國 立 交 通 大 學

應用數學系

碩 士 論 文

一類三次非線性正定問題全分枝性
及確切正解個數

Global Bifurcation and Exact Multiplicity of Positive Solutions for a Positone Problem with Cubic Nonlinearity

研 究 生 ：曾至均
指導教授 ：石至文 教授
王信華 教授

中華民國 一 百 年 七 月

一類三次非線性正定問題全分枝性及確切正解個數

Global Bifurcation and Exact Multiplicity of Positive Solutions for a Positone Problem with Cubic Nonlinearity

研 究 生 ： 曾至均 Student : Chih-Chun Tzeng

指導教授 ： 石至文 教授 Advisors : Chih-Wen Shih

王信華 教授 Shin-Hwa Wang

國立交通大學

應用數學系

碩士論文

A Thesis

Submitted to Department of Applied Mathematics

College of Science

National Chiao Tung University

in Partial Fulfillment of the Requirements

for the Degree of

Master

in

Applied Mathematics

July 2011

Hsinchu, Taiwan, Republic of China.

中華民國 一 百 年 七 月

一類三次非線性正定問題全分枝性
及確切正解個數

學生：曾至均　　　　　　　　　指導教授：石至文 教授

王信華 教授

國立交通大學

應用數學系

摘要

本篇論文主要是探討一類三次非線性正定問題

$$\begin{cases} u''(x)+\lambda f_{\varepsilon}(u)=0,\ -1<x<1,\ u(-1)=u(1)=0, \\ f_{\varepsilon}(u)=-\varepsilon u^{3}+\sigma u^{2}-\kappa u+\rho,\ \lambda,\varepsilon,\sigma,\rho>0,\ 0<\kappa\le\sqrt{\sigma\rho}. \end{cases}$$

的全分支性及正解的確切個數。在適當的條件下，我們利用時間映射(time map)的方法來研究此一問題，並且證明在不同的演化參數下會有不同的分支曲線圖，進一步來說這些分支曲線基本上有兩種，不是單調曲線就是我們所稱的 S 型曲線。

Global Bifurcation and Exact Multiplicity of Positive Solutions for a Positone Problem with Cubic Nonlinearity

Student：Chih-Chun Tzeng　　　　Advisors：Chih-Wen Shih

Shin-Hwa Wang

Department of Applied Mathematics

National Chiao Tung University

Abstract

We study the global bifurcation and exact multiplicity of positive solutions of

$$\begin{cases} u''(x) + \lambda f_\varepsilon(u) = 0, \ -1 < x < 1, \ u(-1) = u(1) = 0, \\ f_\varepsilon(u) = -\varepsilon u^3 + \sigma u^2 - \kappa u + \rho, \ \lambda, \varepsilon, \sigma, \rho > 0, \ 0 < \kappa \le \sqrt{\sigma\rho}. \end{cases}$$

Where λ, $\varepsilon > 0$ are two bifurcation parameters, and σ, $\rho > 0$, $0 < \kappa \le \sqrt{\sigma\rho}$ are constants. We prove the global bifurcation of bifurcation curves for varying $\varepsilon > 0$ by developed some time-map techniques. More precisely, we prove that, for any σ, $\rho > 0$, $0 < \kappa \le \sqrt{\sigma\rho}$, there exists $\tilde{\varepsilon} > 0$ such that, on the $(\lambda, \|u\|_\infty)$-plane, the bifurcation curve is S-shaped for $0 < \varepsilon < \tilde{\varepsilon}$ and is monotone increasing for $\varepsilon \ge \tilde{\varepsilon}$. (We also prove the global bifurcation of bifurcation curves for varying $\lambda > 0$.) Thus we are able to determine the exact number of positive solutions by the values of ε and λ. Our results extend those of Hung and Wang (*Trans. Amer. Math. Soc.,* accepted to appear under minor revision) from $\kappa \le 0$ to $0 < \kappa \le \sqrt{\sigma\rho}$.

