Chapter One Introduction

Given a partial differential equation, if the boundary is allowed to move with time, then we call the problem a Stefan problem. The classical Stefan problem has been modeled to describe the phenomena of melting ice [1]. In this thesis, we get into a Stefan-type problem which describes a liquid contained in a one-dimensional horizontal pipe with a freely moving piston (see [2], [3], [4], [5], [6], [7], [8]). Hence, we can consider the problem as the heat equation with a particular free boundary condition.

Here, we investigate the discontinuous Galerkin approximation of the Stefan-type problem. For that purpose, we apply a coordinate transformation to transform the problem in a fixed domain [9]. Then, we use the semidiscrete scheme and the fully discrete scheme for the problem. We perform the error analysis of the semidiscrete scheme and the fully discrete schemes using the symmetric interior penalty Galerkin (SIPG) method and the optimal orders of convergence in L^2-norm are given. Finally, we test two model problems [2], [3] by using the three types of fully discontinuous Galerkin methods (Nonsymmetric Interior Penalty Galerkin (NIPG) method, Incomplete Interior Penalty Galerkin (IIPG) method and SIPG method). Numerical results given by three types of methods all match our theoretical results. However, the analysis of error estimates for NIPG and IIPG methods will be studied elsewhere.

The rest of the thesis are the followings: in Chapter 2, we introduce the formulation of the Stefan-type problem based on a Landau-type transformation. In Chapter 3, we define the discontinuous Galerkin approximation for the Stefan-type problem. In Chapter 4, the semidiscrete case and a fully discrete discontinuous Galerkin method are given. In Chapter 5, the theoretical error estimates and the numerical experiments are put together to confirm our approach. Finally, the summary and some remarks are given in Chapter 6.

Chapter Two Stefan Problem

In this chapter, we consider the following Stefan-type problem: to find a pair of solutions

$\{(U,S): U = U(x',t'), S = S(t')\}$ satisfying

$$\frac{\partial U}{\partial t'} = \frac{\partial^2 U}{\partial x'^2} + f(x',t'),\ \ 0 < x' < S(t'),\ \ 0 < t' < T, \tag{1}$$

with the following initial conditions

$$\begin{cases} U(x',0) = g(x'), & x' \in (0, S(0)), \\ S(0) = 1, & \dfrac{dS}{dt'}(0) = S_0, \end{cases} \tag{2}$$

and with the following boundary conditions

$$\begin{cases} U(0,t') = 0, & t' \ge 0, \\ \dfrac{\partial U}{\partial x'}(S(t'),t') = F(U(S(t'),t')), & t' \ge 0, \\ U(S(t'),t') = \dfrac{d^2 S}{dt'^2} + H(S), & t' \ge 0, \end{cases} \tag{3}$$

where S_0 is a given constant and the functions f, g, F and H are assumed to be smooth enough to be satisfied by our subsequent error analysis.

Here we use a Landau-type coordinate transformation [9] to fix the free boundary and set

$$x = \frac{x'}{S(t')},\ \ t = t'. \tag{4}$$

Then, the problem (1)-(3) is transformed into the following one: to find a pair of solutions

$\{(u,s): u = U(xS(t),t), s = S(t)\}$ satisfying

$$s^2 \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - s\frac{ds}{dt} x \frac{\partial u}{\partial x} = s^2 f(x,t),\ \ x \in I = (0,1),\ \ 0 < t < T, \tag{5}$$

with the following initial conditions

$$\begin{cases} u(x,0) = g(x), & x \in I, \\ s(0) = 1, & \dfrac{ds}{dt}(0) = S_0, \end{cases} \tag{6}$$

and with the following boundary conditions

$$\begin{cases} u(0,t) = 0, & t \geq 0, \\ \dfrac{\partial u}{\partial x}(1,t) = sF(u(1,t)), & t \geq 0, \\ \dfrac{d^2 s}{dt^2} = u(1,t) - H(s), & t \geq 0. \end{cases} \tag{7}$$

For computation, we rewrite (5) in the conservative form as follows:

$$s^2 \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - s\frac{ds}{dt}\frac{\partial}{\partial x}(xu) + s\frac{ds}{dt}u = s^2 f(x,t). \tag{8}$$

In the thesis, c is a generic constant and is different at different occurrences.

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Chapter Three Discontinuous Galerkin Method

To solve the problem (6)-(8), we define the discontinuous Galerkin method. For that purpose, let $\varepsilon_h = \{x_j\}_{j=0}^{J}$ be a partition of the interval I such that

$$0 = x_0 < x_1 < x_2 < \cdots < x_J = 1 .$$

Let $h_j = x_{j+1} - x_j$, $h = \max_j h_j$ and $I_j = [x_j, x_{j+1}]$ for $j = 0,..., J-1$.

Consider $P_r(I_j)$, the space of polynomials of degree less than or equal to r on each interval $I_j \in \varepsilon_h$. Then the discontinuous finite element space can be defined as

$$V_h = \{v_h \in L^2(I) \mid \ v_h|_{I_j} \in P_r(I_j), j = 0,1,..., J-1\} .$$

For the discontinuous setting, we define the jump and average for $v \in H^1(\varepsilon_h)$ as follows:

$$[v(x_j)] = v(x_j^-) - v(x_j^+), \ \ \{v(x_j)\} = \frac{v(x_j^-) + v(x_j^+)}{2}, \quad j = 1,..., J-1 .$$

As for $j = 0$ and $j = J$, we define the jump and average as follows:

$$[v(x_0)] = -v(x_0^+), \ \ \{v(x_0)\} = v(x_0^+), \ \ [v(x_J)] = v(x_J^-), \ \ \{v(x_J)\} = v(x_J^-),$$

where $v(x_j^+) = \lim_{x \to x_j^+} v(x)$, $v(x_j^-) = \lim_{x \to x_j^-} v(x)$.

Now, we introduce the bilinear form $J_0^{\sigma_0}(v, w)$ which is to penalize the jump of the function by setting

$$J_0^{\sigma_0}(v, w) = \sum_{j=0}^{J-1} \frac{\sigma_0}{h_j} [v(x_j)][w(x_j)],$$

where the parameter σ_0 is a nonnegative real number.

Due to the technical reasons, we did not introduce the penalized term at the boundary $x = 1$, which is different from the usual discontinuous Galerkin method.

Also, we define the discontinuous Galerkin form $A_\varepsilon(v,w)$ to construct a weak formulation for the model problem by setting

$$A_\varepsilon(v,w)=\sum_{j=0}^{J-1}\int_{x_j}^{x_{j+1}}\frac{\partial v}{\partial x}\frac{\partial w}{\partial x}dx-\sum_{j=0}^{J-1}\{\frac{\partial v}{\partial x}(x_j)\}[w(x_j)]+\varepsilon\sum_{j=0}^{J-1}\{\frac{\partial w}{\partial x}(x_j)\}[v(x_j)]+J_0^{\sigma_0}(v,w).$$

Here, the parameter ε in A_ε may take on real values. Notice that for a fixed number σ_0, the bilinear form A_ε is symmetric if $\varepsilon=-1$ and nonsymmetric otherwise. With different values of the parameter ε, the discontinuous Galerkin method is referred to SIPG ($\varepsilon=-1$), NIPG ($\varepsilon=1$) or IIPG ($\varepsilon=0$). In this thesis, we carry out the analysis for the case of SIPG. However, we perform numerical experiments using three types of the discontinuous Galerkin methods in Chapter 5.

Meanwhile, we introduce the bilinear form $B(v,w)$ with the convective term approximated by using a numerical flux $Q(x_j,v(x_j^-),v(x_j^+))$ which is applied from the finite volume method by setting

$$B(v,w)=-\sum_{j=0}^{J-1}\int_{x_j}^{x_{j+1}}xv\frac{\partial w}{\partial x}dx+\sum_{j=0}^{J-1}Q(x_j,v(x_j^-),v(x_j^+))[w(x_j)].$$

Here, we shall assume that the numerical flux Q satisfies the Lipschitz continuity, that is,

$$\left|Q(x,v,w)-Q(x,v^*,w^*)\right|\le c(\left|v-v^*\right|+\left|w-w^*\right|),\text{ for } x\in I,\ \ v,w,v^*,w^*\in\mathbb{R},$$

and the consistency with convective fluxes, namely,

$$Q(x,v,v)=xv \quad\text{for}\quad x\in I\text{, and }\ v\in\mathbb{R}.$$

Note that when $j=0$ or J, we define the numerical flux Q as

$$Q(x_0,v(x_0^-),v(x_0^+))=x_0v(x_0^+)\,,\quad Q(x_J,v(x_J^-),v(x_J^+))=x_Jv(x_J^-)\,.$$

Then, we can write down the weak formulation for the problem as follows: to find

$u(t) \in H^1(\varepsilon_h)$ satisfying

$$s^2(\frac{\partial u(t)}{\partial t}, v) + A_\varepsilon(u(t), v) - s\frac{ds}{dt}B(u(t), v) + sF(u(1))v(1)$$

$$-s\frac{ds}{dt}u(1)v(1) + s\frac{ds}{dt}b(u(t), v) = s^2\ell(v) \quad \text{for} \quad v \in H^1(\varepsilon_h) \tag{9}$$

where

$$b(u(t), v) := \sum_{j=0}^{J-1}\int_{x_j}^{x_{j+1}} uv\ dx,$$

and

$$\ell(v) := \sum_{j=0}^{J-1}\int_{x_j}^{x_{j+1}} f(x,t)v\ dx.$$

Chapter Four Numerical Schemes

Since $V_h \subset H^1(\varepsilon_h)$, we define the semidiscrete approximation solution $u_h(t) \in V_h$ and s_h satisfying the identities:

$$s_h^2(\frac{\partial u_h(t)}{\partial t}, v_h) + A_\varepsilon(u_h(t), v_h) - s_h \frac{ds_h}{dt} B(u_h(t), v_h) + s_h F(u_h(1,t)) v_h(1)$$

$$-s_n \frac{ds_h}{dt} u_h(1) v_h(1) + s_h \frac{ds_h}{dt} b(u_h(t), v_h) = s_h^2 \ell(v_h) \quad \text{for} \quad v_h \in V_h, \quad t \in [0,T], \tag{10}$$

and

$$\frac{d^2 s_h}{dt^2} = u_h(1) + H(s_h), \quad t \in [0,T], \tag{11}$$

with

$$\begin{cases} u_h(0) = \Pi g, \\ s_h(0) = 1, \qquad \dfrac{ds_h}{dt}(0) = S_0. \end{cases} \tag{12}$$

where Π is some projection which will be defined later.

Later on, as for the fully discrete discontinuous Galerkin approximation to the problem (6)-(8), we use the modified backward Euler method in time. Let $\Delta t = T / N$ be the size of time step where N is the number of time step and also set $t_n = n\Delta t$ for $n = 0,1,...,N$. We define the approximation of u at $t = t_n$ as u_h^n and the approximation of s at $t = t_n$ as s_h^n.

Now, given (u_h^n, s_h^n), where $u_h^n \in V_h$, we are looking for (u_h^{n+1}, s_h^{n+1}) where $u_h^{n+1} \in V_h$, the solutions of

$$(s_h^{n+1})^2(\frac{u_h^{n+1}-u_h^n}{\Delta t},v_h)+A_\varepsilon(u_h^{n+1}(t),v_h)-s_h^{n+1}\frac{s_h^{n+1}-s_h^n}{\Delta t}B(u_h^{n+1},v_h)$$

$$-s_h^{n+1}\frac{s_h^{n+1}-s_h^n}{\Delta t}u_h^{n+1}(1)v_h(1)+s_h^{n+1}F(u_h^n(1))v_h(1)+s_h^{n+1}\frac{s_h^{n+1}-s_h^n}{\Delta t}b(u_h^{n+1},v_h)=(s_h^{n+1})^2\ell(v_h), \qquad (27)$$

for $v_h \in V_h$, $n \geq 0$ with

$$u_h^0 = Q_h g, \qquad (28)$$

and

$$\frac{s_h^{n+1}-2s_h^n+s_h^{n-1}}{(\Delta t)^2}=u_h^n(1)-H(s_h^n), \quad n \geq 1, \qquad (29)$$

with

$$s_h^0 = 1, \quad \frac{s^1-s^0}{\Delta t}=S_0. \qquad (30)$$

Note that we define Q_h as the elliptic projection such that $Q_h u \equiv \tilde{u}$.

Chapter Five Theoretical and Numerical Results

We denote errors for (u,s) at t_n by $e_h^n = u(t_n) - u_h^n$ and $e_s^n = s(t_n) - s_h^n$. Similarly, for the semidiscrete scheme, we set $\eta_h^n = \tilde{u}(t_n) - u(t_n)$ and $\xi_h^n = u_h^n - \tilde{u}(t_n)$. Then we have the following error estimates for ξ_h^n and e_s^n:

Theorem 1

There exists a constant c *such that for* $M \ge 0$

$$\left|\xi_h^{M+1}\right|_{L^2(I)}^2 + \left|\frac{e_s^{M+1} - e_s^M}{\Delta t}\right|^2 + \Delta t \sum_{n=0}^{M} \left|\xi_h^{n+1}\right|_{\varepsilon}^2 \le c\left((\Delta t)^2 + h^{2(r+1)}\right), \quad r \ge 1.$$

Now, as in the semidiscrete case, we define the discrete-time discontinuous Galerkin approximation for the pair of solutions $\{U,S\}$ as

$$\begin{cases} U_h^{n+1} = U_h(x', t_{n+1}) = u_h(x, t_{n+1}), \\ S_h^{n+1} = S_h(t_{n+1}) = s_h^{n+1}, \end{cases}$$

where $x' = x s_h^{n+1}$. Then, the error estimates for $U(t_{n+1}) - U_h^{n+1}$ and $S(t_{n+1}) - S_h^{n+1}$ can be stated as follows:

Theorem 2

There exists a constant c *and* $r \ge 1$, $n \ge 0$ *such that*

$$\left|U(t_{n+1}) - U_h^{n+1}\right|_{L^2(I(t_{n+1}))} + \left|\frac{\left(S(t_{n+1}) - S_h^{n+1}\right) - \left(S(t_n) - S_h^n\right)}{\Delta t}\right| \le c\left(\Delta t + h^{r+1}\right).$$

For the proof of Theorems 1 and 2, we refer to see [10].

Here, we illustrate the rates of convergence of our modified backward Euler method developed in Chapter 4 with $\varepsilon = -1$, 0 and 1. In numerical experiments, we consider the uniform meshes in space and denote h the size of the meshes. Two different degrees ($r=1$ or 2) of the approximate functions in V_h are used. Here, the corresponding basis is chosen to be $\{\phi_{j,i}\}$, $j=0,\dots,J-1$ and $i=1,\dots,r+1$. For $r=1$, $\phi_{j,i}$ is of the form

$$\phi_{0,1}(x)=0\,,\quad \phi_{0,2}(x)=x\,,$$

and

$$\phi_{j,i}=\begin{cases}\dfrac{1}{h}\big(x-(j-1)h\big), & i=1,\ j=1,\dots,J-1,\\[2mm] -\dfrac{1}{h}\big(x-jh\big), & i=2,\ j=1,\dots,J-1.\end{cases}$$

As for $r=2$, $\phi_{j,i}$ is of the form

$$\phi_{0,1}(x)=0\,,\quad \phi_{0,2}(x)=\frac{(x-x_0)(x-x_1)}{\left(\dfrac{x_0+x_1}{2}-x_0\right)\left(\dfrac{x_0+x_1}{2}-x_1\right)}\,,$$

$$\phi_{0,3}(x)=\frac{(x-x_0)\left(x-\dfrac{x_0+x_1}{2}\right)}{(x_1-x_0)\left(x_1-\dfrac{x_0+x_1}{2}\right)}\,,$$

and

$$\phi_{j,i}(x)=\begin{cases}\dfrac{\left(x-\dfrac{x_j+x_{j+1}}{2}\right)(x-x_{j+1})}{\left(x_j-\dfrac{x_j+x_{j+1}}{2}\right)(x_j-x_{j+1})}, & i=1,\ j=2,\dots,J-1,\\[6mm] \dfrac{(x-x_j)(x-x_{j+1})}{\left(\dfrac{x_j+x_{j+1}}{2}-x_j\right)\left(\dfrac{x_j+x_{j+1}}{2}-x_{j+1}\right)}, & i=2,\ j=2,\dots,J-1,\\[6mm] \dfrac{(x-x_j)\left(x-\dfrac{x_j+x_{j+1}}{2}\right)}{(x_{j+1}-x_j)\left(x_{j+1}-\dfrac{x_j+x_{j+1}}{2}\right)}, & i=3,\ j=2,\dots,J-1.\end{cases}$$

For numerical fluxes $Q(x_j, v(x_j^-), v(x_j^+))$, we use the following one

$$Q(x_j, v(x_j^-), v(x_j^+)) = \begin{cases} x_j v(x_j^-) & \text{if } -s_h(t^{n+1})\dfrac{s_h(t^{n+1}) - s_h(t^n)}{\Delta t} > 0, \\ x_j v(x_j^+) & \text{if } -s_h(t^{n+1})\dfrac{s_h(t^{n+1}) - s_h(t^n)}{\Delta t} \le 0. \end{cases}$$

Note that the numerical fluxes depend on the sign of $-s\frac{d}{dt}s$. We approximate $-s\frac{d}{dt}s$ by the term $-s_h(t^{n+1})\frac{s_h(t^{n+1}) - s_h(t^n)}{\Delta t}$. Moreover, the defined numerical fluxes Q satisfy the Lipschitz continuity and the consistency.

