On the spectral theory of wave propagation in a weakly range-dependent environment

Kuan-Kin Chan
Institute of Communication Engineering, National Chiao Tung University, Hsinchu, Taiwan, Republic of China

(Received 12 September 1985; accepted for publication 30 January 1986)

The spectral theory of wave propagation in a weakly range-dependent environment is deduced from the corresponding theory for a wedge-shaped structure. The solution obtained is compared to that derived previously, which suggests that some modifications are necessary in the previous formulation in order to bring the solution into a symmetric form.

PACS numbers: 43.30.Bp

INTRODUCTION

The adiabatic mode theory[1] has found wide applications in the analysis of wave propagation in weakly range-dependent environments. Under the assumption of adiabatic approximation, local modes propagate independently with negligible mutual coupling and preserve their identity to adapt themselves smoothly to the slowly changing environments. It fails, however, to apply in a region containing fields of continuous spectrum. A notable example is the wave propagation in a shallow leaky wedge structure, where the trapped adiabatic modal fields may reach their respective cutoffs and begin to radiate across the wedge boundary. Jensen and Kuperman[2] have computed the field numerically via the parabolic equation method, which vividly displayed the picture of the coupling mechanism involved. Based on this observation, recent advances[3,4] make use of the characteristic Green's function, representing the field as a spectral integral, where the adiabatic approximation is extended to include the continuous spectrum contribution by invoking the condition of invariance of local transverse resonance. The spectral integral formulation makes it feasible to treat the modal field constituents, whether discrete or continuous, from a unified point of view. It also substantially broadens the scope of the solution since various well-developed asymptotic techniques can be readily applied. It is most useful, in particular, in deriving a uniform representation of the fields in the transition region which is otherwise difficult to obtain. This aspect has been explored extensively in Ref. 3. However, the resultant solution in Ref. 3 is found to be formally unsatisfactory since it is not symmetric in form. In this paper we show how a symmetric form can be obtained based on the results in Ref. 4.

I. FORMULATION AND SOLUTION FOR A WEDGE REGION

For clarity and completeness, the solution obtained in Ref. 4 is rederived in this section. The model we consider here is a wedge-shaped medium with refractive index n embedded in a semi-infinite half-space as shown in Fig. 1. The cylindrical coordinate system is adopted with the bottom interface taken as abscissas. We then seek a solution of the reduced wave equation (a harmonic time dependence $e^{-i\omega t}$ is assumed and deleted throughout),

$$\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}+k^2\right)G(\boldsymbol{\rho},\boldsymbol{\rho}')=-\delta(\boldsymbol{\rho}-\boldsymbol{\rho}'),\tag{1a}$$

with a unit line source located at the position $\boldsymbol{\rho}=(\rho',\phi')$ and satisfying the boundary conditions,

$$\begin{aligned}&G=0\ \text{ at }\phi=\alpha,\\&G\text{ and }\frac{\partial G}{\partial\phi}\text{ continuous at }\phi=0,\\&\text{edge condition at }\rho=0,\\&\text{radiation condition at }\rho=\infty,\end{aligned}\tag{1b}$$

where k is the wavenumber associated with the wedge region being considered. To confine our attention to a gradually changing environment, we assume that the wedge angle is small and concentrate our interest in a region sufficiently remote from the edge such that the local arc length (which is approximately the local depth in adiabatic mode theory) remains finite and varies slowly with range.

The problem is a nonseparable one. However, we proceed as in the adiabatic mode theory. But instead of directly constructing the adiabatic eigenfunctions, the method of characteristic Green's function[5] is used. The characteristic

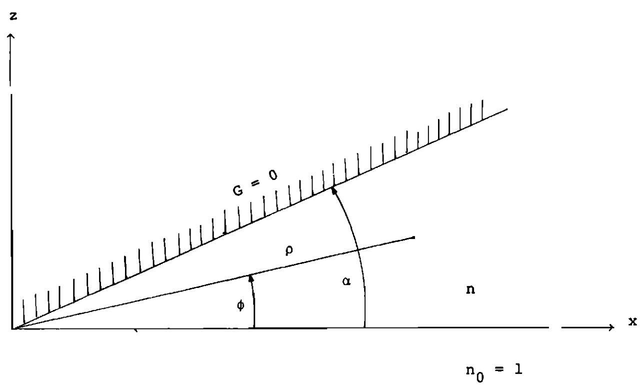

FIG. 1. The wedge geometry.