Key words and phrases: Global bifurcation; Exact multiplicity; Positive solutions; Positone problem; S-shaped bifurcation curve; Time map

Running head: A positone problem with cubic nonlinearity

誌謝

在交大應用數學所求學的過程裡，認識了不少老師和學長姊，不管是在追求學業的嚴謹態度或者是做人處事應對進退的觀念，都時常給予我很大的提醒，特別是要感謝我的指導教授王信華老師以及石至文老師。由於王信華老師的關愛與提攜，我才得以在研究所階段順利的完成學業，這點我必須要再次感謝王老師。

這篇論文的完成，首先要感謝王信華老師以及洪國智學長。王信華老師對於我的指導與鼓勵，甚或給予我人生規劃上的建議都給了我很大的幫助，王老師不僅授予我課業上的知識，對於我們的生活困境與挫折都給予幫助。

洪國智學長在我研究這篇論文的問題過程當中，無私地給予我建議以及方向，對於我的疑惑也都耐心解答。洪國智學長對於做研究的態度與耐心也是值得我學習效法的。

最後要感謝我的家人，爸爸、媽媽、哥哥至弘、大嫂宛嫻以及可愛的寶貝姪子文佑，你們是我的精神支柱，在我承受壓力以及挫折時默默的支持我、鼓勵我，尤其是爸媽時常關心我，讓我得以在學業上專注付出。另外還要感謝啟豪、建賢、家豪、書祥，感謝你們兩年給我的鼓舞和陪伴，讓我在碩士班生活感到多采多姿，謝謝你們。

Contents

1. Introduction

In this paper we study the global bifurcation and exact multiplicity of positive solutions of the positone problem with cubic nonlinearity

$$\begin{cases} u''(x) + \lambda f_\varepsilon(u) = 0, \quad -1 < x < 1, \;\; u(-1) = u(1) = 0, \\ f_\varepsilon(u) = -\varepsilon u^3 + \sigma u^2 - \kappa u + \rho, \;\; \lambda, \varepsilon, \sigma, \rho > 0, \end{cases} \tag{1.1}$$

where λ, ε are *two* bifurcation parameters, and σ, ρ are given constants. Moreover, we mainly consider that κ satisfies

$$0 < \kappa \le \sqrt{\sigma\rho}. \tag{1.2}$$

For any $\varepsilon > 0$, it is easy to see that there exists a positive number β_ε which is the unique positive zero of $f_\varepsilon(u)$, and a positive number $\gamma_\varepsilon = \sigma/(3\varepsilon) < \beta_\varepsilon$, which is the unique (positive) inflection point of $f_\varepsilon(u)$, such that cubic polynomial f_ε satisfies

(i) $f_\varepsilon(0) = \rho > 0$ (positone), $f'_\varepsilon(0) = -\kappa < 0$, $f_\varepsilon(u) > 0$ on $(0, \beta_\varepsilon)$ and $f_\varepsilon(\beta_\varepsilon) = 0$,

(ii) $f_\varepsilon(u)$ is strictly convex on $(0, \gamma_\varepsilon)$ and is strictly concave on $(\gamma_\varepsilon, \infty)$. (So f_ε is convex-concave on $(0, \beta_\varepsilon)$.)

Note that it is easy to see that β_ε is a continuous, strictly decreasing function of $\varepsilon > 0$. In addition, $\lim_{\varepsilon\to 0^+} \beta_\varepsilon = \infty$ and $\lim_{\varepsilon\to\infty} \beta_\varepsilon = 0$. Three possible graphs of $f_\varepsilon(u)$ satisfying (1.1), (1.2) are depicted in Fig. 1.