The choice of the time step Δt is $(\Delta x)^{r+1}$ to verify the theoretical results. The parameter σ_0 is also chosen suitably to stabilize the numerical schemes. Now, two model problems (Problem 1 and Problem 2) [3] are considered respectively.

Problem 1: We choose the exact solutions $\{U(x', t') = e^{t'-x'+1} - e^{t'+1}, S(t') = t' + 1\}$ with setting the data $g(x') = e^{-x'+1} - e^1$, $H(x') = 1 - e^{x'}$, $F(U) = -1$ and $S_0 = 1$. The corresponding right-hand side $f(x', t')$ is $e^{t'+1}$.

Tables 1-12 show the errors in the solutions and free boundaries for different degree of the approximation r $(= 1$ or $2)$ and different values of ε indicating SIPG ($\varepsilon = -1$), IIPG ($\varepsilon = 0$) and NIPG ($\varepsilon = 1$) methods, respectively. Here, we compute the errors between the exact solutions and approximate solutions defined in the fixed domain. The order of convergence is calculated using the following formula:

$$O(h) = \log_2\left(\frac{\|u - u_{h_1}\|_{L^p(I)}}{\|u - u_{h_2}\|_{L^p(I)}}\right), \quad p = 2, \infty .$$

In addition, the graphs of errors in L^2- and L^∞-norms against the mesh size h for the solutions of Problem 1 with different degrees and methods are shown in Figures 1-12. As for the free boundary, the graphs of errors in L^2- and L^∞-norms against the mesh size h with different degrees and methods are shown in Figures 13-18.

Problem 2: We choose the exact solutions $\{U(x',t') = e^{x'+t'-1} - e^{t'-1}, S(t') = 1 - t'\}$ with setting the data $g(x') = e^{x'-1} - e^{-1}$, $H(x') = 1 - e^{-x'}$, $F(U) = 1$ and $S_0 = -1$. The corresponding right-hand side $f(x',t')$ is $e^{t'-1}$. Note that Problem 2 becomes a singular one as the time t approaches to the value 1. Therefore, the numerical simulations are chosen just before the time $t = 1$.

As we did in Problem 1, Tables 13-24 show the errors in the solutions and free boundaries for Problem 2 with different degrees and methods. Also, the graphs of errors in L^2- and L^∞-norms against the mesh size h for the solutions and the free boundaries of Problem 2 with different degrees and methods are presented in Figures 19-37.

6. Summary and Concluding Remarks

For both Problem 1 and Problem 2, we observe that the orders of convergence for the L^2- and L^∞-norm errors in the solutions and free boundaries are approximately 2 if piecewise linear polynomials are used and 3 if piecewise quadratic polynomials are used, which confirm our theoretical results derived in Chpater 4. However, we carry out our L^2-norm analysis in the case of using SIPG methods. For L^∞-norm analysis and two other approaches (IIPG and NIPG) which give us similar numerical results as well, these analyses will be studied somewhere else.

Compared with numerical results which were solved by continuous Galerkin finite element methods in [2], we observe that thanks to more basis applied, discontinuous Galerkin methods give us more accurate solutions when using the same degree of polynomial approximation. However, the optimal orders of convergence for both methods are the same.

For the future work, we will extend this work to deal with higher dimension. Stefan problems are more complicated and close to the problems in the real world applications.

Table 1 The order of convergence for the errors in the solutions for Problem 1 with $r=1$ for $\sigma_0=3$ and $\varepsilon=-1$.

Time	h	L^2	O(h)	L^∞	O(h)
t = 0.125	1/4	9.810386E-03	1.965093E+00	2.201044E-02	1.774466E+00
	1/8	2.512662E-03	1.988602E+00	6.433706E-03	1.880291E+00
	1/16	6.331480E-04	1.996687E+00	1.747581E-03	1.939073E+00
	1/32	1.586509E-04		4.557410E-04	
t = 0.25	1/4	1.674047E-02	1.964284E+00	3.249520E-02	1.909893E+00
	1/8	4.290020E-03	1.988603E+00	8.647373E-03	1.850595E+00
	1/16	1.081011E-03	1.996773E+00	2.397725E-03	1.924999E+00
	1/32	2.708580E-04		6.314180E-04	
t = 0.375	1/4	2.598312E-02	1.973342E+00	4.964502E-02	2.027333E+00
	1/8	6.616922E-03	1.991152E+00	1.217832E-02	1.905478E+00
	1/16	1.664407E-03	1.997456E+00	3.250735E-03	1.916840E+00
	1/32	4.168361E-04		8.609047E-04	
t = 0.50	1/4	3.636488E-02	1.977693E+00	6.783391E-02	2.032588E+00
	1/8	9.232877E-03	1.992292E+00	1.657971E-02	1.928732E+00
	1/16	2.320585E-03	1.997738E+00	4.354826E-03	1.912068E+00
	1/32	5.810568E-04		1.157127E-03	
t = 0.625	1/4	4.758006E-02	1.977434E+00	8.685748E-02	2.035785E+00
	1/8	1.208253E-02	1.992124E+00	2.118239E-02	1.875999E+00
	1/16	3.037168E-03	1.997661E+00	5.770892E-03	1.909541E+00
	1/32	7.605238E-04		1.536080E-03	
t = 0.75	1/4	5.937505E-02	1.972853E+00	1.061515E-01	1.990738E+00
	1/8	1.512572E-02	1.990764E+00	2.670880E-02	1.817961E+00
	1/16	3.805717E-03	1.997260E+00	7.575192E-03	1.908739E+00
	1/32	9.532377E-04		2.017464E-03	
t = 0.875	1/4	7.144345E-02	1.963728E+00	1.246542E-01	1.839859E+00
	1/8	1.831562E-02	1.988180E+00	3.482202E-02	1.819779E+00
	1/16	4.616575E-03	1.996526E+00	9.863836E-03	1.909495E+00
	1/32	1.156926E-03		2.625613E-03	
t = 1.00	1/4	8.354916E-02	1.949789E+00	1.430040E-01	1.661398E+00
	1/8	2.162705E-02	1.984337E+00	4.520822E-02	1.824996E+00
	1/16	5.465780E-03	1.995451E+00	1.275965E-02	1.911887E+00
	1/32	1.370761E-03		3.390810E-03	

Table 2 The order of convergence for the errors in the solutions for Problem 1 with $r=1$ for $\sigma_0=3$ and $\varepsilon=1$.

Time	h	L^2	O(h)	L^∞	O(h)
t = 0.125	1/4	9.914113E-03	1.951722E+00	2.301094E-02	1.808966E+00
	1/8	2.562872E-03	1.966497E+00	6.567218E-03	1.895929E+00
	1/16	6.557713E-04	1.980217E+00	1.764615E-03	1.946181E+00
	1/32	1.662063E-04		4.579217E-04	
t = 0.25	1/4	1.306867E-02	1.985050E+00	2.941344E-02	1.741807E+00
	1/8	3.301199E-03	1.987998E+00	8.794473E-03	1.863770E+00
	1/16	8.321943E-04	1.992398E+00	2.416345E-03	1.930819E+00
	1/32	2.091478E-04		6.337595E-04	
t = 0.375	1/4	1.997807E-02	1.999365E+00	3.822957E-02	1.701883E+00
	1/8	4.996716E-03	1.998136E+00	1.175119E-02	1.845055E+00
	1/16	1.250794E-03	1.998485E+00	3.270881E-03	1.921618E+00
	1/32	3.130272E-04		8.633765E-04	
t = 0.50	1/4	2.850906E-02	2.002233E+00	5.016254E-02	1.685666E+00
	1/8	7.116243E-03	1.999896E+00	1.559349E-02	1.833518E+00
	1/16	1.779189E-03	1.999499E+00	4.375216E-03	1.915822E+00
	1/32	4.449517E-04		1.159523E-03	
t = 0.625	1/4	3.810652E-02	1.999392E+00	6.571815E-02	1.678383E+00
	1/8	9.530647E-03	1.998221E+00	2.053249E-02	1.826641E+00
	1/16	2.385601E-03	1.998547E+00	5.788525E-03	1.912191E+00
	1/32	5.970012E-04		1.537947E-03	
t = 0.75	1/4	4.859254E-02	1.992056E+00	8.479122E-02	1.659759E+00
	1/8	1.221522E-02	1.994319E+00	2.683574E-02	1.823080E+00
	1/16	3.065852E-03	1.996432E+00	7.584236E-03	1.910122E+00
	1/32	7.683608E-04		2.017937E-03	
t = 0.875	1/4	5.987974E-02	1.980217E+00	1.101610E-01	1.660614E+00
	1/8	1.517662E-02	1.988386E+00	3.484443E-02	1.822200E+00
	1/16	3.824824E-03	1.993311E+00	9.853633E-03	1.909365E+00
	1/32	9.606494E-04		2.623133E-03	
t = 1.00	1/4	7.200886E-02	1.963782E+00	1.429388E-01	1.667178E+00
	1/8	1.845987E-02	1.980458E+00	4.500695E-02	1.824042E+00
	1/16	4.677905E-03	1.989228E+00	1.271125E-02	1.909894E+00
	1/32	1.178241E-03		3.382619E-03	

Table 3 The order of convergence for the errors in the solutions for Problem 1 with $r=1$ for $\sigma_0=3$ and $\varepsilon=0$.

Time	h	L^2	O(h)	L^∞	O(h)
t = 0.125	1/4	9.510817E-03	1.966667E+00	2.261961E-02	1.795877E+00
	1/8	2.433281E-03	1.977756E+00	6.514368E-03	1.889819E+00
	1/16	6.177723E-04	1.986958E+00	1.757843E-03	1.943398E+00
	1/32	1.558456E-04		4.570451E-04	
t = 0.25	1/4	1.412342E-02	1.989579E+00	2.896180E-02	1.729379E+00
	1/8	3.556453E-03	1.995003E+00	8.734348E-03	1.858556E+00
	1/16	8.921981E-04	1.997538E+00	2.408515E-03	1.928433E+00
	1/32	2.234305E-04		6.327515E-04	
t = 0.375	1/4	2.196422E-02	1.998779E+00	4.044871E-02	1.791919E+00
	1/8	5.495704E-03	2.000492E+00	1.168109E-02	1.840435E+00
	1/16	1.373458E-03	2.000683E+00	3.261799E-03	1.919548E+00
	1/32	3.432019E-04		8.622153E-04	
t = 0.50	1/4	3.119178E-02	2.000741E+00	5.655420E-02	1.866041E+00
	1/8	7.793941E-03	2.001071E+00	1.551424E-02	1.829525E+00
	1/16	1.947040E-03	2.000874E+00	4.365044E-03	1.914067E+00
	1/32	4.864651E-04		1.158236E-03	
t = 0.625	1/4	4.137553E-02	1.998095E+00	7.335573E-02	1.842911E+00
	1/8	1.035755E-02	1.999442E+00	2.044855E-02	1.823375E+00
	1/16	2.590389E-03	1.999930E+00	5.777928E-03	1.910781E+00
	1/32	6.476285E-04		1.536631E-03	
t = 0.75	1/4	5.232210E-02	1.991254E+00	9.043519E-02	1.757032E+00
	1/8	1.316006E-02	1.996090E+00	2.675582E-02	1.820584E+00
	1/16	3.298944E-03	1.998183E+00	7.574743E-03	1.909119E+00
	1/32	8.257751E-04		2.016814E-03	
t = 0.875	1/4	6.386943E-02	1.979914E+00	1.095916E-01	1.655485E+00
	1/8	1.619122E-02	1.990955E+00	3.478777E-02	1.820635E+00
	1/16	4.073263E-03	1.995623E+00	9.848293E-03	1.908856E+00
	1/32	1.021410E-03		2.622635E-03	
t = 1.00	1/4	7.596027E-02	1.963750E+00	1.426922E-01	1.664584E+00
	1/8	1.947326E-02	1.983908E+00	4.501016E-02	1.823630E+00
	1/16	4.922921E-03	1.992183E+00	1.271579E-02	1.910007E+00
	1/32	1.237417E-03		3.383560E-03	

Table 4 The order of convergence for the errors in the free boundary for Problem 1 with $r=1$ for $\sigma_0=3$ and $\varepsilon=-1$.

h	L^2	O(h)	L^∞	O(h)
1/4	1.149604E-02	2.141289E+00	2.905908E-02	2.013944E+00
1/8	2.605890E-03	2.040143E+00	7.194892E-03	2.006745E+00
1/16	6.335951E-04	2.010523E+00	1.790333E-03	2.002074E+00
1/32	1.572477E-04		4.469402E-04	

Table 5 The order of convergence for the errors in the free boundary for Problem 1 with $r=1$ for $\sigma_0=3$ and $\varepsilon=1$.

h	L^2	O(h)	L^∞	O(h)
1/4	7.733118E-03	2.259352E+00	1.989212E-02	2.114252E+00
1/8	1.615183E-03	2.112140E+00	4.594389E-03	2.069748E+00
1/16	3.735977E-04	2.050115E+00	1.094388E-03	2.037543E+00
1/32	9.021070E-05		2.665693E-04	

Table 6 The order of convergence for the errors in the free boundary for Problem 1 with $r=1$ for $\sigma_0=3$ and $\varepsilon=0$.

h	L^2	O(h)	L^∞	O(h)
1/4	9.095028E-03	2.214414E+00	2.321142E-02	2.077303E+00
1/8	1.959743E-03	2.084216E+00	5.500107E-03	2.046019E+00
1/16	4.621548E-04	2.034770E+00	1.331858E-03	2.024183E+00
1/32	1.127874E-04		3.274299E-04	

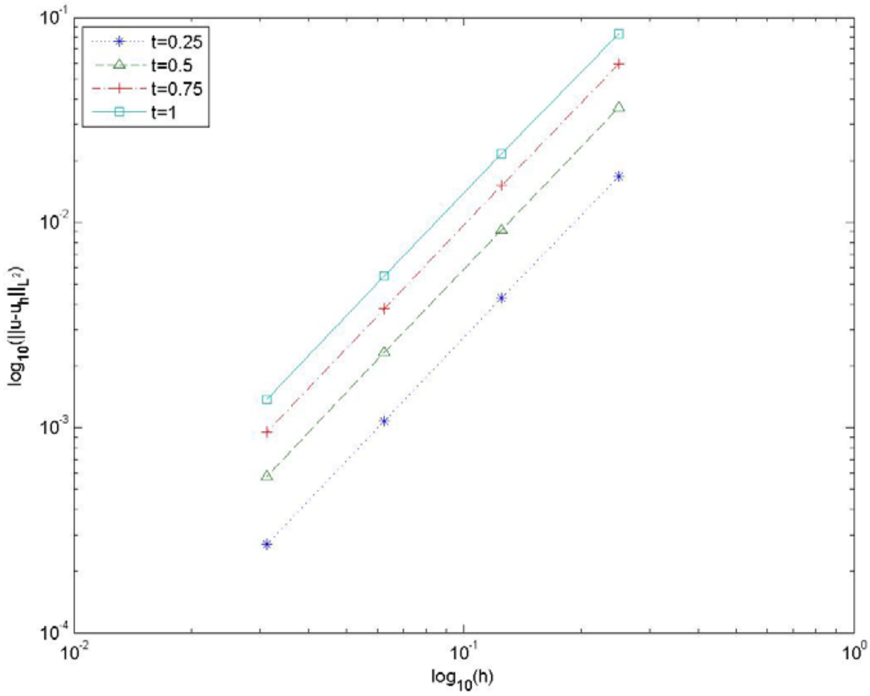

Figure 1 The graph of the L^2-errors against the space step h for solutions of Problem 1 with $r=1$ for $\sigma_0=3$ and $\varepsilon=-1$.

