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Design of Minimum-Phase FIR Digital Filters by Differential Cepstrum

SOO-CHANG PEI AND SHEN-TAN LU

Abstract—**The differential cepstrum is introduced to design equiripple minimum phase FIR filters by using cepstral deconvolution; this fast procedure only takes three FFT computation and avoids the complicated phase wrapping and polynomial root-finding algorithms.**

I. Introduction

In recent years, there has been a considerable interest in the design of finite impulse response (FIR) linear phase digital filters; a powerful computer program [1] is available to efficiently design linear phase FIR filters. However, for some applications, the delay introduced by these linear phase filters is prohibitive; when the linear phase property is not desirable, much lower delay time can be achieved with minimum phase filters satisfying the same specifications on the amplitude. Optimum magnitude minimum phase filters are highly attractive in several applications, such as CTD transversal filters and communication channel filters, etc.

The optimum magnitude minimum phase FIR filter design has been theoretically described by Herrman and Schuessler [2], it transforms equiripple linear phase designs into equiripple minimum-phase designs with half the degree and with attenuation characteristics equal to the square root of the prototype, this method is limited to the numerical root-finding difficulty, Recently, Mian and Nainer proposes a design procedure to overcome this difficulty by using homomorphic deconvolution [3], however the complicated phase wrapping algorithm will be involved in the computation; In this paper, the differential cepstrum is introduced to design equiripple minimum phase FIR filters by using cepstral deconvolution; This fast procedure only takes three FFT computation and avoids the complicated phase wrapping and polynomial root-finding algorithms.

II. The Differential Cepstrum: Definition, Implementation and Property

Consider the convolution

$$x(n)=x_1(n)*x_2(n). \tag{1}$$

Then

$$\frac{\frac{d}{dz}X(z)}{X(z)}=\frac{\frac{d}{dz}X_1(z)}{X_1(z)}+\frac{\frac{d}{dz}X_2(z)}{X_2(z)}. \tag{2}$$

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The differential cepstrum [4], [5] of $x(n)$ is defined as

$$\hat{x}_d(n)\triangleq z^{-1}\left\{\frac{\frac{d}{dz}X(z)}{X(z)}\right\}. \tag{3}$$

Equation (2) becomes

$$\hat{x}_d(n)=\hat{x}_{1d}(n)+\hat{x}_{2d}(n) \tag{4}$$

which can be used as a new tool for homomorphic deconvolution.

$$\therefore\quad \hat{x}_d(n)=\frac{1}{2\pi j}\oint_c\left\{\frac{\frac{d}{dz}X(z)}{X(z)}\right\}z^{n-1}\,dz. \tag{5}$$

This allows for the following DFT implementation:

$$X(k)=\sum_{n=0}^{N-1}x(n)e^{-j2\pi nk/N}$$

$$X'(k)=-j\sum_{n=0}^{N-1}nx(n)e^{-j2\pi nk/N}$$

$$\therefore\quad \hat{x}_d(n)=\frac{-j}{N}\sum_{k=0}^{N-1}\frac{X'(k)}{X(k)}e^{j(2\pi/N)k(n-1)}. \tag{6}$$

The differential cepstrum avoids the complicated logarithm and phase wrapping operations existed in the log cepstrum, and have some interesting properties, such as shift invariance, delay time measurement and scale standardization [1], [2]. The properties of the differential cepstrum can be summarized as follows:

P1: The differential cepstrum is shift and scale invariant. If $y(n)=Ax(n-r)$, then

$$\hat{y}_d(n)=\begin{cases}\hat{x}_d(n), & \text{for } n\neq 1\\ r, & \text{for } n=1\end{cases} \tag{7}$$

where A is a scale factor, r is a delay and $\hat{x}_d(n)$ is the differential cepstrum of a normalized exponential sequence $x(n)$ without delay.

