write the SDE (10) as

$$d\log Z_{z,\pi}(t) = \gamma_\pi(t)dt + \sum_{k=1}^{n} \sigma_{\pi k}(t)dW_k(t) \ , \qquad Z_{z,\pi}(0) = z \ , \tag{12}$$

or, equivalently,

$$Z_{z,\pi}(t) = z \exp\left\{ \int_0^t \gamma_\pi(s)ds + \sum_{k=1}^{n} \int_0^t \sigma_{\pi k}(s)dW_k(s) \right\}, \qquad 0 \le t < \infty. \tag{13}$$

Here

$$\sigma_{\pi k}(t) := \sum_{i=1}^{n} \pi_i(t)\sigma_{ik}(t) \quad \text{ for } k = 1, \cdots, n \tag{14}$$

are the volatility coefficients associated with the portfolio $\pi(\cdot)$, and

$$\gamma_\pi(t) := \sum_{i=1}^{n} \pi_i(t)\gamma_i(t) + \gamma_\pi^*(t) \tag{15}$$

is the growth rate of the portfolio $\pi(\cdot)$, where

$$\gamma_\pi^*(t) := \frac{1}{2} \left(\sum_{i=1}^{n} \pi_i(t)\alpha_{ii}(t) - \sum_{i=1}^{n}\sum_{j=1}^{n} \pi_i(t)\alpha_{ij}(t)\pi_j(t) \right) \tag{16}$$

is the excess growth rate of the portfolio $\pi(\cdot)$.

For an arbitrary portfolio $\pi(\cdot)$, and with e_i denoting the ith unit vector in $\mathbb{R}^n$, let us introduce the quantities

$$\tau_{ij}^{\pi}(t) := \sum_{k=1}^{n} (\sigma_{ik}(t) - \sigma_{\pi k}(t))(\sigma_{jk}(t) - \sigma_{\pi k}(t)) \tag{17}$$

$$= (\pi(t) - e_i)'\alpha(t)(\pi(t) - e_j) = \alpha_{ij}(t) - \alpha_{\pi i}(t) - \alpha_{\pi j}(t) + \alpha_{\pi\pi}(t) \tag{18}$$

for $1 \leq i, j \leq n$, and set

$$\alpha_{\pi i}(t) := \sum_{j=1}^{n} \alpha_{ij}(t)\pi_j(t) \ , \qquad \alpha_{\pi\pi}(t) := \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_{ij}(t)\pi_i(t)\pi_j(t) \ . \tag{19}$$

We shall call the matrix-valued process $\tau^{\pi}(\cdot) = (\tau^{\pi}_{ij}(\cdot))_{1\leq i,j\leq n}$ the process of individual stocks' covariance relative to the portfolio $\pi(\cdot)$. In fact, we have

$$\tau^{\pi}_{ij}(t) = \frac{d}{dt}\left\langle \log\frac{X_i}{Z_{X_i(0),\pi}}, \log\frac{X_j}{Z_{X_j(0),\pi}}\right\rangle(t), \quad 1 \leq i, j \leq n. \tag{20}$$

It satisfies the equations

$$\sum_{j=1}^{n} \tau^{\pi}_{ij}(t)\pi_j(t) = 0, \quad i = 1, \cdots, n. \tag{21}$$

Also we have

$$\gamma^*_{\pi}(t) = \frac{1}{2}\left(\sum_{i=1}^{n} \pi_i(t)\tau^{\eta}_{ii}(t) - \sum_{i=1}^{n}\sum_{j=1}^{n} \pi_i(t)\pi_j(t)\tau^{\eta}_{ij}(t)\right), \tag{22}$$

for any portfolio η. In particular, when $\eta = \pi$, we have

$$\gamma^*_{\pi}(t) = \frac{1}{2}\sum_{i=1}^{n} \pi_i(t)\tau^{\pi}_{ii}(t). \tag{23}$$

Remark 4. *Since* $\langle \log(X_i/Z_{X_i(0),\pi})\rangle(t)$ *is a.s. nondecreasing,*

$$\tau^{\pi}_{ii}(t) \geq 0, \quad t \in [0, \infty), \quad a.s.$$

2.3 Relative Arbitrage

Given any two investment strategies π and ρ, we shall say that π is an arbitrage relative to ρ over $[0, T]$, if we have

$$\mathbb{P}\left(Z_{1,\pi}(T) \geq Z_{1,\rho}(T)\right) = 1 \quad \text{and} \quad \mathbb{P}\left(Z_{1,\pi}(T) > Z_{1,\rho}(T)\right) > 0. \tag{24}$$

We call such relative arbitrage strong, if

$$\mathbb{P}\left(Z_{1,\pi}(T) > Z_{1,\rho}(T)\right) = 1. \tag{25}$$

Existence of Arbitrage Relative to the Market Portfolio.

(A) If there exists a real constant $h > 0$ such that

$$\gamma^*_{\mu}(t) = \frac{1}{2}\left(\sum_{i=1}^{n} \mu_i(t)\alpha_{ii}(t) - \sum_{i=1}^{n}\sum_{j=1}^{n} \mu_i(t)\mu_j(t)\alpha_{ij}(t)\right) \geq h, \quad \forall\ 0 \leq t < \infty \tag{26}$$

holds almost surely, it can be shown that, for a sufficiently large constant $c > 0$, the long-only portfolio

$$\pi_i(t) = \frac{\mu_i(t)(c - \log\mu_i(t))}{\sum_{j=1}^{n} \mu_j(t)(c - \log\mu_j(t))}, \quad i = 1, \cdots, n \tag{27}$$

is a strong arbitrage relative to the market portfolio μ over any time-horizon $[0,T]$ with $T > (2\log n)/h$. (See Fernholz and Karatzas (2009) Example 11.1 for a proof.) Note also that if the market is nondegenerate and diverse, then (26) holds. (Proposition 2.2.2 of Fernholz (2002).)

(B) If there exists a real constant $h > 0$ such that

$$(\mu_1(t)\cdots\mu_n(t))^{1/n}\left[\sum_{i=1}^{n}\alpha_{ii}(t)-\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_{ij}(t)\right]\geq h,\quad \forall\, 0\leq t<\infty, \tag{28}$$

holds almost surely, then for a sufficiently large constant $c > 0$, the long-only portfolio

$$\pi_i(t)=\frac{c}{c+(\mu_1(t)\cdots\mu_n(t))^{1/n}}\frac{1}{n}+\frac{(\mu_1(t)\cdots\mu_n(t))^{1/n}}{c+(\mu_1(t)\cdots\mu_n(t))^{1/n}}\mu_i(t) \tag{29}$$

is a strong arbitrage relative to the market portfolio μ over any time-horizon $[0,T]$ with $T > 2n^{1-(1/n)}/h$. (See Fernholz and Karatzas (2009) Example 11.2 for a proof, we will also give a proof in latter context.)

