國立交通大學

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碩士論文

離散漢米爾頓系統中二階差分方程的廣域吸引子及拓樸混沌

Global attractor and topological chaos of second-order difference equations in discrete Hamiltonian systems

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摘要

於本篇論文中，我們討論一個差分方程式兩種不同的動態：

$$\Delta[p\Delta x(t-1)]+qx(t)=f\big(x(t-1)\big)\text{或}f\big(x(t)\big),\ t\in\mathbb{Z}，$$

其中$\Delta x(t-1)=ax(t)-bx(t-1)$。此兩種動態行為分別為廣域吸引子與拓樸混沌。我們做出了多樣的結果。在參數a,b,p與q的某種條件之下，此方程任意解的軌跡最終都將會收斂到一個廣域吸引子。請參照定理2.2與2.3。在某個特定的參數值之下，若存在一個與f有關的函數且此函數擁有不只一個簡單根或者正拓樸熵，則限制在此方程式解集合上的轉移映射會具有拓樸混沌。請參照定理2.6、2.7、2.8及2.9。最後，我們將此方程式經由變數變換轉變成參數化的連續函數。我們也可將之表示成離散漢米爾頓系統的形式。針對$f\left(x(t)\right)$的情況，定理2.10表示會存在一個與f有關的函數且此函數擁有正拓樸熵使得對應函數具有拓樸混沌。針對$f\left(x(t-1)\right)$的情況，若滿足前面的條件並且此與f有關的函數值域被局部性地限制住範圍，則定理2.11表示此對應函數也會擁有拓樸混沌。

Global attractor and topological chaos of second-order difference equations in discrete Hamiltonian systems

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Abstract

In this thesis, we discuss two distinct dynamics of the difference equation

$$\Delta[p\Delta x(t-1)]+qx(t)=f\big(x(t-1)\big) \text{ or } f\big(x(t)\big),\ t\in\mathbb{Z},$$

where $\Delta x(t-1)=ax(t)-bx(t-1)$. These two dynamics are the behavior of globally attracting and topological chaos. We have several results. Under some conditions of *a*, *b*, *p* and *q*, every orbit of the equation asymptotically converges to a global attractor. See theorems 2.2 and 2.3. If there exists a function relating to *f* which has more than one simple zeros or positive topological entropy at an expected parametric value, then the shift map restricted to the set of solutions of this equation has topological chaos. See theorems 2.6, 2.7, 2.8 and 2.9. Finally, we transform this equation into a parameterized continuous function by changing variables. We can also write it as the form of a discrete Hamiltonian system. For the case $f\big(x(t)\big)$, theorem 2.10 says that there exists a function relating to *f* which has positive topological entropy such that the corresponding function has topological chaos. For the case $f\big(x(t-1)\big)$, with an additional assumption that the function relating to *f* is locally trapping, theorem 2.11 says that the corresponding function has also topological chaos.

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1 Introduction

In this thesis, we mainly discuss the globally attracting and chaotic behavior in view of topological entropy of the nonlinear second-order difference equation $\Delta\left[p\left(t\right)\Delta x\left(t-1\right)\right]+q\left(t\right)x\left(t\right)=u\left(t,x\left(t\right)\right)$, $t\in\mathbb{Z}$. In 2006, Ma and Guo in [9] used variational methods to study the existence of nontrivial homoclinic orbits emanating from 0 of this difference equation. A homoclinic orbit emanating from 0 is a solution $x\left(t\right)$ if $x\left(t\right)\to 0$ as $t\to\pm\infty$. This equation can also be written as an equivalent first order nonlinear nonautonomous discrete Hamiltonian system $\Delta X\left(t\right)=J\nabla H_X\left(t,x\left(t+1\right),y\left(t\right)\right)$, where $X\left(t\right)=\left(x\left(t\right),y\left(t\right)\right)^T$, $y\left(t\right)=p\left(t\right)\Delta x\left(t-1\right)$, the Hamiltonian function $H\left(t,X\left(t\right)\right)=\frac{1}{2p(t)}\left[y\left(t\right)\right]^2+\frac{1}{2}q\left(t\right)\left[x\left(t\right)\right]^2-V\left(t,x\left(t\right)\right)$, $V\left(t,x\right)=\int_0^x u\left(t,s\right)ds$, and the normal symplectic matrix $J=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$. They considered the assumption that the function $u\left(t,x\right)$ grows superlinearly both at origin and at infinity or is odd with respect to $x\in\mathbb{R}$. There are also other assumptions we don't mention here. There were some people discussing solutions of continuous Hamiltonian systems in the past. In 1991, Zelati and Rabinowitz in [6] considered a class of continuous second-order Hamiltonian systems of the form $\ddot{z}-L\left(t\right)z+V_z\left(t,z\right)=0$, where $z\in\mathbb{R}^n$, $L\in C\left(\mathbb{R},\mathbb{R}^{n\times n}\right)$ and $V\in C^2\left(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}\right)$ with other assumptions. They established the existence of infinitely many homoclinic orbits for the class. Tanaka in [13] studied the existence of nontrivial homoclinic orbits emanating from 0 of a first-order Hamiltonian system of the form $\dot{z}=JH_z\left(t,z\right)$, where $z\in\mathbb{R}^{2N}$, $J=\begin{bmatrix}0 & I_N\\ -I_N & 0\end{bmatrix}$, and $H\in C^1\left(\mathbb{R}\times\mathbb{R}^{2N},\mathbb{R}\right)$ with other assumptions. In this thesis, we suppose $p\left(t\right)=p$, $q\left(t\right)=q$ and the difference operator Δ is defined as $\Delta x\left(t-1\right)=ax\left(t\right)-bx\left(t-1\right)$ with two weighted numbers a and b. The function $u\left(t,x\left(t\right)\right)$ is of the forms $u\left(t,x\left(t\right)\right)=f\left(x\left(t-1\right)\right)$ or $f\left(x\left(t\right)\right)$ for some continuous one-variable function f with $f\left(0\right)=0$.

In this paragraph, we briefly introduce the goals of the papers we apply throughout this thesis. In [1], a concept called dynamical networks is discussed. [1] states that each dynamical network is characterized by three factors. These factors are respectively (1) the network's topology (structure of a network), (2) the interactions between the elements (local subsystems) and (3) the intrinsic dynamics of local subsystems. Afraimovich and Bunimovich in [1] used the methods of symbolic dynamics and formalism to analyze the stability of dynamical networks and their subnetworks with these three factors. Their main result gives sufficient conditions to enable dynamical networks to possess a global attractor. In [8, 12], the family of difference equations $\Phi_\lambda\left(y_n,y_{n+1},\cdots,y_{n+m}\right)=0$, $n\in\mathbb{Z}$, with parameters λ in some metric space is discussed. Li and Malkin in [12] proved that if the difference equations have a singular limit of the form $\Phi_{\lambda_0}\left(y_0,y_1,\cdots,y_m\right)=\varphi\left(y_N\right)$ as $\lambda\to\lambda_0$ for some N with $0\leq N\leq m$ and some function φ with k simple zeros, then for λ close enough to λ_0, the difference equation has a topological k-horseshoe. Indeed, there exists a closed (in the product topology) shift-invariant set Γ_λ such that $\sigma|_{\Gamma_\lambda}$ is conjugate to the full shift of k symbols $\sigma|_{\Sigma_k}$. On the other hand, Juang, Li and Malkin in [8] proved for another case that $\Phi_{\lambda_0}(y_0, y_1, \cdots, y_m) = \xi(y_N, y_{N+L})$ for some N, L with $0 \leq N, N+L \leq m$ and a function ξ in two variables as $\lambda \to \lambda_0$, if $\xi(x, y) = 0$ has a branch $y = \varphi(x)$ with $h_{top}(\varphi) > 0$, then for λ in some neighborhood of λ_0, there exists a closed (in the product topology) invariant set to which the restriction of the shift map has topological entropy close arbitrarily to $\frac{h_{top}(\varphi)}{|L|}$. In [11], Li, Lyu and Zgliczyński considered perturbations from a low-dimensional continuous map f to a family of high-dimensional continuous maps F_λ. They proved at $\lambda = 0$, if F_0 satisfies one of two forms: (1) $F_0(x, y) = (f(x), g(x)) \in \mathbb{R} \times \mathbb{R}^n$; (2) $F_0(x, y) = (f(x), g(x, y)) \in \mathbb{R} \times \mathbb{R}^n$ with $g(\mathbb{R} \times U) \subset int(U)$ for some compact set U homeomorphic to the closed unit ball in $\mathbb{R}^n$, then $\liminf_{\lambda \to 0} h_{top}(F_\lambda) \geq h_{top}(f)$. Note that $h_{top}(f)$ here denotes the supremum of topological entropies of f restricted to compact f-invariant sets.

In this thesis, by applying these results, we have several results to find sufficient conditions to make the difference equation mentioned in the beginning of this section have a global attractor at the origin and be topologically chaotic in the set of its solutions with parameter perturbation. Notice that the (difference) equation mentioned in this paragraph means the equation in the beginning of this section if no additional explanation is involved. See the next section. Theorems 2.2 and 2.3 state that under some conditions we find, the equation has the behavior of global attracting in the set of its solutions. Secondly, the remaining results concern with the topological chaos of the equation. Theorems 2.6 and 2.7 connect the dynamic of the shift map restricted to the set of the solutions of the equation involving parameter perturbing with symbolic dynamics of full shift, while theorems 2.8 and 2.9 connect it with the topological entropy of some function relating to the function f. In theorems 2.10 and 2.11, we consider the behavior of topological chaos of the difference equation in terms of its corresponding discrete Hamiltonian system. Moreover, it can be written as the dynamical system of an iterated function. The two theorems state that if one can find a function which relates to the function f and possesses positive topological entropy at an expected value of the parameter we concern, then the iterated system must own topological chaos as the parameter is near enough to this expected value.

This thesis is organized as follows. In section 2, we state some definitions we use in our results and the details of these results. In section 3, we state the preliminary for the proofs of theorems 2.2 and 2.3, mainly from the work of [1]. In section 4, we prove if a, b, p, q and f satisfy some conditions, then our difference equation has a global attractor at the origin. See theorems 2.2 and 2.3. In section 5, we state the main results about the family of difference equations $\Phi_\lambda(y_n, y_{n+1}, \cdots, y_{n+m}) = 0$, $n \in \mathbb{Z}$, from [8, 12]. In section 6, we apply the results in section 5 to the proofs of theorems 2.6, 2.7, 2.8 and 2.9. In section 7, we state the main results in [11] and some corollaries. In section 8, we prove theorems 2.10 and 2.11 by using the corollaries in section 7. In section 9, we find some numerical examples to verify the applicability of our results.

2 Definitions and the statements of main theorems

Following [9], we now consider two special cases of the nonlinear difference equation defined on $\mathbb{R}$

$$\Delta\left[p\left(t\right)\Delta x\left(t-1\right)\right]+q\left(t\right)x\left(t\right)=u\left(t,x\left(t\right)\right),\ t\in\mathbb{Z}. \tag{1}$$

Let $\Delta x\left(t-1\right)=ax\left(t\right)-bx\left(t-1\right)$, $a,b>0$, $p\left(t\right)=p$, $q\left(t\right)=q$, where a,b,p and q are real parameters independent of t, and a continuous function $u\left(t,x\left(t\right)\right)=f\left(x\left(t-1\right)\right)$ or $f\left(x\left(t\right)\right)$ with $f\left(0\right)=0$, which means u is real-valued and dependent only on x. Then one can see that $\Delta x\left(t-1\right)$ is a *weighted difference* with the weights a and b. Let x_n denote $x\left(t\right)$. Then we get the form of recursive sequence of one variable

$$a^2px_{n+2}+b^2px_n-\left(2abp-q\right)x_{n+1}-f\left(x_{n+1}\right)=0,n\in\mathbb{Z} \tag{2}$$

for the case $u\left(t,x\left(t\right)\right)=f\left(x\left(t\right)\right)$ and

$$a^2px_{n+2}+b^2px_n-\left(2abp-q\right)x_{n+1}-f\left(x_n\right)=0,n\in\mathbb{Z} \tag{3}$$

for the case $u\left(t,x\left(t\right)\right)=f\left(x\left(t-1\right)\right)$.

We also call the equations (2) and (3) difference equations in this thesis.

Next, we denote a new sequence $y\left(t-1\right)=p\Delta x\left(t-1\right)$. Then $x\left(t\right)=\frac{b}{a}x\left(t-1\right)+\frac{1}{ap}y\left(t-1\right)$. On the other hand, consider the equation (1). We have two consequences for $y\left(t\right)$ below.

1. if $u\left(t,x\left(t\right)\right)=f\left(x\left(t\right)\right)$, then $y\left(t\right)=-\frac{bq}{a^2}x\left(t-1\right)+\frac{1}{a}\left(b-\frac{q}{ap}\right)y\left(t-1\right)+\frac{1}{a}\times f\left(\frac{b}{a}x\left(t-1\right)+\frac{1}{ap}y\left(t-1\right)\right)$;

2. if $u\left(t,x\left(t\right)\right)=f\left(x\left(t-1\right)\right)$, then $y\left(t\right)=-\frac{bq}{a^2}x\left(t-1\right)+\frac{1}{a}\left(b-\frac{q}{ap}\right)y\left(t-1\right)+\frac{1}{a}\times f\left(x\left(t-1\right)\right)$.