P2: The differential cepstrum $\hat{x}_d(n)$ is related its corresponding log-cepstrum $\hat{c}(n)$ and the original sequence $x(n)$.

$$\hat{c}(n)=\frac{\hat{x}_d(n+1)}{n},\qquad n\neq 0$$

and

$$-(n-1)x(n-1)=\sum_{k=-\infty}^{\infty}\hat{x}_d(k)x(n-k). \tag{8}$$

P3: If $x(n)$ is minimum phase (no poles or zeros outside the

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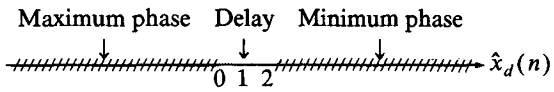
unit circle) then

$$\hat{x}_d(n) = 0, \qquad n < 1. \tag{9}$$

P4: If $x(n)$ is maximum phase (no poles or zeros inside the unit circle) then

$$\hat{x}_d(n) = 0, \qquad n > 1. \tag{10}$$



P5: If $x(n)$ is of finite duration, $\hat{x}_d(n)$ will nevertheless have infinite duration.

III. Minimum Phase Filter Design

Following Herrman and Schuessler's method [2], the relationship between linear phase $H(z)$, second-order zeros $H_1(z)$ and minimum phase $H_2(z)$ is presented as following:

$$H_1(z) = H(z) + \delta_2 z^{-(N-1)/2}$$

$$H_1(z) = z^{-(N-1)/2} H_2(z) \cdot H_2(z^{-1})$$

$$H_2(e^{jw}) = \sqrt{H_1(e^{jw})} \tag{11}$$

where δ_1 and δ_2 are the passband and stopband ripples of the filter $H(z)$. Due to the very simple relationship between the linear phase and minimum phase cepstral sequences $\hat{c}_1(n)$ and $\hat{c}_2(n)$ associated, respectively, with $H_1(z)$ and $H_2(z)$, the minimum phase filter impulse response $h_2(n)$ can be obtained from $\hat{c}_2(n)$ [3].

$$\hat{c}_2(n) = \tfrac{1}{2}[\hat{c}_1(n) + \hat{c}_1(-n)], \qquad n > 0$$

$$h_2(n) = c_2(n) h_2(0) + \sum_{k=0}^{n-1} \frac{k}{n} c_2(k) h(n-k), \qquad 0 \leqslant n \leqslant (N-1)/2. \tag{12}$$

The $\hat{c}_1(n)$ can be efficiently calculated by its corresponding differential cepstrum $\hat{h}_{d1}(n)$, by Property 2

$$\hat{c}_1(n) = \frac{\hat{h}_{d1}(n+1)}{n}, \qquad n \neq 0$$

then

$$\hat{c}_2(n) = -\frac{1}{2n}[\hat{h}_{d1}(n+1) - \hat{h}_{d1}(-n+1)]. \tag{13}$$

In conclusion, the steps necessary to obtain $h_2(n)$ from $h_1(n)$ are the following:

1) Prepare $h_1(n)$ of length N (odd).

2) Choose $\rho > 1$ and $\rho \cong 1$ for avoiding the zeros on the unit circle.

(i) $H_1 \leftarrow (\mathrm{FFT})_L\{\rho^{-n} h_1(n)\}$, $L \gg N$ to reduce the aliasing error.

(ii) $H_1' \leftarrow (\mathrm{FFT})_L\{n\rho^{-n} h_1(n)\}$

(iii) $d(n) \leftarrow (\mathrm{IFFT})_L\{-H_1'/H_1\}$

(iv) $\hat{h}_{d1}(n+1) \leftarrow \rho^n \cdot d(n)$, $-(N+1)/2 < n < (N+1)/2$.

