Shifted-Jacobi Series Analysis of Linear Optimal Control Systems Incorporating Observers

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ABSTRACT: *This paper uses the Jacobi series to analyze linear optimal control systems incorporating observers. The method simplifies the system of equations into the successive solution of a set of linear algebraic equations. An illustrative example is included to demonstrate that only a small number* (m = 6) *of shifted-Jacobi series are needed to obtain an accurate solution.*

I. Introduction

Orthogonal functions, often used to represent an arbitrary time function, have recently been used to solve control problems. Typical examples are the Walsh functions (**1**), block-pulse functions (**2**), Laguerre polynomials (**3**), Legendre polynomials (**4**) and Chebyshev series (**5**).

Stavroulakis and Tzafestas (**6**) first used the Walsh function to analyze an optimal control system incorporating an observer, but the results were derived based on very unrealistic assumptions. These assumptions were corrected by Kawaji and Tada (**7**), where the Walsh series was adopted to solve the optimal control law of linear systems incorporating observers. More recently, Chou and Horng (**5**) applied the shifted-Chebyshev series to approach the same problem.

In the present paper, the shifted-Jacobi series (**8**) is taken to facilitate research on the analysis of linear optimal control systems incorporating an observer.

II. Problem Statement

Consider a linear time-invariant controllable system

$$\dot{\mathbf{X}}(t) = \mathbf{A}\mathbf{X}(t) + \mathbf{B}\mathbf{U}(t) \tag{1a}$$

$$\mathbf{Y}(t) = \mathbf{C}\mathbf{X}(t) \tag{1b}$$

with the performance index

$$J = \int_0^\infty [\mathbf{X}^T(t)\mathbf{Q}\mathbf{X}(t) + \mathbf{U}^T(t)\mathbf{R}\mathbf{U}(t)]\,\mathrm{d}t;\ \mathbf{Q} \geqslant \mathbf{0},\quad \mathbf{R} > \mathbf{0} \tag{2}$$

where $\mathbf{X}(t)$ is the $n \times 1$ state vector, $\mathbf{U}(t)$ is the $q \times 1$ control vector, $\mathbf{Y}(t)$ is the $p \times 1$ output vector, and $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are $n \times n$, $n \times q$, $p \times n$ constant matrices, respectively.

The problem considered in this paper is to find the optimal control law $\mathbf{U}^*(t)$ for the system of Eq. (1), and at the same time minimize the performance index (2) subject to the following constraints (**9**):

(1) an $(n-p)$-dimensional Luenberger observer is constructed to incorporate the system, and
(2) the optimal control $\mathbf{U}^*(t)$ is achieved by using digital computation.

It is well known that the optimal control law is given by (**10**)

$$\mathbf{U}^*(t) = -\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}\mathbf{X}(t) = \mathbf{K}\mathbf{X}(t) \tag{3}$$

where the superscript T denotes transpose, and $\mathbf{P}$ is the unique positive-definite solution of the Riccati equation:

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{Q} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} = \mathbf{0}.$$

But, in general application, only the output vector $\mathbf{Y}(t)$ is available for measurement. In this case, the control signal may be realized with $(n-p)$-dimensional state observers (**11**)

$$\dot{\mathbf{Z}}(t) = \mathbf{D}\mathbf{Z}(t) + \mathbf{G}\mathbf{Y}(t) + \mathbf{H}\mathbf{V}(t) \tag{4a}$$

$$\hat{\mathbf{X}}(t) = \mathbf{M}\mathbf{Y}(t) + \mathbf{N}\mathbf{Z}(t) \tag{4b}$$

$$\mathbf{U}^*(t) = \mathbf{K}\hat{\mathbf{X}}(t) \tag{5}$$

where $\mathbf{Z}(t)$ is a $(n-p) \times 1$ vector and $\mathbf{D}$, $\mathbf{G}$, $\mathbf{H}$, $\mathbf{M}$, $\mathbf{N}$ are matrices of appropriate dimensions. For the dynamic system of (4) to be an observer, the following relationship must be hold (**12**):

$$\mathbf{Z}(t) = \mathbf{U}\mathbf{X}(t) + \mathbf{e}(t) \tag{6}$$

$$\dot{\mathbf{e}}(t) = \mathbf{D}\mathbf{e}(t) \tag{7}$$

where

$$\mathbf{U}\mathbf{A} - \mathbf{D}\mathbf{U} = \mathbf{G}\mathbf{C} \tag{8a}$$

$$\mathbf{H} - \mathbf{U}\mathbf{B} = \mathbf{0} \tag{8b}$$

$$\mathbf{M}\mathbf{C} + \mathbf{N}\mathbf{U} = \mathbf{I}_n \tag{8c}$$

where $\mathbf{I}_n$ stands for an $n \times n$ identity matrix. Substituting Eqs (4b), (6) and (8c) into (5), we can obtain

$$\mathbf{U}^*(t) = \mathbf{K}\mathbf{X}(t) + \mathbf{K}\mathbf{N}\mathbf{e}(t). \tag{9}$$