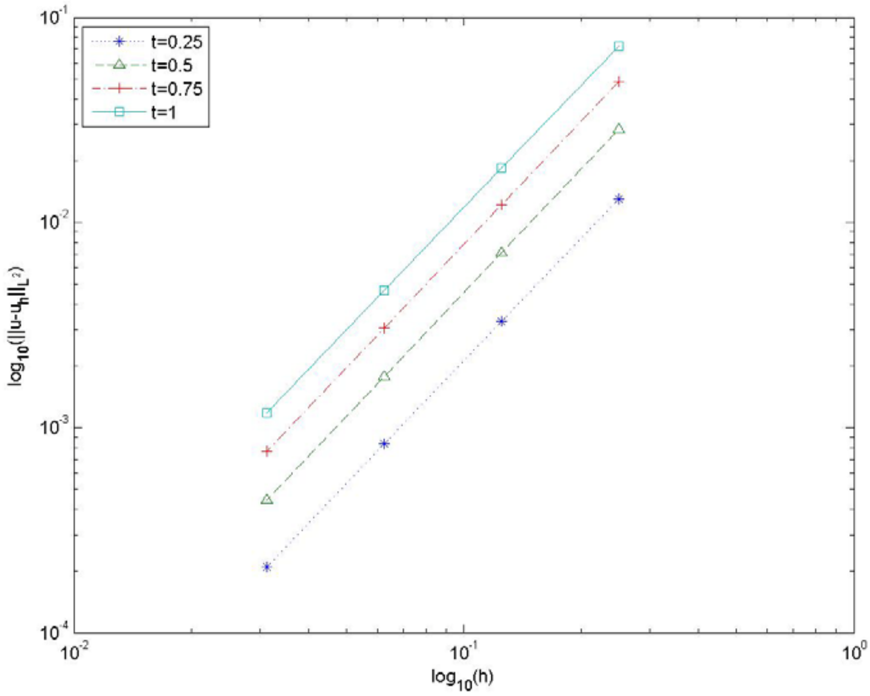

Figure 2 The graph of the L^2-errors against the space step h for solutions of Problem 1 with $r=1$ for $\sigma_0=3$ and $\varepsilon=1$.

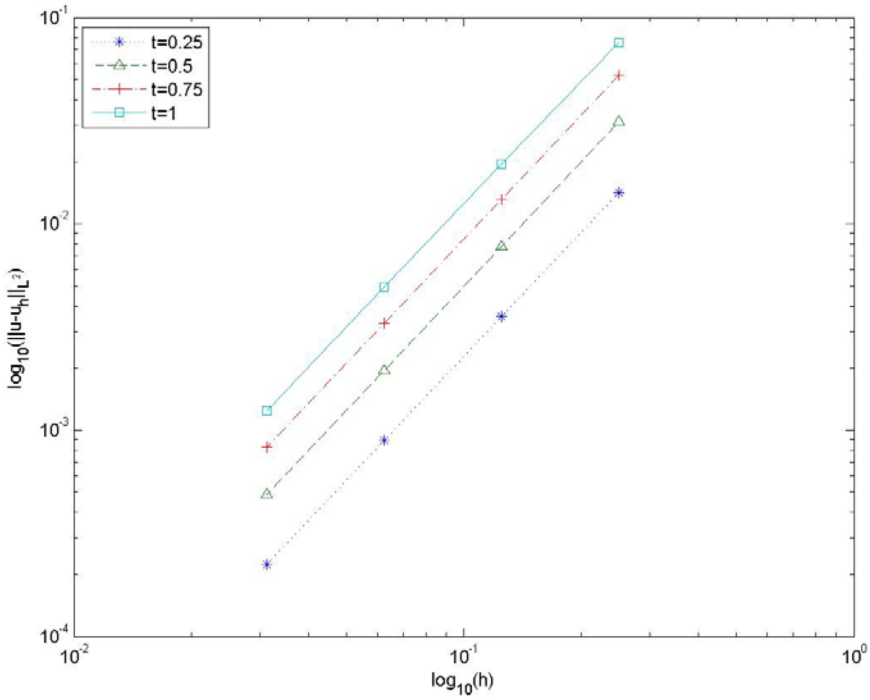

Figure 3 The graph of the L^2-errors against the space step h for solutions of Problem 1 with $r=1$ for $\sigma_0=3$ and $\varepsilon=0$.

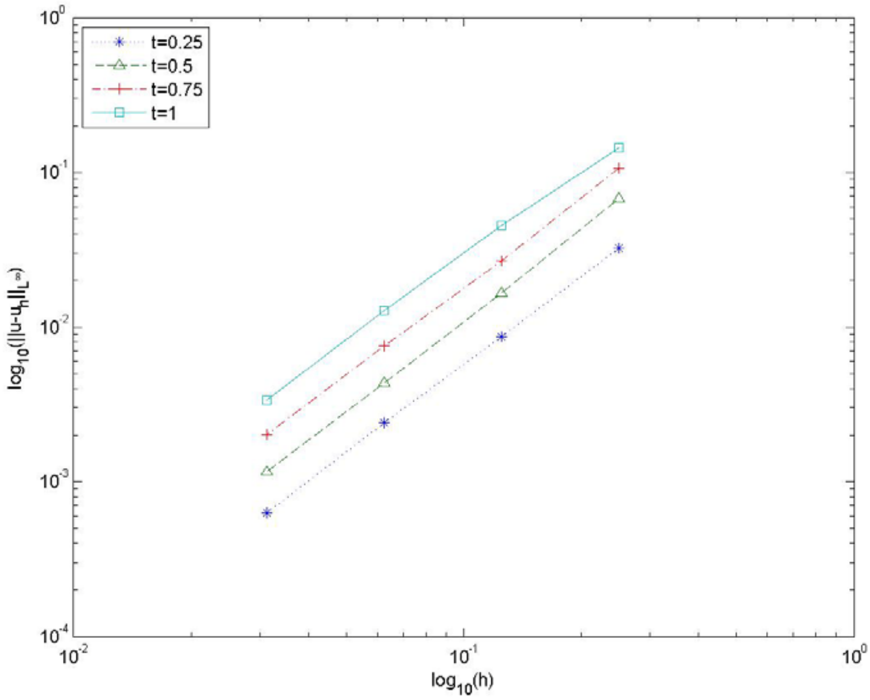

Figure 4 The graph of the L^∞-errors against the space step h for solutions of Problem 1 with $r=1$ for $\sigma_0=3$ and $\varepsilon=-1$.

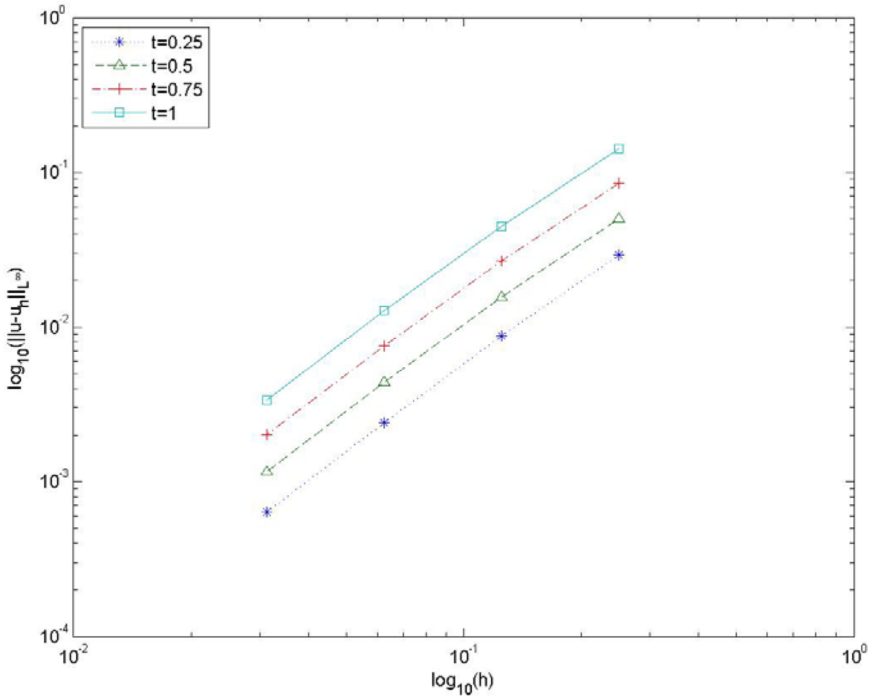

Figure 5 The graph of the L^∞-errors against the space step h for solutions of Problem 1 with $r=1$ for $\sigma_0=3$ and $\varepsilon=1$.

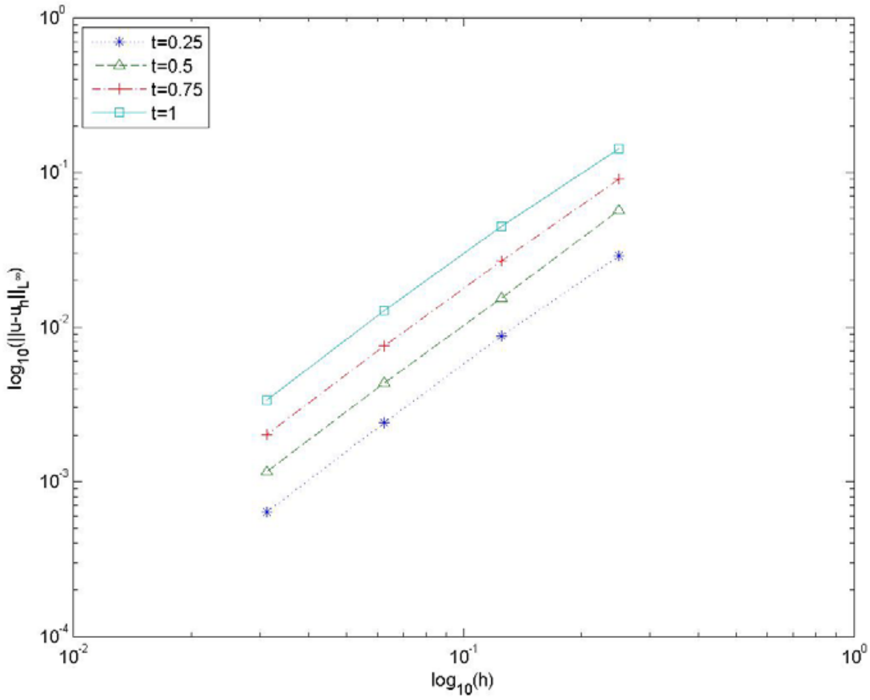

Figure 6 The graph of the L^∞-errors against the space step h for solutions of Problem 1 with $r=1$ for $\sigma_0=3$ and $\varepsilon=0$.

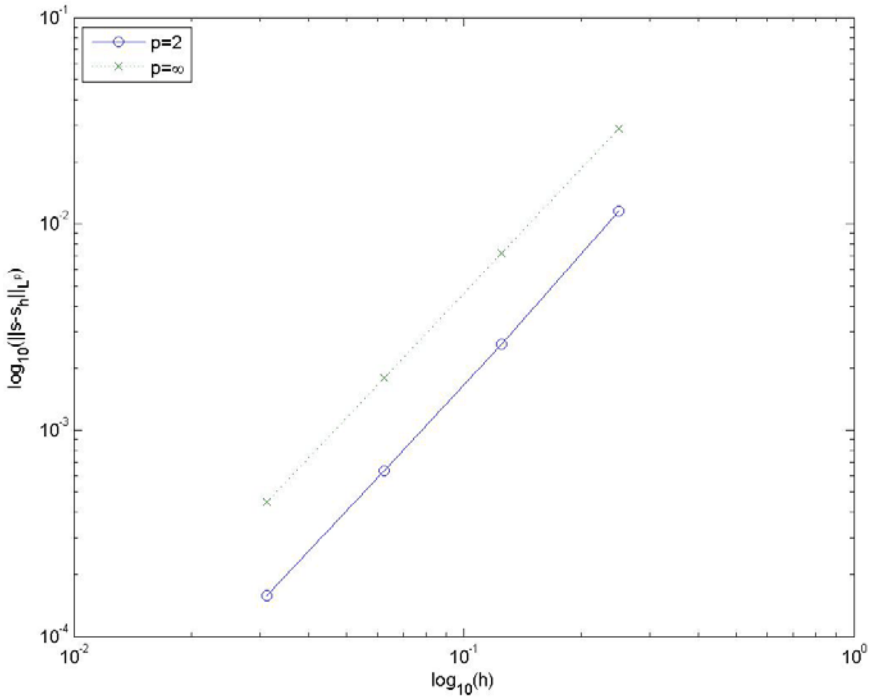

Figure 7 The graph of the L^2- and L^∞-errors against the space step h for free boundary of Problem 1 with $r=1$ for $\sigma_0=3$ and $\varepsilon=-1$.

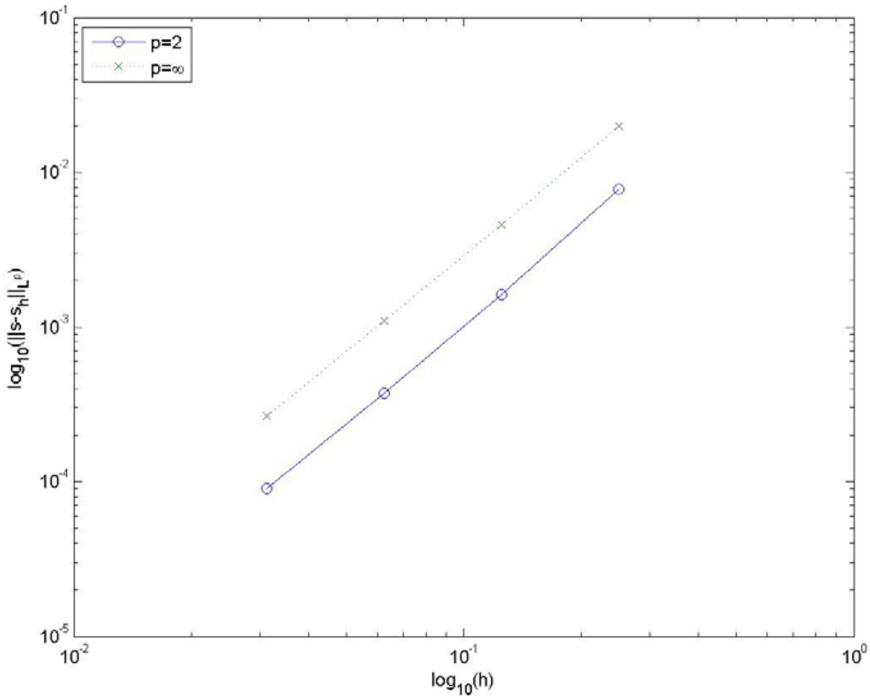

Figure 8 The graph of the L^2- and L^∞-errors against the space step h for free boundary of Problem 1 with $r=1$ for $\sigma_0=3$ and $\varepsilon=1$.

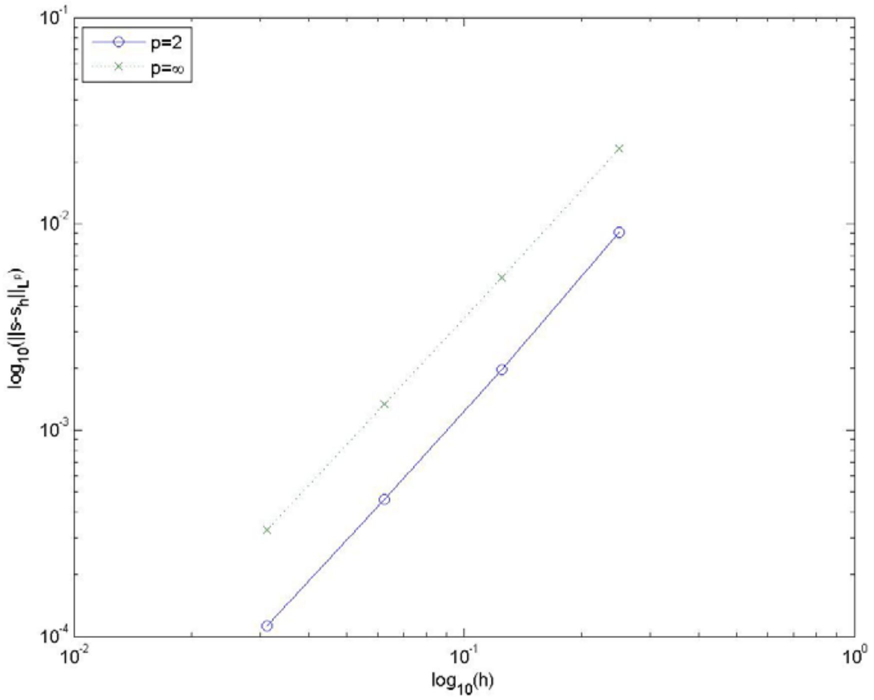

Figure 9 The graph of the L^2- and L^∞-errors against the space step h for free boundary of Problem 1 with $r=1$ for $\sigma_0=3$ and $\varepsilon=0$.

Table 7 The order of convergence for the errors in the solutions for Problem 1 with $r=2$ for $\sigma_0=3$ and $\varepsilon=-1$.