3) Calculate $\hat{c}_2(n)$ from $\hat{h}_{d1}(n)$ by (13).

$$\hat{c}_2(n) \leftarrow -\frac{1}{2n}[\hat{h}_{d1}(n+1) - \hat{h}_{d1}(-n+1)], \qquad 0 < n < \frac{N+1}{2}.$$

4) Obtain $t(n)$ from the recursion formula in (12).

$$t(n) \leftarrow \begin{cases} 0, & n < 0 \\ h_1(n), & n = 0 \\ c_2(n) t(0) + \sum_{k=0}^{N-1} \frac{k}{n} c_2(k) t(n-k), & 0 < n < \frac{N+1}{2}. \end{cases}$$

5) Get the scaling factor R for unity in the passband

$$R \leftarrow \frac{\sqrt{\left|\sum_{n=0}^{N-1} h_1(n)\right|}}{\sum_{n=0}^{M-1} t(n)}, \qquad 0 \leqslant n \leqslant \frac{N-1}{2}, \quad M = \frac{N+1}{2}$$

6) Scaling the impulse response $t(n)$

$$h_2(n) \leftarrow R \cdot t(n), \qquad 0 \leqslant n \leqslant \frac{N-1}{2}.$$

Then $h_2(n)$ is the desired minimum phase filter impulse response of length $(N+1)/2$.

7) The passband and stopband ripples δ_1' and δ_2' of minimum phase filter is obtained as follows [2]:

$$\delta_1' = \sqrt{1 + \frac{\delta_1}{1+\delta_1}} - 1 \simeq \frac{\delta_1}{2}$$

$$\delta_2' = \sqrt{\frac{2\delta_2}{1+\delta_2}} \simeq \sqrt{2\delta_2} \tag{14}$$

A choice $L \geqslant 8N$, $L = 1024$, and $\rho \cong 1.026$ [3] worked quite well in all the examples considered.

IV. Experimental Results

The proposed method has been tested on a large number of filters on a VAX-11 computer. The linear phase filter $H(z)$ were first designed by the popular computer program given in [1]. The accuracy of the procedure has been measured by the deviation of $|H_2(e^{jw})|$ from its theoretical value $\sqrt{|H_1(e^{jw})|}$. Tests carried out have shown that all examples obtain accurate results for 9 filters in Table I using single precision arithmetic; Some of design examples for equiripple minimum phase filters are shown in Figs. 1 and 2.

V. Conclusions

In this paper, the differential cepstrum is introduced to design equiripple minimum phase FIR filters by using cepstral deconvolution; This fast procedure avoids the complex logarithm, phase wrapping and polynomial root-finding algorithm, and requires only three FFT computation and a few other operations. Some design examples are illustrated to show this effective approach using differential cepstrum.