For these two consequences, we get two dynamical systems which is defined on $\mathbb{R}^2$. For all $n\in\mathbb{Z}$,

$$\begin{aligned} x_{n+1} &= \frac{b}{a}x_n+\frac{1}{ap}y_n \\ y_{n+1} &= -\frac{bq}{a^2}x_n+\frac{1}{a}\left(b-\frac{q}{ap}\right)y_n+\frac{1}{a}f\left(\frac{b}{a}x_n+\frac{1}{ap}y_n\right) \end{aligned} \tag{4}$$

and

$$\begin{aligned} x_{n+1} &= \frac{b}{a}x_n+\frac{1}{ap}y_n \\ y_{n+1} &= -\frac{bq}{a^2}x_n+\frac{1}{a}\left(b-\frac{q}{ap}\right)y_n+\frac{1}{a}f\left(x_n\right) \end{aligned} \tag{5}$$

respectively. Transform them into the dynamics of maps in the form $X_{n+1} = F(X_n)$, where $X_n = (x_n, y_n) \in \mathbb{R}^2$ and F is a continuous function with parameters a, b, p, q such that

$$F(x,y) = \left(\frac{b}{a}x + \frac{1}{ap}y, -\frac{bq}{a^2}x + \frac{1}{a}\left(b - \frac{q}{ap}\right)y + \frac{1}{a}f\left(\frac{b}{a}x + \frac{1}{ap}y\right)\right) \tag{6}$$

and

$$F(x,y) = \left(\frac{b}{a}x + \frac{1}{ap}y, -\frac{bq}{a^2}x + \frac{1}{a}\left(b - \frac{q}{ap}\right)y + \frac{1}{a}f(x)\right). \tag{7}$$

Let $\gamma : D \subset E \to E$ be a function, where E is a metric space endowed with a metric ρ. Recall that γ is *Lipschitz* if $L = \sup_{x \neq y} \frac{\rho(\gamma(x),\gamma(y))}{\rho(x,y)} < \infty$. Such L is called the *Lipschitz constant*. We state the definition of a global attractor as follows.

Definition 2.1. *A function $\gamma : E \to E$ has a global attractor $x_0 \in E$ if $\rho(\gamma^n(x), \gamma^n(x_0)) \to 0$ as $n \to \infty$ for all $x \in E$.*

Given a differentiable dynamical system $\Gamma : D \to D$, where $D \subset \mathbb{R}^n$. we know that if the derivative of Γ at a point x_0 has all eigenvalues with absolute values less than one, then x_0 is an attractor. But in this thesis, by Contraction Mapping Principle, we use the main result in [1] to guarantee each of the systems (4) and (5) without differentiability to have a global attractor in terms of contraction. We state the conditions and results in theorems 2.2 and 2.3 and the proofs of them are showed in section 4.

Theorem 2.2. *Let $p, q \in \mathbb{R}$, $p \neq 0$, and $M > 0$ be the Lipschitz constant of a continuous function f. If $\max\left(\frac{b}{a}, \left|\frac{1}{ap}\right|\right) + \max\left(\left|\frac{bq}{a^2}\right| + \frac{bM}{a^2}, \frac{1}{a}\left|b - \frac{q}{ap}\right| + \frac{M}{|a^2 p|}\right) < 1$, then the dynamical system (6) has a global attractor.*

Theorem 2.3. *Let $p, q \in \mathbb{R}$, $p \neq 0$, and $M > 0$ be a Lipschitz constant of the continuous function f . If $\max\left(\frac{b}{a}, \left|\frac{1}{ap}\right|\right) + \max\left(\left|\frac{bq}{a^2}\right| + \frac{M}{a}, \frac{1}{a}\left|b - \frac{q}{ap}\right|\right) < 1$, then the dynamical system (7) has a global attractor.*

Secondly, we define a simple zero of a function. Let γ be a C^1 function on a subset of $\mathbb{R}$. We say a point x_0 is a *simple zero* if $\gamma(x_0) = 0$ and $\gamma'(x_0) \neq 0$. This means that x_0 is a zero of γ with multiplicity one. We recall the definition of topological entropy of a continuous map $\gamma : X \to X$, where (X, ρ) is a compact metric space. The main work comes from Bowen [5]. Let $n \in \mathbb{N}$ and $\varepsilon > 0$. First define a metric $\rho_n : X \times X \to \mathbb{R}$ by $\rho_n(x,y) = \max_{0 \le i < n} \rho(\gamma^i(x), \gamma^i(y))$ for any $x, y \in X$.

Definition 2.4 ([5]).

1. *A set $S \subset X$ is said to be (n, ε)-separated if $\rho_n(x,y) \geq \varepsilon$ for any distinct points $x, y \in S$.*

2. *Denote the maximum cardinality of an (n,ε)-separated set for γ by $sep(n,\varepsilon,\gamma)$. Since X is compact, $sep(n,\varepsilon,\gamma)$ is finite. The topological entropy of γ is*

$$h_{top}(\gamma) = \lim_{\varepsilon\to 0}\left[\limsup_{n\to\infty}\frac{1}{n}\ln sep(n,\varepsilon,\gamma)\right].$$

We define topological chaos of a system γ as the following statement, which can be found in page 137 of [10].

Definition 2.5 ([10])**.** *We say $\gamma : X \to X$ exhibits topological chaos if it has positive topological entropy.*

From now on, we discuss the property of topological chaos of systems (6) and (7).

In [12], Li and Malkin proved that a difference equation has the same dynamic as its corresponding map. (in [12], definition 3.1 describes how a difference equation corresponds a map. Furthermore, the item (iii) of theorem 3.3 describes a commutative diagram which claims that the topological entropies of the difference equation and the map are identical.) Since the maps (6) and (7) correspond to the difference equations (2) and (3) respectively, we just exhibit the dynamics of the maps in the results of theorems 2.6, 2.7, 2.8 and 2.9.

Theorems 2.6 and 2.7 below show us how to find some special parameters and make a small perturbation such that the difference equations (2) and (3) both possess chaotic behavior in view of simple zeros of some function associated with f and by applying the main theorems in [12].

Theorem 2.6. *Let f be C^1 on $Q = I\backslash V$ for some compact interval $I \subset \mathbb{R}$ and some open set $V \subset I$. Suppose that $-qx + f(x)$ has $k \geq 2$ simple zeros in $int(Q)$. Then there exists $\delta > 0$ such that for any $p \in (0,\delta)$, there exists a closed σ-invariant subset Γ_p of Y_p, the set of solutions of the difference equation (2) with the topology of pointwise convergence, such that $\sigma|_{\Gamma_p}$ is topologically conjugate to $\sigma|_{\Sigma_k}$. In particular, $h_{top}\left(\sigma|_{Y_p}\right) \geq \log k$ and thus (6) exhibits topological chaos.*

Theorem 2.7. *Let f be C^1 on $Q = I\backslash V$ for some compact interval $I \subset \mathbb{R}$ and some open set $V \subset I$. If f has $k \geq 2$ simple zeros in $int(Q)$, then there exists $\eta > 0$ such that for p, q satisfying $p \neq 0$ and $\sqrt{p^2+q^2} < \eta$, there exists a closed σ-invariant subset $\Pi_{p,q}$ of $Y_{p,q}$, the set of solutions of difference equation (3) with the topology of pointwise convergence, such that $\sigma|_{\Pi_{p,q}}$ is topologically conjugate to $\sigma|_{\Sigma_k}$. In particular, $h_{top}\left(\sigma|_{Y_{p,q}}\right) \geq \log k$ and thus (7) exhibits topological chaos.*

The constants δ and η in theorems 2.6 and 2.7 are mainly chosen as δ_0 in theorem 5.1.

Different from theorems 2.6 and 2.7. Theorems 2.8 and 2.9 below show the chaos property of (2) and (3) in view of topological entropy of some function associated with f and by applying the main theorems in [8].

Theorem 2.8. *Let $b, p \neq 0$ $(a, p \neq 0)$. Suppose that*

1. *f is analytic on $Q = I \backslash V$ for some compact interval $I = [\alpha, \beta] \subset \mathbb{R}$, $\alpha < \beta$, and some set V which is a union of finitely many open subintervals in I;*

2. *$-\frac{q}{b^2 p} x + \frac{1}{b^2 p} f(x)$ ($-\frac{q}{a^2 p} x + \frac{1}{a^2 p} f(x)$): $Q \to I$ has positive topological entropy.*

Then there exists $\delta > 0$ such that for any $a \in (0, \delta)$ (or $b \in (0, \delta)$), $\sigma|_{\Gamma_a}$ (or $\sigma|_{\Gamma_b}$) has positive topological entropy for some closed (in the product topology) σ-invariant subset Γ_a (or Γ_b) of the set of solutions of (2). Thus, the dynamical system (6) exhibits topological chaos.

Theorem 2.9. *Let $q \neq 0$ be a constant. Suppose that*

1. *f is analytic on $Q = I \backslash V$ for some compact interval $I = [\alpha, \beta] \subset \mathbb{R}$, $\alpha < \beta$, and some set V which is a union of finitely many open subintervals in I*

2. *$\frac{1}{q} f : Q \to I$ has positive topological entropy.*

Then there exists $\eta > 0$ such that for any p with $0 < |p| < \eta$, $\sigma|_{\Pi_p}$ has positive topological entropy for some closed (in the product topology) σ-invariant subset Π_p of the set of solutions of (3). Thus, the dynamical system (7) exhibits topological chaos.

The constants δ and η in theorems 2.8 and 2.9 are mainly chosen as δ in theorem 5.3.

Finally, discuss the chaos of multidimensional-function form of (1) directly (refer to (6) and (7)). The following theorems tell us that some lower-dimensional function associated with f affects the higher-dimensional function F.

Theorem 2.10. *Let $p \neq 0$ be constant. Suppose that f is a continuous function and $\frac{-q}{a^2 p} y + \frac{1}{a} f\left(\frac{1}{ap} y\right)$ has positive topological entropy. Then there exists $\delta_0 > 0$ such that the dynamical system (6) exhibits topological chaos for $0 < b < \delta_0$.*

Theorem 2.11. *Let $p \neq 0$ be constant. Suppose that f is continuous and $\frac{-1}{a^2 p} f(-U_1) \subset int(U_1)$ and $\frac{1}{a} f\left(\frac{1}{ap} U_2\right) \subset int(U_2)$ for some compact intervals $U_1 = [\alpha_1, \beta_1] \subset \mathbb{R}$ and $U_2 = [\alpha_2, \beta_2]$, where $\alpha_1 < \beta_1$ and $\alpha_2 < \beta_2$. If $\max\left(h_{top}\left(\frac{1}{a^2 p} f\right), h_{top}(g)\right) > 0$ with $g(y) = \frac{1}{a} f\left(\frac{1}{ap} y\right)$, then there exists $\eta_0 > 0$ such that the dynamical system (7) exhibits topological chaos for $0 < \sqrt{b^2 + q^2} < \eta_0$.*

3 Preliminary I

In this section, we introduce the preliminary of proving theorems 2.2 and 2.3 which mainly appears in [1].

Let $T_i : \mathbb{R} \to \mathbb{R}$, $i \in \mathbb{Z}$, be a family of maps and each T_i satisfies the Lipschitz condition, i.e., $L_i = \sup_{x \neq y} \frac{|T_i x - T_i y|}{|x-y|} < \infty$. Next, define a function $H : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}}$, $\mathbb{R}^{\mathbb{Z}} = \{(\cdots r_{-1} r_0 r_1 \cdots) : r_j \in \mathbb{R},\ j \in \mathbb{Z}\}$, which satisfies the following statements:

1. for each $i \in \mathbb{Z}$, there exists a finite set $\mathbb{Z} \supset K_i \ni i$;

2. for any $i \in \mathbb{Z}$, there exists a continuous function $H_i : \prod_{j \in K_i} X_j \to X_i$, $X_j = \mathbb{R}$ for all j, which satisfies the Lipschitz condition of the form

$$|H_i(x_{j \in K_i}) - H_i(y_{j \in K_i})| \leq \Lambda_i \sum_{j \in K_i} |x_j - y_j|$$

   for any $x_j, y_j \in X_j$ and for some constant $\Lambda_i > 0$.

   If $K_i = \{i\}$, we suppose H_i is an identity map.

3. $(H(x_j))_i = H_i(x_{j \in K_i})$ for all $i \in \mathbb{Z}$.

Let $T : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}}$ be defined by $(T(x))_i = T_i(x_i)$ for $x \in \mathbb{R}^{\mathbb{Z}}$ and $\mathfrak{F} : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}}$ be defined by $\mathfrak{F} = H \circ T$. $\mathfrak{F}$ is called the *dynamical network* [1].

Set a finite subset B of $\mathbb{Z}$. A graph $G = G(B, H)$ means that it contains elements in B called vertices and edges $i \to j$ (starting from i and ending to j) if and only if $i \in K_j$. We say G is *directed* if and only if every edge of G is directive. A path is called a *simple path* if each vertex on the path appears exactly once. Hereafter we assume G is connected, i.e., for any pair of vertices i and j, there exists a simple path from i to j without the directivity of G. For this graph, we can make a representation of it by a transition matrix $A = [a_{ij}]$ with $a_{ij} = 1$ if there is an edge $i \to j$ and $a_{ij} = 0$ otherwise. Next, we define a chain called *Topological Markov Chain* (TMC) as $\Sigma_A^+ = \left\{(i_0 i_1 i_2 \cdots) : a_{i_{j-1} i_j} = 1, j \in \mathbb{N}\right\}$ with a left-shift map $\sigma : \Sigma_A^+ \to \Sigma_A^+$ with $\sigma(i_0 i_1 i_2 \cdots) = (i_1 i_2 i_3 \cdots)$. We embed a metric $\underline{d}$ to Σ_A^+ by for any $\underline{i}, \underline{j} \in \Sigma_A^+$,

$$\underline{d}\left(\underline{i}, \underline{j}\right) = \sum_{n=0}^{\infty} \frac{|i_n - j_n|}{q^n},$$

where $q > 1$ is a constant.

Let $[i_0 i_1 \cdots i_r] = \left\{(j_0 j_1 \cdots) \in \Sigma_A^+ :\ j_0 = i_0, \cdots, j_r = i_r\right\}$ be a subset of the TMC $\left(\Sigma_A^+, \sigma\right)$, which is called a *cylinder*. A word $(i_0 \cdots i_r)$ is *allowable* if the cylinder $[i_0 i_1 \cdots i_r] \neq \emptyset$. For two vertices i, j of G, we can define a partial order and a relation of equivalence on them. We say $i \prec j$ if there exists a cylinder $[i_0 i_1 \cdots i_r] \neq \emptyset$ with $i_0 = j$ and $i_r = i$ for some $r \in \mathbb{N}$ and $i \sim j$ if $i \prec j \prec i$. By this relation, the set of the vertices of G

can be divided into classes of equivalence. Let symbols $1, 2, \cdots, N$ be the vertices of the connected graph G. Then $\{1, 2, \cdots, N\} = E_1 \cup E_2 \cup \cdots \cup E_s$ is a partition of classes of equivalence "$\sim$" for some $s \in \mathbb{N}$. We also define a partial order on E_m's: $E_m \prec E_k$ if and only if for every $p \in E_m$ and $q \in E_k$, $p \prec q$.