 0001-4966/86/051402-04$00.80

Green's function in the angular domain satisfies

$$\left(\frac{d^2}{d\phi^2}+k^2\lambda\right)g_\phi(\phi,\phi';\lambda)=-\delta(\phi-\phi'),\quad 0\leqslant\phi,\phi'\leqslant\alpha, \tag{2}$$

and is given by

$$g_\phi(\phi,\phi';\lambda)=\frac{\sin k\sqrt{\lambda}\,(\alpha-\phi_>)}{k\sqrt{\lambda}}\times\frac{e^{ik\sqrt{\lambda}(\alpha-\phi_<)}+\Gamma(\lambda;\rho)e^{ik\sqrt{\lambda}(\alpha+\phi_<)}}{1+\Gamma(\lambda;\rho)\,e^{2ik\sqrt{\lambda}\alpha}},$$

$$\phi_>=\max(\phi,\phi'),\quad \phi_<=\min(\phi,\phi'), \tag{3}$$

where λ is a spectral variable and $\Gamma(\lambda;\rho)$ is the reflection coefficient resulting from the local boundary condition where the assumption has been made to ignore the backscattering angular wave under the wedge. A simple evaluation invoking boundary conditions yields

$$\Gamma(\lambda;\rho)=\left[\sqrt{\frac{\lambda}{\rho^2}}-\sqrt{\frac{\lambda}{\rho^2}-\left(1-\frac{1}{n^2}\right)}\right]\Bigg/\left[\sqrt{\frac{\lambda}{\rho^2}}+\sqrt{\frac{\lambda}{\rho^2}-\left(1-\frac{1}{n^2}\right)}\right] \tag{4a}$$

or

$$\Gamma(\lambda;\rho)=\left[\sqrt{\frac{\lambda}{\rho^2}}-i\sqrt{\left(1-\frac{1}{n^2}\right)-\frac{\lambda}{\rho^2}}\right]\Bigg/\left[\sqrt{\frac{\lambda}{\rho^2}}+i\sqrt{\left(1-\frac{1}{n^2}\right)-\frac{\lambda}{\rho^2}}\right]. \tag{4b}$$

The local completeness relation is then given by the spectral integral

$$\delta(\phi-\phi')=-\frac{k^2}{2\pi i}\int_{C_\lambda}d\lambda\, g_\phi(\phi,\phi';\lambda), \tag{5}$$

where the integration contour C_λ encircles all the singularities of the integrand g_ϕ in the proper spectral plane (Fig. 2). The pole singularities λ_m are evaluated through the local

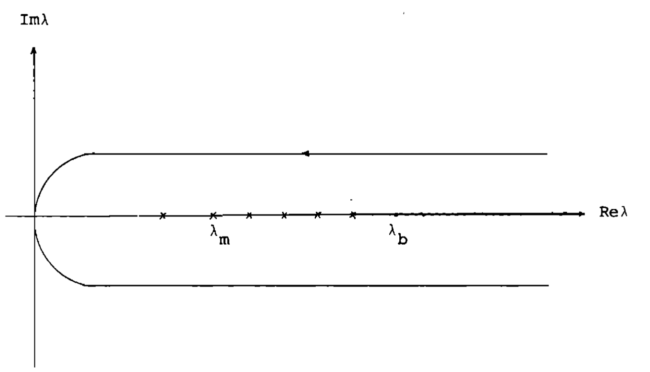

FIG. 2. Contour of integration in the complex λ plane.

resonance equation in the denominator in Eq. (3)

$$1+\Gamma(\lambda_m;\rho)e^{2ik\sqrt{\lambda}_m\alpha}=0 \tag{6}$$

or, alternatively, by making use of Eq. (4b)

$$\tan\left(k\sqrt{\frac{\lambda_m}{\rho^2}}\,s_\alpha(\rho)\right)=\frac{-\sqrt{\lambda_m/\rho^2}}{\sqrt{(1-1/n^2)-\lambda_m/\rho^2}},$$

$$s_\alpha(\rho)=\alpha\rho=\text{arc length at }\rho\text{ extended by the wedge}. \tag{7}$$

The correct adiabatic modes can be derived from the residue contribution at the poles. In addition, there exists a branch point singularity at

$$\lambda_b=\rho^2(1-1/n^2), \tag{8}$$

which is responsible for the field radiated across the bottom interface and which does not appear in the adiabatic mode theory. It is, however, inherent in the present approach. The local resonance in Eq. (6) is extended to a spectral variable while maintaining the range dependence through the invariance relation,

$$1+\Gamma(\lambda;\rho)e^{2ik\sqrt{\lambda}\alpha}=1+\Gamma(\lambda';\rho')e^{2ik\sqrt{\lambda'}\alpha}, \tag{9}$$

which is required to construct a solution that is consistent with the adiabatic mode theory.