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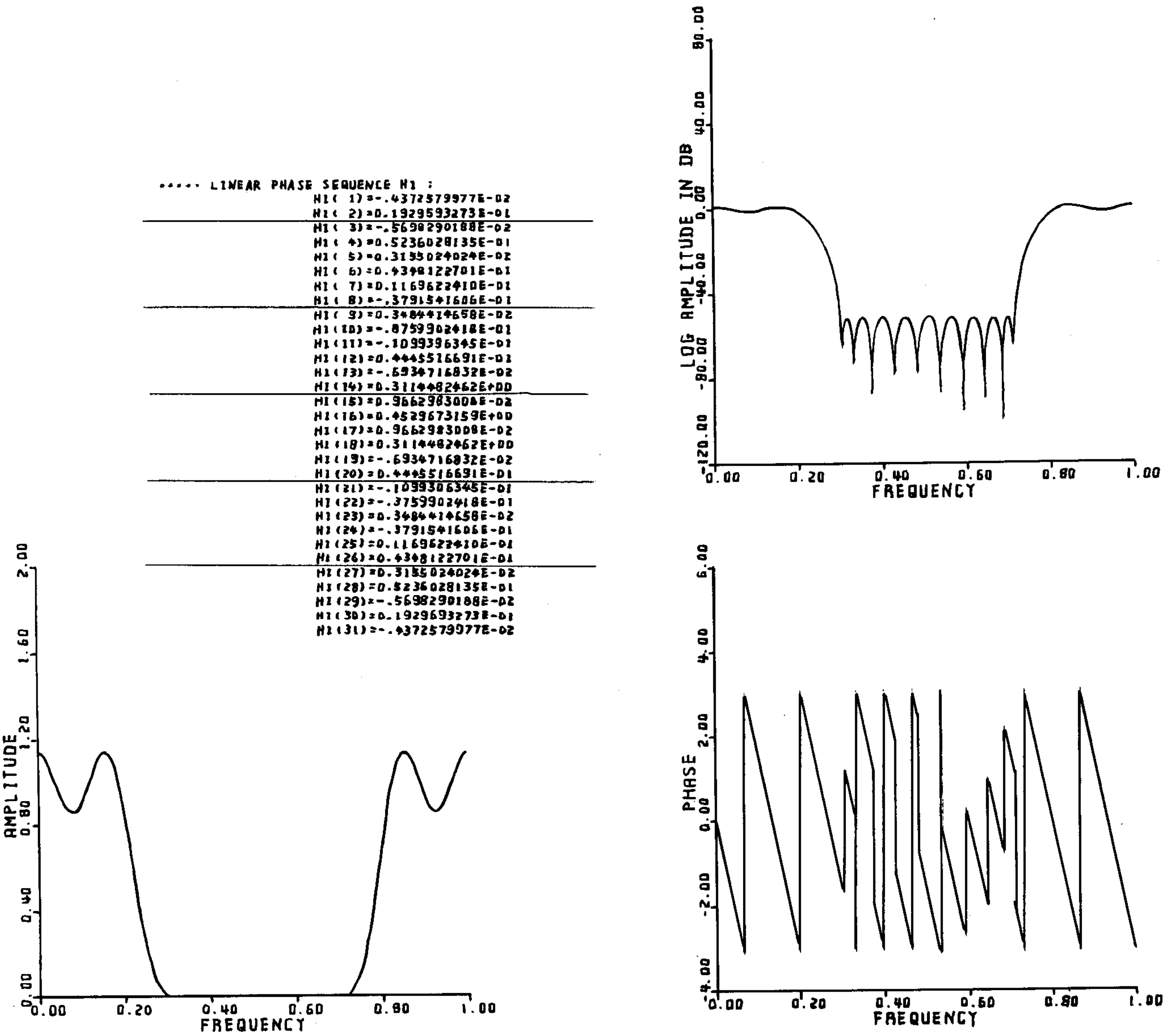

(a) (b)