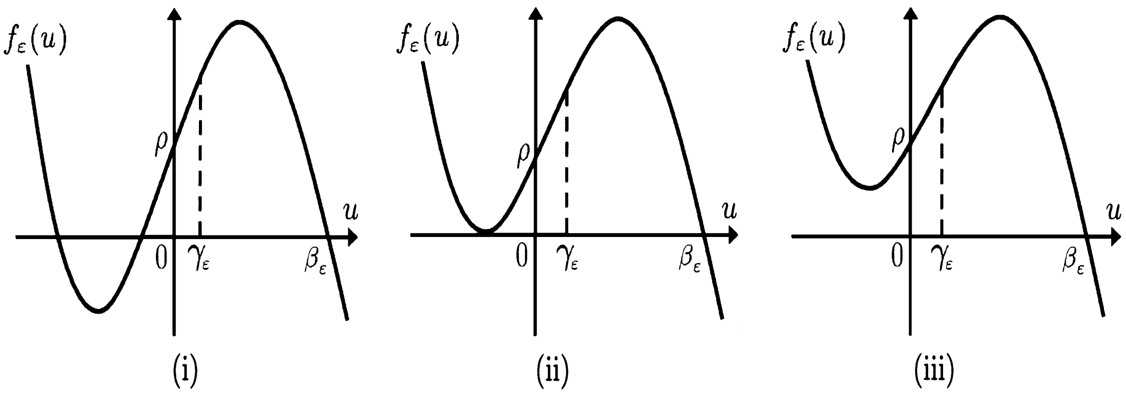

Fig. 1. Three possible graphs of $f_\varepsilon(u)$ satisfying (1.1), (1.2).

For any $\varepsilon > 0$, on the $(\lambda, \|u\|_\infty)$-plane, we study the shape and structure of bifurcation curves S_ε of positive solutions of (1.1) with $\kappa \le \sqrt{\sigma\rho}$, defined by

$$S_\varepsilon \equiv \{(\lambda, \|u_\lambda\|_\infty) : \lambda > 0 \text{ and } u_\lambda \text{ is a positive solution of (1.1) with } \kappa \le \sqrt{\sigma\rho}\}.$$

We say that, on the $(\lambda, \|u\|_\infty)$-plane, the bifurcation curve S_ε is S-shaped if S_ε is a continuous curve and there exist two *positive* numbers $\lambda_* < \lambda^*$ such that S_ε has *exactly two* turning points at some points $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ and $(\lambda_*, \|u_{\lambda_*}\|_\infty)$, and

(i) $\lambda_* < \lambda^*$ and $\|u_{\lambda^*}\|_\infty < \|u_{\lambda_*}\|_\infty$,

(ii) at $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ the bifurcation curve S_ε turns to the *left*,

(iii) at $(\lambda_*, \|u_{\lambda_*}\|_\infty)$ the bifurcation curve S_ε turns to the *right*.

See Fig. 2(i) for example.

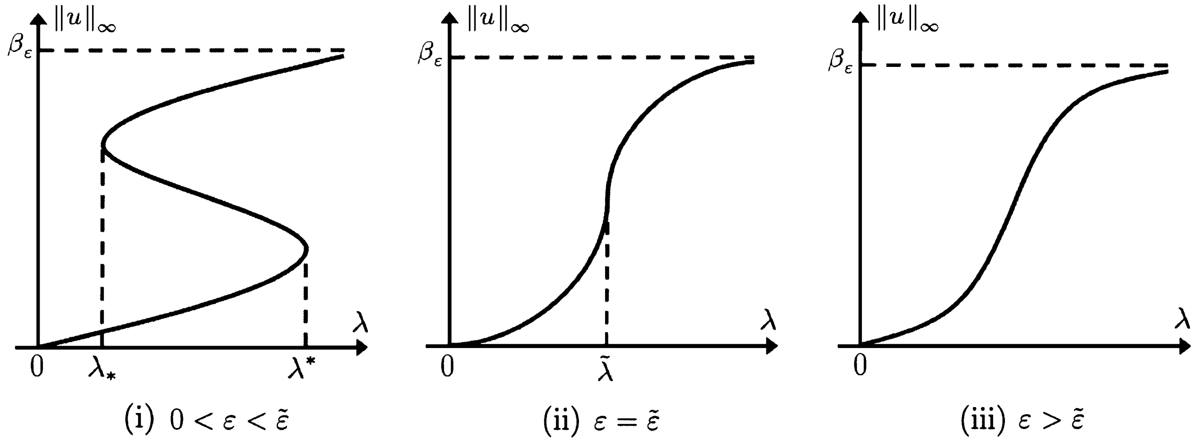

Fig. 2. Global bifurcation of bifurcation curves S_ε with varying $\varepsilon > 0$.