(C) Suppose there exists a continuous, strictly increasing function $\Gamma : [0,\infty) \to [0,\infty)$ with $\Gamma(0) = 0$, $\Gamma(\infty) = \infty$ and such that

$$\Gamma(t)\leq\int_0^t\gamma_\mu^*(s)ds<\infty,\qquad \text{for all } 0\leq t\leq\infty \tag{30}$$

holds almost surely, then, for a sufficiently large constant $c > 0$, the portfolio

$$\pi_i(t)=\frac{c\mu_i(t)-\mu_i(t)\log\mu_i(t)}{c-\sum_{j=1}^{n}\mu_j(t)\log\mu_j(t)},\qquad i=1,\cdots,n \tag{31}$$

is a strong arbitrage relative to the market portfolio μ over any time-horizon $[0,T]$ with $T > T^*$, where

$$T^*:=\Gamma^{-1}\left(-\sum_{j=1}^{n}\mu_j(0)\log\mu_j(0)\right). \tag{32}$$

(See Fernholz and Karatzas (2009) Remark 11.4 .)
Open Question: *Does any one of the sufficient condition above guarantee the existence of strong relative arbitrage opportunities over arbitrary time-horizons?*

3 Some Useful Properties

3.1 Relative Return Process

It is frequently of interest to measure the performance of stocks or portfolios relative to a given benchmark in the market portfolio, consisting of all the shares of all the stocks in the market.

For any two portfolios π and η, the relative return process of π versus η is defined by

$$\log\left(\frac{Z_\pi(t)}{Z_\eta(t)}\right),\quad t\in[0,\infty).$$

Then, by (1) and (12), we have

$$d\log Z_\pi(t) = \sum_{i=1}^{n} \pi_i(t) d\log X_i(t) + \gamma_\pi^*(t)dt, \quad \text{a.s.}, \tag{33}$$

for $t \in [0, \infty)$, so we also have

$$d\log\left(\frac{Z_\pi(t)}{Z_\eta(t)}\right) = \sum_{i=1}^{n} \pi_i(t) d\log\left(\frac{X_i(t)}{Z_\eta(t)}\right) + \gamma_\pi^*(t)dt, \quad \text{a.s.}, \tag{34}$$

for $t \in [0, \infty)$.

In particular, when $\eta = \mu$, the market portfolio, this equation can be expressed in an especially useful form :

$$d\log\left(\frac{Z_\pi(t)}{Z_\mu(t)}\right) = \sum_{i=1}^{n} \pi_i(t) d\log \mu_i(t) + \gamma_\pi^*(t)dt, \quad \text{a.s.}, \tag{35}$$

for $t \in [0, \infty)$.

3.2 Portfolio Generating Functions

Functionally generated portfolios were introduced by Fernholz, in Fernholz (1999). For this class of portfolios one can derive a decomposition of their relative return which proves useful in the construction and study of arbitrages relative to the market. This decomposition is so powerful because it does not involve stochastic integrals, and opens the possibility for making probability-one comparisons over given fixed time-horizon.

Specifically speaking, given S a positive C^2 function defined on some open neighborhood U of Δ^n such that for all $i = 1, \cdots, n$, $x \mapsto x_i D_i \log S(x)$ is bounded on U. Consider also the portfolio $\pi(\cdot)$ with weights

$$\pi_i(t) = \left(D_i \log S(\mu(t)) + 1 - \sum_{j=1}^{n} \mu_j(t) D_j \log S(\mu(t))\right) \mu_i(t), \qquad 1 \le i \le n. \tag{36}$$

We call this the portfolio generated by $S(\cdot)$. It can be shown that the relative wealth process of the portfolio, with respect to the market, is given by the master formula

$$\log\left(\frac{Z_\pi(T)}{Z_\mu(T)}\right) = \log\left(\frac{S(\mu(T))}{S(\mu(0))}\right) + \int_0^T \Theta(t)dt \qquad 0 \le T < \infty, \tag{37}$$

where the so-called drift process $\Theta(\cdot)$ is given by

$$\Theta(t) = \frac{-1}{2S(\mu(t))} \sum_{i,j=1}^{n} D_{ij} S(\mu(t)) \mu_i(t) \mu_j(t) \tau_{ij}^{\mu}(t). \tag{38}$$

(For a proof, see Fernholz(2002) p.46)

Remark 5. *The generated portfolio weights depend only on the market weights $\mu_1(t), \cdots, \mu_n(t)$, not on the covariance structure of the market. Hence, such portfolio can be implemented easily.*

Remark 6. *Suppose the function $S(\cdot)$ is concave, or, more precisely, its Hessian $D^2S(x) = (D^2_{ij}S(x))_{1\le i,j\le n}$ has at most one positive eigenvalue for each $x \in U$ and, if a positive eigenvalue exists, the corresponding eigenvector is orthogonal to Δ^n. Then the generated portfolio corresponding to this S is long-only with the drift term $\Theta(\cdot)$ being non-negative; if rank$(D^2S(x)) > 1$ holds for each $x \in U$, then $\Theta(\cdot)$ is positive.*

Here are a few examples of simple generating functions and the portfolios they generate.

1. $S(x) \equiv w$, a positive constant, generates the market portfolio with $\Theta(\cdot) \equiv 0$;

2. $S(x) = w_1x_1 + \cdots + w_nx_n$ generates the passive portfolio that buys at time $t = 0$, and holds up until time $t = T$, a fixed number of shares w_i in each stock $i = 1, \cdots, n$(the market portfolio corresponds to the special case $w_1 = \cdots = w_n = w$ of equal numbers of shares across assets);

3. $S(x) = x_1^{p_1} \cdots x_n^{p_n}$, where $p_1, \cdots, p_n$ are constants and $p_1 + \cdots + p_n = 1$, generates the constant-weighted portfolio with weights $\pi_i(t) = p_i$ and $d\Theta(t) = \gamma^*_\pi(t)dt$. Indeed, for this generating function, we have

$$\log S(x) = \sum_{i=1}^n p_i \log x_i \Rightarrow D_i \log S(x) = \frac{p_i}{x_i} \Rightarrow \sum_{j=1}^n \mu_j(t) D_j \log S(\mu(t)) = \sum_{j=1}^n p_j = 1$$

and

$$D_iS(x) = \frac{p_i}{x_i}S(x), \quad D_{ii}S(x) = \frac{p_i(p_i-1)}{x_i^2}S(x), \quad D_{ij}S(x) = \frac{p_ip_j}{x_ix_j}S(x), \text{ for } i \ne j.$$