Fig. 1. (a) Impulse and magnitude responses of an equiripple linear phase bandstop filter of length 31. (b) Log magnitude and phase responses of an equiripple linear phase bandstop filter of length 31.

```
MINIMUM PHASE SEQUENCE H2 :
                    H2( 1)=0.2150656134E+00
                    H2( 2)=0.1349139959E-01
                    H2( 3)=0.4390382171E+00
                    H2( 4)=0.1152965985E-01
                    H2( 5)=0.4662240124E+00
                    H2( 6)=-.1621417888E-01
                    H2( 7)=0.9569749236E-01
                    H2( 8)=-.2666388266E-01
                    H2( 9)=-.1420112262E+00
                    H2(10)=0.3234097601E-02
                    H2(11)=-.9216353297E-01
                    H2(12)=0.2556440234E-01
                    H2(13)=0.5818791928E-01
                    H2(14)=0.9289475158E-02
                    H2(15)=0.9098791331E-01
                    H2(16)=-.2033299953E-01
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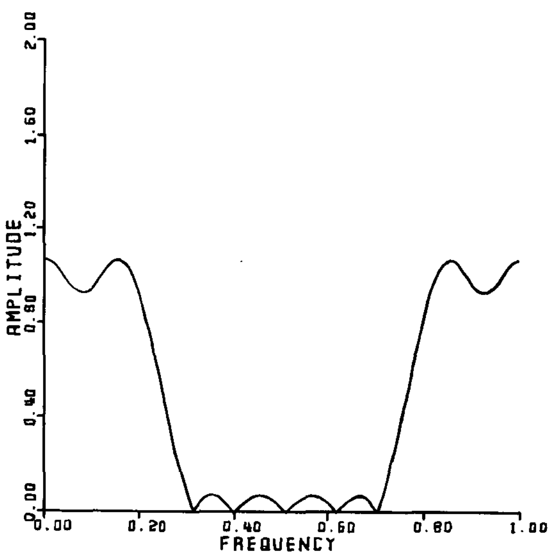
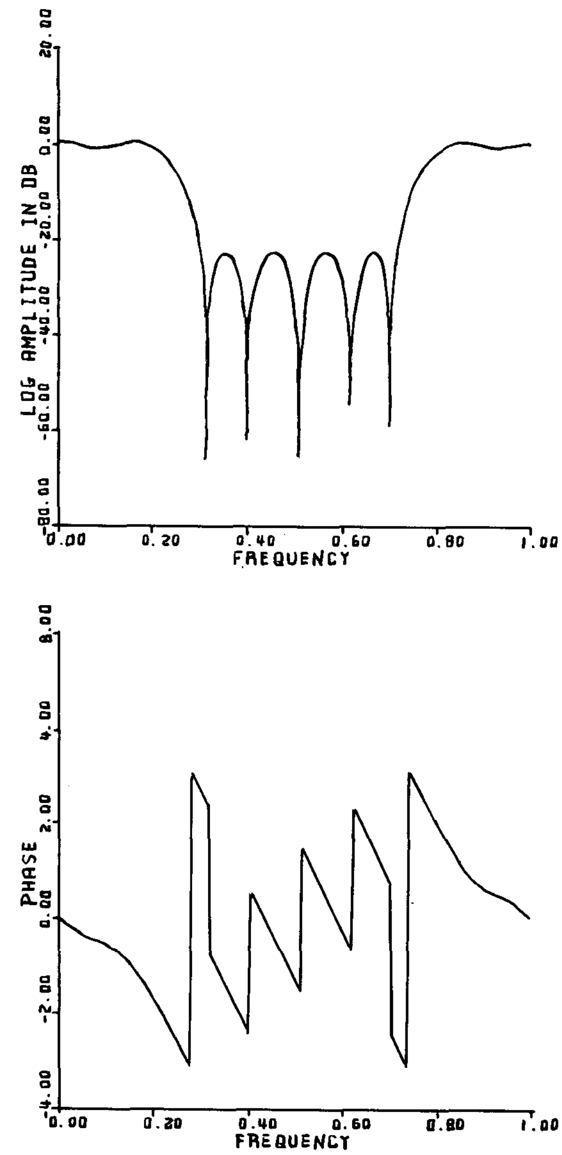
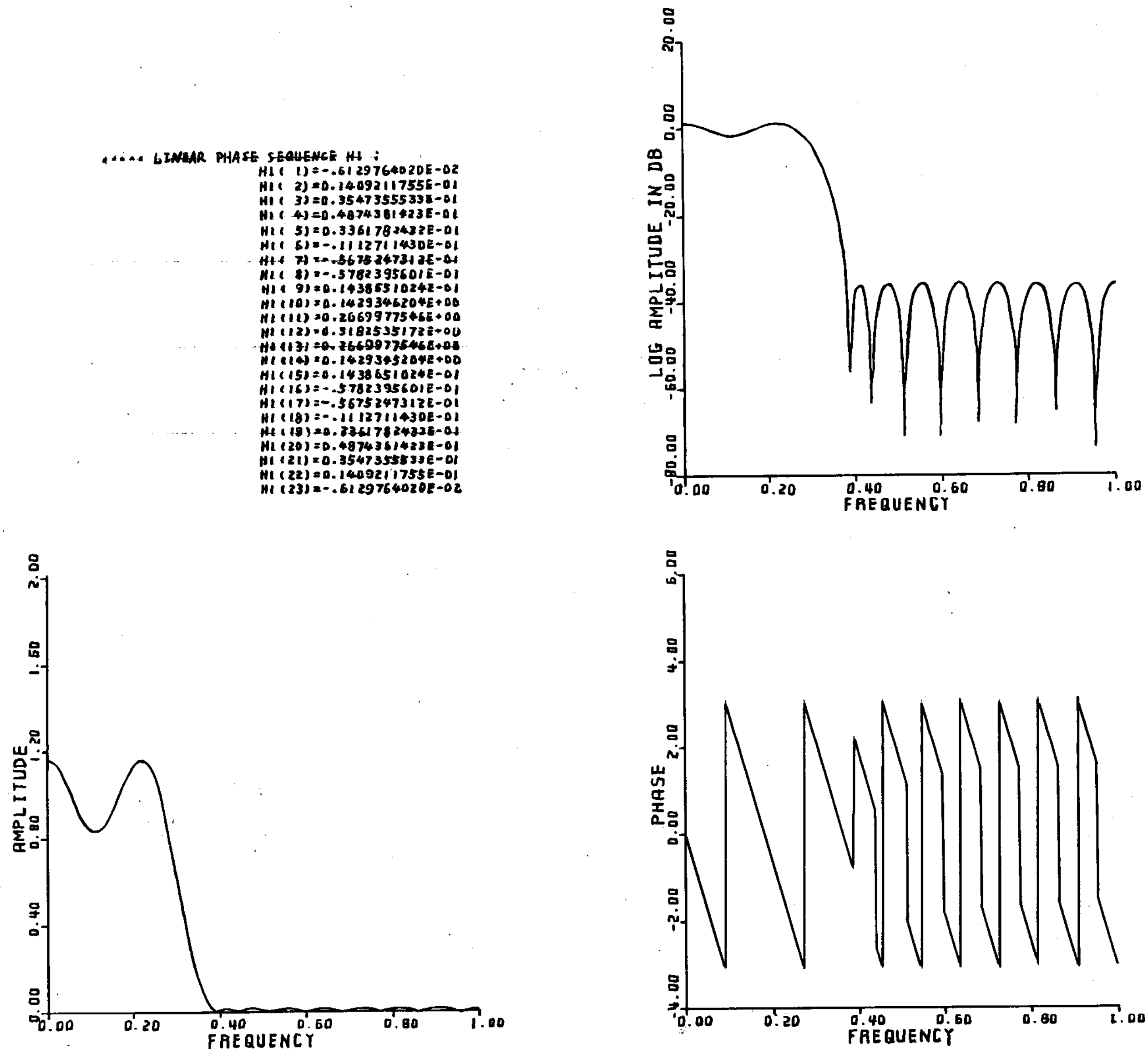


(c)



(d)

(c) Impulse and magnitude responses of an equiripple minimum phase band stop filter of length 16. (d) Log magnitude and phase responses of an equiripple minimum phase bandstop filter of length 16.



(a) (b)

Fig. 2. Impuse and magnitude responses of an equiripple linear phase low-pass filter of length 23. (b) Log magnitude and phase responses of an equiripple linear phase low-pass filter of length 23. (c) Impulse and magnitude responses of an equiripple minimum phase low-pass filter of length 12. (d) Log magnitude and phase responses of an equiripple minimum phase low-pass filter of length 12.