Our results in this paper are extensions of those of Hung and Wang [3]. Hung and Wang [3] developed some time-map techniques to study S-shaped bifurcation curve S_ε of problem (1.1) with

$$\kappa \le 0. \tag{1.3}$$

For problem (1.1), (1.3), Hung and Wang [3, Theorem 2.1] proved that there exists a positive number $\tilde{\varepsilon} = \tilde{\varepsilon}(\sigma, \kappa, \rho)$ satisfying

$$\left(\frac{25}{32}\left(\frac{\sigma^3}{27\rho}\right)\right)^{1/2} < \tilde{\varepsilon} < \left(\frac{\sigma^3}{27\rho}\right)^{1/2}$$

such that, on the $(\lambda, \|u\|_\infty)$-plane,

(i) For $0 < \varepsilon < \tilde{\varepsilon}$, the bifurcation curve S_ε of (1.1), (1.3) is S-shaped (see Fig. 2(i)).

(ii) For $\varepsilon = \tilde{\varepsilon}$, the bifurcation curve $S_{\tilde{\varepsilon}}$ of (1.1), (1.3) is monotone increasing. Moreover, problem (1.1), (1.3) has exactly one (cusp type) degenerate positive solution $u_{\tilde{\lambda}}$ (see Fig. 2(ii)).

(iii) For $\varepsilon > \tilde{\varepsilon}$, the bifurcation curve S_ε of (1.1), (1.3) is monotone increasing. Moreover, all positive solutions u_λ of (1.1), (1.3) are *nondegenerate* (see Fig. 2(iii)).

In Theorem 2.1 stated below for (1.1), (1.2) with varying $\varepsilon > 0$, we prove the same global bifurcation results of bifurcation curve S_ε. Hence we are able to determine the exact number of positive solutions by the values of ε and λ. In addition, we give lower and upper bounds of the critical bifurcation value $\tilde{\varepsilon}$. See Fig. 2.

While for any $\lambda > 0$, on the $(\varepsilon, \|u\|_\infty)$-plane, it is interesting to study the shape and structure of bifurcation curves Σ_λ of positive solutions of (1.1) with $\kappa \le \sqrt{\sigma\rho}$, defined by

$$\Sigma_\lambda \equiv \{(\varepsilon, \|u_\varepsilon\|_\infty) : \varepsilon > 0 \text{ and } u_\varepsilon \text{ is a positive solution of (1.1) with } \kappa \le \sqrt{\sigma\rho}\}.$$

(Note that we allow that bifurcation curve Σ_λ consists of two (or more) connected components.) We say that, on the $(\varepsilon, \|u\|_\infty)$-plane, the bifurcation curve Σ_λ is *reversed* S-shaped if Σ_λ is a continuous curve and there exist two numbers $\varepsilon_* < \varepsilon^*$ such that S_ε has *exactly two* turning points at some points $(\varepsilon_*, \|u_{\varepsilon_*}\|_\infty)$ and $(\varepsilon^*, \|u_{\varepsilon^*}\|_\infty)$, and

(i) $\varepsilon_* < \varepsilon^*$ and $\|u_{\varepsilon_*}\|_\infty < \|u_{\varepsilon^*}\|_\infty$,

(ii) at $(\varepsilon_*, \|u_{\varepsilon_*}\|_\infty)$ the bifurcation curve Σ_λ turns to the *right*,

(iii) at $(\varepsilon^*, \|u_{\varepsilon^*}\|_\infty)$ the bifurcation curve Σ_λ turns to the *left*.

See Fig. 3(iii) for example.