Therefore,

$$\pi_i(t) = \left(\frac{p_i}{\mu_i(t)} + 1 - 1\right)\mu_i(t) = p_i,$$

and

$$\begin{aligned} d\Theta(t) &= \frac{-1}{2S(\mu(t))}\left(\sum_{i=1}^n p_i(p_i-1)S(\mu(t))\tau^\mu_{ii}(t) + \sum_{1\le i,j\le n, i\ne j} p_ip_jS(\mu(t))\tau^\mu_{ij}(t)\right)dt \\ &= \frac{-1}{2}\left(\sum_{i=1}^n\sum_{j=1}^n p_ip_j\tau^\mu_{ij}(t) - \sum_{i=1}^n p_i\tau^\mu_{ii}(t)\right)dt \\ &= \frac{1}{2}\left(\sum_{i=1}^n \pi_i\tau^\mu_{ii}(t) - \sum_{i=1}^n\sum_{j=1}^n \pi_i\pi_j\tau^\mu_{ij}(t)\right)dt = \gamma^*_\pi(t)dt. \end{aligned}$$

(In the last line, we use equation (22).)

4. The modified entropy function $H_c(x) = c - \sum_{i=1}^n x_i \log x_i$ generates the portfolio in (31). The drift process for this portfolio is $d\Theta(t) = \gamma^*_\mu(t)/H_c(\mu(t))dt$. Since $c < H_c(x) \le c + \log n$, for $x \in \Delta^n$, similar argument as in 3 show that the portfolio is a relative arbitrage to the market portfolio over sufficiently large time.

Proof of (B) in section 2.3 : Consider the equally weighted portfolio $\varphi_i(\cdot) \equiv 1/n$, $i = 1\cdots, n$, then condition (28) can be written as

$$(\mu_i(t)\cdots\mu_n(t))^{1/n}\gamma^*_\varphi(t) \geq \frac{h}{2n}.$$

From 3 above, we know this portfolio is generated by $S(x) = (x_1 \ldots x_n)^{1/n}$. For this $S(\cdot)$ and any $c > 0$, consider another portfolio

$$\varphi_i^c(t) = \frac{S(\mu(t))}{c + S(\mu(t))} \cdot \frac{1}{n} + \frac{c}{c + S(\mu(t))} \cdot \mu_i(t), \quad i = 1, \cdots, n.$$

It is not hard to see that this portfolio is generated by $S_c(x) = c + S(x)$, and with $n^{-1/n} \geq S(x) \geq 0$, we can derive the following equation

$$\begin{aligned}
\log\left(\frac{Z_{\varphi^c}(T)}{Z_\mu(T)}\right) &= \log\left(\frac{c + S(\mu(T))}{c + S(\mu(0))}\right) + \int_0^T \frac{S(\mu(t))\gamma^*_\varphi(t)}{c + S(\mu(t))} dt \\
&\geq \log\left(\frac{c}{c + n^{-1/n}}\right) + \frac{hT}{2n(c + n^{-1/n})}
\end{aligned}$$

Moreover,

$$\log\left(\frac{c}{c + n^{-1/n}}\right) + \frac{hT}{2n(c + n^{-1/n})} > 0 \Longleftrightarrow T > \frac{2n}{h}(c + n^{-1/n}) \log\left(\frac{c + n^{-1/n}}{c}\right),$$

and

$$\begin{aligned}
\lim_{c\to\infty}(c + n^{-1/n}) \log\left(\frac{c + n^{-1/n}}{c}\right) &= \lim_{c\to\infty} \frac{\log\left(\frac{c+n^{-1/n}}{c}\right)}{\frac{1}{c+n^{-1/n}}} = \lim_{c\to\infty} \frac{\frac{c}{c+n^{-1/n}} \cdot \left(\frac{-n^{-1/n}}{c^2}\right)}{\frac{-1}{(c+n^{-1/n})^2}} \\
&= \lim_{c\to\infty} \frac{n^{-1/n}(c + n^{-1/n})}{c} = n^{-1/n}.
\end{aligned}$$

Thus, if $T > 2n^{1-(1/n)}/h$, then there exists a $c > 0$ large enough such that the corresponding portfolio φ^c is a strong arbitrage relative to the market.

□

3.3 Concave Functions

We list some important results here which will be used in the later context.

Lemma 1. *If f is concave on $[A, B]$, and $a_1, \cdots, a_m \in [A, B]$, then*

$$\sum_{i=1}^m f(a_i) \leq m f\left(\frac{1}{m}\sum_{i=1}^m a_i\right). \tag{39}$$

Proof. Since f is concave on $[A, B]$, for any $\lambda \in (0, 1)$ and $a_1, a_2 \in [A, B]$,

$$(1 - \lambda)f(a_1) + \lambda f(a_2) \leq f\left((1 - \lambda)a_1 + \lambda a_2\right), \tag{40}$$

by induction, we have

$$\sum_{i=1}^{m} w_i f(a_i) \le f\left(\sum_{i=1}^{m} w_i a_i\right), \tag{41}$$

where $a_i \in [A,B]$ for all $i = 1, \cdots, n$ and $w = (w_1, \cdots, w_n) \in \Delta^n$.
Finally, replace w_i with $\frac{1}{m}$ for each i, it follows the result.

□

Lemma 2. *Suppose that f is concave on $[A,B]$, and $a_1, \cdots, a_m \in [A,B]$ are chosen such that $\sum_{i=1}^{m} a_i - (m-1)A \le B$. Then*

$$(m-1)f(A) + f\left(\sum_{i=1}^{m} a_i - (m-1)A\right) \le \sum_{i=1}^{m} f(a_i). \tag{42}$$

Proof. For any $a_1, \cdots, a_m \in [A,B]$ with $\sum_{i=1}^{m} a_i - (m-1)A \le B$, define

$$\lambda_j = \frac{a_j - A}{\sum_{i=1}^{m} a_i - mA} = \frac{a_j - A}{(a_1 - A) + \cdots + (a_m - A)}$$

It is straightforward to check that $0 \le \lambda_j \le 1$ and that

$$(1-\lambda_j)A + \lambda_j \left(\sum_{i=1}^{m} a_i - (m-1)A\right) = a_j \ .$$