Time	h	L^2	O(h)	L^∞	O(h)
t = 0.125	1/4	2.014879E-03	2.984121E+00	3.610391E-03	3.020235E+00
	1/8	2.546472E-04	2.998771E+00	4.450131E-04	3.014743E+00
	1/16	3.185802E-05		5.506108E-05	
t = 0.25	1/4	3.980470E-03	2.991156E+00	6.585879E-03	3.007972E+00
	1/8	5.006181E-04	2.999720E+00	8.186982E-04	3.010853E+00
	1/16	6.258943E-05		1.015703E-04	
t = 0.375	1/4	6.042610E-03	2.994299E+00	9.789750E-03	3.010256E+00
	1/8	7.583170E-04	3.000391E+00	1.215050E-03	3.011926E+00
	1/16	9.476396E-05		1.506309E-04	
t = 0.50	1/4	8.229399E-03	2.995714E+00	1.328618E-02	3.016942E+00
	1/8	1.031735E-03	3.000845E+00	1.641384E-03	3.014745E+00
	1/16	1.288914E-04		2.030867E-04	
t = 0.625	1/4	1.052773E-02	2.995886E+00	1.708876E-02	3.025659E+00
	1/8	1.319724E-03	3.001158E+00	2.098439E-03	3.018573E+00
	1/16	1.648332E-04		2.589497E-04	

t = 0.75	1/4	1.287855E-02	2.994805E+00	2.115928E-02	3.034845E+00
	1/8	1.615626E-03	3.001355E+00	2.581794E-03	3.023963E+00
	1/16	2.017636E-04		3.174081E-04	
t = 0.875	1/4	1.516058E-02	2.992077E+00	2.538950E-02	2.989195E+00
	1/8	1.905509E-03	3.001434E+00	3.197547E-03	2.960818E+00
	1/16	2.379521E-04		4.106973E-04	
t = 1.00	1/4	1.716064E-02	2.986707E+00	3.067436E-02	2.887242E+00
	1/8	2.164936E-03	3.001364E+00	4.145998E-03	2.919990E+00
	1/16	2.703612E-04		5.478032E-04	

Table 8 The order of convergence for the errors in the solutions for Problem 1 with $r = 2$ for $\sigma_0 = 50$ and $\varepsilon = 1$.

Time	h	L^2	O(h)	L^∞	O(h)
t = 0.125	1/4	2.187381E-03	2.837308E+00	3.105560E-03	2.820629E+00
	1/8	3.060623E-04	2.727505E+00	4.395880E-04	2.704028E+00
	1/16	4.621138E-05		6.746092E-05	#DIV/0!
t = 0.25	1/4	4.173314E-03	2.896330E+00	5.858830E-03	2.882484E+00
	1/8	5.605300E-04	2.818599E+00	7.945050E-04	2.798452E+00
	1/16	7.945406E-05		1.142033E-04	#DIV/0!
t = 0.375	1/4	6.251520E-03	2.917358E+00	8.736922E-03	2.905374E+00
	1/8	8.275105E-04	2.851896E+00	1.166148E-03	2.833834E+00
	1/16	1.146218E-04		1.635628E-04	#DIV/0!
t = 0.50	1/4	8.449661E-03	2.927851E+00	1.177859E-02	2.917308E+00
	1/8	1.110371E-03	2.868525E+00	1.559179E-03	2.851864E+00
	1/16	1.520394E-04	#DIV/0!	2.159729E-04	#DIV/0!
t = 0.625	1/4	1.074927E-02	2.934131E+00	1.497220E-02	2.924904E+00
	1/8	1.406429E-03	2.878606E+00	1.971522E-03	2.863024E+00
	1/16	1.912367E-04	#DIV/0!	2.709852E-04	#DIV/0!
t = 0.75	1/4	1.308268E-02	2.938545E+00	1.825295E-02	2.930621E+00
	1/8	1.706501E-03	2.886006E+00	2.394022E-03	2.871306E+00
	1/16	2.308512E-04	#DIV/0!	3.271740E-04	#DIV/0!
t = 0.875	1/4	1.531473E-02	2.942308E+00	2.148598E-02	2.935756E+00
	1/8	1.992445E-03	2.892833E+00	2.808049E-03	2.878725E+00
	1/16	2.682606E-04	#DIV/0!	3.817877E-04	#DIV/0!
t = 1.00	1/4	1.720819E-02	2.946338E+00	2.443028E-02	2.941165E+00
	1/8	2.232540E-03	2.900887E+00	3.180896E-03	2.886287E+00
	1/16	2.989132E-04	#DIV/0!	4.302198E-04	#DIV/0!

Table 9 The order of convergence for the errors in the solutions for Problem 1 with $r=2$ for $\sigma_0=50$ and $\varepsilon=0$.

Time	h	L^2	O(h)	L^∞	O(h)
t = 0.125	1/4	2.096240E-03	2.908489E+00	2.959613E-03	2.897519E+00
	1/8	2.791892E-04	2.841872E+00	3.971868E-04	2.825107E+00
	1/16	3.894126E-05	#DIV/0!	5.604704E-05	#DIV/0!
t = 0.25	1/4	4.066364E-03	2.942519E+00	5.685857E-03	2.933446E+00
	1/8	5.289562E-04	2.899172E+00	7.442877E-04	2.886138E+00
	1/16	7.090584E-05	#DIV/0!	1.006762E-04	#DIV/0!
t = 0.375	1/4	6.125053E-03	2.954693E+00	8.530144E-03	2.946943E+00
	1/8	7.900573E-04	2.918944E+00	1.106211E-03	2.907679E+00
	1/16	1.044646E-04	#DIV/0!	1.474143E-04	#DIV/0!
t = 0.50	1/4	8.300963E-03	2.960824E+00	1.153285E-02	2.954166E+00
	1/8	1.066182E-03	2.928610E+00	1.488141E-03	2.918468E+00
	1/16	1.400335E-04	#DIV/0!	1.968329E-04	#DIV/0!
t = 0.625	1/4	1.057703E-02	2.964475E+00	1.468414E-02	2.958895E+00
	1/8	1.355090E-03	2.934414E+00	1.888567E-03	2.925134E+00
	1/16	1.772643E-04	#DIV/0!	2.486447E-04	#DIV/0!
t = 0.75	1/4	1.288766E-02	2.966963E+00	1.792180E-02	2.962553E+00
	1/8	1.648274E-03	2.938658E+00	2.299134E-03	2.930109E+00
	1/16	2.149835E-04	#DIV/0!	3.016571E-04	#DIV/0!
t = 0.875	1/4	1.510091E-02	2.968954E+00	2.111499E-02	2.965837E+00
	1/8	1.928675E-03	2.942564E+00	2.702620E-03	2.932350E+00
	1/16	2.508759E-04	#DIV/0!	3.540460E-04	#DIV/0!
t = 1.00	1/4	1.698427E-02	2.970902E+00	2.402912E-02	2.964690E+00
	1/8	2.166288E-03	2.947149E+00	3.078061E-03	2.937845E+00
	1/16	2.808898E-04	#DIV/0!	4.016962E-04	#DIV/0!

Table 10 The order of convergence for the errors in the free boundary for Problem 1 with $r=2$ for $\sigma_0=3$ and $\varepsilon=-1$.

h	L^2	O(h)	L^∞	O(h)
1/4	1.738071E-03	3.046281E+00	4.801854E-03	3.007609E+00
1/8	2.103999E-04	3.010627E+00	5.970744E-04	3.005858E+00
1/16	2.610698E-05		7.433187E-05	

Table 11 The order of convergence for the errors in the free boundary for Problem 1 with $r = 2$ for $\sigma_0 = 50$ and $\varepsilon = 1$.

h	L^2	O(h)	L^∞	O(h)
1/4	1.788841E-03	2.908554E+00	4.919621E-03	2.876211E+00
1/8	2.382373E-04	2.792906E+00	6.700479E-04	2.798867E+00
1/16	3.437647E-05		9.628594E-05	

Table 12 The order of convergence for the errors in the free boundary for Problem 1 with $r = 2$ for $\sigma_0 = 50$ and $\varepsilon = 0$.

h	L^2	O(h)	L^∞	O(h)
1/4	1.738644E-03	2.962499E+00	4.789736E-03	2.927023E+00
1/8	2.230538E-04	2.883459E+00	6.297816E-04	2.885124E+00
1/16	3.022748E-05		8.524739E-05	

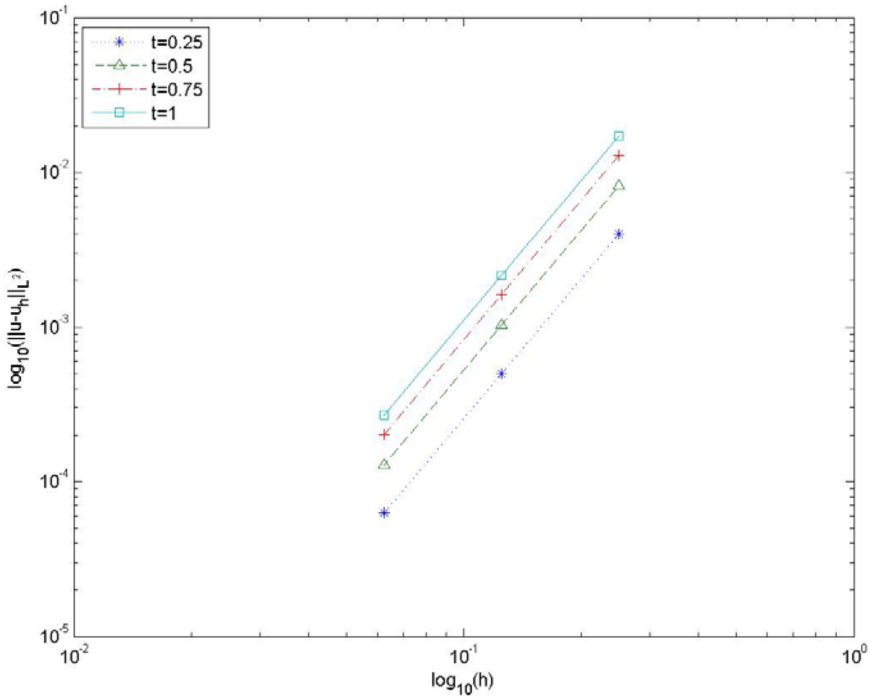

Figure 10 The graph of the L^2-errors against the space step h for solutions of Problem 1 with $r = 2$ for $\sigma_0 = 3$ and $\varepsilon = -1$.

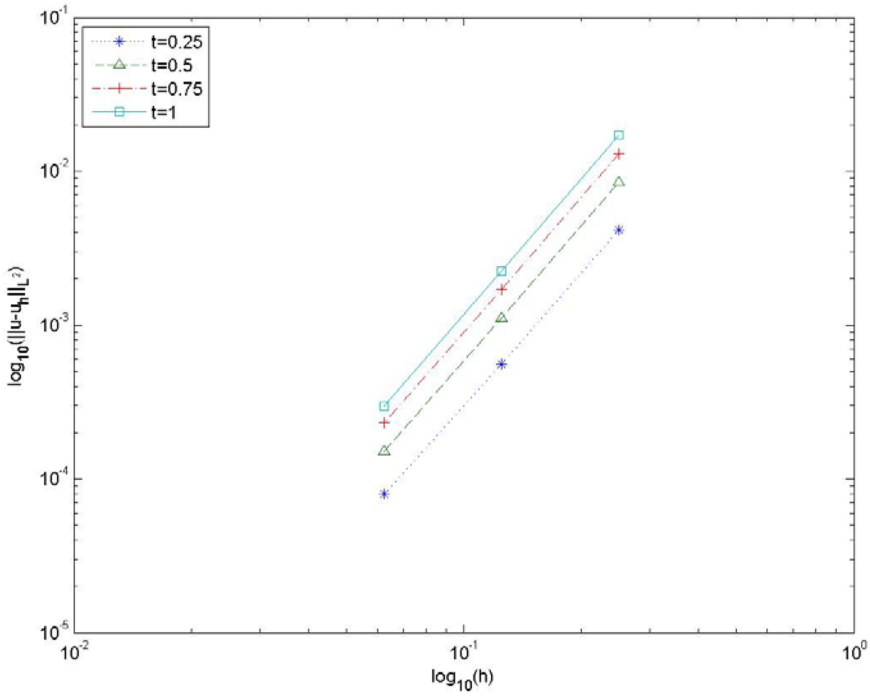

Figure 11 The graph of the L^2-errors against the space step h for solutions of Problem 1 with $r=2$ for $\sigma_0=50$ and $\varepsilon=1$.

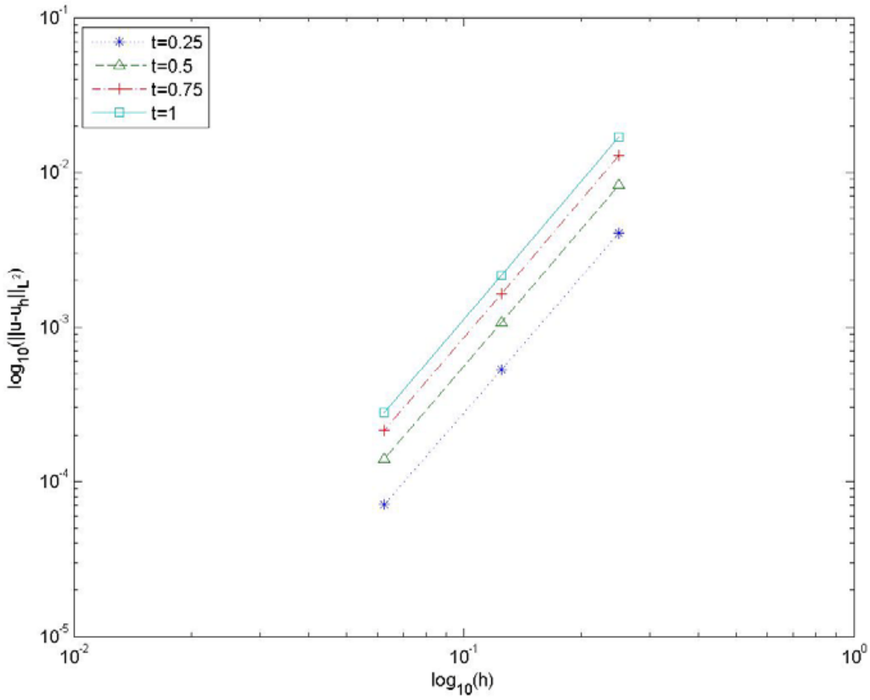

Figure 12 The graph of the L^2-errors against the space step h for solutions of Problem 1 with $r=2$ for $\sigma_0=50$ and $\varepsilon=0$.

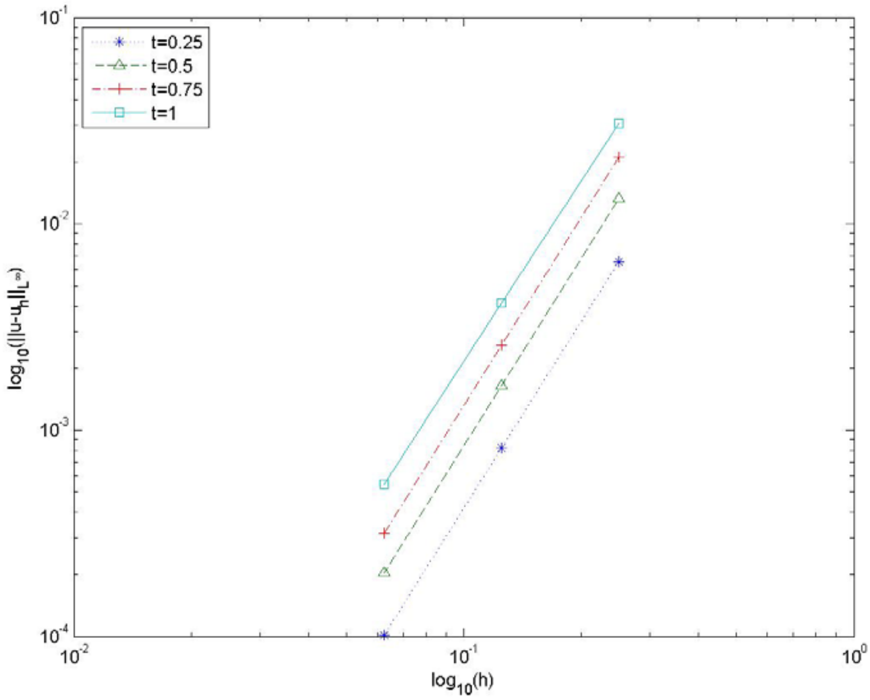

Figure 13 The graph of the L^∞-errors against the space step h for solutions of Problem 1 with $r=2$ for $\sigma_0=3$ and $\varepsilon=-1$.