```
MINIMUM PHASE SEQUENCE H2 :
                    H2( 1)=0.2117003053E+00
                    H2( 2)=0.2868788540E+00
                    H2( 3)=0.3227372165E+00
                    H2( 4)=0.2449565828E+00
                    H2( 5)=0.9156860411E-01
                    H2( 6)=-.4998296872E-01
                    H2( 7)=-.1062501371E+00
                    H2( 8)=-.6862803549E-01
                    H2( 9)=0.9852915071E-02
                    H2(10)=0.6833804399E-01
                    H2(11)=0.1057921499E+00
                    H2(12)=-.2894456126E-01
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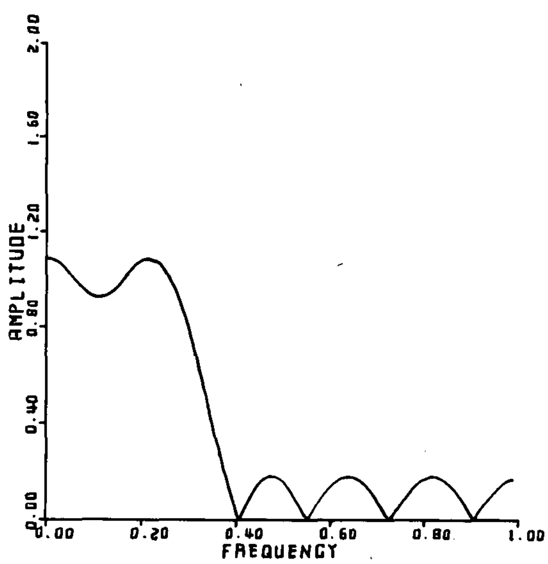
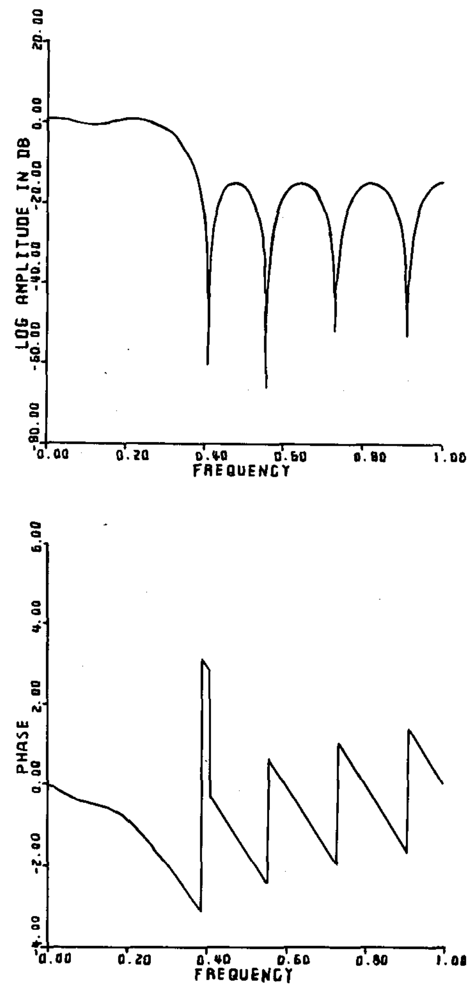