Inserting Eq. (9) into (1) yields

$$\dot{\mathbf{X}}(t) = (\mathbf{A} + \mathbf{B}\mathbf{K})\mathbf{X}(t) + \mathbf{B}\mathbf{K}\mathbf{N}\mathbf{e}(t) \triangleq \hat{\mathbf{A}}\mathbf{X}(t) + \hat{\mathbf{B}}\mathbf{e}(t). \tag{10}$$

It follows from Eq. (9) that the solution of (7) and (10) is necessary for the determination of the control law. In the next section, we adopt the shifted-Jacobi series to carry out the solution of these equations. This approach will result in an algebraic matrix equation which is conveniently available for digital computation.

III. Shifted-Jacobi Series Approach

The Jacobi polynomial can be represented in terms of a hypergeometric function in the interval $-1 \leqslant Z \leqslant 1$

$$P_n^{(a,b)}(Z) = \frac{(a+1)n}{n!}\, {}_2F_1\left(\begin{matrix} -n & n+a+b+1 \\ & b+1 \end{matrix} \middle| \frac{1-Z}{2}\right)$$
$$= (-1)^n \frac{(b+1)n}{n!}\, {}_2F_1\left(\begin{matrix} -n & n+a+b+1 \\ & b+1 \end{matrix} \middle| \frac{1+Z}{2}\right). \quad (11)$$

In the form, the subscripts 2 and 1 become clear. The leading subscript 2 indicates that two Pochhammer symbols in the numerator and the final subscript 1 indicates one Pochhammer symbol in the denominator. We try to transform the independent variable into values between 0 and t_f, and let

$$\frac{t}{t_f} = \frac{1+Z}{2}.$$

Then, the shifted-Jacobi polynomials become

$$J_n^{(a,b)}(t) = (-1)^n \frac{(b+1)n}{n!}\, {}_2F_1\left(\begin{matrix} -n & n+a+b+1 \\ & b+1 \end{matrix} \middle| \frac{t}{t_f}\right)$$

where t_f is the final time and a and b are parameters with $a \geqslant -1$, $b \geqslant -1$ and

$$(a+1)n = (a+1)(a+2)\dots(a+n)$$
$$(a+1)_0 = 1.$$

Let $c = a+b+1$, then

$${}_2F_1\left(\begin{matrix} -n & n+c \\ & b+1 \end{matrix} \middle| \frac{t}{t_f}\right) = \sum_{k=0}^{\infty} \frac{(-1)_k(n+c)_k}{(b+1)_k \cdot k!}\left(\frac{t}{t_f}\right)^k. \quad (12)$$

The recursive formulas for shifted-Jacobi polynomials are

$$J_0(t) = 1$$
$$J_1(t) = -(b+1)+(c+1)\left(\frac{t}{t_f}\right)$$
$$J_2(t) = \tfrac{1}{2}\left[(b+1)_2 - 2(b+2)(c+2)\left(\frac{t}{t_f}\right) + (c+2)_2\left(\frac{t}{t_f}\right)^2\right]$$
$$\vdots$$
$$J_n(t) = \frac{(-1)^n(b+1)}{n!}\left[1 + \sum_{k=1}^{n} (-1)^k \frac{(c+n)_k}{(b+1)_k}\left(\frac{t}{t_f}\right)^K\right] \quad (13)$$

with

$$2n(n+a+b)(2n+a+b-2)J_n(t)$$
$$= (2n+a+b-1)\left[(2n+a+b)(2n+a+b-2)\left(2\frac{t}{t_f}-1\right)+a^2-b^2\right]J_{n-1}(t)$$
$$-2(n+a-1)(n+b-1)(2n+a+b)J_{n-2}(t). \quad (14)$$