Since $\sum_{i=1}^{m} a_i - (m-1)A \in [A,B]$ and f is concave, we have

$$(1-\lambda_j)f(A) + \lambda_j f\left(\sum_{i=1}^{m} a_i - (m-1)A\right) \le f\left((1-\lambda_j)A + \lambda_j\left(\sum_{i=1}^{m} a_i - (m-1)A\right)\right) = f(a_j).$$

Summing these inequalities form $j = 1$ to $j = m$ gives

$$(m-1)f(A) + f\left(\sum_{i=1}^{m} a_i - (m-1)A\right) \le \sum_{i=1}^{m} f(a_i).$$

□

Lemma 3. *Suppose that f is concave on $[0,1]$. For $x = (x_1, \cdots, x_n) \in \Delta^n$, we have*

$$(n-1)f(0) + f(1) \le (n-1)f\left(x_{(n)}\right) + f\left(1 - (n-1)x_{(n)}\right) \le \sum_{i=1}^{n} f(x_i), \tag{43}$$

and

$$\sum_{i=1}^{n} f(x_i) \le f\left(x_{(n)}\right) + (n-1)f\left(\frac{1 - x_{(n)}}{n-1}\right) \le nf\left(\frac{1}{n}\right). \tag{44}$$

where $x_{(n)} = \min\{x_1, \cdots, x_n\}$.

Proof. For $x = (x_1, \cdots, x_n) \in \Delta^n$, without loss of generality, we may assume $x_{(n)} = x_n$. By Lemma 1 with $A = 0$, $B = 1$, $m = n-1$ and $a_i = x_i$ for $i = 1, \cdots, m$, we have

$$\sum_{i=1}^{n} f(x_i) = f(x_n) + \sum_{i=1}^{n-1} f(x_i) \leq f(x_n) + (n-1) f\left(\frac{1}{n-1}\sum_{i=1}^{n-1} x_i\right) = f(x_n) + (n-1) f\left(\frac{1-x_n}{n-1}\right).$$

Reapplying Lemma 1, with $m = n$, $a_1 = \cdots = a_{n-1} = \frac{1-x_n}{n-1}$, and $a_n = x_n$, then we have

$$f(x_n) + (n-1) f\left(\frac{1}{n-1}\sum_{i=1}^{n-1} x_i\right) \leq n f\left(\frac{1}{n}\left[x_n + (n-1)\frac{1}{n-1}\sum_{i=1}^{n-1} x_i\right]\right) = n f\left(\frac{1}{n}\right).$$

Hence,

$$\sum_{i=1}^{n} f(x_i) \leq f(x_n) + (n-1) f\left(\frac{1-x_n}{n-1}\right) \leq n f\left(\frac{1}{n}\right). \tag{45}$$

On the other hand, by Lemma 2 with $m = n$, $A = x_n$, $B = 1$, and $a_i = x_i$, for $i = 1, \cdots, n$, we find

$$(n-1) f(x_n) + f(1 - (n-1)x_n) = (n-1) f(x_n) + f\left(\sum_{i=1}^{n} x_i - (n-1)x_n\right) \leq \sum_{i=1}^{n} f(x_i).$$

Reapplying Lemma 2 with $A = 0$, $B = 1$, $m = n$, $a_1 = \cdots = a_{n-1} = x_n$ and $a_n = 1 - (n-1)x_n$,then

$$(n-1) f(0) + f(1) \leq (n-1) f(x_n) + f(1 - (n-1)x_n).$$

Hence,

$$(n-1) f(0) + f(1) \leq (n-1) f(x_n) + f(1 - (n-1)x_n) \leq \sum_{i=1}^{n} f(x_i). \tag{46}$$

□

Remark 7. *For any $r \in (0, 1/n]$, obviously, $1 - (n-1)r \geq r$, then choose $x = (1 - (n-1)r, r, \cdots, r) \in \Delta^n$, $x_{(n)} = r$ and by Lemma 3, we have*

$$(n-1) f(0) + f(1) \leq (n-1) f(r) + f(1 - (n-1)r) \leq \sum_{i=1}^{n} f(x_i).$$

and

$$\sum_{i=1}^{n} f(x_i) \leq f(r) + (n-1) f\left(\frac{1-r}{n-1}\right) \leq n f\left(\frac{1}{n}\right).$$

4 Main Results

We say that a bounded portfolio π is a strong relative arbitrage opportunity over the time horizon $[0, T]$, if

$$\mathbb{P}(Z_{1,\pi}(T) > Z_{1,\mu}(T)) = 1.$$

Proposition 1. *There exists a strong relative arbitrage opportunity over any time horizon* $[0,T]$ *in any market model of the form (1) satisfying the following conditions :*

(A) There exists a positive differentiable function l *defined on* $[0,\frac{1}{n}]$*, such that*

$$\tau^{\mu}_{m(t)m(t)}(t) \geq l(\mu_{m(t)}(t)), \text{ for all } t \geq 0$$

almost surely, where $m(t)$ *denotes the index of the stock of minimum weights at time* t*, namely,* $\mu_{m(t)}(t) = \min\{\mu_1(t), \cdots, \mu_n(t)\}$*.*

(B) For this $l(\cdot)$*, there exists a family of* C^3 *functions* $F = \{f_\alpha\}_{\alpha \in I}$*, where* I *is some nonempty index set, such that*
(a) $f_\alpha(x) > 0, \forall x \in (0,1]$ *and* $f_\alpha(0) = 0$ *for any* $\alpha \in I$*.*
(b) $xf'_\alpha(x)$ *is bounded on* $(0,1)$ *for each* $\alpha \in I$*.*
(c) $f'_\alpha(x) \geq 0$ *for all* $x \in (0,1)$ *and* $\alpha \in I$*.*
(d) $f''_\alpha(x) < 0$ *for all* $x \in (0,1)$ *and* $\alpha \in I$*.*
(e) $\frac{d}{dx}[-f''_\alpha(x)x^2 l(x)] \leq 0$ *for all* $x \in (0,1/n)$ *and* $\alpha \in I$*.*
(f) For each $T > 0$*, there exist* $\beta \in I$*, such that*

$$\int_0^{\frac{1}{n}} \frac{f'_\beta(x)}{-f''_\beta(x)x^2 l(x)} dx \leq T.$$

Proof. Given any $T > 0$, we want to find a strong relative arbitrage to the market portfolio over the time horizon $[0,T]$, that is, find a bounded portfolio π, such that $\log\left(\frac{Z_\pi(T)}{Z_\mu(T)}\right) > 0$ almost surely.