Next, we state the definition of a *nonwandering point* and *topological transitivity* of a dynamical system $\gamma : X \to X$, where X is a phase space.

Definition 3.1. *A point $x_0 \in X$ is nonwandering if for any neighborhood O of x_0, there exists a point $y \in O$ and $m \in \mathbb{N}$ such that $\gamma^m(y) \in O$; otherwise, x_0 is wandering.*

Definition 3.2. *$\gamma : X \to X$ is topologically transitive if for any nonempty open sets $O_1, O_2 \subset X$, there exists $n \geq 0$ such that $\gamma^n(O_1) \cap O_2 \neq \emptyset$.*

Notice that y in definition 3.1 is possibly chosen to be x_0. We then have the following well-known theorem.

Theorem 3.3 (Spectral Decomposition Theorem [1, 4])**.**

1. *Let NW be the set of nonwandering points of (Σ_A^+, σ). Then NW has a decomposition*

$$NW = \Sigma^1 \cup \Sigma^2 \cup \cdots \cup \Sigma^s$$

such that for any $k = 1, \cdots, s$,

   i. *$\sigma|_{\Sigma^k}$, σ restricted to Σ^k, is a TMC corresponding to E_k and Σ^k has a corresponding transition matrix $A(k)$,*

   ii. *$\sigma|_{\Sigma^k}$ is topologically transitive,*

   iii. *define a partial order $\prec$ on Σ^k's by $\Sigma^k \prec \Sigma^m$ if and only if $E_k \prec E_m$. Then it is well-defined.*

2. *Let W be the set of wandering points of (Σ_A^+, σ). Then W can be written as a composition*

$$W = \bigcup_{k,m=1}^{s} W_{km},$$

where

   i. *$W_{km} \neq \emptyset$ if and only if $k \neq m$ and $\Sigma^k \succ \Sigma^m$,*

   ii. *if $\underline{i} = (i_0 i_1 \cdots) \in W_{km}$, then $i_0 \in E_k$ and $\sigma^n(\underline{i}) \in \Sigma^m$ for some $n \in \mathbb{N}$,*

   iii. *if $\Sigma^k \succ \Sigma^m$, then for any $\underline{j} \in \Sigma^k$, $\underline{i} \in \Sigma^m$ and $\epsilon > 0$ there exist $\underline{w} \in W_{km}$ and $n \in \mathbb{N}$ such that $\underline{d}(\underline{w}, \underline{j}) < \epsilon$ and $\sigma^n(\underline{w}) = \underline{i}$.*

We use these two decompositions of NW and W to define two kinds of sets. For each $k \in \{1, \cdots, s\}$, let $P_k = \{m : E_m \succ E_k\} \cup \{k\}$ and $\Psi_k = \left(\bigcup_{m \in P_k} \Sigma^m\right) \cup \left(\bigcup_{m_1 \neq m_2, m_2 \in P_k} W_{m_1 m_2}\right)$. Then $\sigma|_{\Psi_k}$ is a TMC and denote it by $\left(\Sigma^+_{R_k}, \sigma\right)$ with the corresponding transition matrix R_k.

Let G be a directed graph with vertices $1, 2, \cdots, N$. Define metric spaces $Y^{(k)} = \prod_{m \in P_k, i \in E_m} X_i$ with sup-metric $d(x, y) = \sup_j |x_j - y_j|$ for all $k \in \{1, \cdots, s\}$. We can show that $Y^{(k)}$ is $\mathfrak{F}$ *-invariant* in the sense that for $m \in P_k$, $(\mathfrak{F}(x))_j = (\mathfrak{F}(y))_j$ if $j \in E_m$ for all $x, y \in \mathbb{R}^{\mathbb{Z}}$ with $x_i = y_i$ if $i \in E_m$.

Lemma 3.4 ([1]). *Let $x, y \in \mathbb{R}^{\mathbb{Z}}$ with $x_i = y_i$ for any $m \in P_k$ and $i \in E_m$. Then $(\mathfrak{F}(x))_i = (\mathfrak{F}(y))_i$ for any $m' \in P_k$ and $i \in E_{m'}$. Thus, $Y^{(k)}$ is said to be $\mathfrak{F}$-invariant.*

Proof. Let $m \in P_k$ and $i \in E_m$. If $j \in K_i$, then the class of equivalence, say $E_{m'}$, containing j is a predecessor of E_m in the order $\succ$, i.e., $E_{m'} \succ E_m$. Since $m \in P_k$, $E_m \succ E_k$ and then $m' \in P_k$. This implies $x_j = y_j$ for any $j \in K_i$.

$$(\mathfrak{F}(x))_i = (H \circ T(x))_i = H_i\left((T(x))_{j \in K_i}\right) = H_i\left((T(x))_{j \in K_i}\right) = H_i(\{T_j(x_j) : j \in K_i\})$$
$$= H_i(\{T_j(y_j) : j \in K_i\}) = (\mathfrak{F}(y))_i.$$ □

By this lemma, $\mathfrak{F}$ restricted to $Y^{(k)}$ is well-defined. We denote it by $\mathfrak{F}_k$.

By the lemma 3 in [1], Afraimovich and Bunimovich estimated the Lipschitz constant of $\mathfrak{F}^n_k$, $n \in \mathbb{N}$, with this consequence:

$$d\left(\mathfrak{F}^n_k(x), \mathfrak{F}^n_k(y)\right) \leq \left(\sum_{(i_0 \cdots i_n)} \prod_{l=0}^{n-1} \lambda_{i_l i_{l+1}}\right) d(x, y), \tag{8}$$

where the sum is taken over the number of all the allowable words of length $(n+1)$ in $\Sigma^+_{R_k}$, $i_n \in E_k$, and $\lambda_{i_l i_{l+1}} = L_{i_l} \Lambda_{i_{l+1}}$.

Define a function $\varphi : \Sigma^+_{R_k} \to \mathbb{R}$ by $\varphi(i_0 i_1 i_2 \cdots) = \ln \lambda_{i_0 i_1}$. We rewrite the estimated Lipschitz constant $\Gamma_k(n, \varphi)$ of $\mathfrak{F}^n_k$ showed in (8) as

$$\Gamma_k(n, \varphi) = \sum_{(i_0 \cdots i_n)} \prod_{l=0}^{n-1} \lambda_{i_l i_{l+1}} = \sum_{\underline{w} \in [i_0 \cdots i_n]} \exp \sum_{l=0}^{n-1} \varphi\left(\sigma^l(\underline{w})\right),$$

where the sum is taken over the same set as in (8) and for each cylinder $[i_0 \cdots i_n]$ we choose only one sequence $\underline{w}$ as a representation. Next, define the topological pressure $P^k(\varphi)$ of φ over the TMC $\left(\Sigma^+_{R_k}, \sigma\right)$ by

$$P^k(\varphi) = \lim_{n \to \infty} \frac{\ln \Gamma_k(n, \varphi)}{n}.$$

This limit exists by proposition 2.5.1 in [3]. Therefore, we have a definition and a theorem as follows.

Definition 3.5. *Let $\gamma : D \subset \mathbb{R}^n \to \mathbb{R}^n$ be a function. γ is said to be a contraction if there exists a constant $0 \le \zeta < 1$ such that for any points $x, y \in D$, $|\gamma(x) - \gamma(y)| \le \zeta |x - y|$.*

Theorem 3.6 ([1])**.** *If $P^k(\varphi) < 0$, then there exists $n_0 \in \mathbb{N}$ such that $\mathfrak{F}_k^n$ is a contraction for all $n > n_0$.*

Finally, to evaluate the topological pressure, in [1], we may simplify the formula of the pressure by making the function φ which depends originally on the first two symbols of a sequence depend only on the first one symbol. Define a new transition matrix $\overline{A}$ whose symbols are the edges of G and which transits (ij) to (lm) iff $j = l$. Then $\left(\Sigma_{\overline{A}}^+, \sigma\right)$ forms a new TMC. From [4], we know that the two TMCs $\left(\Sigma_A^+, \sigma\right)$ and $\left(\Sigma_{\overline{A}}^+, \sigma\right)$ are topologically conjugate. Let $\Sigma_{\overline{R}_k}^+$ be the image of $\Sigma_{R_k}^+$ and then $\left(\Sigma_{\overline{R}_k}^+, \sigma\right)$ is the corresponding TMC. Denote the symbols of $\overline{A}$ by $1, 2, \cdots, \overline{N}$. For each $m = 1, 2, \cdots, \overline{N}$, there is a corresponding edge (ij) of G and then we set $\phi(m) = \varphi(ij)$. Define a new function $\phi : \Sigma_{\overline{R}_k}^+ \to \mathbb{R}$ by $\phi(i_0 i_1 \cdots) = \phi(i_0)$. Set $\phi(m) = \ln \mu_m$, $m = 1, 2, \cdots, \overline{N}$. Then $\Gamma_k(n, \varphi) = \Gamma_k(n, \phi) = \sum_{(i_0 \cdots i_n)} \prod_{m=0}^{n} \mu_m$ [1, 3]. We use such method of evaluation to prove our main result in this section.

Proposition 3.7 ([1])**.** *$\Gamma_k(n, \phi) = R\left(\overline{R}_k diag(\mu_1, \cdots, \mu_{\overline{N}})\right)^n E^T$, where $R = (\mu_1, \cdots, \mu_{\overline{N}})$ and $E = (1, 1, \cdots, 1)$.*

Corollary 3.8 ([1])**.** *$P^k(\varphi) = \ln r_k$, where r_k is the maximal absolute value of all the eigenvalues of $\overline{R}_k diag(\mu_1, \cdots, \mu_{\overline{N}})$.*

4 Proofs of theorems 2.2 and 2.3

In this section, we show that the dynamical systems (6) and (7) have global attractors individually as follows.

Proof of theorem 2.2. For any $i \in \mathbb{Z}$, let $T_i : \mathbb{R} \to \mathbb{R}$ be defined by $T_i(z) = z$, for all $z \in \mathbb{R}$. Set $K_1 = K_2 = \{1, 2\}$ and $K_i = \{i\}$ for all $i \neq 1, 2$. Define $H_i : \prod_{j \in K_i} X_j \to X_i$, $i \in \mathbb{Z}$ and $X_i = \mathbb{R}$, by

$$\begin{aligned} H_1(z_1, z_2) &= \frac{b}{a} z_1 + \frac{1}{ap} z_2, \\ H_2(z_1, z_2) &= -\frac{bq}{a^2} z_1 + \frac{1}{a}\left(b - \frac{q}{ap}\right) z_2 + \frac{1}{a} f\left(\frac{b}{a} z_1 + \frac{1}{ap} z_2\right), \\ H_i(z_i) &= z_i, i \neq 1, 2. \end{aligned}$$

Then H_i is continuous and $L_i = \sup_{x \neq y} \frac{|T_i x - T_i y|}{|x-y|} = 1$ for all i. Next, we want to find constants $\Lambda_i > 0$ satisfying $|H_i(z_1, z_2) - H_i(w_1, w_2)| \leq \Lambda_i \sum_{j \in K_i} |z_j - w_j|$. Let $z_i, w_i \in X_i$,

$$\begin{aligned} |H_1(z_1, z_2) - H_1(w_1, w_2)| &= \left| \left(\frac{b}{a} z_1 + \frac{1}{ap} z_2\right) - \left(\frac{b}{a} w_1 + \frac{1}{ap} w_2\right) \right| \\ &\leq \frac{b}{a} |z_1 - w_1| + \left|\frac{1}{ap}\right| |z_2 - w_2| \\ &\leq \max\left(\frac{b}{a}, \left|\frac{1}{ap}\right|\right) \sum_{j \in K_1} |z_j - w_j| \\ |H_2(z_1, z_2) - H_2(w_1, w_2)| &= \left| \left[-\frac{bq}{a^2} z_1 + \frac{1}{a}\left(b - \frac{q}{ap}\right) z_2 + \frac{1}{a} f\left(\frac{b}{a} z_1 + \frac{1}{ap} z_2\right)\right] \right. \\ &\quad \left. - \left[-\frac{bq}{a^2} w_1 + \frac{1}{a}\left(b - \frac{q}{ap}\right) w_2 + \frac{1}{a} f\left(\frac{b}{a} w_1 + \frac{1}{ap} w_2\right)\right] \right| \\ &= \left| -\frac{bq}{a^2}(z_1 - w_1) + \frac{1}{a}\left(b - \frac{q}{ap}\right)(z_2 - w_2) + \frac{1}{a}\left[f\left(\frac{b}{a} z_1 + \frac{1}{ap} z_2\right) \right.\right. \\ &\quad \left.\left. - f\left(\frac{b}{a} w_1 + \frac{1}{ap} w_2\right)\right] \right| \\ &\leq \left|\frac{bq}{a^2}\right| |z_1 - w_1| + \frac{1}{a}\left|b - \frac{q}{ap}\right| |z_2 - w_2| + \frac{M}{a}\left| \left(\frac{b}{a} z_1 + \frac{1}{ap} z_2\right) \right. \\ &\quad \left. - \left(\frac{b}{a} w_1 + \frac{1}{ap} w_2\right) \right| \\ &\leq \left|\frac{bq}{a^2}\right| |z_1 - w_1| + \frac{1}{a}\left|b - \frac{q}{ap}\right| |z_2 - w_2| + \frac{bM}{a^2} |z_1 - w_1| \\ &\quad + \left|\frac{M}{a^2 p}\right| |z_2 - w_2| \end{aligned}$$

$$\le \max\left(\left|\frac{bq}{a^2}\right| + \frac{bM}{a^2}, \frac{1}{a}\left|b - \frac{q}{ap}\right| + \frac{M}{|a^2 p|}\right) \sum_{j\in K_2} |z_j - w_j|$$

$$|H_i(z_i) - H_i(w_i)| = |z_i - w_i| = \sum_{j\in K_i} |z_j - w_j|, i \neq 1, 2.$$