Substitution of $\lambda(\rho)$ into the radial equation leads to

$$\left(\frac{d}{d\rho}\rho\frac{d}{d\rho}+k^2\rho-\frac{k^2\lambda(\rho)}{\rho}\right)g_\rho=-\delta(\rho-\rho'). \tag{10}$$

To solve Eq. (10) let

$$g_\rho=\rho^{-1/2}f_\rho.$$

One finds

$$\left[\frac{d^2}{d\rho^2}+k^2\left(1-\frac{\lambda(\rho)}{\rho^2}+\frac{1}{4k^2\rho^2}\right)\right]f_\rho=-(\rho')^{-1/2}\delta(\rho-\rho'). \tag{11}$$

Equation (11) can be solved asymptotically by WKB approximation to obtain

$$g_\rho\sim\exp\left(ik\int_{\rho_<}^{\rho_>}d\rho\sqrt{1-\frac{\lambda(\rho)}{\rho^2}+\frac{1}{4k^2\rho^2}}\right)\times\left[-2ik(\rho')^{1/2}\rho^{1/2}\left(1-\frac{\lambda(\rho)}{\rho^2}+\frac{1}{4k^2\rho^2}\right)^{1/4}\times\left(1-\frac{\lambda(\rho')}{(\rho')^2}+\frac{1}{4k^2(\rho')^2}\right)^{1/4}\right]^{-1}, \tag{12}$$

where we have ignored the backscattering wave field from the edge, due to the fact that this field will be strongly attenuated in a leaky wedge structure before reflecting back to the region concerned.

Recall that the exact solution for the Green's function in a separable problem is given by[5]

$$G(\boldsymbol{\rho},\boldsymbol{\rho}')=-\frac{k^2}{2\pi i}\int_{C_\lambda} g_\rho(\rho,\rho';\lambda)g_\phi(\phi,\phi';\lambda)\,d\lambda\,. \tag{13}$$

A simple substitution of Eqs. (3) and (12) would not be justified due to the fact that the solution thus obtained would violate the principle of reciprocity. To remedy this, we rewrite Eq. (3) as

$$g_\phi=\frac{\sin k\sqrt{\lambda_1}(\alpha-\phi_1)}{k(\lambda\lambda')^{1/4}}\times\frac{e^{ik\sqrt{\lambda_2}(\alpha-\phi_2)}+\Gamma(\lambda_2;\rho_2)e^{ik\sqrt{\lambda_2}(\alpha+\phi_2)}}{1+\Gamma(\lambda;\rho)e^{2ik\sqrt{\lambda}\alpha}}, \tag{14}$$

where $\phi_1=\phi_>$, $\phi_2=\phi_<$ are either the source point or the observation point while λ_1, λ_2 are associated with ϕ_1, ϕ_2, respectively.

Substituting Eqs. (12) and (14) into (13), we obtain

$$G(\boldsymbol{\rho},\boldsymbol{\rho}')=-\frac{k^2}{2\pi i}\int_{C_\lambda} d\lambda\sqrt{\frac{\partial L/\partial\lambda}{\partial L/\partial\lambda'}}\frac{\sin k\sqrt{\lambda_1}(\alpha-\phi_1)}{k(\lambda\lambda')^{1/4}}\frac{e^{ik\sqrt{\lambda_2}(\alpha-\phi_2)}+\Gamma(\lambda_2;\rho_2)\,e^{ik\sqrt{\lambda_2}(\alpha+\phi_2)}}{1+\Gamma(\lambda;\rho)\,e^{2ik\sqrt{\lambda}\alpha}}$$
$$\times\exp\left(ik\int_{\rho_<}^{\rho_>}d\rho''\sqrt{1-\frac{\lambda(\rho'')}{(\rho'')^2}+\frac{1}{4k^2(\rho'')^2}}\right)$$
$$\times\left[-2ik\rho^{1/2}(\rho')^{1/2}\left(1-\frac{\lambda(\rho')}{(\rho')^2}+\frac{1}{4k^2(\rho')^2}\right)^{1/4}\left(1-\frac{\lambda(\rho)}{\rho^2}+\frac{1}{4k^2\rho^2}\right)^{1/4}\right]^{-1}. \tag{15}$$