Consider $S(x) = S(x_1, \cdots, x_n) = f(x_1) + \cdots + f(x_n)$ with $f \in F$. Since f satisfies *(B.a)* and *(B.b)* so that S satisfies the statements in section 3.2. Apply the master formula to this S, it follows that

$$\log\left(\frac{Z_\pi(T)}{Z_\mu(T)}\right) = \log S(\mu(T)) - \log S(\mu(0)) + \int_0^T \frac{1}{2S(\mu(s))} \sum_{i=1}^n -f''(\mu_i(s))\mu_i^2(s)\tau^{\mu}_{ii}(s)ds \quad (47)$$

The reason we consider $S(x) = f(x_1) + \cdots + f(x_n)$ is that with S being this form, we can have $D_{ij}S(\cdot) = 0$, for $i \neq j$. If $D_{ij}S(\cdot) \neq 0$ for some $i \neq j$, then we have to deal with the term $\tau^{\mu}_{ij}(t)$, for $i \neq j$ — which may be both positive and negative. It will make the whole estimation much harder.

Moreover, even though $\log S(\mu(T)) - \log S(\mu(0))$ may be negative, but by Remark 4 $\tau^{\mu}_{ii}(t) > 0$, for all $t > 0$, and we know $f'' < 0$ from *(B.d)*; these facts ensure that the last term of (47) is always positive, the question is that does this positive term large enough to offset the negative part.

From *(B.d)*, we also know that f is a convex function, with this observation, we can use the convexity to give estimations to $f(x_1) + \cdots + f(x_n)$, or $S(x)$. The upper bound is well-known, as for the lower bound is a important step to this proof. Combine (47) with upper bound (45) and lower bound (46), we have

$$\log\left(\frac{Z_\pi(T)}{Z_\mu(T)}\right) \geq \log S(\mu(T)) - \log S(\mu(0)) + \int_0^T \frac{1}{2S(\mu(s))}\left(-f''(y_s)\right)\tau^{\mu}_{m(s)m(s)}(s)y_s^2 ds$$

$$\geq \log\left[(n-1)f(y_T) + f(1-(n-1)y_T)\right] - \log\left(nf\left(\frac{1}{n}\right)\right) + \int_0^T \frac{-f''(y_s)\tau^{\mu}_{m(s)m(s)}(s)y_s^2}{2\left[f(y_s) + (n-1)f\left(\frac{1-y_s}{n-1}\right)\right]}ds,$$

where $y_s = \mu_{(n)}(s)$.

From *(A)*, we know that there exist a positive function l defined on $[0,\frac{1}{n}]$, such that

$$\tau^{\mu}_{m(t)m(t)}(t) \geq l(\mu_{m(t)}(t)) = l(y_t), \text{ for all } t \geq 0$$

almost surely.

It follows that

$$\log\left(\frac{Z_\pi(T)}{Z_\mu(T)}\right) \geq \log\left[(n-1)f(y_T) + f(1-(n-1)y_T)\right] - \log\left(nf\left(\frac{1}{n}\right)\right) + \int_0^T \frac{-f''(y_s)l(y_s)y_s^2}{2\left[f(y_s)+(n-1)f\left(\frac{1-y_s}{n-1}\right)\right]}ds$$

$$= S_1(y_T) - \log\left(nf\left(\frac{1}{n}\right)\right) + \int_0^T \Theta_1(y_s)ds, \tag{48}$$

where

$$\begin{aligned} S_1(x) &= \log\left[(n-1)f(x) + f(1-(n-1)x)\right] \quad , \\ \Theta_1(x) &= \frac{-f''(x)l(x)x^2}{2\left[f(x)+(n-1)f\left(\frac{1-x}{n-1}\right)\right]} \quad . \end{aligned}$$

Next, consider the question : given $T > t_0 > 0$, and f as above, does there exist a deterministic function $h(\cdot)$ such that, for all $t \in [t_0, T]$,

$$\begin{cases} S_1(h(t)) + \displaystyle\int_{t_0}^t \Theta_1(s)ds = \log\left(nf\left(\frac{1}{n}\right)\right) \quad , \\ h(t_0) = \dfrac{1}{n}. \end{cases} \tag{49}$$

If so, what is the connection between $h(t)$ and y_t ?

To answer these questions, we assume first that such a function $h(\cdot)$ exist, then, differentiate the first equation of (49) with respect to t, we find

$$0 = \frac{d}{dt}\left[S_1(h(t)) + \int_{t_0}^t \Theta_1(h(s))ds\right] = S_1'(h(t))h'(t) + \Theta_1(h(t)) \quad ,$$

Moreover, if the function $h(t)$ is monotone, and let $g(t)$ be the inverse function of $h(t)$, then by *Inverse Function Theorem*, we have

$$g'(h(t)) = \frac{1}{h'(t)} = -\frac{S_1'(h(t))}{\Theta_1(h(t))} \Rightarrow g'(x) = -\frac{S_1'(x)}{\Theta_1(x)} \quad , \tag{50}$$

Combine with the initial condition $h(t_0) = \frac{1}{n}$, or $g(\frac{1}{n}) = t_0$, we can derive

$$g(x) = t_0 + \int_{\frac{1}{n}}^x -\frac{S_1'(r)}{\Theta_1(r)}dr \quad . \tag{51}$$

Conversely, if $g(\cdot)$ is defined as above, and g is a continuous monotone function, then define $h = g^{-1}$, h will be a solution of (49). Therefore, here comes another question — when is g monotone ? Since $f''(x) < 0$, $f(x) > 0$ and $l(x) > 0$, for $x \in (0, 1/n]$, it is not hard to see that $\Theta_1(x) > 0$, and $S_1'(x) > 0$, for $x \in (0, 1/n]$, it follows that $g(x)$ is decreasing on [0,1/n]. It remains to check $g(0) < \infty$.

To see $g(0)$ is well defined, note that by choosing f properly, *(B.c)*, *(B.f)* and Remark 7 implies that

$$\begin{aligned} g(0) &:= t_0 + \int_{\frac{1}{n}}^{0} -\frac{S_1'(r)}{\Theta_1(r)} dr \\ &= t_0 + \int_0^{\frac{1}{n}} \frac{(n-1)f'(r) - (n-1)f'(1-(n-1)r)}{(n-1)f(r) + f(1-(n-1)r)} \frac{2\left(f(r) + (n-1)f\left(\frac{1-r}{n-1}\right)\right)}{-f''(r)l(r)r^2} dr \\ &\leq t_0 + \int_0^{\frac{1}{n}} \frac{(n-1)f'(r)}{f(1)} \frac{2nf\left(\frac{1}{n}\right)}{-f''(r)l(r)r^2} dr \\ &\leq t_0 + 2n(n-1) \int_0^{\frac{1}{n}} \frac{f'(r)}{-f''(r)l(r)r^2} dr < \infty. \end{aligned}$$

Hence, h is a continuous decreasing function defined on $[t_0, g(0)]$ with $h(t_0) = \frac{1}{n}$ and $h(g(0)) = 0$.