Then $\Lambda_1 = \max\left(\frac{b}{a}, \left|\frac{1}{ap}\right|\right)$, $\Lambda_2 = \max\left(\left|\frac{bq}{a^2}\right| + \frac{bM}{a^2}, \frac{1}{a}\left|b - \frac{q}{ap}\right| + \frac{M}{|a^2p|}\right)$ and $\Lambda_i = 1$ if $i \neq 1, 2$. Thus,

$$\lambda_{11} = \lambda_{21} = \Lambda_1 = \max\left(\frac{b}{a}, \left|\frac{1}{ap}\right|\right),$$
$$\lambda_{12} = \lambda_{22} = \Lambda_2 = \max\left(\left|\frac{bq}{a^2}\right| + \frac{bM}{a^2}, \frac{1}{a}\left|b - \frac{q}{ap}\right| + \frac{M}{|a^2p|}\right).$$

Now let $B = \{1, 2\}$ be a finite subset of $\mathbb{Z}$ and consider the connected graph $G = G(B, H)$ corresponding to our H defined above. Then its transition matrix is $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. So the two symbols in B are in the same class of equivalence, say E_1. We have that $P_1 = \{m : E_m \succ E_1\} \cup \{1\} = \{1\}$, $\Psi_1 = \left(\bigcup_{m\in P_1} \Sigma^m\right) \cup \left(\bigcup_{m_1\neq m_2, m_2\in P_1} W_{m_1m_2}\right) = \Sigma^1 \cup \emptyset = \Sigma^1 = \Sigma_A^+$, $R_1 = A$, and $Y^{(1)} = \prod_{m\in P_1, i\in E_m} X_i = X_1 \times X_2 = \mathbb{R}^2$. Let $I = (11), II = (12), III = (21)$ and $IV = (22)$ be the new symbols of the new TMC $\left(\Sigma_{\overline{A}}^+, \sigma\right)$ corresponding to $\overline{A} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. Clearly, $\overline{R}_1 = \overline{A}$, $\mu_I = \mu_{III} = \Lambda_1$ and $\mu_{II} = \mu_{IV} = \Lambda_2$.

$$\overline{R}_1 diag(\mu_I, \mu_{II}, \mu_{III}, \mu_{IV}) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \Lambda_1 & 0 & 0 & 0 \\ 0 & \Lambda_2 & 0 & 0 \\ 0 & 0 & \Lambda_1 & 0 \\ 0 & 0 & 0 & \Lambda_2 \end{bmatrix} = \begin{bmatrix} \Lambda_1 & \Lambda_2 & 0 & 0 \\ 0 & 0 & \Lambda_1 & \Lambda_2 \\ \Lambda_1 & \Lambda_2 & 0 & 0 \\ 0 & 0 & \Lambda_1 & \Lambda_2 \end{bmatrix}. \tag{9}$$

The eigenvalues of (9) are 0 and $\Lambda_1 + \Lambda_2$. Since

$$P^{(1)}(\varphi) = \ln|\Lambda_1 + \Lambda_2|$$
$$= \ln\left[\max\left(\frac{b}{a}, \left|\frac{1}{ap}\right|\right) + \max\left(\left|\frac{bq}{a^2}\right| + \frac{bM}{a^2}, \frac{1}{a}\left|b - \frac{q}{ap}\right| + \frac{M}{|a^2p|}\right)\right]$$
$$< 0$$

by the hypotheses and corollary 3.8, $\mathfrak{F}^n$ is a contraction on $X_1 \times X_2 = \mathbb{R}^2$ if $n > n_1$ for some $n_1 \in \mathbb{N}$. Hence, by Contraction Mapping Principle, the dynamical system (6) has a global attractor. □

Remark 4.1. *Let $p, q \in \mathbb{R}$, $p \neq 0$, and $M > 0$ be a Lipschitz constant of a continuous function f on $\mathbb{R}$. If $q = 0 \vee 2abp$ and $\frac{b}{a} + \left|\frac{1}{ap}\right| + \max\left(\left|\frac{bq}{a^2}\right| + \frac{bM}{a^2}, \frac{M}{a^2 p}\right) < 1$, then (6) has a global attractor.*

Proof of theorem 2.3. For any $i \in \mathbb{Z}$, let $T_i : \mathbb{R} \to \mathbb{R}$ be defined by $T_i(z) = z$, for all $z \in \mathbb{R}$. Set $K_1 = K_2 = \{1, 2\}$ and $K_i = \{i\}$ for all $i \neq 1, 2$. Define $H_i : \prod_{j \in K_i} X_j \to X_i$, $i \in \mathbb{Z}$ and $X_i = \mathbb{R}$, by

$$\begin{aligned} H_1(z_1, z_2) &= \frac{b}{a} z_1 + \frac{1}{ap} z_2, \\ H_2(z_1, z_2) &= -\frac{bq}{a^2} z_1 + \frac{1}{a}\left(b - \frac{q}{ap}\right) z_2 + \frac{1}{a} f(z_1), \\ H_i(z_i) &= z_i, i \neq 1, 2. \end{aligned}$$

Then H_i is continuous and $L_i = \sup_{x \neq y} \frac{|T_i x - T_i y|}{|x - y|} = 1$ for all i. Next, we can find $\Lambda_1 = \max\left(\frac{b}{a}, \left|\frac{1}{ap}\right|\right)$, $\Lambda_2 = \max\left(\left|\frac{bq}{a^2}\right| + \frac{M}{a}, \frac{1}{a}\left|b - \frac{q}{ap}\right|\right)$ and $\Lambda_i = 1$ for all $i \neq 1, 2$.

Let $B = \{1, 2\}$. Since the graph of H and its corresponding transition matrix are the same as the ones in the proof of theorem 2.2, $Y^{(1)} = \prod_{m \in P_1, i \in E_m} X_i = X_1 \times X_2 = \mathbb{R}^2$ and

$$P^{(1)}(\varphi) = \ln|\Lambda_1 + \Lambda_2| = \ln\left[\max\left(\frac{b}{a}, \left|\frac{1}{ap}\right|\right) + \max\left(\frac{bq}{a^2} + \frac{M}{a}, \frac{1}{a}\left|b - \frac{q}{ap}\right|\right)\right] < 0.$$

By theorem 3.6 and Contraction Mapping Principle, (7) has a global attractor. □

Remark 4.2. *Let $p, q \in \mathbb{R}$, $p \neq 0$, and $M > 0$ be a Lipschitz constant of f. If $q = 0 \vee 2abp$ and $\frac{b}{a} + \left|\frac{1}{ap}\right| + \frac{|bq|}{a^2} + \frac{M}{a} < 1$, then (7) has a global attractor.*

5 Preliminary II

In this section, we introduce the preliminary of proving theorems 2.6, 2.7, 2.8 and 2.9.

At first, we must know how a difference equation

$$\Gamma\left(y_n, y_{n+1}, \cdots, y_{n+m}\right) = 0, \tag{10}$$

$n \in \mathbb{Z}$ and $\Gamma : \prod_{i=1}^{m+1} D_i \to \mathbb{R}$, $D_1, D_2, \cdots, D_{m+1} \subset E$ for some metric space E, has chaotic behavior. A bi-sequence $\{y_n\}_{n\in\mathbb{Z}}$ is a solution of the difference equation (10) iff for all $n \in \mathbb{Z}$, $(y_n, y_{n+1}, \cdots, y_{n+m})$ is a zero of Γ. Moreover, (10) has chaotic behavior iff the left-shift map σ restricted to the set of solutions for (10) has chaotic behavior. In the next step, we consider a special kind of parametric difference equations.

Let $S_\infty = \left\{\underline{y} = (\cdots y_{-2}y_{-1}y_0y_1\cdots) \in \mathbb{R}^{\mathbb{Z}} : \left\|\underline{y}\right\| = \sup_{n\in\mathbb{Z}} |y_n| < \infty\right\}$ be the space of all bounded sequences with the topology of uniform convergence and let σ be the left-shift map on S_∞. Let $\Phi_\lambda\left(y_n, y_{n+1}, \cdots, y_{n+m}\right) = 0$ be a family of difference equations corresponding to parameters $\lambda \in [\lambda_0, \lambda_1]$ for some real numbers $\lambda_0 < \lambda_1$. For every $\lambda \in [\lambda_0, \lambda_1]$, $\Phi_\lambda : Q^{m+1} \to \mathbb{R}$ is a C^1 function of $(m+1)$ variables, where $Q = I\backslash V$ for some compact nondegenerate interval $I \subset \mathbb{R}$ and some open subset V of I. Moreover, Φ_λ and every partial derivative $\partial_i\Phi_\lambda$ with respect to the i-th variable, $1 \le i \le m+1$, are also continuous in λ. We set $Y_\lambda = \left\{(\cdots y_{-2}y_{-1}y_0y_1\cdots) \in Q^{\mathbb{Z}} : \Phi_\lambda\left(y_n, y_{n+1}, \cdots, y_{n+m}\right) = 0 \text{ for all } n \in \mathbb{Z}\right\}$ to be the set of solutions of $\Phi_\lambda\left(y_n, y_{n+1}, \cdots, y_{n+m}\right) = 0$. Then Y_λ is closed in S_∞. Endow $I^{\mathbb{Z}}$ and Y_λ with the product topology, i.e., the topology of pointwise convergence, and denote such spaces by $I^{\mathbb{Z}}_{prod}$ and $Y_{\lambda,prod}$. Then $Y_{\lambda,prod}$ is a closed subset of $I^{\mathbb{Z}}_{prod}$ and is compact by the Tychonoff's Theorem. Tychonoff's theorem states that the product of any collection of compact topological spaces is compact.

For all $\lambda \in [\lambda_0, \lambda_1]$, $Y_{\lambda,prod}$ is σ-invariant and we may define the topological entropy of $\sigma|_{Y_{\lambda,prod}}$ by $h_{top}\left(\sigma|_{Y_{\lambda,prod}}\right)$. Li and Malkin in [12] proved an important result about the chaos of $\Phi_\lambda\left(y_n, y_{n+1}, \cdots, y_{n+m}\right) = 0$, a part of whose detail is shown below.

Theorem 5.1 ([12]). *Let*

$$\Phi_\lambda\left(y_n, y_{n+1}, \cdots, y_{n+m}\right) = 0, n \in \mathbb{Z} \tag{11}$$

be a family of difference equations with parameters $\lambda \in [\lambda_0, \lambda_1]$ and $\Phi_\lambda : Q^{m+1} \to \mathbb{R}$ with $Q = I\backslash V$ for some compact interval $I \subset \mathbb{R}$ and some open set $V \subset I$ satisfies

1. *for each λ, Φ_λ is C^1 on Q^{m+1},*

2. *Φ_λ is continuous in λ and*

3. *each partial derivative $\partial_i\Phi_\lambda$,which corresponds to the i-th variable, is also continuous in λ, $i = 1, \cdots, m+1$.*

Suppose that $\Phi_{\lambda_0}(x_1, \cdots, x_{m+1}) = \varphi(x_N)$ for some C^1 function $\varphi : Q \to \mathbb{R}$ which has k simple zeros in $int(Q)$, $k \in \mathbb{N}$, and some $N \in \mathbb{N}$ with $1 \le N \le m+1$. Then there exists $\delta_0 > 0$ such that for any $\lambda \in [\lambda_0, \lambda_0 + \delta_0]$ there exists a closed σ-invariant subset $\Gamma_\lambda \subset Y_{\lambda,prod}$ such that $\sigma|_{\Gamma_\lambda}$ is topologically conjugate to the full shift of k symbols $\sigma|_{\Sigma_k}$. In particular, $h_{top}(\sigma|_{Y_\lambda}) \ge \log k$.

Remark 5.2. *In fact, we may extend the space of λ to a general metric space E. At the moment, $[\lambda_0, \lambda_0 + \delta_0]$ is replaced by a δ_0-ball $B(\lambda_0, \delta_0)$ in E.*

In [8], there is another conclusion about the chaotic behavior of the difference equation (11). Indeed, at a specific value $\lambda = \lambda_0$, (11) is of the form $\Phi_{\lambda_0}(x_1, \cdots, x_{m+1}) = \xi(x_N, x_{N+L})$ for some distinct integers $1 \le N, N+L \le m+1$. Under a certain situation of ξ, a perturbation on λ is able to force (11) to obtain topological chaos. We write it in detail in theorem 5.3.

Theorem 5.3 ([8]). *Consider the family of difference equations (11), with parameters λ in a neighborhood of a specific value λ_0 in a metric space, satisfying the following assumptions:*

1. *for each λ, $\Phi_\lambda : Q^{m+1} \to \mathbb{R}$ with $Q = I \backslash V$ for some compact nondegenerate interval $I \subset \mathbb{R}$ and some set V which is a union of finitely many open subintervals in I,*
2. *for each λ, Φ_λ is C^1 in Q^{m+1},*
3. *Φ_λ and $\partial_i \Phi_\lambda$, $i = 1, \cdots, m+1$, are continuous in λ and*
4. *$\Phi_{\lambda_0}(x_1, \cdots, x_{m+1}) = \xi(x_N, x_{N+L})$ for some $1 \le N, N+L \le m+1$.*

Suppose that there exists a piecewise analytic function $\varphi : Q \to I$ with $h_{top}(\varphi) > 0$ such that $\xi(x, \varphi(x)) = 0$ for all $x \in Q$. Then for any $\varepsilon > 0$, there exists $\delta > 0$ such that for each λ in the δ-neighborhood of λ_0, $h_{top}(\sigma|_{\Gamma_\lambda}) > \frac{1}{|L|} h_{top}(\varphi) - \varepsilon$ for some closed (in the product topology) σ-invariant subset Γ_λ of $Y_{\lambda,prod}$.

Remark 5.4. *Let $D \subset \mathbb{R}$ be the domain of γ. We say a function $\gamma : D \to \mathbb{R}$ is analytic (on D) if for any $x_0 \in D$, there exists a sequence of real numbers $\{a_k\}_{k=0}^{\infty}$ such that $\gamma(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k$ in a neighborhood of x_0. γ is said to be piecewise analytic (on D) if D is a union of finitely many disjoint sets D_i and γ is analytic on each D_i. In particular, an analytic function is also piecewise analytic. For instance, all of polynomials are analytic on $\mathbb{R}$ and $\gamma(x) = |x|$ is piecewise analytic on $(-1, 1)$.*

6 Proofs of theorems 2.6, 2.7, 2.8 and 2.9

In this section, we prove theorems 2.6, 2.7, 2.8 and 2.9. The following two proofs are shown according to theorem 5.1.