(c)



(d)

TABLE I

	LINEAR PHASE DEVIATION		MINIMUM PHASE DEVIATION			
			ESTIMATION		MEASUREMENT	
	pass band	stop band	pass band	stop band	pass band	stop band
1	0.144	.00288	0.072	0.076	0.071	0.075
2	0.170	.01671	0.085	0.183	0.086	0.181
3	0.072 0.086	0.004	0.036 0.043	0.089	0.038 0.044	0.090
4	.0006 .00011 .00051	.0006	.0003 .00006 .00026	0.035	----- ----- -----	0.035
5	0.005	0.001	.0025	.0447	-----	0.044
6	0.097	0.003	.0485	0.775	0.049	0.780
7	0.097	0.003	.0485	0.775	0.049	0.780
8	0.072 0.086	0.003	0.035 0.043	0.775	0.038 0.044	0.780
9	0.200	0.150	0.100	0.548	0.096	0.551

P.S. ----- : Too small to measure

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Accurate Predistorted Design of Single Op Amp Active Filters

JACEK M. ZURADA

Abstract —**A general method of forcing the transfer function of a second-order active filter to have poles in the desired location is presented. Closed-form formulas for predistorted design of commonly used filters have been derived and tabulated for designer's reference.**

INTRODUCTION

Most of the previous work on the finite gain–bandwidth (GB) effects in active filters pioneered by [1] has centered mainly on the analysis of the pole shift from its nominal position to an undesired position. In such an analytical approach, the loci of actual poles or incremental sensitivities of nominal pole due to finite GB may be found [2].