Now, given $t_0 = T/2$, by *(B.6)*, we may assume this f satisfies

$$\int_0^{\frac{1}{n}} \frac{f'(r)}{-f''(r)l(r)r^2} dr \leq \frac{T}{4n(n-1)} \quad .$$

This implies $g(0) \leq T$. Then, define a stopping time η as

$$\eta = \inf\{t \geq t_0 | y_t > h(t)\} \quad , \tag{52}$$

Note that $t_0 \leq \eta \leq g(0) \leq T$ a.s., the fact is easy to observe from the path behavior, and by the path continuity, we have $y_\tau = h(\tau)$.

With this stopping time η, define a corresponding portfolio $\tilde{\pi}(\cdot)$ by setting

$$\tilde{\pi}(t) = \begin{cases} \pi(t), & t < \eta, \\ \mu(t), & t \geq \eta. \end{cases} \tag{53}$$

$$\begin{aligned} (48) \Rightarrow \log\left(\frac{Z_{\tilde{\pi}}(T)}{Z_\mu(T)}\right) &= \log\left(\frac{Z_\pi(\eta)}{Z_\mu(\eta)}\right) \geq S_1(y_\eta) - \log\left(nf\left(\frac{1}{n}\right)\right) + \int_0^\eta \Theta_1(y_s) ds \\ &= S_1(h(\eta)) - \log\left(nf\left(\frac{1}{n}\right)\right) + \int_0^{t_0} \Theta_1(y_s) ds + \int_{t_0}^\eta \Theta_1(y_s) ds \end{aligned}$$

It remains to connect $h(t)$ with y_t, the key step is to choose f such that $\Theta_1(x)$ is decreasing on $[0, 1/n]$, if we can do this, then combine with (49), we have

$$\begin{aligned} \log\left(\frac{Z_{\tilde{\pi}}(T)}{Z_\mu(T)}\right) &= S_1(h(\eta)) - \log\left(nf\left(\frac{1}{n}\right)\right) + \int_0^{t_0} \Theta_1(y_s) ds + \int_{t_0}^\eta \Theta_1(y_s) ds \\ &\geq S_1(h(\eta)) - \log\left(nf\left(\frac{1}{n}\right)\right) + \int_0^{t_0} \Theta_1\left(\frac{1}{n}\right) ds + \int_{t_0}^\eta \Theta_1(h(s)) ds \\ &= t_0 \Theta_1\left(\frac{1}{n}\right) = \frac{T}{2} \Theta_1\left(\frac{1}{n}\right) > 0. \end{aligned}$$

Hence, for any $T > 0$, a strong relative arbitrage opportunity exists over the time horizon $[0, T]$.

To check that $\Theta_1(\cdot)$ is decreasing, observe that the denominator of $\Theta_1(\cdot)$ is increasing, and by *(B.e)*, the numerator of $\Theta_1(\cdot)$ is decreasing, then $\Theta_1(\cdot)$ is indeed decreasing.

□

Remark 8. *η defined in (52) is a stopping time since $y_t = \mu_{(n)}(t)$ is a continuous adapted process and $h(t)$ is a deterministic continuous function, moreover, $\mathbb{F}$ is a right-continuous filtration. With these facts, for each $t \geq 0$, we have*

$$\{\eta < t\} = \bigcup_{t_0 \leq s < t, s \in \mathbb{Q}} \{y_s > h(s)\} \in \mathcal{F}(t).$$

This implies that η is a stopping time. Furthermore, for the reason that η is a stopping time, $\tilde{\pi}$ defined in (53) is indeed a portfolio.

Theorem 1. *For any $T > 0$, a strong relative arbitrage opportunity exists over the time horizon $[0, T]$ in any market of the form (1) satisfying the condition*

$$\tau^{\mu}_{m(t)m(t)}(t) \geq \frac{C}{\mu^p_{m(t)}(t)}, \quad \textit{for all } t > 0 \tag{54}$$

almost surely, where $C > 0$ is a constant and p is a constant with $0 < p \leq 1$.

Proof. Consider $l(x) = Cx^{-p}$ and the family of functions $\{f_\alpha(\cdot)\}_{\alpha \geq 1}$ defined by the formula

$$f_\alpha(y) = \begin{cases} \dfrac{1}{p} \displaystyle\int_{-p \log y}^{\infty} e^{-r} r^\alpha dr, & if\ 0 < y \leq 1, \\ 0, \quad if\ y = 0. \end{cases}$$

Then we have

$$\tau^{\mu}_{m(t)m(t)}(t) \geq l(\mu_{m(t)}(t)), \text{ for all } t \geq 0$$

almost surely. Also,
(a) $f_\alpha(x) > 0$, for all $x \in (0, 1]$ and $f_\alpha(0) = 0$, for any $\alpha \geq 1$.
(b) $x f'_\alpha(x) = x^p(-p \log x)^\alpha$ for all $x \in (0, 1)$ and $\alpha \geq 1$
By *L'Hôpital's rule*

$$\lim_{x \to 0} x^p (-\log x)^\alpha = \lim_{x \to 0} \frac{(-\log x)^\alpha}{x^{-p}} = \lim_{x \to 0} \frac{\alpha(-\log x)^{\alpha - 1}}{p x^{-p}} = \cdots = 0$$

Hence $x f'_\alpha(x)$ is bounded on $(0, 1)$.
(c) $f'_\alpha(x) = x^{p-1}(-p \log x)^\alpha > 0$, for all $x \in (0, 1)$ and $\alpha \geq 1$. (d) For any $\alpha \geq 1$,

$$\begin{aligned} f''_\alpha(x) &= \frac{d}{dx}\left[x^{p-1}(-p \log x)^\alpha\right] \\ &= (p-1)x^{p-2}(-p \log x)^\alpha + x^{p-1}\alpha(-p \log x)^{\alpha-1}(-p x^{-1}) \\ &= (p-1)x^{p-2}(-p \log x)^\alpha - p x^{p-2}\alpha(-p \log x)^{\alpha-1} < 0, \quad \text{for all } x \in (0, 1). \end{aligned}$$