Proof of theorem 2.6. Let a, b and q be fixed numbers and $\Phi_p(x_1, x_2, x_3) = a^2px_3 + b^2px_1 - (2abp - q)x_2 - f(x_2)$. It's easy to check that (i) for each $p \in \mathbb{R}$, Φ_p is C^1 on Q^3 and (ii) Φ_p and $\partial_i\Phi_p$ are continuous in p since f is C^1 on Q. By applying $p = 0$, we get that $\Phi_0(x_1, x_2, x_3) = qx_{n+1} - f(x_{n+1})$, which is a C^1 function of x_{n+1}.

Let Y_p be the set of solutions of Φ_p with the topology of pointwise convergence. Since $-qx + f(x)$ has $k \geq 2$ simple zeros in $int(Q)$, by theorem 5.1, there exists $\delta > 0$ such that for any $p \in (0, \delta)$, there exists a closed subset Γ_p of Y_p such that $\sigma|_{\Gamma_p}$ is conjugate to $\sigma|_{\Sigma_k}$ and so system (6) exhibits topological chaos. □

Proof of theorem 2.7. Let a and b be fixed numbers. If $p = q = 0$, then $\Phi_{p,q}(x_n, x_{n+1}, x_{n+2}) \overset{let}{=} a^2px_{n+2} + b^2px_n - (2abp - q)x_{n+1} - f(x_n) = -f(x_n)$, which is a C^1 function of one variable. f has $k \geq 2$ simple zeros in $int(Q)$, and so does $-f$. Let $Y_{p,q}$ be the set of solutions of $\Phi_{p,q}$ with the topology of pointwise convergence. By theorem 5.1, there exists $\eta > 0$ such that if $p \neq 0$ and $\sqrt{p^2 + q^2} < \eta$ then for some closed σ-invariant subset $\Pi_{p,q}$ of $Y_{p,q}$, $\sigma|_{\Pi_{p,q}}$ is conjugate to $\sigma|_{\Sigma_k}$, and $h_{top}\left(\sigma|_{Y_{p,q}}\right) \geq \log k$. Thus, system (7) is topologically chaotic. □

The two proofs below are shown according to theorem 5.3.

Proof of theorem 2.8. We discuss the case the constants $b, p \neq 0$. Denote $\Phi_a(x_1, x_2, x_3) = a^2px_3 + b^2px_1 - (2abp - q)x_2 - f(x_2)$. Since f is analytic on Q, Φ_a is also analytic and so C^1 on Q^3. $\Phi_0(x_1, x_2, x_3) = b^2px_1 + qx_2 - f(x_2) \overset{set}{=} \xi(x_1, x_2)$. For the equation $\xi(x_1, x_2) = 0$, x_1 can be expressed as $x_1 = -\frac{q}{b^2p}x_2 + \frac{1}{b^2p}f(x_2) \overset{set}{=} \varphi(x_2)$, which is analytic on Q.

Since $-\frac{q}{b^2p}x + \frac{1}{b^2p}f(x)$ has positive topological entropy, by theorem 5.3, for any $\varepsilon > 0$, there exists $\delta > 0$ such that for each $a \in (0, \delta)$, $h_{top}(\sigma|_{\Gamma_a}) > h_{top}(\varphi) - \varepsilon$ for some closed (in the product topology) σ-invariant subset Γ_a of the set of solutions for Φ_a. Therefore, if $\varepsilon > 0$ is chosen to be sufficiently small, then (6) has topological chaos. □

Proof of theorem 2.9. Define $\Phi_p(x_1, x_2, x_3) = a^2px_3 + b^2px_1 - (2abp - q)x_2 - f(x_1)$. Then $\Phi_0(x_1, x_2, x_3) = qx_2 - f(x_1) \overset{set}{=} \xi(x_1, x_2)$. The equation $\xi(x_1, x_2) = 0$ has an implicit-functioned solution $x_2 = \frac{1}{q}f(x_1)$. Assume $q \neq 0$. Since $\frac{1}{q}f$ is analytic on Q, it is also C^1 on Q.

By the hypothesis that $h_{top}\left(\frac{1}{q}f\right) > 0$ and theorem 5.3, for any $\varepsilon > 0$, there exists $\eta > 0$ such that for each p with $0 < |p| < \eta$, $h_{top}\left(\sigma|_{\Pi_p}\right) > \frac{1}{|L|}h_{top}(\varphi) - \varepsilon$ for some closed (in the product topology) σ-invariant subset Π_p of the set of solutions for Φ_p such that $h_{top}\left(\sigma|_{\Pi_p}\right) > h_{top}\left(\frac{1}{q}f\right) - \varepsilon$. If $\varepsilon > 0$ is small enough, then (7) has topological chaos. □

7 Preliminary III

In this section, we introduce the preliminary of proving theorems 2.10 and 2.11. In [11], it mainly discuss the multidimensional perturbations to a family of high-dimensional functions F_λ on $\mathbb{R}\times\mathbb{R}^n$ with parameters $\lambda\in\mathbb{R}^k$ at a specific value λ_0. For simplicity, set $\lambda_0=0$. Next, suppose F_0 has two forms: $F_0(x,y)=(f(x),g(x))$ and $F_0(x,y)=(f(x),g(x,y))$. The following two theorems (see the beginning of section 2 in [11]) explain the relation between f and F_λ. Moreover, two corollaries implied by these two theorems respectively follow immediately and we apply the results of them to our dynamical systems of function form. Notice that topological entropy of a map T here means the supremum of topological entropies of T restricted to compact T-invariant sets.

Theorem 7.1 ([11]). *Let F_λ be a family of continuous functions on $\mathbb{R}\times\mathbb{R}^n$ with parameters λ. Suppose that $F_\lambda(x,y)$ is continuous as a function jointly of $\lambda\in\mathbb{R}^k$ and $(x,y)\in\mathbb{R}\times\mathbb{R}^n$ and $F_0(x,y)=(f(x),g(x))$ with $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}^n\to\mathbb{R}^n$. Then $\liminf_{\lambda\to 0} h_{top}(F_\lambda)\geq h_{top}(f)$.*

Corollary 7.2. *Let F_λ be a family of continuous functions on $\mathbb{R}\times\mathbb{R}$ with parameters λ. Suppose that $F_\lambda(x,y)$ is continuous as a function jointly of $\lambda\in\mathbb{R}^k$ and $(x,y)\in\mathbb{R}\times\mathbb{R}$ and $F_0(x,y)=(f(y),g(y))$ with $f,g:\mathbb{R}\to\mathbb{R}$. Then $\liminf_{\lambda\to 0} h_{top}(F_\lambda)\geq h_{top}(g)$.*

Proof. Define a new family of functions $\widetilde{F}_\lambda$ by $\widetilde{F}_\lambda=L^{-1}\circ F_\lambda\circ L$, where $L:(x,y)\to(-y,x)$ is a linear map. Then $\widetilde{F}_0(x,y)=(g(x),-f(x))$ and $h_{top}\left(\widetilde{F}_\lambda\right)=h_{top}(F_\lambda)$ for all λ. Hence, $\liminf_{\lambda\to 0} h_{top}(F_\lambda)=\liminf_{\lambda\to 0} h_{top}\left(\widetilde{F}_\lambda\right)\geq h_{top}(g)$. □

Theorem 7.3 ([11]). *Let F_λ be a family of continuous functions on $\mathbb{R}\times\mathbb{R}^n$ with parameters λ. Suppose that $F_\lambda(x,y)$ is continuous as a function jointly of $\lambda\in\mathbb{R}^k$ and $(x,y)\in\mathbb{R}\times\mathbb{R}^n$ and $F_0(x,y)=(f(x),g(x,y))$ with $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$ which satisfies $g(\mathbb{R}\times U)\subset int(U)$ for some compact set $U\subset\mathbb{R}^n$ homeomorphic to the closed unit ball of $\mathbb{R}^n$. Then $\liminf_{\lambda\to 0} h_{top}(F_\lambda)\geq h_{top}(f)$.*

Corollary 7.4. *Let F_λ be a family of continuous functions on $\mathbb{R}\times\mathbb{R}$ with parameters λ. Suppose that $F_\lambda(x,y)$ is continuous as a function jointly of $\lambda\in\mathbb{R}^k$ and $(x,y)\in\mathbb{R}\times\mathbb{R}$ and $F_0(x,y)=(f(x,y),g(y))$ with $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ which satisfies $-f((-U)\times\mathbb{R})\subset int(U)$ for some compact set $U\subset\mathbb{R}$ homeomorphic to $[-1,1]$. Then $\liminf_{\lambda\to 0} h_{top}(F_\lambda)\geq h_{top}(g)$.*

Proof. Define $\widetilde{F}_\lambda$ in the same way as in the proof of corollary 7.2 and then $\widetilde{F}_0(x,y)=(g(x),-f(-y,x))$. Thus, $\liminf_{\lambda\to 0} h_{top}(F_\lambda)=\liminf_{\lambda\to 0} h_{top}\left(\widetilde{F}_\lambda\right)\geq h_{top}(g)$. □

Remark 7.5.

1. *Note that $rU=\{ru: u\in U\}$ for any $r\in\mathbb{R}$.*

2. *Such compact set U is actually a compact interval $[\alpha, \beta]$ for some $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, since any continuous map on the Euclidean spaces keeps the compactness and the connectedness of a set.*

8 Proofs of theorems 2.10 and 2.11

In this section, we regard F (see (6) and (7)) as a map with one or two real parameters. In order to avoid misunderstanding, we write the parameter as the subscript of F at appropriate moments.

For the first proof, we regard F as a family of maps $F_b : \mathbb{R}^2 \to \mathbb{R}^2$ by $F(x,y) = F_b(x,y) = \left(\frac{b}{a}x + \frac{1}{ap}y, -\frac{bq}{a^2}x + \frac{1}{a}\left(b - \frac{q}{ap}\right)y + \frac{1}{a}f\left(\frac{b}{a}x + \frac{1}{ap}y\right)\right)$, where b is a real parameter and $a > 0, p \neq 0, q$ are constants. Then F_b corresponds to the weighted difference equation (1) for case 1 (see (6)). The proof below is applied by corollary 7.2.

Proof of theorem 2.10. First, we have $F_0(x,y) = \left(\frac{1}{ap}y, \frac{-q}{a^2p}y + \frac{1}{a}f\left(\frac{1}{ap}y\right)\right)$. Clearly, F_0 is of the form in corollary 7.2. By corollary 7.2, $\liminf_{b\to 0} h_{top}(F_b) \geq h_{top}(g)$, where $g(y) = \frac{-q}{a^2p}y + \frac{1}{a}f\left(\frac{1}{ap}y\right)$.

Let $h_{top}(g) > 0$. Given $\varepsilon > 0$,

$$\begin{aligned} h_{top}(g) - \varepsilon &< h_{top}(g) \\ &\leq \liminf_{b\to 0} h_{top}(F_b) \\ &= \sup_{\delta>0}\left[\inf_{0<b<\delta} h_{top}(F_b)\right]. \end{aligned}$$

Then there exists $\delta_0 > 0$ such that $\inf_{0<b<\delta_0} h_{top}(F_b) > h_{top}(g) - \varepsilon$. Thus, if $0 < b < \delta_0$, then $h_{top}(F_b) \geq \inf_{0<b<\delta_0} h_{top}(F_b) > h_{top}(g) - \varepsilon$. Choose $\varepsilon < h_{top}(g)$, we get $h_{top}(F_b) > 0$ for $0 < b < \delta_0$ and so the result holds. □

Next, regard F as another family of maps

$$F(x,y) = F_{b,q}(x,y) = \left(\frac{b}{a}x + \frac{1}{ap}y, -\frac{bq}{a^2}x + \frac{1}{a}\left(b - \frac{q}{ap}\right)y + \frac{1}{a}f(x)\right)$$

with constants $a, p \neq 0$. Think about the twice iteration of $F_{b,q}$, denoted by $F_{b,q}^2$. Especially, $F_{0,0}^2(x,y) = F_{0,0}\left(\frac{1}{ap}y, \frac{1}{a}f(x)\right) = \left(\frac{1}{a^2p}f(x), \frac{1}{a}f\left(\frac{1}{ap}y\right)\right)$ when $b = q = 0$.

Before prove theorem 2.11, we recall a well-known property about topological entropy.

Proposition 8.1. *Let T be a map defined on compact metric space. Then $h_{top}(T^n) = n \cdot h_{top}(T)$ for all $n \geq 0$.*

The proof below is applied by corollary 7.4 and proposition 8.1.

Proof of theorem 2.11. Let a, p be fixed. Both $F_{b,q}$ and $F_{b,q}^2$ are continuous in (b,q) and $(x,y) \in \mathbb{R}^2$. Denote $\widehat{f}(x,y) = \widetilde{f}(x) = \frac{1}{a^2p}f(x)$ and $\widehat{g}(x,y) = \widetilde{g}(y) = \frac{1}{a}f\left(\frac{1}{ap}y\right)$. Since $-\widehat{f}((-U_1)\times\mathbb{R}) = \frac{-1}{a^2p}f(-U_1) \subset int(U_1)$ and $\widehat{g}(\mathbb{R}\times U_2) = \frac{1}{a}f\left(\frac{1}{ap}U_2\right) \subset int(U_2)$, by theorem 7.3 and corollary 7.4, $\liminf_{b,q\to 0} h_{top}\left(F_{b,q}^2\right) \geq \max\left(h_{top}\left(\widetilde{f}\right), h_{top}(\widetilde{g})\right) > 0$.