Here λ is a spectral integration variable whereas λ' and λ'' are related to λ through the invariance relation,

$$L(\lambda;\rho)=\Gamma(\lambda;\rho)e^{2ik\sqrt{\lambda}\alpha}=\Gamma(\lambda';\rho')e^{2ik\sqrt{\lambda'}\alpha}=\Gamma(\lambda'';\rho'')e^{2ik\sqrt{\lambda''}\alpha}\,. \tag{16}$$

A scalar factor $[(\partial L/\partial\lambda)/(\partial L/\partial\lambda')]^{1/2}$ has been inserted in order to preserve the symmetry of the solution.

II. TRANSFORMATION INTO LOCAL MODE DESCRIPTION

Although we have adopted a cylindrical coordinate system for our discussion, all of the relations obtained can be easily cast into forms suitable for a local stratification under the assumption of weak range dependence. Specifically, we first transform all of the angular variables into linear ones via

$$\xi=\lambda/\rho^2,\qquad s(\rho)=\rho\phi,\qquad s_\alpha(\rho)=\rho\alpha\,. \tag{17}$$

Next we make use of the following approximations under the assumption of weak range dependence:

$$\rho\sim x,\qquad s(\rho)\sim z,$$

$s_\alpha(\rho)\sim H(x)$, the local channel depth.

As a result we have for the reflection coefficient

$$\Gamma(\lambda;\rho)\to\Gamma(\xi)=\frac{\sqrt{\xi}-i\sqrt{\xi_b-\xi}}{\sqrt{\xi}+i\sqrt{\xi_b-\xi}}=\exp[-2i\tan^{-1}(\sqrt{\xi_b-\xi}/\sqrt{\xi})]\,;\quad \xi_b=1-1/n^2; \tag{18a}$$

the local resonance equation

$$1+\Gamma(\lambda_m;\rho)e^{2ik\sqrt{\lambda_m}\alpha}\to 1+\exp\left[i\left(2k\sqrt{\xi_m}H(x)-2\tan^{-1}\frac{\sqrt{\xi_b-\xi_m}}{\sqrt{\xi_m}}\right)\right]=0; \tag{18b}$$

the Green's functions

$$g_\rho\to\frac{1}{\sqrt{xx'}}\frac{\exp[ik\int_{x_<}^{x_>}dx\sqrt{1-\xi(x)}]}{-2ik[1-\xi(x)]^{1/4}[1-\xi(x')]^{1/4}}, \tag{18c}$$

$$g_\phi\to\frac{\sin k\sqrt{\xi_1}[H(x_1)-z_1]}{k(xx')^{1/2}(\xi\xi')^{1/4}}\times\frac{e^{ik\sqrt{\xi_2}[H(x_2)-z_2]}+\Gamma(\xi_2)e^{ik\sqrt{\xi_2}[H(x_2)+z_2]}}{1+\Gamma(\xi)e^{2ik\sqrt{\xi}H(x)}}; \tag{18d}$$

and the range-invariant relation

$$L(\lambda;\rho)\to\Gamma(\xi)e^{2ik\sqrt{\xi}H(x)}=\Gamma(\xi')e^{2ik\sqrt{\xi'}H(x')}, \tag{18e}$$

where the higher-order terms have been neglected in Eq. (18c). Futhermore,

$$d\lambda\sqrt{\frac{\partial L/\partial\lambda}{\partial L/\partial\lambda'}}=d(\xi\rho^2)\sqrt{\frac{(\partial L/\partial\xi)(d\xi/d\lambda)}{(\partial L/\partial\xi')(d\xi'/d\lambda')}}\to d\xi(xx')\sqrt{\frac{\partial L/\partial\xi}{\partial L/\partial\xi'}}\,. \tag{18f}$$