(e) For any $\alpha \geq 1$,

$$\begin{aligned} \frac{d}{dx}[-f''_\alpha(x) x^2 l(x)] &= \frac{d}{dx}[-f''_\alpha(x) C x^{2-p}] \\ &= \frac{d}{dx}\left[C(1-p)(-p \log x)^\alpha + C p \alpha (-p \log x)^{\alpha - 1}\right] \\ &= -C(1-p)p\alpha x^{-1}(-p \log x)^{\alpha-1} - C p^2 \alpha(\alpha - 1) x^{-1} (-p \log x)^{\alpha - 2} \leq 0, \end{aligned}$$

for all $x \in (0,1)$.
(f) Observe that

$$-f''_\alpha(x)x^2 l(x) = C(1-p)(-p\log x)^\alpha + Cp\alpha(-p\log x)^{\alpha-1} \geq Cp\alpha(-p\log x)^{\alpha-1}.$$

Also, $f'_\alpha(x) > 0$ for all $x \in (0,1)$ and $\alpha \geq 1$, these imply

$$\frac{f'_\alpha(x)}{-f''_\alpha(x)x^2 l(x)} \leq \frac{f'_\alpha(x)}{Cp\alpha(-p\log x)^{\alpha-1}} = \frac{x^{p-1}(-p\log x)^\alpha}{Cp\alpha(-p\log x)^{\alpha-1}} = \frac{1}{Cp\alpha}x^{p-1}(-p\log x),$$

for all $x \in (0,1)$.
Moreover,

$$\begin{aligned}
\int_0^{\frac{1}{n}} \frac{f'_\alpha(x)}{-f''_\alpha(x)x^2 l(x)}dx &\leq \int_0^{\frac{1}{n}} \frac{1}{Cp\alpha}x^{p-1}(-p\log x)dx \\
\textit{let } x^p &= y \\
&= \int_0^{n^{-p}} \frac{1}{Cp^2\alpha}(-\log y)dy \\
&= \frac{1}{Cp^2\alpha}(-y\log y + y)|_0^{n^{-p}} \\
&= \frac{1}{Cp^2\alpha}\frac{p\log n}{n^p}.
\end{aligned}$$

Hence, for each $T > 0$, choose

$$\beta = \left\lceil \frac{1}{Cp^2T}\frac{p\log n}{n^p} \right\rceil \geq 1,$$

then

$$\int_0^{\frac{1}{n}} \frac{f'_\beta(x)}{-f''_\beta(x)x^2 l(x)}dx \leq T.$$

□

Remark 9. *Propositions 3.1 and 3.8 of Fernholz and Karatzas (2005) state that strong relative arbitrage opportunities exist over long enough time horizons in any market satisfying the condition*

$$\Gamma(t) \leq \int_0^t \gamma^*_{\mu,p}(s)ds < \infty, \qquad a.s. \tag{55}$$

for some $p > 0$ and continuous, strictly increasing function $\Gamma : [0,\infty) \to [0,\infty)$ with $\Gamma(0) = 0$ and $\Gamma(\infty) = \infty$, where $\gamma^*_{\mu,p}(\cdot)$ is the generalized excess growth rate of the market and defined as

$$\gamma^*_{\mu,p}(t) = \frac{1}{2}\sum_{i=1}^n (\mu_i(t))^p \tau^\mu_{ii}(t).$$

For markets that satisfying (54) for some $p \in (0,1]$, we have

$$\gamma^*_{\mu,p}(t) = \frac{1}{2}\sum_{i=1}^n (\mu_i(t))^p \tau^\mu_{ii}(t) \geq \frac{1}{2}(\mu_{m(t)}(t))^p \tau^\mu_{m(t)m(t)}(t) \geq \frac{C}{2},$$

it follows that

$$\int_0^t \gamma^*_{\mu,p}(s)ds \geq \frac{Ct}{2}, \qquad a.s.$$

Hence, we have a long-term strong relative arbitrage.

Conversely, if we have the stronger condition

$$\gamma^*_{\mu,p}(t) \geq C \quad \forall\ t,\ a.s., \tag{56}$$

for some $p \in (0,1]$, then for $n = 2$, by (21) we have $\mu_1^2(t)\tau_{11}^{\mu}(t) = \mu_2^2(t)\tau_{22}^{\mu}(t)$. Without loss of generality, we may assume $\mu_2(t) \geq \mu_1(t)$, it follows that

$$\mu_1^p(t)\tau_{11}^{\mu}(t) \geq \frac{1}{2}\left[\mu_1^p(t) + \left(\frac{\mu_1(t)}{\mu_2(t)}\right)^{2-p}\mu_1^p(t)\right]\tau_{11}^{\mu}(t) = \frac{1}{2}\left(\mu_1^p(t)\tau_{11}^{\mu}(t) + \mu_2^p(t)\tau_{22}^{\mu}(t)\right) \geq C.$$

Therefore, the stronger condition (56) leads to short-term relative arbitrage for $n = 2$, but we are still unable to show that short-term relative arbitrage exist for $n \geq 3$ under condition (56).

Remark 10. *Our sufficient condition (54) is weaker than that in Banner and Fernholz (2008). In fact in the next section, we provide a market model which may not satisfy the sufficient condition in Banner and Fernholz (2008). However, it satisfies our sufficient condition (54).*

5 Generalized Volatility-Stabilized Market Model

We consider

$$d\log X_i(t) = \frac{\delta}{2\mu_i^p(t)}dt + \frac{1}{\mu_i^{p/2}(t)}dW_i(t), \quad i = 1, \cdots, n, \tag{57}$$

where $\delta \geq 0$ and $p \in (0,1]$ are both constants .
When $p = 1$, the theory developed by Bass and Perkins (2002) shows that the resulting system of stochastic differential equations determines the distribution of the Δ^n-valued diffusion process $(X_1(t), \cdots, X_n(t))$ uniquely.