Given $\varepsilon > 0$,

$$\max\left(h_{top}\left(\widetilde{f}\right), h_{top}\left(\widetilde{g}\right)\right) - \varepsilon \;<\; \liminf_{b,q\to 0} h_{top}\left(F_{b,q}^2\right) = \sup_{\eta>0}\left[\inf_{0<|(b,q)|<\eta} h_{top}\left(F_{b,q}^2\right)\right].$$

Then $\inf_{0<\sqrt{b^2+q^2}<\eta_0} h_{top}\left(F_{b,q}^2\right) > \max\left(h_{top}\left(\widetilde{f}\right), h_{top}\left(\widetilde{g}\right)\right) - \varepsilon$ for some $\eta_0 > 0$. This implies that $h_{top}\left(F_{b,q}^2\right) > P_0 = \max\left(h_{top}\left(\widetilde{f}\right), h_{top}\left(\widetilde{g}\right)\right) - \varepsilon > 0$ for $0 < \sqrt{b^2+q^2} < \eta_0$ and $\varepsilon > 0$ is small enough. By the definition of supremum and for any b, q with $0 < \sqrt{b^2+q^2} < \eta_0$, we can find a compact $F_{b,q}^2$-invariant set $\Lambda_{p,q}$ such that $h_{top}\left(F_{b,q}^2|_{\Lambda_{p,q}}\right) > P_0$. Let $\Lambda'_{p,q} = \Lambda_{p,q} \cup F_{b,q}\left(\Lambda_{p,q}\right)$. Since $F_{b,q}$ is continuous, $\Lambda'_{p,q}$ is compact. Moreover, $F_{b,q}\left(\Lambda'_{p,q}\right) = F_{b,q}\left(\Lambda_{p,q}\right) \cup F_{b,q}^2\left(\Lambda_{p,q}\right) = F_{b,q}\left(\Lambda_{p,q}\right) \cup \Lambda_{p,q} = \Lambda'_{p,q}$. So $\Lambda'_{p,q}$ is $F_{b,q}$-invariant and also $F_{b,q}^2$-invariant. By proposition 8.1, $0 < P_0 < h_{top}\left(F_{b,q}^2|_{\Lambda'_{p,q}}\right) = 2 \cdot h_{top}\left(F_{b,q}|_{\Lambda'_{p,q}}\right)$. Thus, $h_{top}\left(F_{b,q}\right) > 0$ and so $F_{b,q}$ has topological chaos for $0 < \sqrt{b^2+q^2} < \eta_0$. $\square$

9 Numerical verification

In this section, we give some examples and their figures of iterations which agree with our main theorems mentioned in section 2. Moreover, these examples make our theorems applicable. From now on, we show their results by running 1500 iterations of F (see (6) and (7)) numerically and printing the points of the last 1000 times in the xy-plane. Finally, they are arranged into three subsections. Notice that case 1 and 2 mentioned in the subsections mean the hypothesis of u of (1), i.e., $u(t, x(t)) = f(x(t))$ and $u(t, x(t)) = f(x(t-1))$ respectively.

9.1 Examples for theorems 2.2 and 2.3

First, consider the dynamic of an example for theorem 2.2. Let $a = 1$, $b = 0.1$, $p = 10$ and $q = 0$. Define $f(x) = \sin x$. It's clear that f is Lipschitz on $\mathbb{R}$ and the Lipschitz constant $M = 1$. We check whether a, b, p and q satisfy the hypothesis of theorem 2.2.

$\max\left(\frac{b}{a}, \left|\frac{1}{ap}\right|\right) + \max\left(\left|\frac{bq}{a^2}\right| + \frac{bM}{a^2}, \frac{1}{a}\left|b - \frac{q}{ap}\right| + \frac{M}{|a^2 p|}\right) = 0.3 < 1$. So the hypothesis is satisfied. Set the initial point $(x_0, y_0) = (2000, 1000)$ and let F iterate with it. Figure 1(a) exhibits that $(0, 0)$ is the global attractor. One can see that the first and second components of F both converge to 0 quickly as the iteration increases in figure 1(b) and 1(c) (notice that variable n represents the times of iteration).

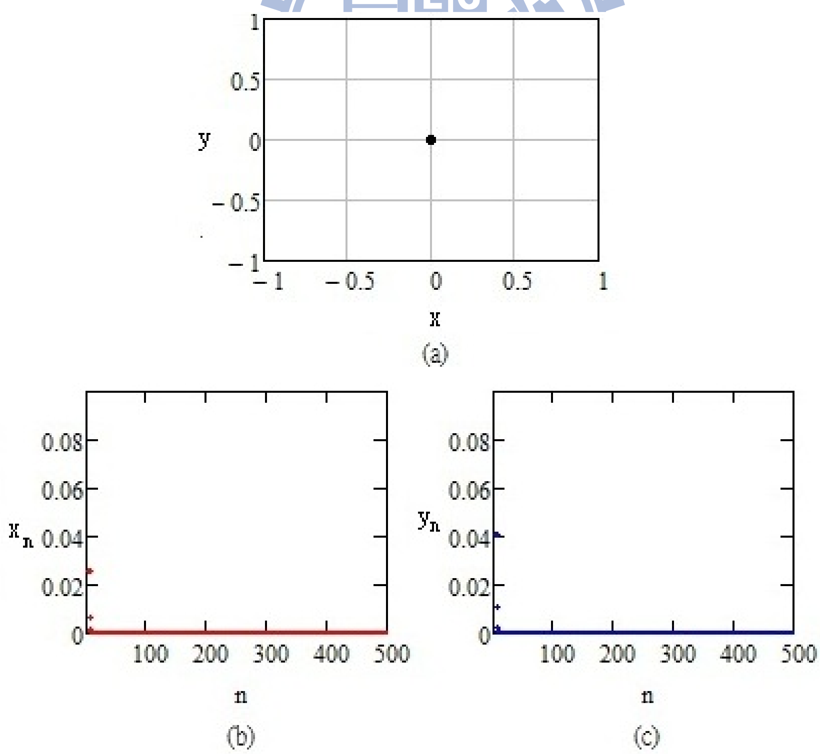

Figure 1: the dynamic diagram and the iterate diagrams of two components of the example for theorem 2.2

n	x_n	y_n
0	$2\cdot10^3$	$1\cdot10^3$
1	300	99
2	39.9	10.708
3	5.061	0.131
4	0.519	0.509
5	0.103	0.154
6	0.026	0.041
7	$6.664\cdot10^{-3}$	0.011
8	$1.743\cdot10^{-3}$	$2.819\cdot10^{-3}$
9	$4.562\cdot10^{-4}$	$7.381\cdot10^{-4}$
10	$1.194\cdot10^{-4}$	$1.932\cdot10^{-4}$
11	$3.127\cdot10^{-5}$	$5.059\cdot10^{-5}$
12	$8.186\cdot10^{-6}$	$1.325\cdot10^{-5}$
13	$2.143\cdot10^{-6}$	$3.468\cdot10^{-6}$
14	$5.611\cdot10^{-7}$	$9.078\cdot10^{-7}$
15	...	...

table 1 : iterations of the function (6) from 0 to 14 at (2000,1000)

Next, consider the dynamic of another example for theorem 2.3. Define $f(x) = |x|$ and let $a = 2$, $b = 0.1$, $p = 10$ and $q = 0$. Then $M = 1$ and $\max\left(\frac{b}{a}, \left|\frac{1}{ap}\right|\right) + \max\left(\frac{bq}{a^2} + \frac{M}{a}, \frac{1}{a}\left|b - \frac{q}{ap}\right|\right) = 0.55 < 1$, which satisfies the hypothesis of theorem 2.3. We also set the initial point $(x_0, y_0) = (2000, 1000)$. Figure 2(a) exhibits $(0,0)$ is the global attractor and figure 2(b),(c) exhibit the situation that two components of F converge.

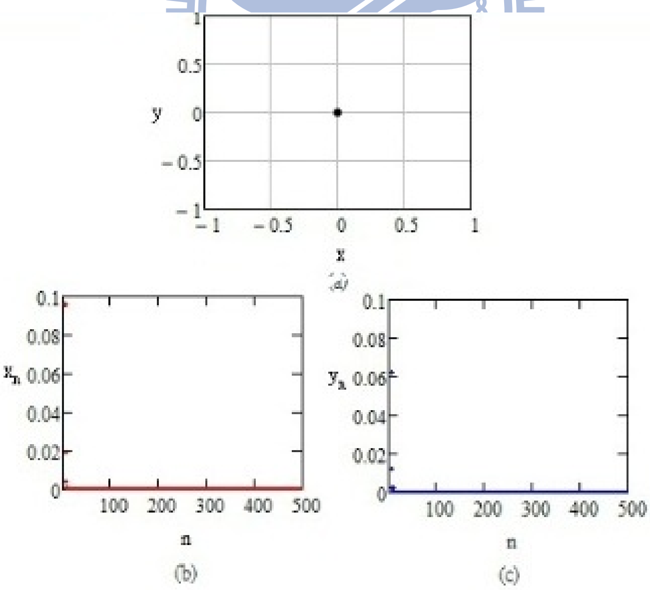

Figure 2: the dynamic diagram and the iterate diagrams of two components of the example for theorem 2.3

n	x_n	y_n
0	$2\cdot10^{3}$	$1\cdot10^{3}$
1	150	$1.05\cdot10^{3}$
2	60	127.5
3	9.375	36.375
4	2.288	6.506
5	0.44	1.469
6	0.095	0.293
7	0.019	0.062
8	$4.091\cdot10^{-3}$	0.013
9	$8.464\cdot10^{-4}$	$2.687\cdot10^{-3}$
10	$1.767\cdot10^{-4}$	$5.576\cdot10^{-4}$
11	$3.671\cdot10^{-5}$	$1.162\cdot10^{-4}$
12	$7.647\cdot10^{-6}$	$2.417\cdot10^{-5}$
13	$1.591\cdot10^{-6}$	$5.032\cdot10^{-6}$
14	$3.311\cdot10^{-7}$	$1.047\cdot10^{-6}$
15	...	...

table 2 : iterations of the map (7) from 0 to 14 at (2000,1000)

Therefore, the two examples verify the validity and the practicability of theorems 2.2 and 2.3.

9.2 Examples for theorems 2.6, 2.7, 2.8 and 2.9

First, we produce an example for theorem 2.6. Choose a set of special values of a, b, p, q and f as $a = 1$, $b = 0.1$, $p = 0.01$, $q = 0$ and $f(x) = 0.95\pi \sin x$. Now we check whether such values and f can satisfy the hypotheses of the theorem. Clearly, $0.95\pi \sin x$ has countably many simple zeros on $\mathbb{R}$. We choose $I = \left[\frac{-\pi}{2}, \frac{3\pi}{2}\right]$ and $V = \emptyset$. Then $-qx + f(x) = f(x)$ is C^1 on $\left[\frac{-\pi}{2}, \frac{3\pi}{2}\right]$ and has two simple zeros $0, \pi$ in $int\left(\left[\frac{-\pi}{2}, \frac{3\pi}{2}\right]\right) = \left(\frac{-\pi}{2}, \frac{3\pi}{2}\right)$. The result $p \in (0, \delta)$ means that p approaches to 0 very closely. Since the system (6) is undefined when $p = 0$, we just consider the dynamic for $p = 0.01$. Observe the dynamic in figure 3 for theorem 2.6 with an initial point $(x_0, y_0) = (0.01, 0.02)$. We see that there is an irregular graph in the xy-plane. We can also see the graphs of iteration of x- and y-components of (6) in figures 4 and 5.

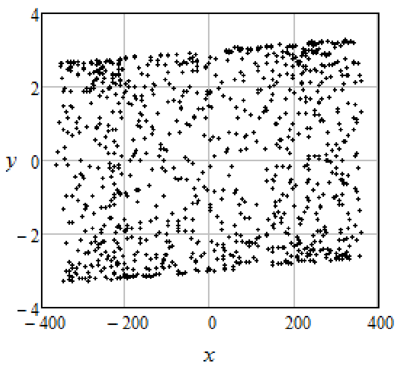

Figure 3: the dynamic diagram the example for theorem 2.6

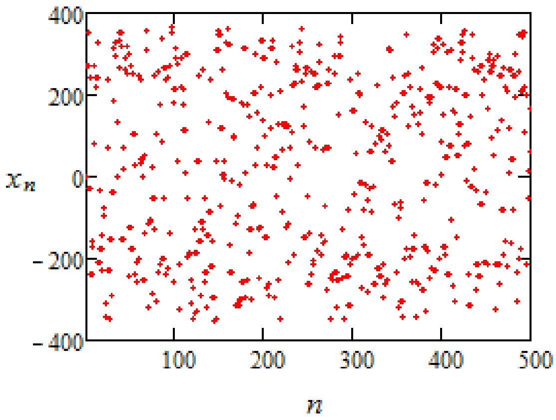

Figure 4: the graph of iteration of x-component of (6) with iterating times from 1 to 500 for theorem 2.6

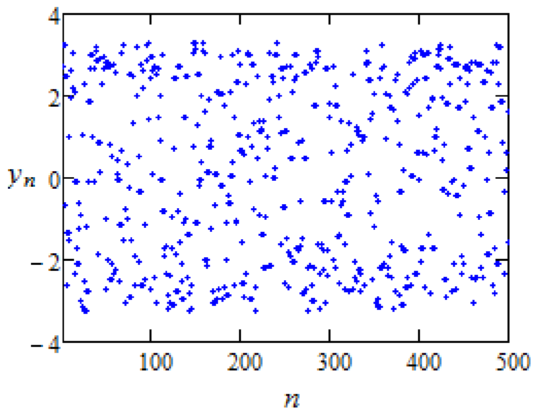

Figure 5: the graph of iteration of y-component of (6) with iterating times from 1 to 500 for theorem 2.6

To observe whether the perturbation of parameter does work, figure 6 shows the bifurcation about p around $p = 0.01$. We choose the interval of variation of p as $[0.001, 1]$ and also fix $q = 0$. With a different value of p, the system begins with a randomly-selected initial point. Afterward, print the second component of the 1000th to 1500th iterations.

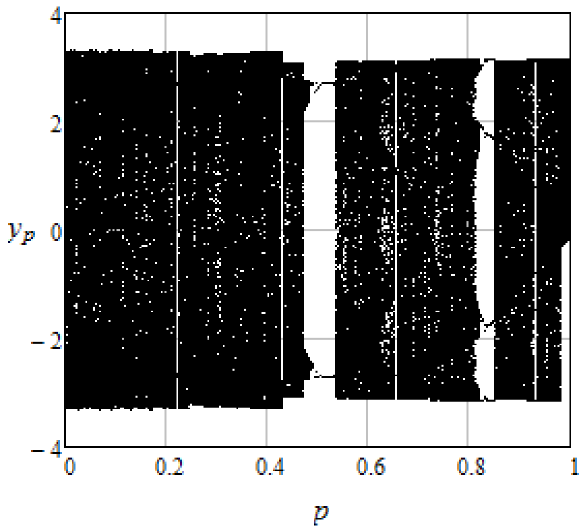

Figure 6: the bifurcation diagram of 2nd component of the map (6) about p of the example for theorem 2.6

Next, for theorem 2.7, we let $a = 6, b = 1, p = 0.01, q = 0$ and define $f(x) = x(1 - x)$. Then f is C^1 on $[-0.1, 1.1]$ and has two simple zeros $0, 1$ in $int([-0.1, 1.1]) = (-0.1, 1.1)$. Figure 7 informs us that the system (7) also exhibits a messy diagram with an initial point $(0.01, 0.02)$. We also show the graphs of iteration of two components of (7) in figures 8 and 9.