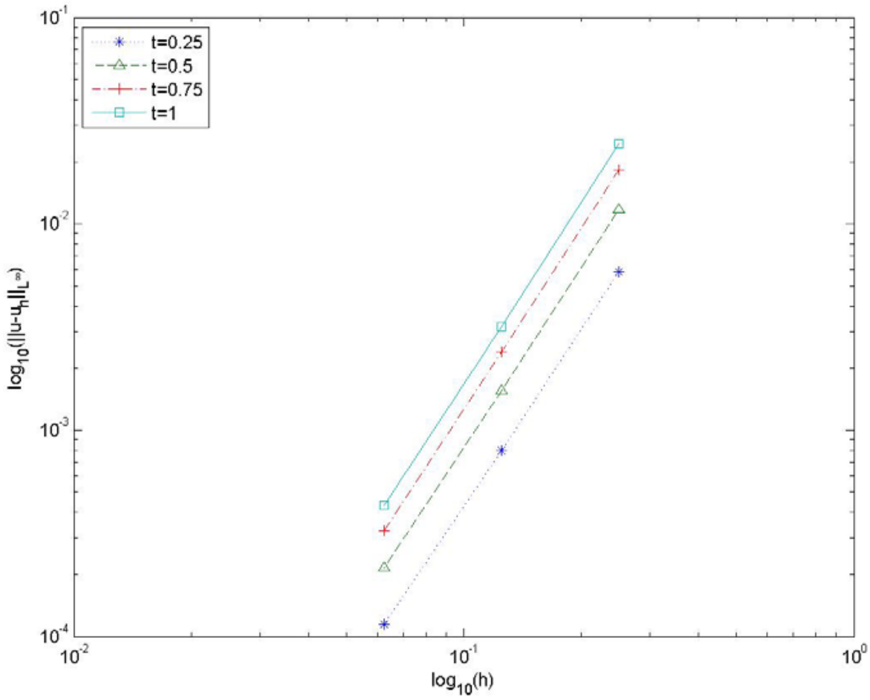

Figure 14 The graph of the L^∞-errors against the space step h for solutions of Problem 1 with $r=2$ for $\sigma_0=50$ and $\varepsilon=1$.

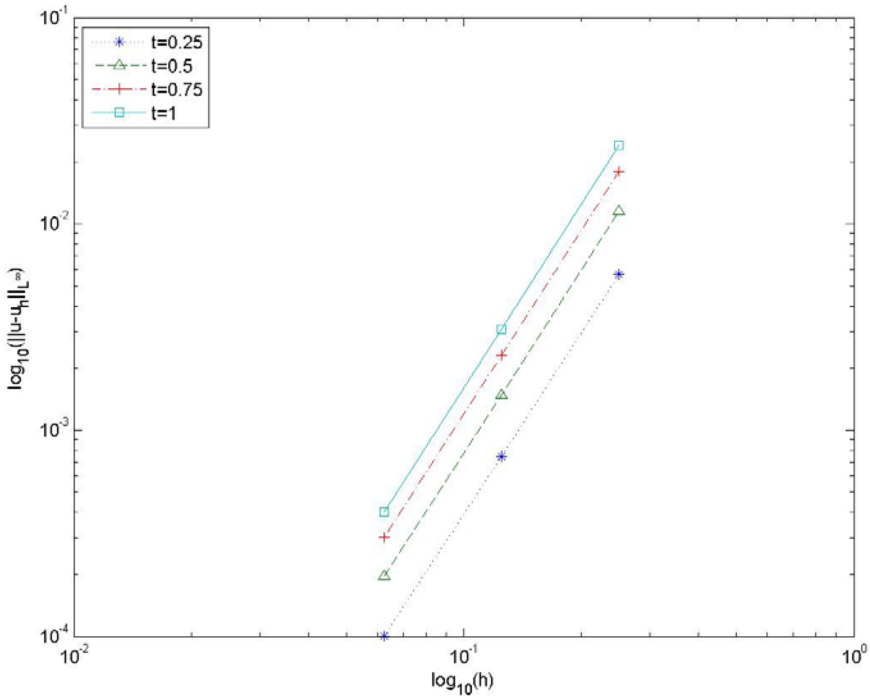

Figure 15 The graph of the L^∞-errors against the space step h for solutions of Problem 1 with $r = 2$ for $\sigma_0 = 50$ and $\varepsilon = 0$.

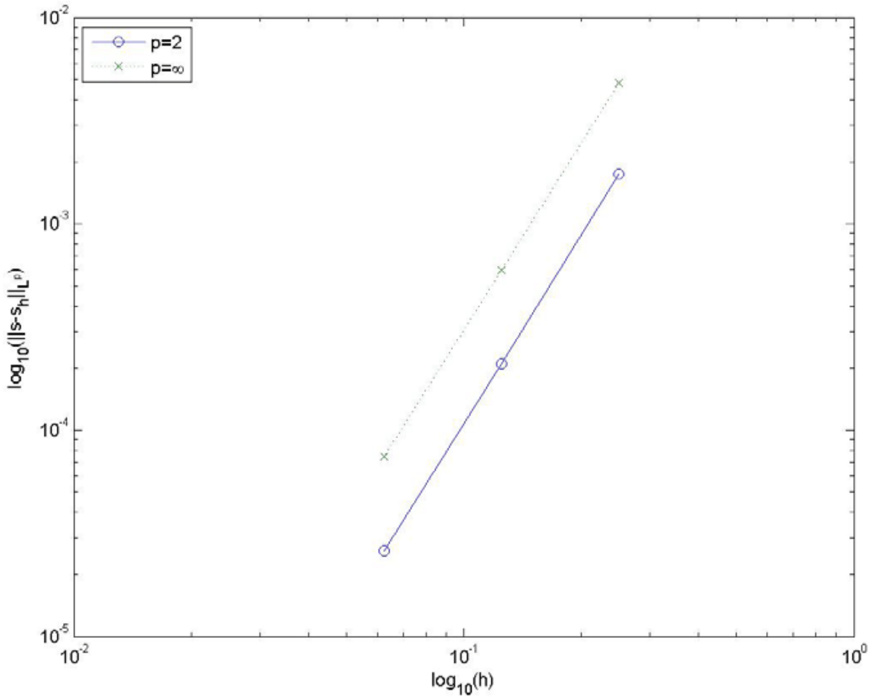

Figure 16 The graph of the L^2- and L^∞-errors against the space step h for free boundary of Problem 1 with $r = 2$ for $\sigma_0 = 3$ and $\varepsilon = -1$.

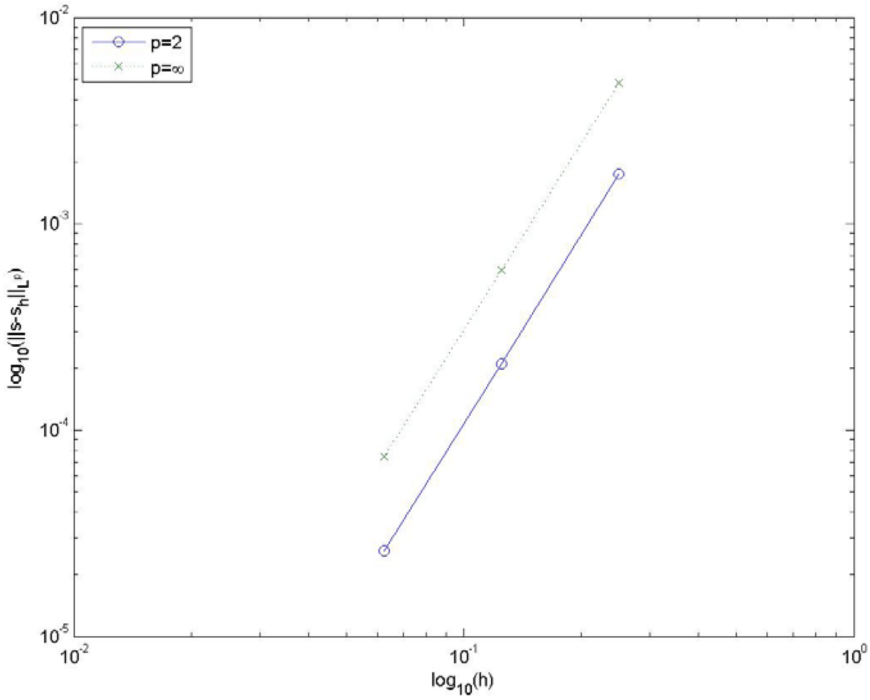

Figure 17 The graph of the L^2- and L^∞-errors against the space step h for free boundary of Problem 1 with $r = 2$ for $\sigma_0 = 50$ and $\varepsilon = 1$.

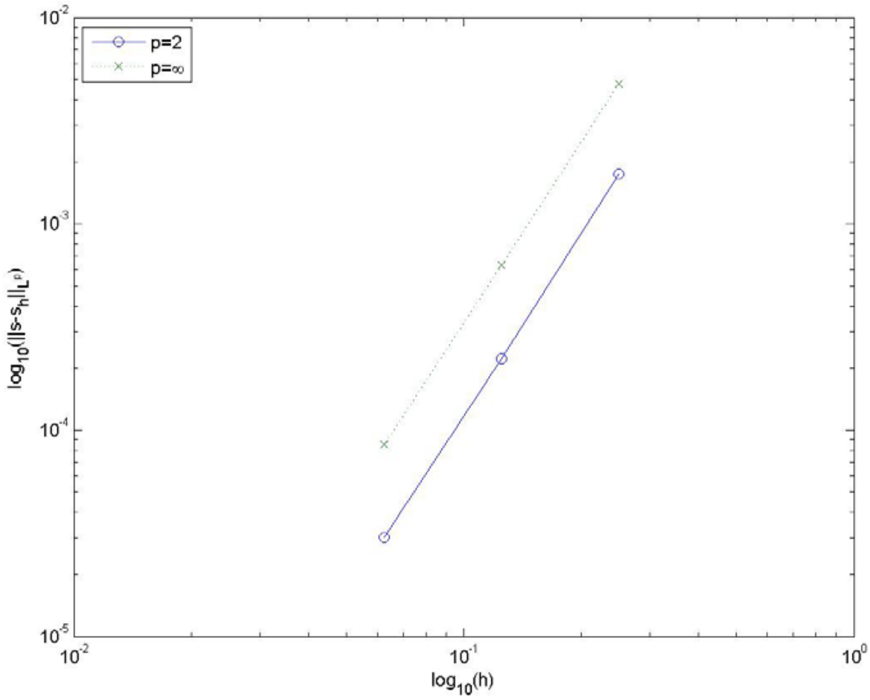

Figure 18 The graph of the L^2- and L^∞-errors against the space step h for free boundary of Problem 1 with $r = 2$ for $\sigma_0 = 50$ and $\varepsilon = 0$.

Table 13 The order of convergence for the errors in the solutions for Problem 2 with $r = 1$ for $\sigma_0 = 50$ and $\varepsilon = -1$.

Time	h	L^2	O(h)	L^∞	O(h)
t = 0.125	1/4	3.090489E-03	1.990089E+00	5.752849E-03	1.883131E+00
	1/8	7.779482E-04	1.997335E+00	1.559567E-03	1.941227E+00
	1/16	1.948467E-04	1.999320E+00	4.061031E-04	1.970355E+00
	1/32	4.873464E-05		1.036336E-04	
t = 0.25	1/4	2.297792E-03	1.993417E+00	4.164902E-03	1.901001E+00
	1/8	5.770751E-04	1.998131E+00	1.115184E-03	1.948969E+00
	1/16	1.444558E-04	1.999516E+00	2.888340E-04	1.973929E+00
	1/32	3.612606E-05		7.352524E-05	
t = 0.375	1/4	1.483469E-03	2.010337E+00	2.608350E-03	1.924430E+00
	1/8	3.682194E-04	2.002721E+00	6.871549E-04	1.956055E+00
	1/16	9.188143E-05	2.000687E+00	1.771020E-04	1.976288E+00
	1/32	2.295942E-05		4.500922E-05	
t = 0.50	1/4	7.892418E-04	2.050255E+00	1.298335E-03	1.975981E+00
	1/8	1.905556E-04	2.014003E+00	3.300329E-04	1.967623E+00
	1/16	4.717875E-05	2.003599E+00	8.438082E-05	1.977917E+00
	1/32	1.176530E-05		2.142059E-05	
t = 0.625	1/4	3.582272E-04	2.045254E+00	6.608046E-04	1.915909E+00
	1/8	8.679119E-05	2.009080E+00	1.751165E-04	1.947288E+00
	1/16	2.156166E-05	2.002065E+00	4.540827E-05	1.972431E+00
	1/32	5.382704E-06		1.157109E-05	
t = 0.75	1/4	1.758967E-04	1.950792E+00	4.430670E-04	1.920419E+00
	1/8	4.549994E-05	1.985941E+00	1.170485E-04	1.986118E+00
	1/16	1.148638E-05	1.996423E+00	2.954504E-05	1.995281E+00
	1/32	2.878724E-06		7.410462E-06	
t = 0.875	1/4	5.122326E-05	2.070880E+00	1.254917E-04	2.073944E+00
	1/8	1.219187E-05	2.019604E+00	2.980546E-05	2.029551E+00
	1/16	3.006831E-06	2.004963E+00	7.300288E-06	2.008259E+00
	1/32	7.491259E-07		1.814654E-06	
t = 1.00	1/4	1.885642E-05	1.315722E+00	3.266041E-05	1.315725E+00
	1/8	7.575085E-06	1.863137E+00	1.312045E-05	1.863139E+00
	1/16	2.082223E-06	1.967274E+00	3.606518E-06	1.967275E+00
	1/32	5.324989E-07		9.223152E-07	

Table 14 The order of convergence for the errors in the solutions for Problem 2 with $r = 1$ for $\sigma_0 = 3$ and $\varepsilon = 1$.

Time	h	L^2	O(h)	L^∞	O(h)
t = 0.125	1/4	2.118910E-03	2.040764E+00	3.607428E-03	1.966634E+00
	1/8	5.149693E-04	2.018550E+00	9.229576E-04	1.976403E+00
	1/16	1.270976E-04	2.008884E+00	2.345444E-04	1.986164E+00
	1/32	3.157933E-05		5.920114E-05	
t = 0.25	1/4	1.542377E-03	2.042627E+00	2.485132E-03	1.988495E+00
	1/8	3.743679E-04	2.017122E+00	6.262571E-04	1.982889E+00
	1/16	9.248780E-05	2.007674E+00	1.584322E-04	1.988355E+00
	1/32	2.299929E-05		3.992904E-05	
t = 0.375	1/4	9.923820E-04	2.054447E+00	1.603250E-03	1.988478E+00
	1/8	2.389068E-04	2.019794E+00	4.040264E-04	1.957422E+00
	1/16	5.891285E-05	2.008226E+00	1.040320E-04	1.984149E+00
	1/32	1.464447E-05		2.629532E-05	
t = 0.50	1/4	5.941513E-04	2.013905E+00	1.362532E-03	1.855424E+00
	1/8	1.471130E-04	2.001845E+00	3.765375E-04	1.945183E+00
	1/16	3.673124E-05	2.001166E+00	9.777996E-05	1.977384E+00
	1/32	9.175391E-06		2.483122E-05	
t = 0.625	1/4	4.205135E-04	1.864274E+00	1.184071E-03	1.854669E+00
	1/8	1.154988E-04	1.951441E+00	3.273910E-04	1.947887E+00
	1/16	2.986312E-05	1.982312E+00	8.485833E-05	1.979192E+00
	1/32	7.557877E-06		2.152278E-05	
t = 0.75	1/4	3.473091E-04	1.798275E+00	8.782589E-04	1.843796E+00
	1/8	9.985769E-05	1.929246E+00	2.446721E-04	1.941917E+00
	1/16	2.621927E-05	1.971826E+00	6.368091E-05	1.975820E+00
	1/32	6.684080E-06		1.618930E-05	
t = 0.875	1/4	3.038105E-04	1.749007E+00	6.156142E-04	1.777412E+00
	1/8	9.038560E-05	1.904877E+00	1.795785E-04	1.913696E+00
	1/16	2.413649E-05	1.959865E+00	4.766224E-05	1.963229E+00
	1/32	6.204347E-06		1.222317E-05	
t = 1.00	1/4	3.602265E-04	1.720697E+00	6.238840E-04	1.720623E+00
	1/8	1.092935E-04	1.893277E+00	1.892975E-04	1.893252E+00
	1/16	2.942126E-05	1.954776E+00	5.095880E-05	1.954769E+00
	1/32	7.589534E-06		1.314544E-05	

Table 15 The order of convergence for the errors in the solutions for Problem 2 with $r=1$ for $\sigma_0=3$ and $\varepsilon=0$.