For this model, we have $\alpha_{ij}(t) = \delta_{ij}\mu_i^{-p}(t)$, hence the relative variances $\tau_{ii}^{\mu}(\cdot)$ are given by

$$\begin{aligned}
\tau_{ii}^{\mu}(t) &= \alpha_{ii}(t) - \sum_{j=1}^{n}\alpha_{ij}(t)\mu_j(t) - \sum_{j=1}^{n}\alpha_{ij}(t)\mu_j(t) + \sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_{ij}(t)\mu_i(t)\mu_j(t) \\
&= \frac{1}{\mu_i^p(t)} - \frac{1}{\mu_i^p(t)}\mu_i(t) - \frac{1}{\mu_i^p(t)}\mu_i(t) + \sum_{j=1}^{n}\frac{1}{\mu_j^p(t)}\mu_j^2(t) \\
&\geq \frac{1}{\mu_i^p(t)}\left(1 - 2\mu_i(t) + \mu_i^2(t)\right) = \frac{1}{\mu_i^p(t)}\left(1 - \mu_i(t)\right)^2,
\end{aligned}$$

for all $i = 1, \cdots, n$ and for all $t > 0$. Hence, for $n \geq 2$, we obtain

$$\tau_{m(t)m(t)}^{\mu}(t) \geq \frac{1}{\mu_{m(t)}^p(t)}\left(1 - \mu_{m(t)}(t)\right)^2 \geq \left(\frac{n-1}{n}\right)^2\frac{1}{\mu_{m(t)}^p(t)}, \qquad \text{for all } t > 0.$$

By Theorem 1, we have the following :

Theorem 2. *Assume that $n \geq 2$, then the generalized volatility-stabilized market as (57) has a strong relative arbitrage opportunity over any time horizon $[0,T]$ for any $T > 0$.*

Remark 11. *For generalized volatility-stabilized market with $p \in (0,1)$, the following may not be true for any $C > 0$:*

$$\tau^{\mu}_{m(t)m(t)}(t) \geq \frac{C}{\mu_{m(t)}(t)}, \text{ for all } t \geq 0 \quad \text{almost surely.} \tag{58}$$

Indeed, we have

$$\begin{aligned} \mu_{m(t)}(t)\tau^{\mu}_{m(t)m(t)}(t) &= \mu^{1-p}_{m(t)}(t)\left(1 - 2\mu_{m(t)}(t)\right) + \mu_{m(t)}(t)\sum_{j=1}^{n}\frac{1}{\mu^{p}_{j}(t)}\mu^2_j(t) \\ &\leq \mu^{1-p}_{m(t)}(t) + n\mu_{m(t)}(t) \leq (n+1)\mu^{1-p}_{m(t)}(t), \end{aligned}$$

and if for any $C > 0$,

$$\mathbb{P}\left(\mu^{1-p}_{m(t)}(t) < \frac{C}{n+1} \text{ for some } t \geq 0\right) > 0, \tag{59}$$

then

$$\mathbb{P}\left(\tau^{\mu}_{m(t)m(t)}(t) \geq \frac{C}{\mu_{m(t)}(t)} \text{ for all } t \geq 0\right) < 1, \qquad \text{for any } C > 0.$$

Therefore, these model may not satisfy the sufficient condition given in (58), however, we give a weaker sufficient condition here to make sure short term arbitrage for any $p \in (0,1)$.

6 Some Related Results

6.1 Diffusion Models

1. In Fernholz and Karatzas(2010), the authors considered a diffusion model as the following:

$$dX_i(t) = X_i(t)\left(b_i(\mathfrak{X}(t))dt + \sum_{k=1}^{n} s_{ik}(\mathfrak{X}(t))dW_k(t)\right), \quad X_i(0) = x_i > 0, \quad i = 1, \cdots, n, \tag{60}$$

where $\mathfrak{X}(t) := (X_1(t), \cdots, X_n(t))$.
Let

$$a_{ij}(x) := \sum_{k=1}^{n} s_{ik}(x)s_{jk}(x) \ , \qquad \forall\ 1 \leq i,j \leq n \ ,$$

and define

$$U(T,x) := \inf\left\{w > 0 | \exists \pi(\cdot) \in \mathcal{H} \text{ s.t } Z_{wx,\pi}(T) \geq \frac{X(T)}{X(0)}x, \text{ a.s.}\right\}.$$

the smallest relative amount of initial capital x, starting with which one can match or exceed the market portfolio at time T. (Note that if $U(T,x) < 1$, then it means that we can have strong relative arbitrage (investment strategy) with respect to market portfolio μ at time T.) Then, under some appropriate conditions, they showed U satisfies

$$\begin{cases} \dfrac{\partial U}{\partial \tau}(\tau,x) = \dfrac{1}{2}\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j a_{ij}(x) D^2_{ij}U(\tau,x) + \sum_{i=1}^{n} x_i\left(\sum_{j=1}^{n}\frac{x_j a_{ij}(\mathbf{x})}{x_1 + \cdots + x_n}\right) D_i U(\tau,x) \ , \\ U(0+,x) = 1 \ , \qquad x \in (0,\infty)^n \ . \end{cases} \tag{61}$$

Moreover, under some further conditions, they proved that if

$$U(T,x) < 1 \ , \qquad \text{for some} \quad (T,x) \in (0,\infty) \times (0,\infty)^n,$$

then

$$U(T,x) < 1 \ , \qquad \forall \ (T,x) \in (0,\infty) \times (0,\infty)^n \ .$$

This shows the existence of long-term relative arbitrage implies the existence of short-term arbitrage. It is worth noting that such an arbitrage opportunity may be made by an investment strategy rather than a bounded portfolio.

6.2 Arbitrage and Diversity

In Fernholz and Karatzas (2009), they claimed that in weakly diverse markets, i.e.

$$\frac{1}{T}\int_0^T \max_{1\le i\le n} \mu_i(t)dt \le 1-\delta \qquad \text{a.s. for some } \delta \in (0,1),$$

which satisfy the strict non-degeneracy condition, that is

$$x'\alpha(t)x \ge \epsilon\|x\|^2 \text{ for all } t \in [0,\infty) \text{ and } x \in \mathbb{R}^n \text{ for some } \epsilon > 0,$$

one can construct simple long-only portfolios $\mu^{(p)}(\cdot)$, for some fixed $p \in (0,1)$, which lead to strong arbitrage relative to the market portfolio over $[0,T]$, where

$$\mu_i^{(p)}(t) := \frac{(\mu_i(t))^p}{\sum_{j=1}^n (\mu_j(t))^p}, \quad \forall \ i = 1, \cdots, n \quad \text{and} \quad T \ge \frac{2}{p\epsilon\delta}\log n.$$

Furthermore, under these same conditions, they can even construct long-only portfolios $\eta(\cdot)$, which achieve strong relative arbitrage to the market over arbitrarily short time horizon, where

$$\eta_i(t) := \frac{1}{\frac{qV^\mu(t)}{(\mu_1(0))^q} - V^{\widehat{\pi}}(t)}\left(\frac{qV^\mu(t)}{(\mu_1(0))^q} - \widehat{\pi}(t)V^{\widehat{\pi}}(t)\right)$$

and

$$\widehat{\pi}(t) := qe_1 + (1-q)\mu(t), \quad 0 \le t \le \infty \quad \text{with} \quad q > 1 + \frac{2}{\epsilon\delta^2 T}\log\left(\frac{1}{\mu_1(0)}\right).$$

Note that these portfolios may be unbounded, which makes such portfolios more difficult for implementation in reality.

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