The orthogonality condition is

$$\int_0^{t_f} t^b(t_f-t)^a J_n(t)J_m(t)\,\mathrm{d}t = \begin{cases} 0 & n \neq m \\ \left[\dfrac{\Gamma(n+a+1)\Gamma(n+b+1)}{(2n+c)(n!)\Gamma(n+c)}\right](t_f)^{a+b+1} & n = m \end{cases} \quad (15)$$

where $\Gamma(\cdot)$ stands for the gamma function. Note that an arbitrary time function $f(t)$ can be approximated by the Jacobi polynomials as

$$f(t) = \sum_{n=0}^{\infty} f_n J_n(t). \quad (16)$$

For practical application, we use only the finite-term of the series to approximate $f(t)$. That is

$$f(t) \simeq \sum_{n=0}^{m-1} f_n J_n(t) = \mathbf{f}^T\mathbf{J}(t) \quad (17)$$

where

$$\mathbf{f}^T = [f_0, \quad f_1, \ldots, f_{m-1}]$$

and

$$\mathbf{J}^T = [J_0(t), \quad J_1(t), \ldots, J_{m-1}(t)].$$

The Jacobi coefficient f_n is obtained by minimizing the integral square error

$$E = \int_0^{t_f} t^b(t_f-t)^a\left[f(t')-\sum_{n=0}^{m-1} f_n J_n(t')\right]^2 \mathrm{d}t'. \quad (18)$$

Using the necessary condition of minimizing E

$$\frac{\partial E}{\partial f_j} = 0 \quad j = 0, 1, 2, \ldots, m-1 \quad (19)$$

we obtain

$$f_n = \frac{(2n+c)(n!)\Gamma(n+c)}{\Gamma(n+a+1)\Gamma(n+b+1)t_f^{(a+b+1}}\int_0^{t_f} t^b(t_f-t)^a f(t)J_n(t)\,\mathrm{d}t$$
$$n = 0, 1, 2, \ldots, m-1. \quad (20)$$

The integration for the shifted-Jacobi series can be represented by (8)

$$\int_0^t J_n(t')\,\mathrm{d}t' = t_f\left[\frac{(n+c)}{(2n+c+1)(2n+c)}J_{n+1}(t) + \frac{(a-b)}{(2n+c+1)(2n+c-1)}J_n(t) - \frac{(n+a)(n+b)}{(2n+c)(2n+c-1)(2n+c+1)}J_{n-1}(t) + \frac{(-1)^n\Gamma(n+b+1)}{(n+c-1)(n+1)!\Gamma(b)}J_0(t)\right],$$

$$n = 0, 1, \ldots, m-1 \tag{21}$$

or in vector form

$$\int_0^t J(t')\,\mathrm{d}t' = FJ(t) \tag{22}$$

where F is the operational matrix of integration, given by (23) on the next page.

Now, we wish to represent the state vector $\mathbf{X}(t)$ and error vector $\mathbf{e}(t)$ by shifted-Jacobi polynomials:

$$\mathbf{e}(t) = \sum_{n=0}^{m-1} E_n J_n(t) = \mathbf{E}^T\mathbf{J}(t) \tag{24}$$

$$\mathbf{X}(t) = \sum_{n=0}^{m-1} X_n J_n(t) = \mathbf{X}^T\mathbf{J}(t) \tag{25}$$

where E_n and X_n are the coefficients of the Jacobi series for $\mathbf{e}(t)$ and $\mathbf{X}(t)$, respectively. If E_n and X_n can be determined, the desired control law can be expanded in terms of shifted Jacobi series as:

$$\mathbf{U}^*(t) = (\mathbf{K}\mathbf{X}^T + \mathbf{K}\mathbf{N}\mathbf{E}^T)\mathbf{J}(t). \tag{26}$$

Using these identities

$$\int_0^t \dot{\mathbf{X}}(t')\,\mathrm{d}t' = \mathbf{X}(t) - \mathbf{x}(0) \tag{27}$$

$$\int_0^t \dot{\mathbf{e}}(t')\,\mathrm{d}t' = \mathbf{e}(t) - \mathbf{e}(0) \tag{28}$$

we can obtain

$$\mathbf{E}^T = [\mathbf{e}(0), \quad \mathbf{0}, \ldots, \mathbf{0}] + \mathbf{D}\mathbf{E}^T\mathbf{F} \tag{29}$$

and

$$\mathbf{X}^T = [\mathbf{X}(0), \quad \mathbf{0}, \ldots, \mathbf{0}] + \hat{\mathbf{A}}\mathbf{X}^T\mathbf{F} + \hat{\mathbf{B}}\mathbf{E}^T\mathbf{F}. \tag{30}$$