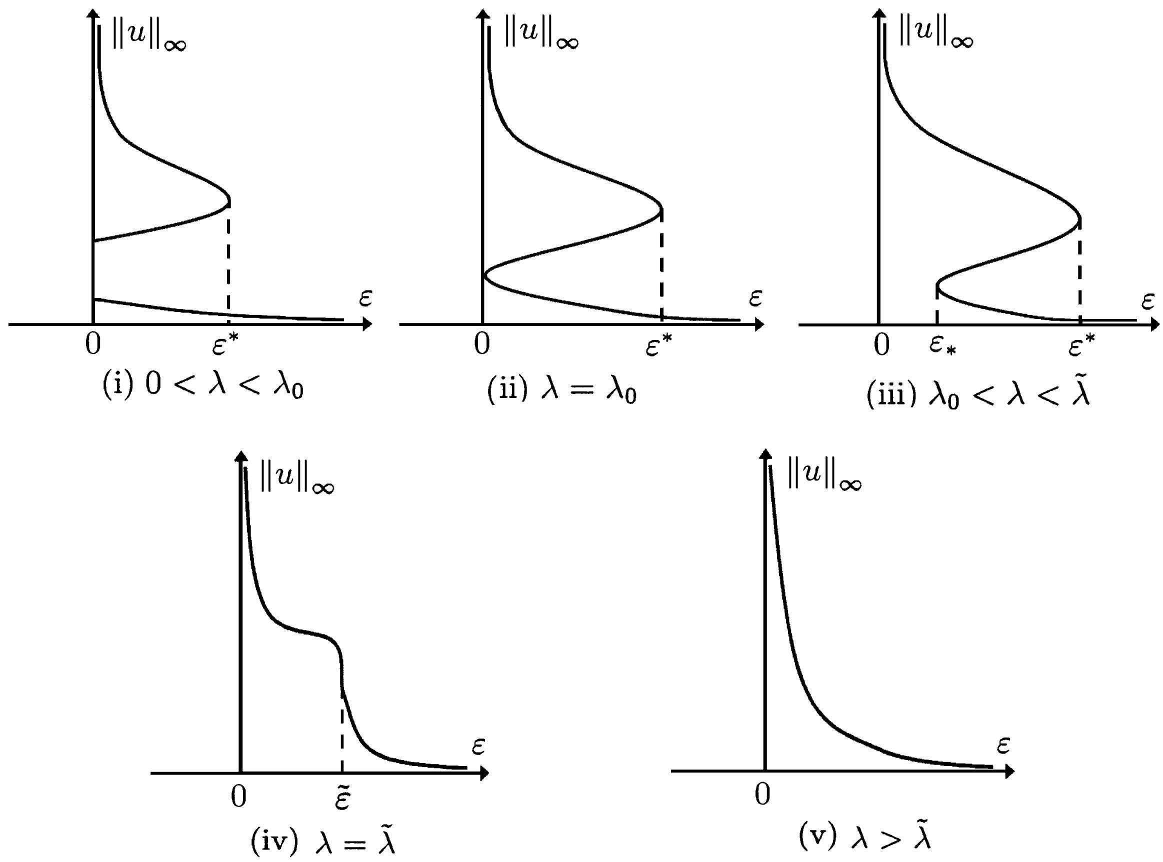

Fig. 3. Global bifurcation of bifurcation curves Σ_λ with varying $\lambda > 0$.

Hung and Wang [3, Theorem 2.3] proved that there exist two positive numbers λ_0 $(= \lambda_0(\sigma, \kappa, \rho)) < \tilde{\lambda}$ $(= \tilde{\lambda}(\sigma, \kappa, \rho))$ such that, on the $(\varepsilon, \|u\|_\infty)$-plane,

(i) For $0 < \lambda < \lambda_0$, the bifurcation curve Σ_λ of (1.1), (1.3) has two disjoint connected components, the upper branch is $\supset$-shaped with exactly one turning point, and the lower branch is a monotone decreasing curve (see Fig. 3(i)).

(ii) For $\lambda = \lambda_0$, the bifurcation curve Σ_{λ_0} of (1.1), (1.3) has two disjoint connected components, the upper branch is $\supset$-shaped with exactly one turning point, and the lower branch is a monotone decreasing curve (see Fig. 3(ii)).

(iii) For $\lambda_0 < \lambda < \tilde{\lambda}$, the bifurcation curve Σ_λ of (1.1), (1.3) is reversed S-shaped (see Fig. 3(iii)).

(iv) For $\lambda = \tilde{\lambda}$, the bifurcation curve $\Sigma_{\tilde{\lambda}}$ of (1.1), (1.3) is monotone decreasing. Moreover, problem (1.1), (1.3) has exactly one (cusp type) degenerate positive solution $u_{\tilde{\varepsilon}}$ (see Fig. 3(iv)).