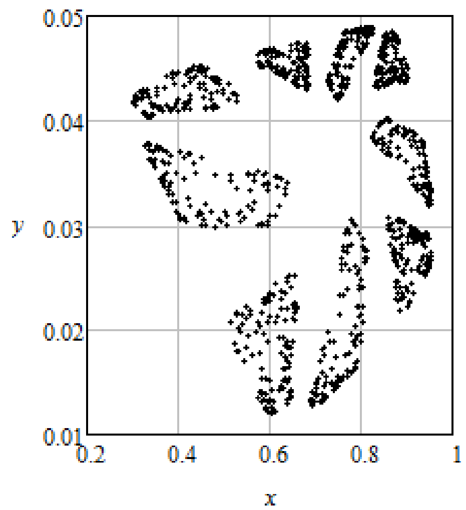

Figure 7: the dynamic diagram the example for theorem 2.7

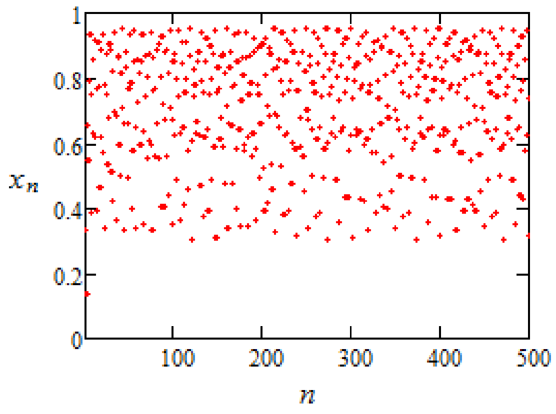

Figure 8: the graph of iteration of x-component of (7) with iterating times from 1 to 500 for theorem 2.7

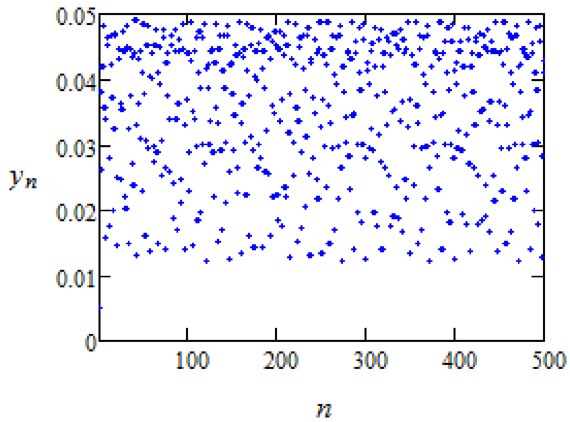

Figure 9: the graph of iteration of y-component of (7) with iterating times from 1 to 500 for theorem 2.7

Additionally, we also show the bifurcation diagrams of the second component about p and q individually as follows (see figure 10 for p and figure 11 for q with random initial points). Choose the intervals of variation of p and q as $[0.009, 0.012]$ and $[-0.01, 0.01]$ respectively.

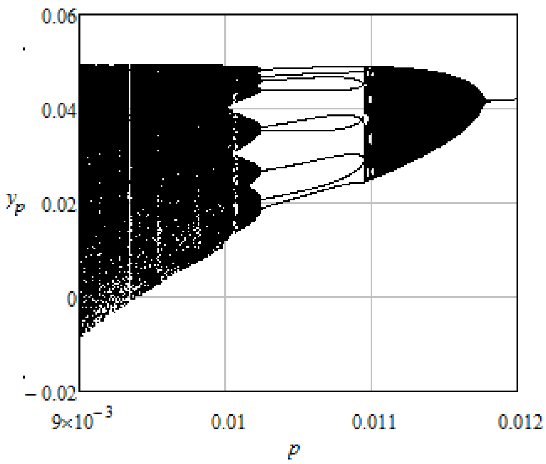

Figure 10: the bifurcation diagram of 2nd component of the map (7) about p of the example for theorem 2.7 and $q = 0$ fixed

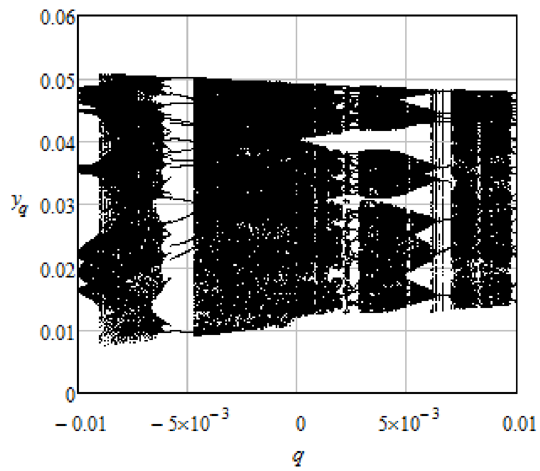

Figure 11: the bifurcation diagram of 2nd component of the map (7) about q of the example for theorem 2.7 and $p = 0.01$ fixed

For theorem 2.8, since a cannot be zero, we consider the case that $a = 0.0025$, $b = 0.001$, $p = 1$, $q = 0$ and $f(x) = 1.1\pi \times 10^{-5} \sin x$. It's clear that f is analytic on $[0, \pi] \setminus (r_1, r_2)$ for $0 < r_1 < r_2 < \pi$ and $11\pi \sin r_1 = 11\pi \sin r_2 = \pi$. The parametric map $g_u(x) = u \sin x$ has period-doubling property (see figure 12 with u from 0 to 35). We have $-\frac{q}{b^2 p}x + \frac{1}{b^2 p} f(x) = 11\pi \sin x \approx 34.56 \sin x$ and $1.1\pi \sin([0, \pi] \setminus (r_1, r_2)) = [0, \pi]$. Since $-\frac{q}{b^2 p}x + \frac{1}{b^2 p} f(x)$ has a 3-periodic point, whose period is not a power of 2, by theorem A in [7], we get that $h_{top}\left(-\frac{q}{b^2 p}x + \frac{1}{b^2 p} f(x)\right) > 0$. Thus, the hypothesis 2 of theorem 2.8 is satisfied.

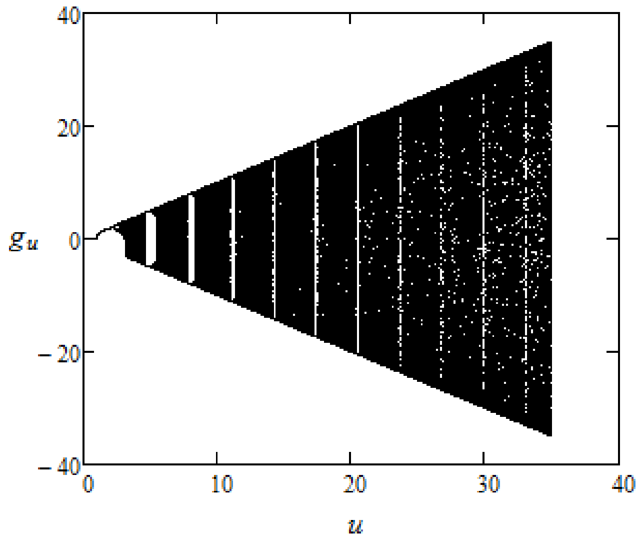

Figure 12: the bifurcation diagram of g_u with u from 0 to 35

Next, its dynamic diagram with the initial point $(0.001, 0.002)$ is shown in figure 13. We can see that its shape curls and the points on the graph are not uniformly dense.

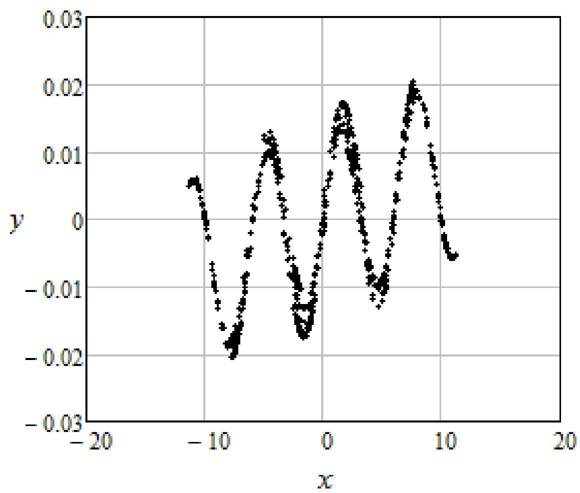

Figure 13: the dynamic diagram the example for theorem 2.8

The graphs of the two components of this example are shown in figures 14 and 15. It is observable that the situations of iterating of them act not so regularly.

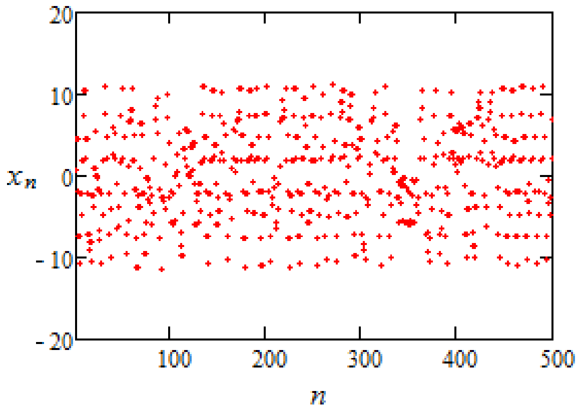

Figure 14: the graph of iteration of x-component of (6) with iterating times from 1 to 500 for theorem 2.8

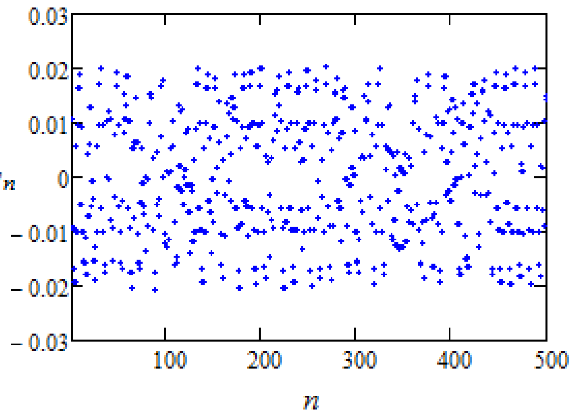

Figure 15: the graph of iteration of y-component of (6) with iterating times from 1 to 500 for theorem 2.8

For theorem 2.9, consider $a = 5$, $b = 1$, $p = 0.01$, $q = \frac{1}{1.1\pi}$ and $f(x) = \sin x$. Then f is analytic on $[0, \pi] \setminus (r_1, r_2)$ for $0 < r_1 < r_2 < \pi$ and $1.1\pi \sin r_1 = 1.1\pi \sin r_2 = \pi$. Moreover, since $\frac{1}{q} f(x) = 1.1\pi \sin x$, $\frac{1}{q} f([0, \pi] \setminus (r_1, r_2)) = [0, \pi]$ and $h_{top}\left(\frac{1}{q} f\right) > 0$. Similarly, we also show its dynamic diagram and the graphs of iteration of components with the initial point $(0.001, 0.002)$ (see figures 16, 17 and 18).

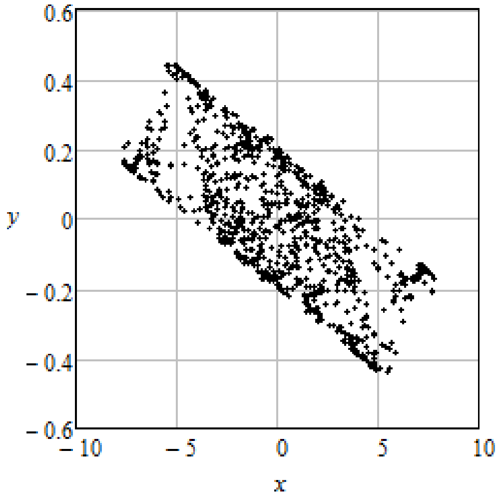

Figure 16: the dynamic diagram the example for theorem 2.9

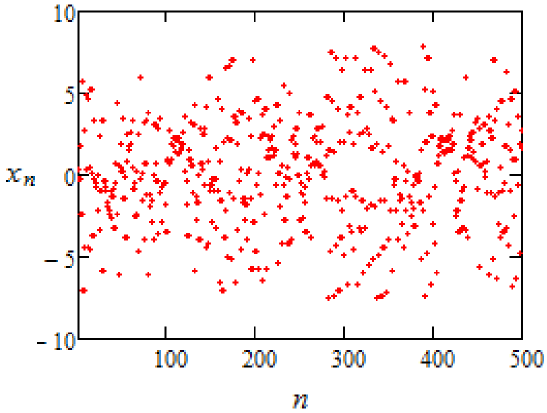

Figure 17: the graph of iteration of x-component of (7) with iterating times from 1 to 500 for theorem 2.9

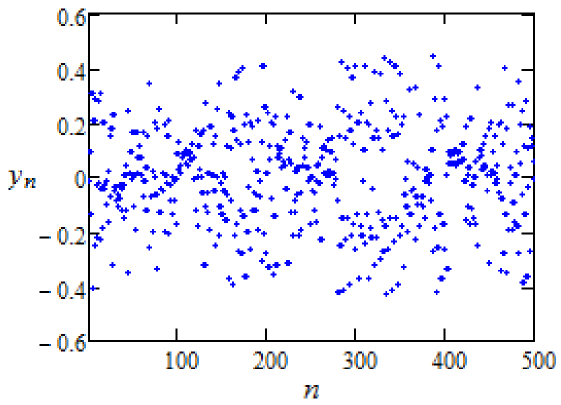

Figure 18: the graph of iteration of y-component of (7) with iterating times from 1 to 500 for theorem 2.9

We set its bifurcation diagram about the parameter p in the interval of variation $[0.009, 0.02]$ in figure 19. The system (7) has the similar dynamic for $a = 5$, $b = 1$, $q = \frac{1}{1.1\pi}$ and $f(x) = \sin x$ when p is around 0.01.