Time	h	L^2	O(h)	L^∞	O(h)
t = 0.125	1/4	2.429108E-03	2.028140E+00	4.302955E-03	1.927626E+00
	1/8	5.955465E-04	2.012590E+00	1.131080E-03	1.958176E+00
	1/16	1.475930E-04	2.005988E+00	2.910875E-04	1.977335E+00
	1/32	3.674543E-05		7.392418E-05	
t = 0.25	1/4	1.777987E-03	2.031921E+00	3.034776E-03	1.947544E+00
	1/8	4.347697E-04	2.012316E+00	7.867875E-04	1.965208E+00
	1/16	1.077684E-04	2.005353E+00	2.014981E-04	1.980158E+00
	1/32	2.684234E-05		5.107214E-05	
t = 0.375	1/4	1.133508E-03	2.052345E+00	1.796892E-03	1.990129E+00
	1/8	2.732795E-04	2.018217E+00	4.523072E-04	1.978195E+00
	1/16	6.746262E-05	2.007070E+00	1.147988E-04	1.984354E+00
	1/32	1.678320E-05		2.901266E-05	
t = 0.50	1/4	6.263876E-04	2.060393E+00	1.105826E-03	1.869123E+00
	1/8	1.501769E-04	2.018135E+00	3.027086E-04	1.958637E+00
	1/16	3.707524E-05	2.006617E+00	7.787826E-05	1.989697E+00
	1/32	9.226395E-06		1.960911E-05	
t = 0.625	1/4	3.709502E-04	1.916795E+00	9.854351E-04	1.851855E+00
	1/8	9.824329E-05	1.965769E+00	2.730009E-04	1.948124E+00
	1/16	2.515056E-05	1.987496E+00	7.074902E-05	1.979786E+00
	1/32	6.342371E-06		1.793682E-05	
t = 0.75	1/4	2.750849E-04	1.811621E+00	7.230944E-04	1.853169E+00
	1/8	7.836362E-05	1.933254E+00	2.001408E-04	1.945375E+00
	1/16	2.051856E-05	1.973476E+00	5.196599E-05	1.977181E+00
	1/32	5.224822E-06		1.319862E-05	
t = 0.875	1/4	2.075124E-04	1.750750E+00	4.407968E-04	1.789042E+00
	1/8	6.166173E-05	1.901167E+00	1.275508E-04	1.913646E+00
	1/16	1.650849E-05	1.956941E+00	3.385464E-05	1.961843E+00
	1/32	4.252160E-06		8.690497E-06	
t = 1.00	1/4	2.250686E-04	1.704014E+00	3.898121E-04	1.703968E+00
	1/8	6.908055E-05	1.881061E+00	1.196492E-04	1.881045E+00
	1/16	1.875426E-05	1.947905E+00	3.248321E-05	1.947901E+00
	1/32	4.860960E-06		8.419421E-06	

Table 16 The order of convergence for the errors in the free boundaries for Problem 2 with $r=1$ for $\sigma_0=50$ and $\varepsilon=-1$.

h	L^2	O(h)	L^∞	O(h)
1/4	3.991214E-05	1.612093E+00	5.679100E-05	1.640215E+00
1/8	1.305619E-05	1.913580E+00	1.821905E-05	1.916900E+00
1/16	3.465546E-06	1.978946E+00	4.824822E-06	1.979516E+00
1/32	8.791229E-07		1.223454E-06	

Table 17 The order of convergence for the errors in the free boundaries for Problem 2 with $r=1$ for $\sigma_0=3$ and $\varepsilon=1$.

h	L^2	O(h)	L^∞	O(h)
1/4	3.139523E-04	1.766554E+00	6.236846E-04	1.720299E+00
1/8	9.227368E-05	1.901953E+00	1.892795E-04	1.893152E+00
1/16	2.469068E-05	1.955576E+00	5.095749E-05	1.954742E+00
1/32	6.365696E-06		1.314535E-05	

Table 18 The order of convergence for the errors in the free boundaries for Problem 2 with $r=1$ for $\sigma_0=3$ and $\varepsilon=0$.

h	L^2	O(h)	L^∞	O(h)
1/4	1.899253E-04	1.749905E+00	3.897342E-04	1.703766E+00
1/8	5.646884E-05	1.887671E+00	1.196420E-04	1.880982E+00
1/16	1.526030E-05	1.947245E+00	3.248268E-05	1.947884E+00
1/32	3.957164E-06		8.419386E-06	

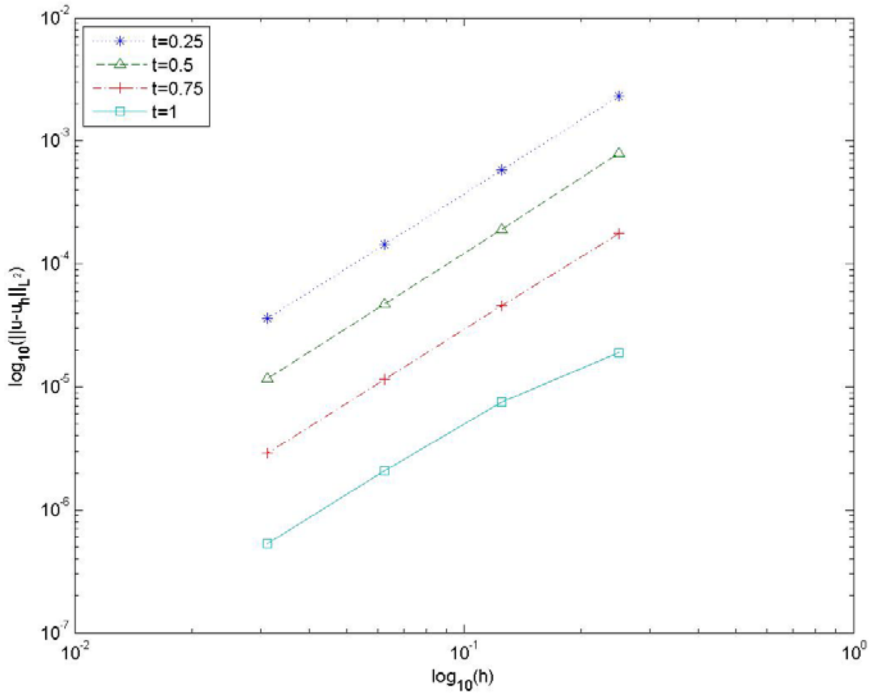

Figure 19 The graph of the L^2-errors against the space step h for solutions of Problem 2 with $r=1$ for $\sigma_0=50$ and $\varepsilon=-1$.

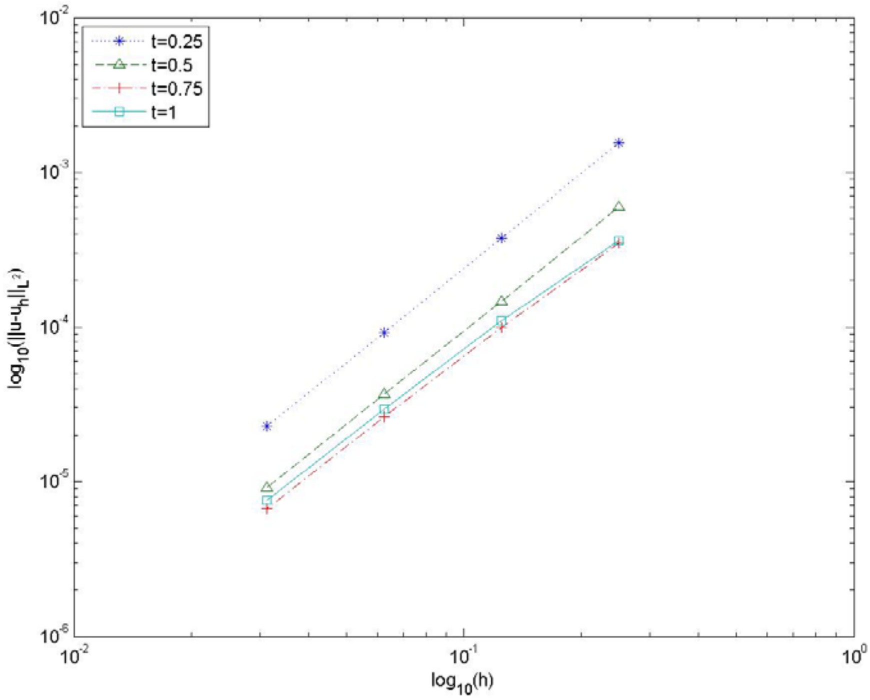

Figure 20 The graph of the L^2-errors against the space step h for solutions of Problem 2 with $r=1$ for $\sigma_0=3$ and $\varepsilon=1$.

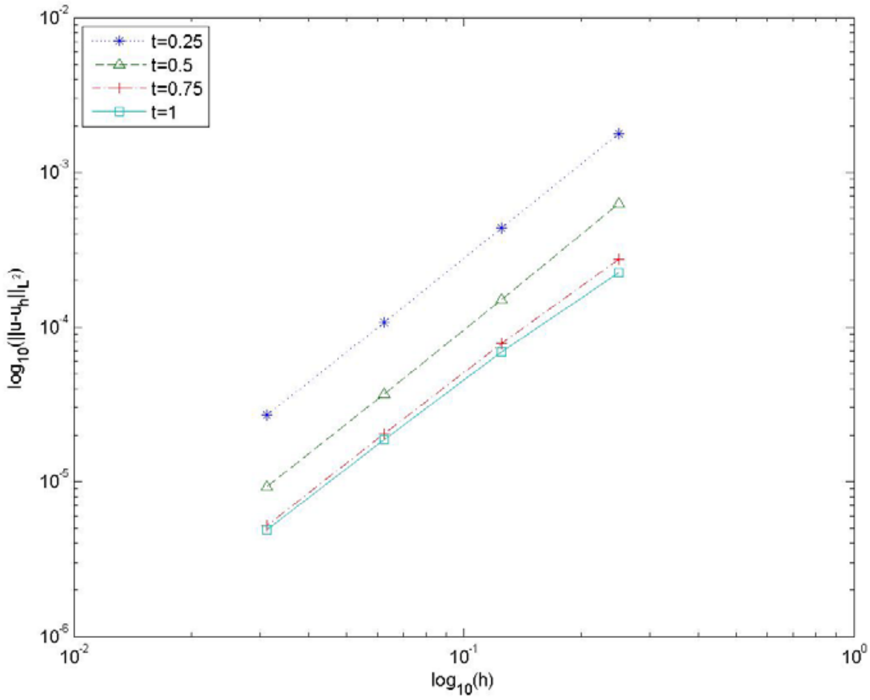

Figure 21 The graph of the L^2-errors against the space step h for solutions of Problem 2 with $r=1$ for $\sigma_0=3$ and $\varepsilon=0$.

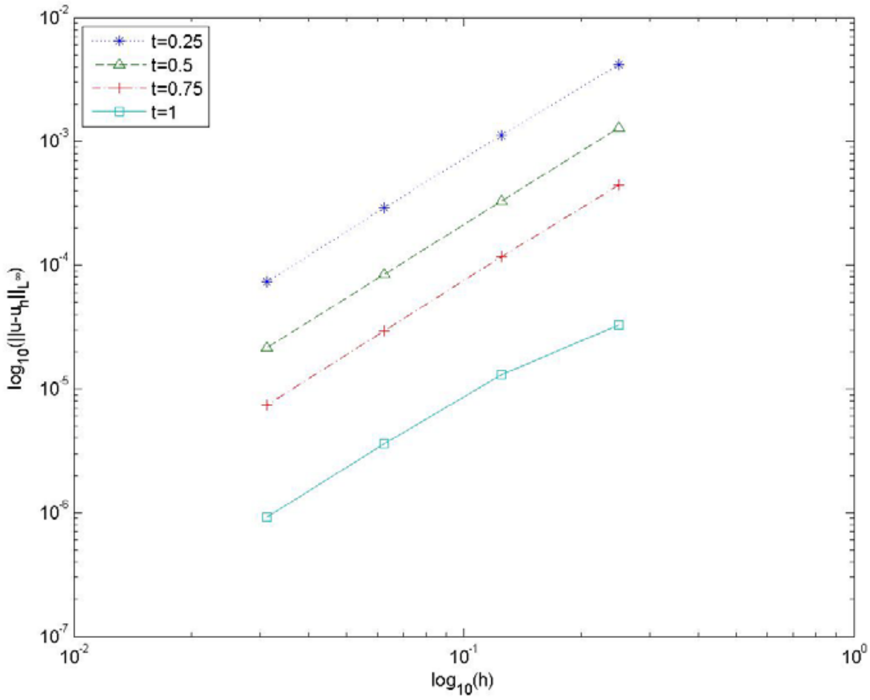

Figure 22 The graph of the L^∞-errors against the space step h for solutions of Problem 2 with $r=1$ for $\sigma_0=50$ and $\varepsilon=-1$.

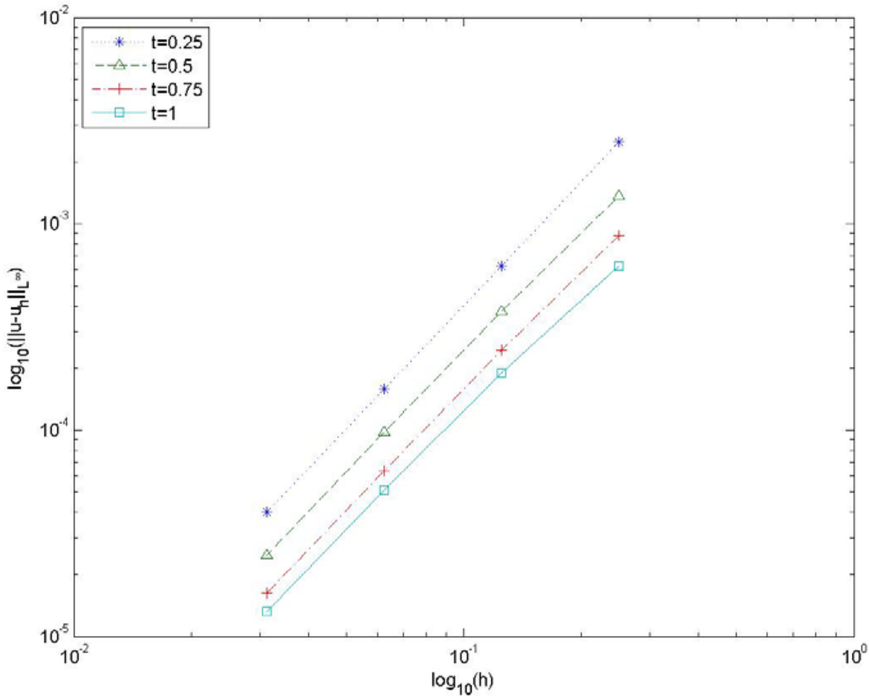

Figure 23 The graph of the L^{∞}-errors against the space step h for solutions of Problem 2 with $r=1$ for $\sigma_0 = 3$ and $\varepsilon = 1$.

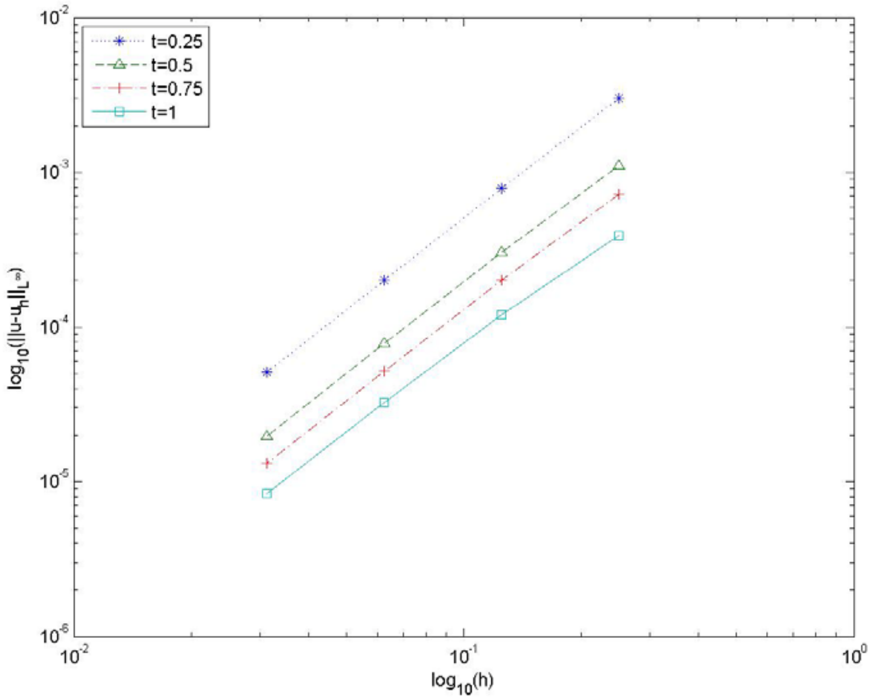

Figure 24 The graph of the L^{∞}-errors against the space step h for solutions of Problem 2 with $r=1$ for $\sigma_0 = 3$ and $\varepsilon = 0$.