The technique outlined below is aimed toward the exact predistorted design, and represents a synthetic approach. It is based on a general method of forcing a characteristic equation of order n to have n desired solutions by adjusting its polynomial coefficients [3].

In recent literature, predistortion problem of active networks has been dealt with in [4]–[6]. Given the GB values of op amps and desired value of pole frequency ω_n and selectivity q, the designed filter may be predistorted so that the actual filter has the desired pole as well as unavoidable parasitic pole(s). The approach in this paper yields accurate predistorted design formulas and does not involve numerical methods for solving resulting nonlinear equations.

The pole defining equation for a practical filter has the form

$$s^n + \sum_{k=0}^{n-1} C_k p_k(s) s^k = 0 \tag{1}$$

where the parasitic effects are represented by the nonideal multipliers $p_k(s)$ and the coefficients C_k are to be chosen so as to force (1) to have n desired solutions. The nonideal multipliers are unity for the case of ideal op amps having GB at infinity [3]. For a second-order filter with desired solutions at

$$Z_1 = Z_2^* = -\frac{\omega_n}{2q} + j\omega_n\sqrt{1-\frac{1}{4q^2}}. \tag{2}$$

Equation (1) may be replaced with two simultaneous equations:

$$Z_i^2 + p_1(Z_i)C_1 Z_i + p_0(Z_i)C_0 = 0, \qquad i = 1,2. \tag{3}$$

The set above can be used to calculate two required coefficients C_1, C_0. For 2nd order filters discussed below, expressions for predistorted natural pole frequency ω_n and selectivity q are developed, rather than C_1, C_0.

PREDISTORTION OF A SINGLE COMPLEX PAIR OF ROOTS

If each filter op-amp is characterized by the transfer function $G(s) = -GB/s$, then a practical pole yielding equation of a 2nd order filter can be written as

$$\sum_{k=3}^{n} h_k s^k + h_2 s^2 + h_1 \frac{\omega_0}{q_0} s + \omega_0^2 = 0 \tag{4}$$

where $n-2$ is the number of op-amps used to implement the filter. Obviously, $h_3 = h_4 \cdots = 0$ and $h_2 = h_1 = 1$ for the ideal op amp case.

Equation (4) can always be rewritten to the form similar to (3)

$$s^2 + p_1(s)\frac{\omega_0}{q_0}s + p_0(s)\omega_0^2 = 0 \tag{5}$$

where $p_1(s), p_0(s)$ are nonideal multipliers which are defined as follows

$$p_0(s) \triangleq \left(h_2 + \sum_{k=3}^{n} h_k s^{k-2}\right)^{-1} \tag{6a}$$

$$p_1(s) \triangleq h_1 p_0(s). \tag{6b}$$

In all equations above ω_0, q_0 denote pole frequency and selectivity of an apparent pole which is not to be practically achieved for finite GB values. For a second-order filter with desired ω_n, q values, one has to force two roots of the equation (5) to be at z and z^* where

$$z = -\omega_n/2q + j\omega_n\sqrt{1-1/4q^2}. \tag{7}$$

Then (5) will hold for $s = z$, and

$$z^2 + p_1(z)\omega_0/q_0 z + p_0(z)\omega_0^2 = 0. \tag{8}$$

From the above equation, q_0 may be easily calculated as

$$q_0 = -\frac{\alpha\omega_0 z}{z^2 + \beta\omega_0^2} \tag{9}$$

where $\alpha \triangleq p_1(z)$, $\beta \triangleq p_0(z)$.

Because q_0 is the real number, condition $q_0 = q_0^*$ can be used to solve for ω_0 as follows:

$$\omega_0^2 = \frac{\mathrm{Im}(\alpha^* z)}{\mathrm{Im}(\alpha z \beta^*)}\omega_n^2. \tag{10}$$

Expressions (10) and (9) give the predistorted values of ω_n, q, being targeted pole parameters expressed in standard design values of ω_0, q_0. These ω_0, q_0 values are describing the predistorted or apparent pole value of the system with desired poles z,

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