$$
F = (t_f)
\begin{array}{cccc}
 & \frac{b+1}{c+1} & \frac{1}{c+1} & 0 \\
 & \frac{-(b+1)(a+1)}{c(c+1)(c+2)} - \frac{\Gamma(b+2)}{2!c\Gamma(b)} & \frac{a-b}{(c+1)(c+3)} & \frac{c+1}{(c+2)(c+3)} \\
 & \frac{\Gamma(b+3)}{(c+1)3!\Gamma(b)} & \frac{-(b+2)(a+2)}{(c+1)(c+3)(c+4)} & \frac{a-b}{(c+3)(c+5)} \\
 & \frac{-\Gamma(b+4)}{(c+2)4!\Gamma(b)} & 0 & \frac{-(b+3)(a+3)}{(c+2)(c+5)(c+6)} \\
 & \vdots & \vdots & \vdots \\
 & \frac{(-1)^{m-2}\Gamma(b+m-1)}{(c+m-3)(m-1)!\Gamma(b)} & 0 & 0 \\
 & \frac{(-1)^{m-1}\Gamma(b+m)}{(c+m-2)(m)!\Gamma(b)} & 0 & 0 \\
\cdots & 0 & 0 & 0 \\
\cdots & 0 & 0 & 0 \\
\cdots & 0 & 0 & 0 \\
\cdots & 0 & 0 & 0 \\
 & \vdots & \vdots & \vdots \\
\cdots & \frac{(b+m-2)(a+m-2)}{(c+2m-4)(c+2m-5)(c+m-3)} & \frac{(a-b)}{(c+2m-3)(c+2m-5)} & \frac{(c+m-2)}{(c+m-3)(c+2m-4)} \\
\cdots & 0 & \frac{(b+m-1)(a+m-1)}{(c+2m-2)(c+2m-3)(c+m-2)} & \frac{(a-b)}{(c+2m-1)(c+2m-3)}
\end{array}
\quad (23)
$$

Equations (29) and (30) can be rewritten as

$$\mathbf{E} = (\mathbf{D} \otimes \mathbf{F}^T)\mathbf{E} + \mathbf{E}(0) \tag{31}$$

$$\mathbf{X} = (\mathbf{A} \otimes \mathbf{F}^T)\mathbf{X} + (\hat{\mathbf{B}} \otimes \mathbf{F}^T)\mathbf{E} + (\hat{\mathbf{A}} \otimes \mathbf{I})\mathbf{X}(0) \tag{32}$$

where

$$\mathbf{E}(0) = [\mathbf{e}(0), \mathbf{0}, \ldots, \mathbf{0}]^T$$

$$\mathbf{X}(0) = [\mathbf{X}(0), \mathbf{0}, \ldots, \mathbf{0}]^T$$

and the operation $\hat{\mathbf{A}} \otimes \mathbf{F}^T$ is a Kronecker product (**13**)

$$\dot{Z}(t) = -1.5Z(t) - 1.25Y(t) - U(t), \quad Z(0) = 0.5$$

$$\hat{\mathbf{X}}(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} Z(t) + \begin{pmatrix} 1 \\ 1.5 \end{pmatrix} Y(t)$$

where $\mathbf{U} = [-1.5, 1]$. One can identify that

$$\hat{\mathbf{A}} = \mathbf{A} + \mathbf{BK} = \begin{pmatrix} 0 & 1 \\ -0.5, & -1 \end{pmatrix}, \quad \hat{\mathbf{B}} = \mathbf{BKN} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

The numerical solutions of **X** and **E** are shown as follows:

(i) when $a = 0$, $b = 1$, $m = 6$ and $t_f = 5$ sec

$$\mathbf{E}^T = [-0.0266730, \quad 0.0646811, \quad -0.0686554$$
$$0.0468477, \quad -0.0233069, \quad -0.0100624]$$

$$\mathbf{X}^T = \begin{bmatrix} 0.188055, & 0.081498, & -0.179148, & +0.051450 \\ 0.026011, & -0.211355, & 0.096915, & 0.025337 \end{bmatrix}$$
$$\begin{matrix} 0.007720, & -0.009478 \\ -0.032898, & 0.014465 \end{matrix}\Bigg].$$

Hence the optimal control is

$$U^*(t) = 0.281420J_0(t) - 0.024427J_1(t) - 0.240462J_2(t)$$
$$+ 0.149376J_3(t) - 0.044626J_4(t) + 0.01031J_5(t).$$