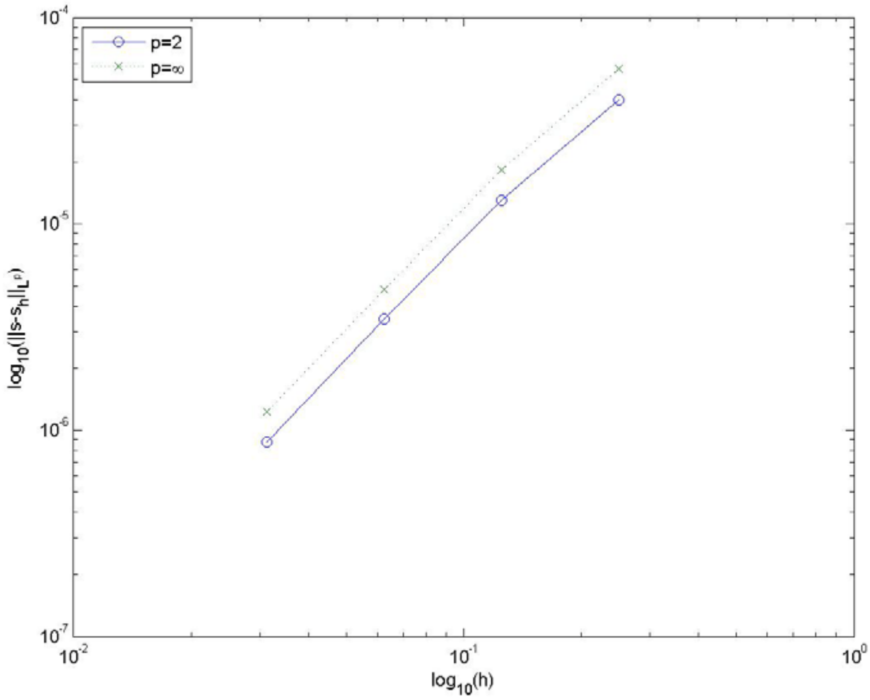

Figure 25 The graph of the L^2- and L^∞-errors against the space step h for free boundary of Problem 2 with $r=1$ for $\sigma_0=50$ and $\varepsilon=-1$.

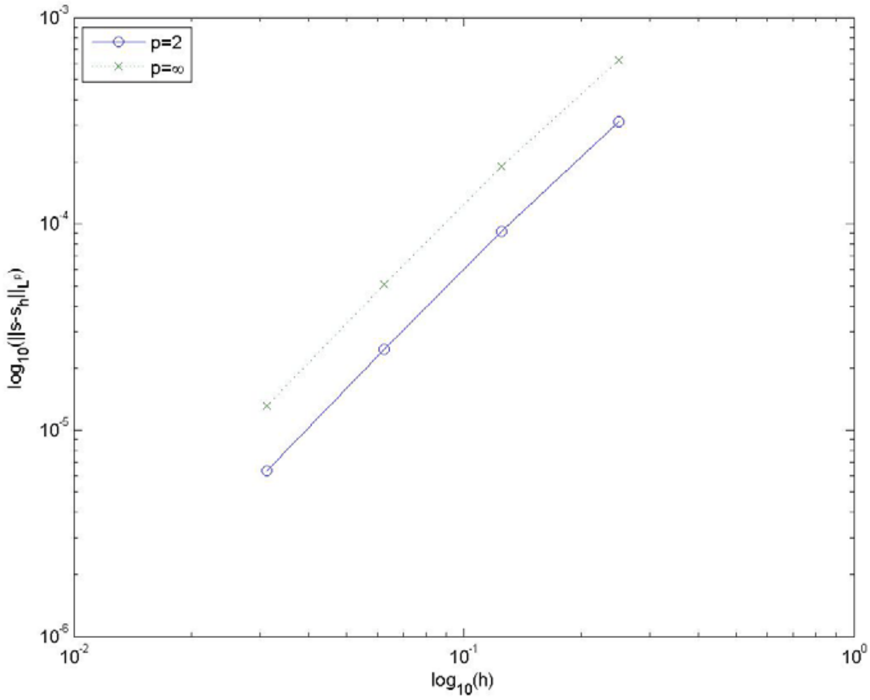

Figure 26 The graph of the L^2- and L^∞-errors against the space step h for free boundary of Problem 2 with $r=1$ for $\sigma_0=3$ and $\varepsilon=1$.

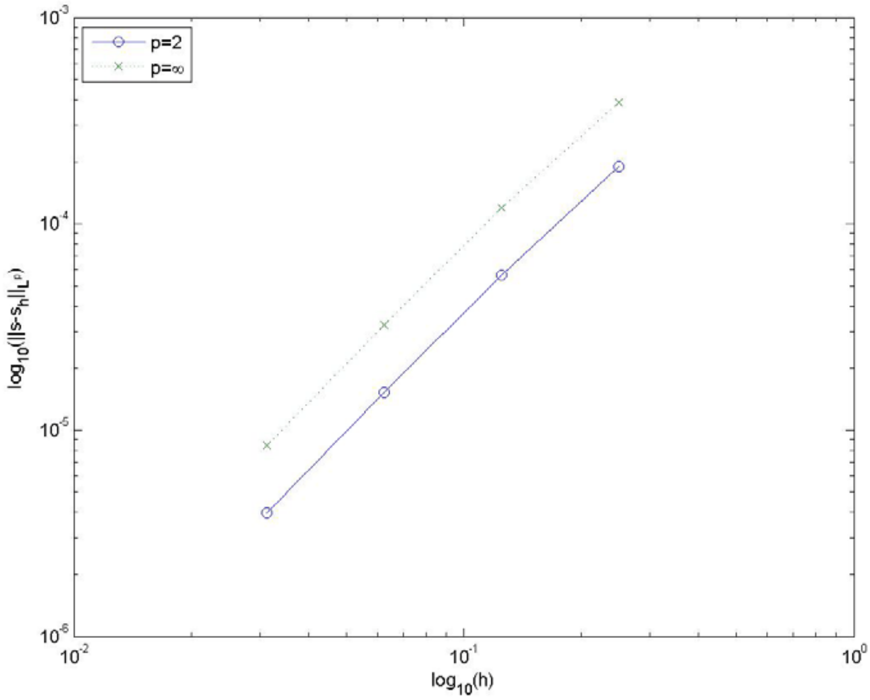

Figure 27 The graph of the L^2- and L^∞-errors against the space step h for free boundary of Problem 2 with $r=1$ for $\sigma_0=3$ and $\varepsilon=0$.

Table 19 The order of convergence for the errors in the solutions for Problem 2 with $r=2$ for $\sigma_0=3$ and $\varepsilon=-1$.

Time	h	L^2	O(h)	L^∞	O(h)
t = 0.125	1/4	2.192610E-04	3.003292E+00	5.917580E-04	2.863377E+00
	1/8	2.734515E-05	3.017400E+00	8.131711E-05	2.940259E+00
	1/16	3.377166E-06		1.059439E-05	
t = 0.25	1/4	3.393073E-04	2.985250E+00	6.516272E-04	2.877784E+00
	1/8	4.284927E-05	3.003171E+00	8.865431E-05	2.957831E+00
	1/16	5.344398E-06		1.141048E-05	
t = 0.375	1/4	3.909493E-04	2.979828E+00	6.382409E-04	2.882467E+00
	1/8	4.955674E-05	3.000002E+00	8.655170E-05	2.965629E+00
	1/16	6.194583E-06		1.107981E-05	
t = 0.50	1/4	3.640103E-04	2.979520E+00	5.442463E-04	2.882668E+00
	1/8	4.615183E-05	2.999069E+00	7.379482E-05	2.967670E+00
	1/16	5.772702E-06		9.433396E-06	
t = 0.625	1/4	2.744209E-04	2.984971E+00	4.028435E-04	2.929613E+00
	1/8	3.466182E-05	2.999411E+00	5.287313E-05	2.964271E+00
	1/16	4.334496E-06		6.774863E-06	

t = 0.75	1/4	1.687669E-04	2.992813E+00	2.587859E-04	2.991500E+00
	1/8	2.120122E-05	3.000585E+00	3.253938E-05	2.987341E+00
	1/16	2.649078E-06		4.103270E-06	
t = 0.875	1/4	1.024747E-04	2.988200E+00	1.712626E-04	2.987245E+00
	1/8	1.291454E-05	3.001088E+00	2.159794E-05	3.001091E+00
	1/16	1.613100E-06		2.697700E-06	
t = 1.00	1/4	1.032030E-04	2.982928E+00	1.787488E-04	2.982900E+00
	1/8	1.305393E-05	3.001323E+00	2.261001E-05	3.001319E+00
	1/16	1.630246E-06		2.823668E-06	

Table 20 The order of convergence for the errors in the solutions for Problem 2 with $r=2$ for $\sigma_0=50$ and $\varepsilon=1$.

Time	h	L^2	O(h)	L^∞	O(h)
t = 0.125	1/4	1.543119E-04	3.485639E+00	2.708642E-04	3.437552E+00
	1/8	1.377582E-05	3.986461E+00	2.500030E-05	3.204751E+00
	1/16	8.691068E-07		2.711560E-06	
t = 0.25	1/4	2.927892E-04	3.176376E+00	4.402756E-04	3.160256E+00
	1/8	3.238699E-05	3.525417E+00	4.924849E-05	3.528136E+00
	1/16	2.812641E-06		4.268921E-06	
t = 0.375	1/4	3.582760E-04	3.090596E+00	5.130781E-04	3.061277E+00
	1/8	4.205868E-05	3.274167E+00	6.146773E-05	3.305290E+00
	1/16	4.347437E-06		6.218072E-06	
t = 0.50	1/4	3.429506E-04	3.055531E+00	4.927751E-04	3.052979E+00
	1/8	4.125010E-05	3.179451E+00	5.937594E-05	3.204226E+00
	1/16	4.553180E-06		6.442320E-06	
t = 0.625	1/4	2.615123E-04	3.050370E+00	3.815193E-04	3.069209E+00
	1/8	3.156743E-05	3.150923E+00	4.545613E-05	3.172030E+00
	1/16	3.553996E-06		5.043318E-06	
t = 0.75	1/4	1.588419E-04	3.087564E+00	2.411288E-04	3.109819E+00
	1/8	1.868598E-05	3.220998E+00	2.793190E-05	3.277806E+00
	1/16	2.004005E-06		2.879934E-06	
t = 0.875	1/4	9.069157E-05	3.204803E+00	1.502179E-04	3.223711E+00
	1/8	9.836151E-06	3.588221E+00	1.608007E-05	3.662592E+00
	1/16	8.178301E-07		1.269807E-06	
t = 1.00	1/4	8.688916E-05	3.306046E+00	1.504936E-04	3.306021E+00
	1/8	8.785100E-06	4.061218E+00	1.521621E-05	4.061216E+00
	1/16	5.262574E-07		9.115044E-07	

Table 21 The order of convergence for the errors in the solutions for Problem 2 with $r = 2$ for $\sigma_0 = 50$ and $\varepsilon = 0$.

Time	h	L^2	O(h)	L^∞	O(h)
t = 0.125	1/4	1.755908E-04	3.183526E+00	3.070850E-04	3.148829E+00
	1/8	1.932702E-05	3.544446E+00	3.462315E-05	3.517901E+00
	1/16	1.656458E-06		3.022547E-06	
t = 0.25	1/4	3.102978E-04	3.064264E+00	4.683958E-04	3.027963E+00
	1/8	3.709739E-05	3.210462E+00	5.742559E-05	3.231248E+00
	1/16	4.007727E-06		6.115089E-06	
t = 0.375	1/4	3.710514E-04	3.027417E+00	5.332798E-04	2.987075E+00
	1/8	4.550831E-05	3.120289E+00	6.725986E-05	3.130046E+00
	1/16	5.233476E-06		7.682777E-06	
t = 0.50	1/4	3.513064E-04	3.012874E+00	5.068426E-04	3.004679E+00
	1/8	4.352318E-05	3.081682E+00	6.315016E-05	3.082522E+00
	1/16	5.140934E-06		7.454918E-06	
t = 0.625	1/4	2.667270E-04	3.014312E+00	3.906637E-04	3.023337E+00
	1/8	3.301175E-05	3.069963E+00	4.804941E-05	3.074861E+00
	1/16	3.931132E-06		5.702467E-06	
t = 0.75	1/4	1.629831E-04	3.036003E+00	2.486927E-04	3.045680E+00
	1/8	1.987076E-05	3.100636E+00	3.011772E-05	3.124692E+00
	1/16	2.316487E-06		3.452996E-06	
t = 0.875	1/4	9.569751E-05	3.083021E+00	1.591889E-04	3.090060E+00
	1/8	1.129324E-05	3.229779E+00	1.869443E-05	3.251100E+00
	1/16	1.203811E-06		1.963512E-06	
t = 1.00	1/4	9.383623E-05	3.118149E+00	1.625258E-04	3.118123E+00
	1/8	1.080723E-05	3.346361E+00	1.871862E-05	3.346358E+00
	1/16	1.062574E-06		1.840432E-06	

Table 22 The order of convergence for the errors in the free boundaries for Problem 2 with $r = 2$ for $\sigma_0 = 3$ and $\varepsilon = -1$.

h	L^2	O(h)	L^∞	O(h)
1/4	8.223724E-05	3.006826E+00	1.787327E-04	2.982786E+00
1/8	1.023113E-05	3.004805E+00	2.260976E-05	3.001305E+00
1/16	1.274639E-06		2.823664E-06	

Table 23 The order of convergence for the errors in the free boundaries for Problem 2 with $r=2$ for $\sigma_0=50$ and $\varepsilon=1$.

h	L^2	O(h)	L^∞	O(h)
1/4	6.781682E-05	3.381150E+00	1.504822E-04	3.305923E+00
1/8	6.508935E-06	4.266261E+00	1.521610E-05	4.061205E+00
1/16	3.382497E-07		9.115040E-07	

Table 24 The order of convergence for the errors in the free boundaries for Problem 2 with $r=2$ for $\sigma_0=50$ and $\varepsilon=0$.

h	L^2	O(h)	L^∞	O(h)
1/4	7.394556E-05	3.161417E+00	1.625125E-04	3.118018E+00
1/8	8.264769E-06	3.404395E+00	1.871845E-05	3.346346E+00
1/16	7.805588E-07	#DIV/0!	1.840430E-06	#DIV/0!

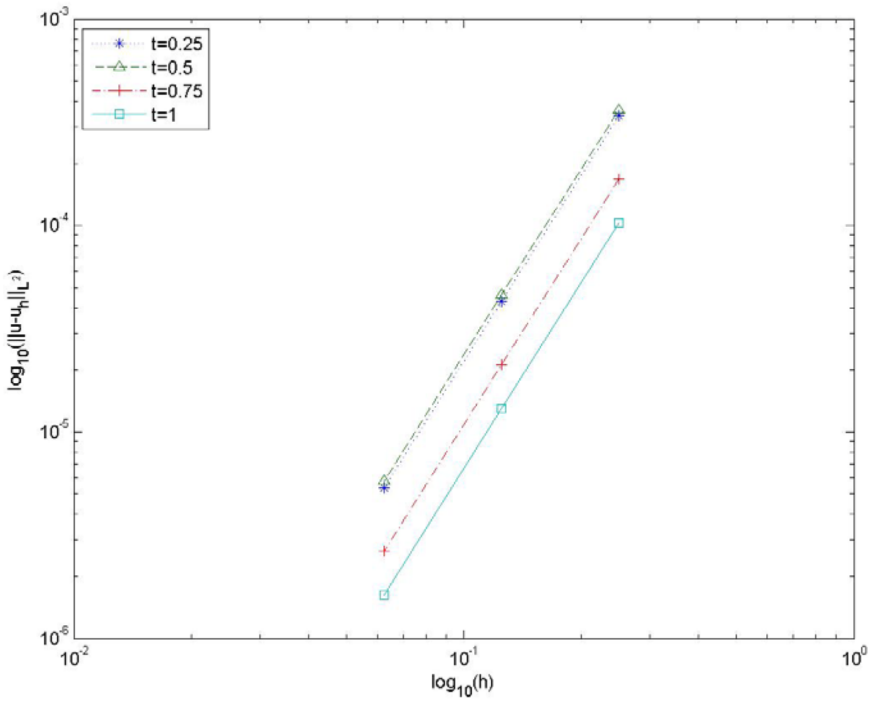

Figure 28 The graph of the L^2-errors against the space step h for solutions of Problem 2 with $r=2$ for $\sigma_0=3$ and $\varepsilon=-1$.