Equation (15) reduces, via substitution of Eqs. (18c), (18d), and (18f) to

$$G(\boldsymbol{\rho},\boldsymbol{\rho}')\sim-\frac{k^2}{2\pi i}\int_{C_\xi}d\xi\sqrt{\frac{\partial L/\partial\xi}{\partial L/\partial\xi'}}\times\frac{\sin k\sqrt{\xi_1}[H(x_1)-z_1]}{k(\xi\xi')^{1/4}}\times\frac{e^{ik\sqrt{\xi_2}[H(x_2)-z_2]}+\Gamma(\xi_2)e^{ik\sqrt{\xi_2}[H(x_2)+z_2]}}{1+\Gamma(\xi)e^{2ik\sqrt{\xi}H(x)}}\times\frac{\exp[ik\int_{x_<}^{x_>}\sqrt{1-\xi(x)}dx]}{-2ik[1-\xi(x)]^{1/4}[1-\xi(x')]^{1/4}}, \tag{19}$$

which is the desired solution pertinent to wave propagation in a region with slowly varying boundary interfaces.

III. DISCUSSION AND CONCLUSION

Equation (19) is essentially the same as that given in Ref. 3. It is not surprising to see this coincidence, since both derivations follow the same line of philosophy by assuming that the invariance of transverse resonance remains valid throughout the entire spectrum as an extension of the adiabatic mode theory. One notices, however, difference exits between them regarding the definitions of the depth coordinates $z_>$ and $z_<$. Literally the $z_>$ and $z_<$ are defined respectively as the greater and lesser values of the source and observation depth locations. While these are meaningful in a range-independent situation, ambiquity may arise for the two depths located in different ranges when the channel depth is a function of the range. Such is the case when one considers the situation as shown in Fig. 3, which shows if the source is specified in a location Q, the depth of an arbitrary observation point P in the shaded region will always appear to be $z_>$, no matter how one looks from the side of the lower boundary (with the depth axis upward) or from the side of the upper boundary (with the depth axis downward). In other words, the solution obtained in Ref. 3 is coordinate dependent, i.e., it will depend upon whether the upper or the lower boundary is chosen as the range coordinate axis. A modification is therefore necessary in order to bring the symmetry into the solution. On the other hand, our present study shows that the $z_>$ and $z_<$ are determined via a transformation from their relative orientations $\phi_>$ and $\phi_<$ in the angular domain. They are defined uniquely in a scaled sense such that the $z_>$ and $z_<$ represent the depth values with the greater and the lesser values of the scaled depths $z/H(x)$ and $z'/H(x')$, respectively.

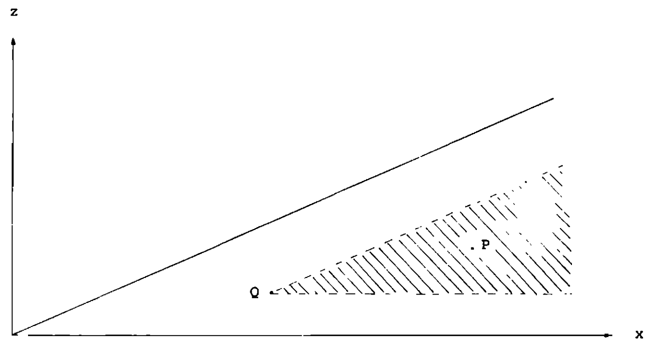

FIG. 3. The ambiguity region for $z_>$.

To conclude our discussion, the spectral theory for a weakly range-dependent environment[3] while correctly retaining the symmetry of the solution in range, has ignored the symmetric relation in depth and is therefore formally unsatisfactory. This can be readily remedied, as we discussed in this paper, by proper interpretation of the location parameters.

[1]A. D. Pierce, "Extension of the Method of Normal Modes to Sound Propagation in an Almost-Stratified Medium," J. Acoust. Soc. Am. 37, 19–27 (1965).

[2]F. B. Jensen and W. A. Kuperman, "Sound Propagation in a Wedge-Shaped Ocean with a Penetrable Bottom," J. Acoust. Soc. Am. 67, 564–1566 (1980).

[3]K. Kamel and L. B. Felsen, "Spectral Theory of Sound Propagation in an Ocean Channel with Weakly Sloping Bottom," J. Acoust. Soc. Am. 73, 1120–1130 (1983).

[4]K. K. Chan, "Radiation of a Line Source in a Leaky Wedge Structure," J. Chin. Inst. Eng. 7, 147–150 (1984).

[5]L. B. Felsen and N. Marcuritz, *Radiation and Scattering of Waves* (Prentice-Hall, Englewood Cliffs, NJ, 1973), pp. 273–288.