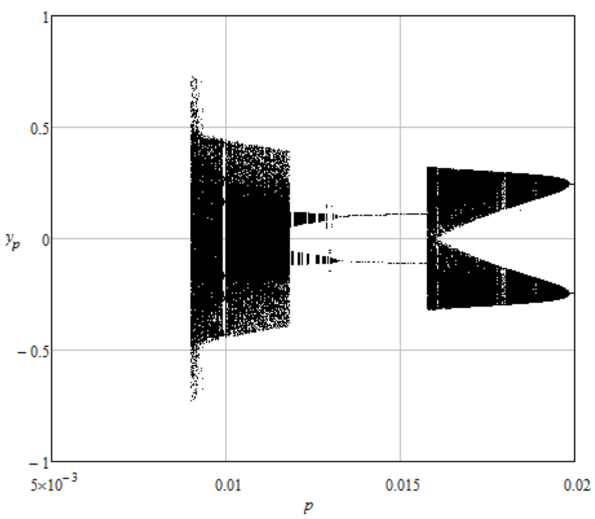

Figure 19: the bifurcation diagram of 2nd component of the map (7) about p of the example for theorem 2.9

Therefore, theorems 2.8 and 2.9 are applicable.

9.3 Examples for theorems 2.10 and 2.11

We give two examples for theorems 2.10 and 2.11 respectively in this subsection.

For theorem 2.10, consider the case $a = \frac{1}{3.9}, b = 0, p = 3.9, q = 0$ and $f(x) = x(1-x)$. Then we see that f is continuous on $\mathbb{R}$ and $\frac{-q}{a^2 p} y + \frac{1}{a} f\left(\frac{1}{ap} y\right) = 3.9y(1-y)$. It's known that $3.9y(1-y)$ has positive topological entropy (it has 3-periodic points).

Now we see the dynamic diagram in figure 20 and the graphs of iteration of two components in figures 21 and 22. Choose the initial point $(x_0, y_0) = (0.01, 0.02)$.

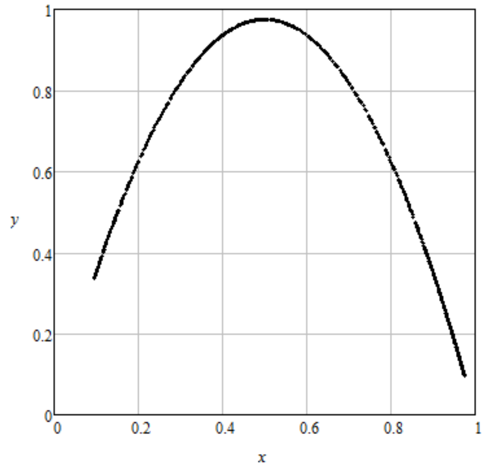

Figure 20: the dynamic diagram the example for theorem 2.10

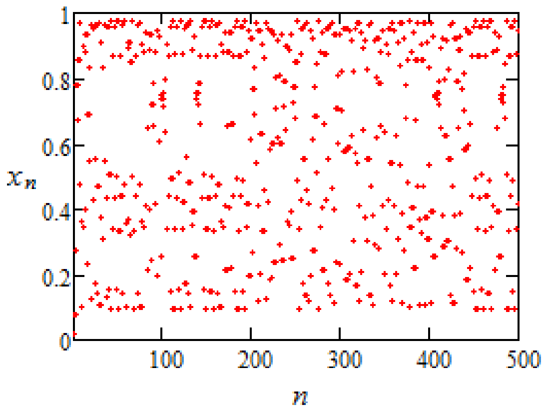

Figure 21: the graph of iteration of x-component of (6) with iterating times from 1 to 500 for theorem 2.10

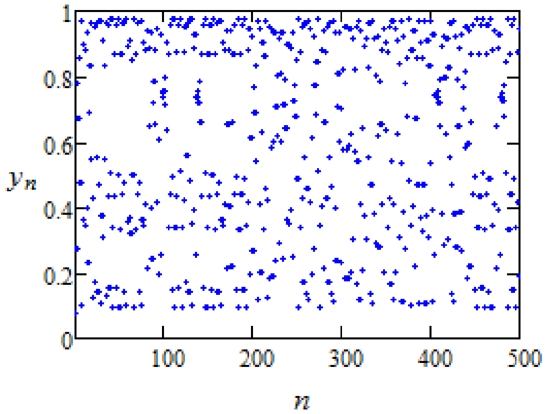

Figure 22: the graph of iteration of y-component of (6) with iterating times from 1 to 500 for theorem 2.10

The shape of figure 20 is like a part of a parabola and its distribution is not uniform. So it satisfies some characteristics of chaotic graphs. Figure 23 is the bifurcation diagram about b in $[-0.01, 0.01]$.

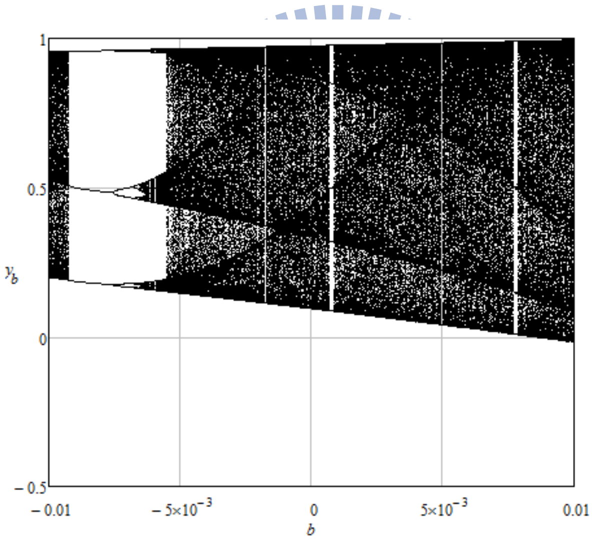

Figure 23: the bifurcation diagram of 2nd component of the map (6) about b of the example for theorem 2.10

For theorem 2.11, consider another case $a = 1, b = 0, p = 1, q = 0$ and $f(x) = \frac{19}{20}\pi \sin x$. Clearly, f is continuous on $\mathbb{R}$. Since $\frac{-1}{a^2 p} f(-x) = -f(-x) = f(x)$ and $\frac{1}{a} f\left(\frac{1}{ap}x\right) = f(x)$, we have that $f\left(\left[\frac{\pi}{21}, \frac{20\pi}{21}\right]\right) = \left[f\left(\frac{\pi}{21}\right), \frac{19\pi}{20}\right] \subset \left(\frac{\pi}{21}, \frac{20\pi}{21}\right)$ ($f\left(\frac{\pi}{21}\right) = f\left(\frac{20\pi}{21}\right) \approx 0.445$, $\frac{\pi}{21} \approx 0.15$). The parametric map $g_u(x) = u \sin(x)$ has also period-doubling property and $h_{top}(f) > 0$ (it has 3-periodic points). Thus, the hypothesis of theorem 2.11 is satisfied. Next, see the dynamic diagram in figure 24. Moreover, figures 25 and 26 are the graphs of iteration of x- and y-components of (7).

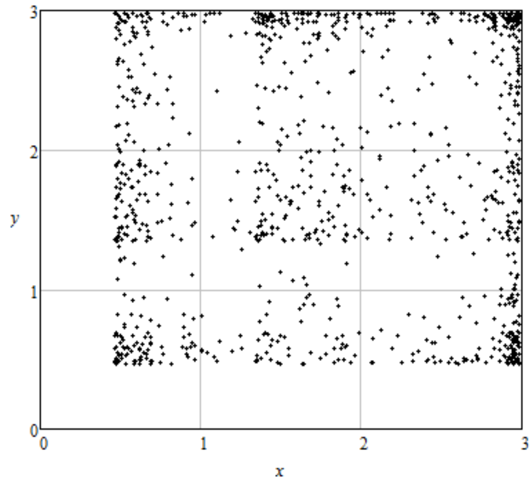

Figure 24: the dynamic diagram the example for theorem 2.11

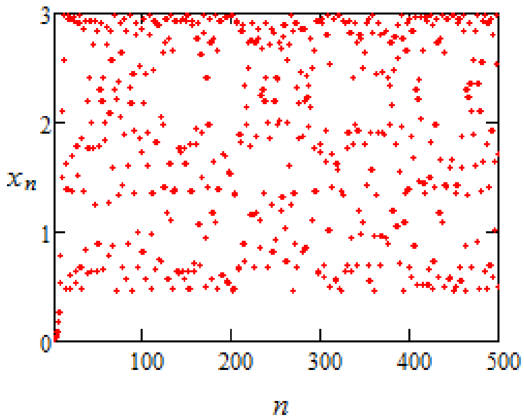

Figure 25: the graph of iteration of x-component of (7) with iterating times from 1 to 500 for theorem 2.11

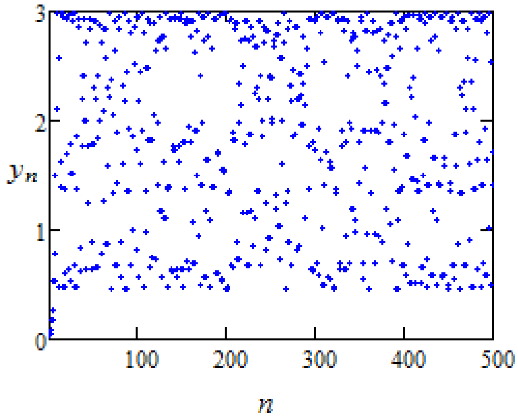

Figure 26: the graph of iteration of y-component of (7) with iterating times from 1 to 500 for theorem 2.11

Due to the result $\sqrt{b^2+q^2} < \eta_0$ for some $\eta_0 > 0$, we print two bifurcation diagrams about b and q respectively (see figures 27 and 28). So we can observe the chaotic property.

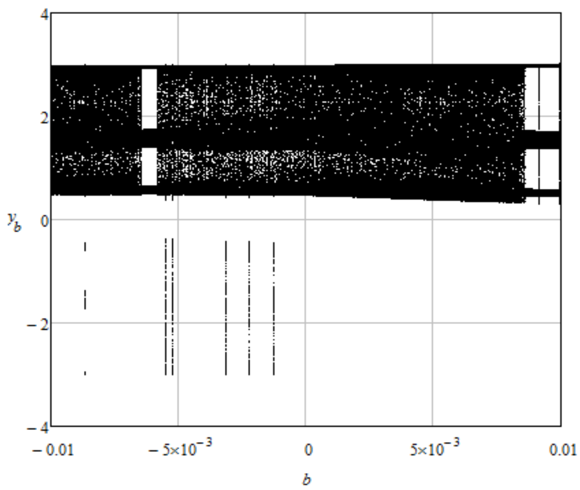

Figure 27: the bifurcation diagram of 2nd component of the map (7) about b of the example for theorem 2.11 and $q = 0$

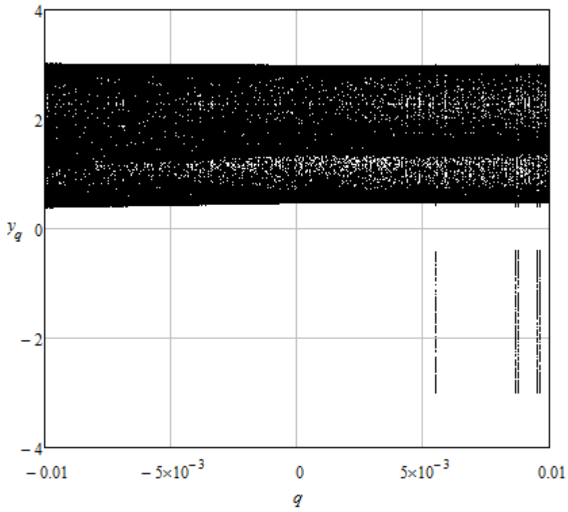

Figure 28: the bifurcation diagram of 2nd component of the map (7) about q of the example for theorem 2.11 and $b = 0$

Conclusively, theorems 2.10 and 2.11 do work.

References

[1] V S Afraimovich and L A Bunimovich 2007 Dynamical networks: interplay of topology, interactions and local dynamics *Nonlinearity* **20** 1761–71

[2] V S Afraimovich, L A Bunimovich and S V Moreno 2010 Dynamical networks: continuous time and general discrete time models *Regular and Chaotic Dynamics Vol. 15 Nos. 2–3* 127–45

[3] V S Afraimovich and S-B Hsu 2003 Lectures on chaotic dynamical systems *AMS Studies in Advance Mathematics* vol 28 (Providence, RI: American Mathematical Society)

[4] V M Alekseyev 1976 Symbolic dynamics *11th Summer Mathematical School* (Kiev: Mathematical Institute of UkAS) (in Russian)

V M Alekseyev 1981 *Translations of the AMS (Series 2) vol 116* (Providence, RI: American Mathematical Society) (Engl. Transl.)

[5] R Bowen 1971 Entropy for group endomorphisms and homogeneous spaces *Trans. Amer. Math. Soc.* **153** 401-14

[6] V Coti Zelati and P H Rabinowitz 1991 Homoclinic orbits for a second order Hamiltonian systems possessing superquadratic potentials *J. Amer. Math. Soc.* **4** 693–727

[7] J Hu and C Tresser 1998 Period doubling, entropy, and renormalization *Fundamenta Mathematicae* **155** 237-49

[8] J Juang, M-C Li and M Malkin 2008 Chaotic difference equations in two variables and their multidimensional perturbations *Nonlinearity* **21** 1019–40

[9] M Ma and Z Guo 2006 Homoclinic orbits for second order self-adjoint difference equations *J. Math. Anal. Appl.* **323** 513–21

[10] T Mitra 2001 A Sufficient Condition for Topological Chaos with an Application to a Model of Endogenous Growth *Journal of Economic Theory* **96** 133-52

[11] M-C Li, M-J Lyu and P Zgliczyński 2008 Topological entropy for multidimensional perturbations of snap-back repellers and one-dimensional maps *Nonlinearity* **21** 2555–67

[12] M-C Li and M Malkin 2006 Topological horseshoes for perturbations of singular difference equations *Nonlinearity* **19** 795–811

[13] K Tanaka 1991 Homoclinic orbits in a first-order superquadratic Hamiltonian system: Convergence of subharmonic orbits *J. Differential Equations* **94** 315–39