(ii) When $a = 0$, $b = 1$, $m = 8$ and $t_f = 5$ sec

$$\mathbf{E}^T = [-0.026548, \quad 0.064379, \quad -0.068371, \quad 0.046578$$
$$-0.023388, \quad 0.009301, \quad -0.003051, \quad 0.000898]$$

$$\mathbf{X}^T = \begin{bmatrix} 0.1869905, & 0.0824952, & -0.1792262, & 0.0514082 \\ 0.0265524, & -0.2115233, & 0.0967997, & 0.0252193 \end{bmatrix}$$
$$\begin{matrix} 0.0077628, & -0.0087817, & 0.0030914, & -0.0008142 \\ -0.0329158, & 0.0140254, & -0.0041601, & 0.0011105 \end{matrix}\Bigg].$$

The optimal control is

$$U^* = 0.280490 - 0.023416J_1(t) - 0.240401J_2(t) + 0.148910J_3(t) - 0.044660J_4(t) + 0.010154J_5(t) - 0.002574J_6(t) + 0.000787J_7(t).$$

(iii) When $a = 0, b = 0, m = 6$ and $t_f = 5$ sec (i.e. shifted-Legendre series)

$$\mathbf{E}^T = [-0.0999381, \quad 0.2202352, \quad -0.2060435 \quad +0.1292673, \quad -0.0606371, \quad 0.0252655]$$

$$\mathbf{X}^T = \begin{bmatrix} 0.0767032, & 0.3308101, & -0.3451541 & 0.0648212, & 0.0318698, & -0.0240565 \\ 0.1486299, & -0.3061541, & 0.8152940 & 0.1120718, & -0.0866035, & +0.0358856 \end{bmatrix}.$$

The optimal control is

$$\mathbf{U}^* = 0.1637466 + 0.3502961J_1(t) - 0.6422453J_2(t) + 0.3385710J_3(t) - 0.0994358J_4(t) + 0.0250663J_5(t).$$

The exact solution is

$$e(t) = 0.75 \exp(-1.5t)$$

and

$$U^*(t) = -0.75 \exp(-1.5t) - 0.55 \exp(-0.5t) \cos(0.5t) + 1.9 \exp(-0.5t) \sin(0.5t).$$

As can be seen from Tables I and II, the approximate solutions obtained by the

TABLE I
Numerical solution of e(t)

t	Exact	$a = 0, b = 1$ approx. ($m = 6$)	$a = 0, b = 1$ $m = 8$	$a = b = 0$ $m = 6$
0.0	−0.750000	−0.726300	−0.747995	−0.741487
0.1	−0.645531	−0.633493	−0.644946	−0.643566
0.2	−0.555614	−0.551735	−0.555728	−0.557281
0.3	−0.478221	−0.479311	−0.478995	−0.481435
0.4	−0.411609	−0.415364	−0.411996	−0.415001
0.5	−0.354275	−0.359094	−0.354563	−0.357025
0.6	−0.304927	−0.309751	−0.305083	−0.306620
0.7	−0.262453	−0.266637	−0.262488	−0.262963
0.8	−0.225896	−0.229106	−0.225845	−0.225295
0.9	−0.194430	−0.196555	−0.194336	−0.192918
1.0	−0.167348	−0.168429	−0.167246	−0.165188

TABLE II
Numerical solution of control variable U*(t)

t	Exact	$a = 0, b = 1$ approx. ($m = 6$)	$a = 0, b = 1$ $m = 8$	$a = b = 0$ $m = 6$
0.0	−1.300000	−1.273610	−1.298087	−1.291868
0.5	−0.403210	−0.404463	−0.403487	−0.406736
1.0	0.092392	0.094090	0.092489	0.094081
1.5	0.342625	0.346473	0.342546	0.344550
2.0	0.441502	0.443857	0.441448	0.441086
2.5	0.449262	0.449924	0.449331	0.448570
3.0	0.405873	0.406494	0.405880	0.407104
3.5	0.337983	0.339328	0.337918	0.340412
4.0	0.262931	0.263803	0.262931	0.263013
4.5	0.191353	0.190635	0.191352	0.187880
5.0	0.129093	0.131591	0.129305	0.134020

shifted-Jacobi series are very close to the exact solution, even when a small number ($m = 6$) of shifted-Jacobi polynomials is used.

IV. Conclusions

In this paper, shifted-Jacobi polynomials are adopted to solve optimal control systems incorporating observers. The proposed technique simplifies the system of equations into the successive solution of a set of linear algebraic equations. Thus, the computation is effective and straightforward. Moreover, only a small number of the shifted-Jacobi series ($m = 6$) are needed to obtain a satisfactory solution, hence it is seen that the method does not, in general, need an excessive capacity of memory and computing time.

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