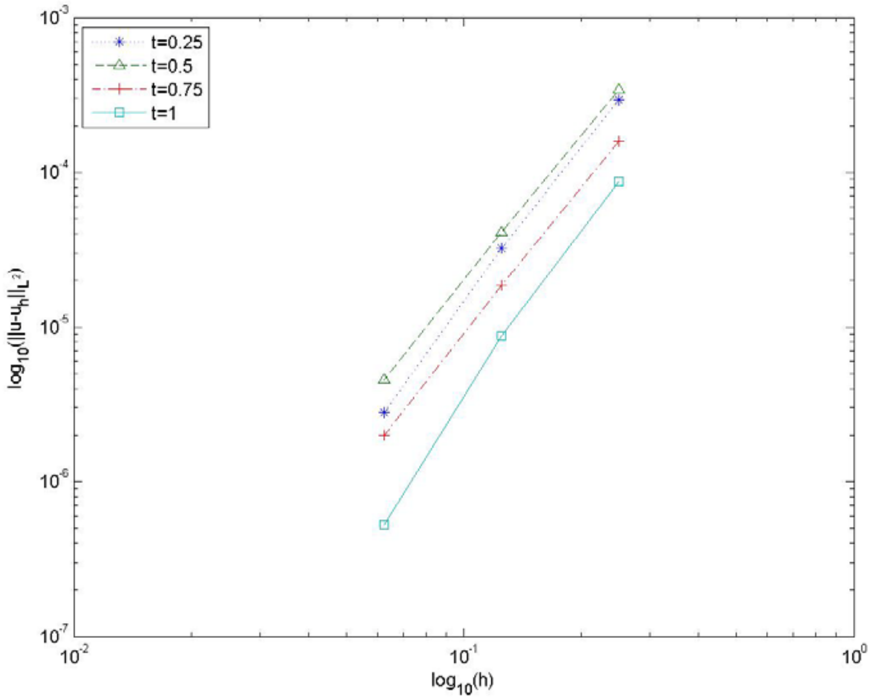

Figure 29 The graph of the L^2-errors against the space step h for solutions of Problem 2 with $r = 2$ for $\sigma_0 = 50$ and $\varepsilon = 1$.

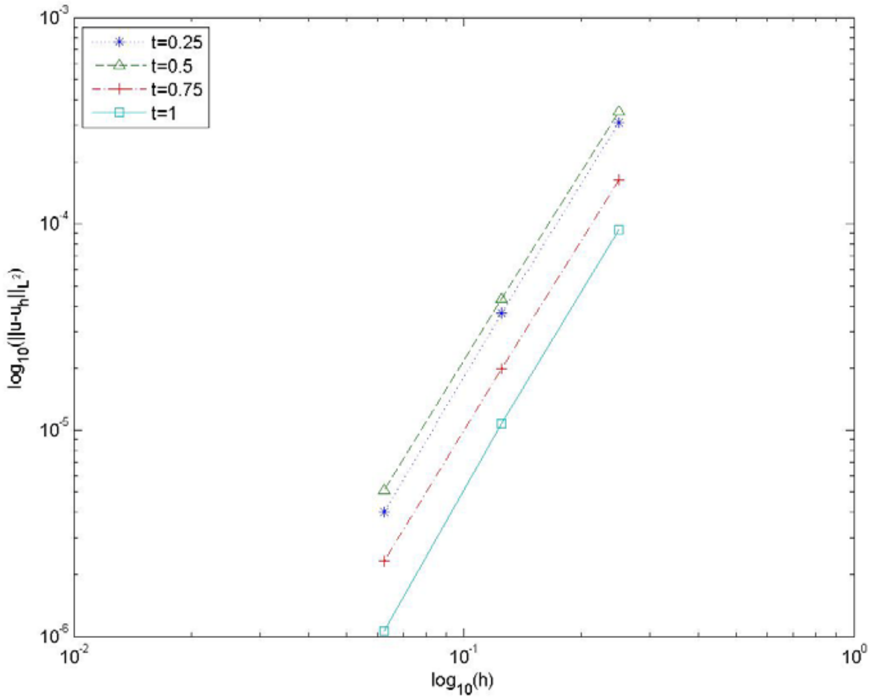

Figure 30 The graph of the L^2-errors against the space step h for solutions of Problem 2 with $r = 2$ for $\sigma_0 = 50$ and $\varepsilon = 0$.

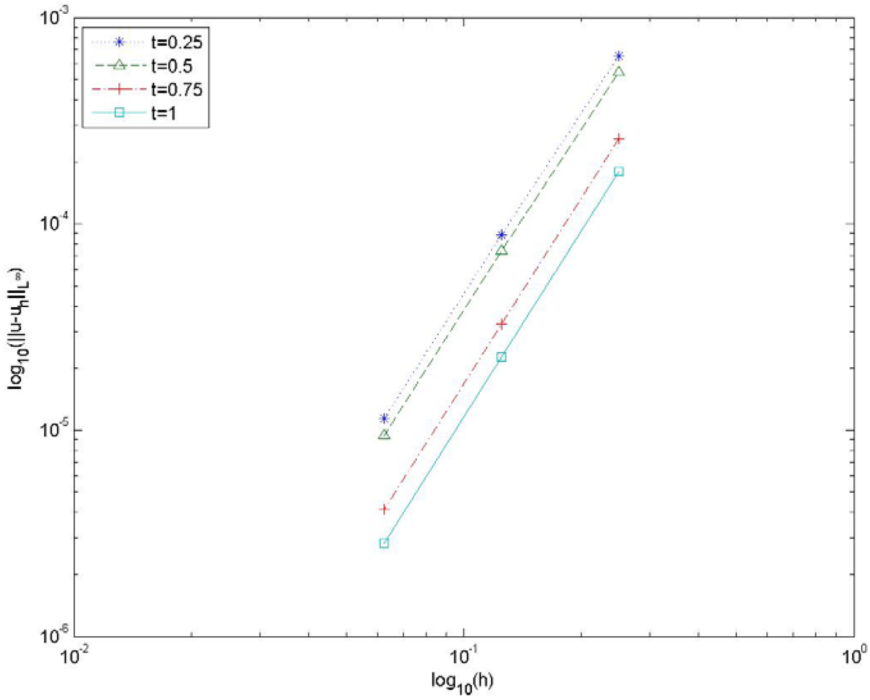

Figure 31 The graph of the L^∞-errors against the space step h for solutions of Problem 2 with $r=2$ for $\sigma_0=3$ and $\varepsilon=-1$.

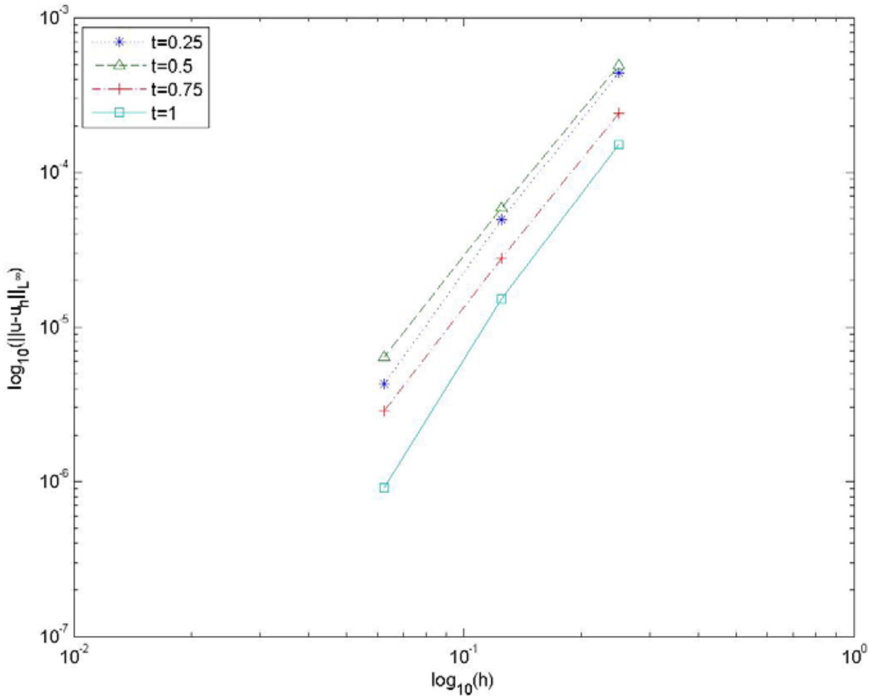

Figure 32 The graph of the L^∞-errors against the space step h for solutions of Problem 2 with $r=2$ for $\sigma_0=50$ and $\varepsilon=1$.

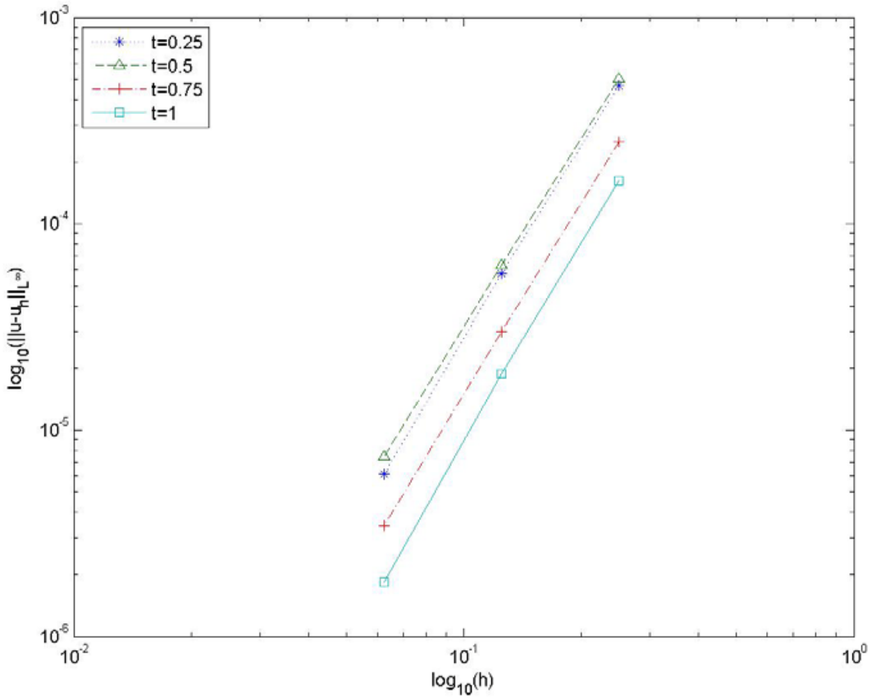

Figure 33 The graph of the L^∞-errors against the space step h for solutions of Problem 2 with $r=2$ for $\sigma_0=50$ and $\varepsilon=0$.

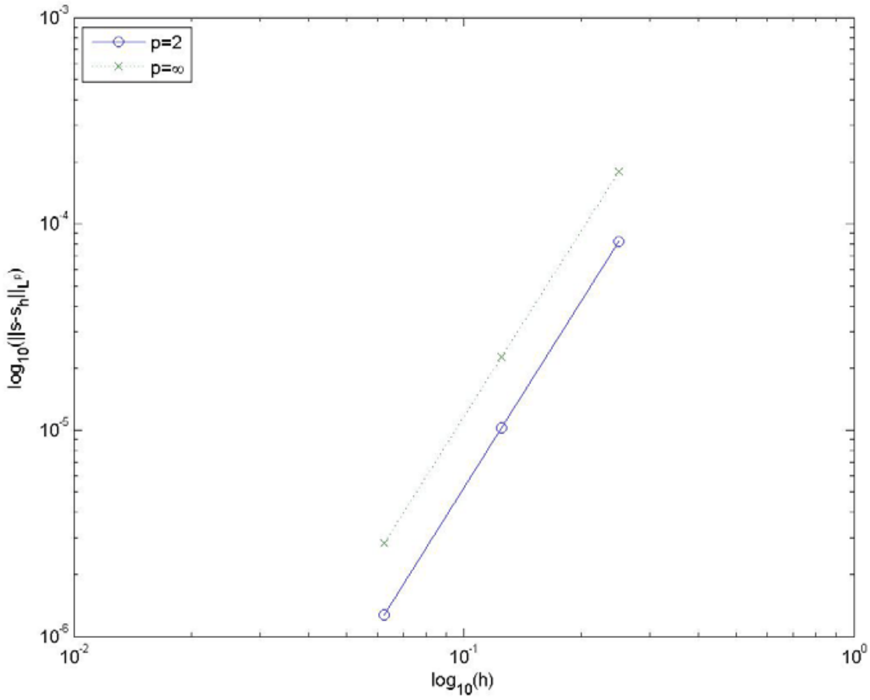

Figure 34 The graph of the L^2- and L^∞-errors against the space step h for free boundary of Problem 2 with $r=2$ for $\sigma_0=3$ and $\varepsilon=-1$.

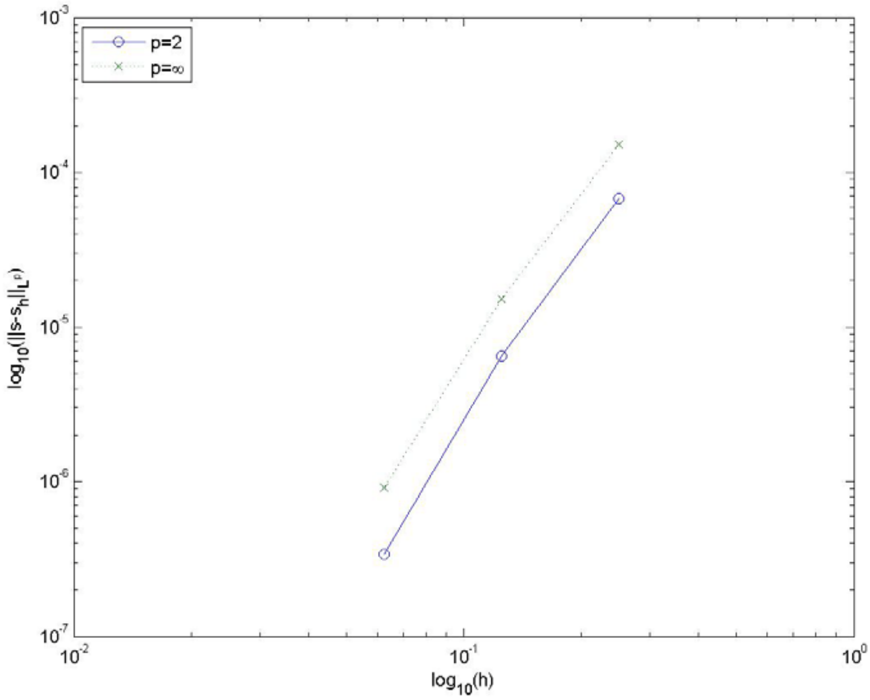

Figure 35 The graph of the L^2- and L^∞-errors against the space step h for free boundary of Problem 2 with $r = 2$ for $\sigma_0 = 50$ and $\varepsilon = 1$.

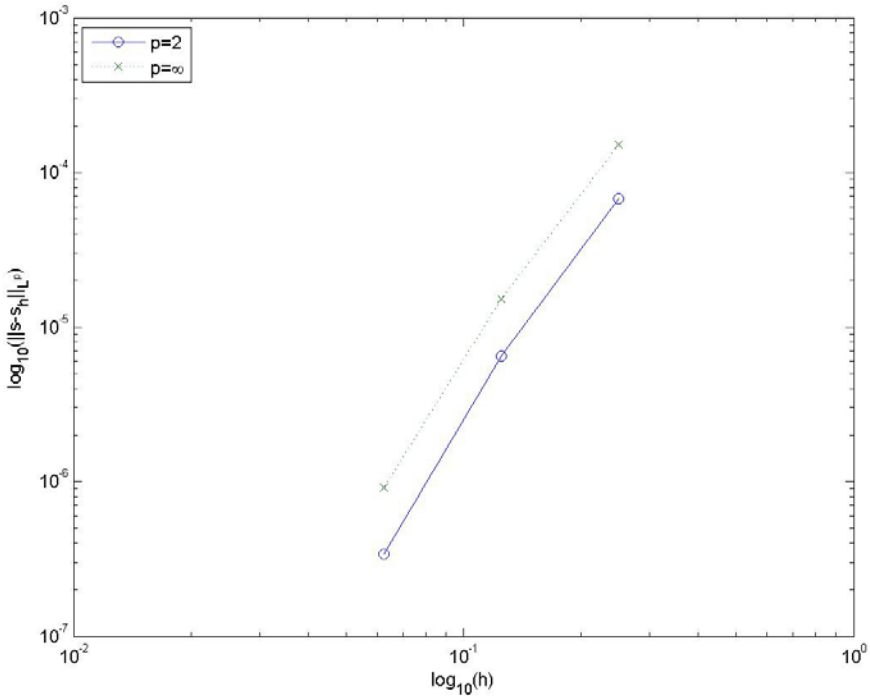

Figure 36 The graph of the L^2- and L^∞-errors against the space step h for free boundary of Problem 2 with $r = 2$ for $\sigma_0 = 50$ and $\varepsilon = 0$.

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