(v) For $\lambda > \tilde{\lambda}$, the bifurcation curve Σ_λ of (1.1), (1.3) is monotone decreasing. Moreover, all positive solutions u_ε of (1.1), (1.3) are *nondegenerate* (see Fig. 3(v)).

In Theorem 2.2 stated below for (1.1), (1.2) with varying $\lambda > 0$, we prove the same global bifurcation results of bifurcation curve Σ_λ. Hence we are able to determine the exact number of positive solutions by the values of λ and ε. See Fig. 3.

We study, in the $(\varepsilon, \lambda, \|u\|_\infty)$-space, the shape and structure of the *bifurcation surface* Γ of positive solutions of (1.1), (1.2), defined by

$$\Gamma \equiv \left\{(\varepsilon, \lambda, \|u_{\varepsilon,\lambda}\|_\infty) : \varepsilon, \lambda > 0 \text{ and } u_{\varepsilon,\lambda} \text{ is a positive solution of (1.1) with } \kappa \leq \sqrt{\sigma\rho}\right\}$$

which has the appearance of a folded surface with the *fold curve*

$$C_\Gamma \equiv \left\{(\varepsilon, \lambda, \|u_{\varepsilon,\lambda}\|_\infty) : \varepsilon, \lambda > 0 \text{ and } u_{\varepsilon,\lambda} \text{ is a } \textit{degenerate} \text{ positive solution of (1.1) with } \kappa \leq \sqrt{\sigma\rho}\right\}.$$

Let F_q denote the first quadrant of the (ε, λ)-parameter plane. We also study, on F_q, the *bifurcation set*

$$B_\Gamma \equiv \{(\varepsilon, \lambda) : \varepsilon, \lambda > 0 \text{ and } u_{\varepsilon,\lambda} \text{ is a } \textit{degenerate} \text{ positive solution of (1.1) with } \kappa \leq \sqrt{\sigma\rho}\}$$

which is the projection of the fold curve C_Γ onto F_q. Let M denote the bounded, open connected subset of F_q, which is 'inside' B_Γ.

Hung and Wang [3, Theorem 2.4] proved that the following assertions (i)–(v) (see Figs. 4 and 5):

(i) The fold curve C_Γ is a continuous curve in the $(\varepsilon, \lambda, \|u\|_\infty)$-space. Moreover, $C_\Gamma = C_1 \cup C_2$ where

$$C_1 \equiv \left\{(\varepsilon, \lambda_*(\varepsilon), \left\|u_{\varepsilon,\lambda_*(\varepsilon)}\right\|_\infty) : 0 < \varepsilon \leq \tilde{\varepsilon}\right\} \quad \text{and} \quad C_2 \equiv \left\{(\varepsilon, \lambda^*(\varepsilon), \left\|u_{\varepsilon,\lambda^*(\varepsilon)}\right\|_\infty) : 0 < \varepsilon \leq \tilde{\varepsilon}\right\}.$$

(ii) The bifurcation set $B_\Gamma = B_1 \cup B_2$ where

$$B_1 \equiv \{(\varepsilon, \lambda_*(\varepsilon)) : 0 < \varepsilon \leq \tilde{\varepsilon}\} \quad \text{and} \quad B_2 \equiv \{(\varepsilon, \lambda^*(\varepsilon)) : 0 < \varepsilon \leq \tilde{\varepsilon}\}.$$

(iii) $\lambda_*(\varepsilon)$ and $\lambda^*(\varepsilon)$ are both continuous, strictly increasing on $(0, \tilde{\varepsilon}]$.

(iv) Problem (1.1), (1.3) has exactly three positive solutions for $(\varepsilon, \lambda) \in M$, exactly two positive solutions for $(\varepsilon, \lambda) \in B_\Gamma \setminus \{(\tilde{\varepsilon}, \tilde{\lambda})\}$, and exactly one positive solution for $(\varepsilon, \lambda) \in (F_q \setminus (B_\Gamma \cup M)) \cup \{(\tilde{\varepsilon}, \tilde{\lambda})\}$.

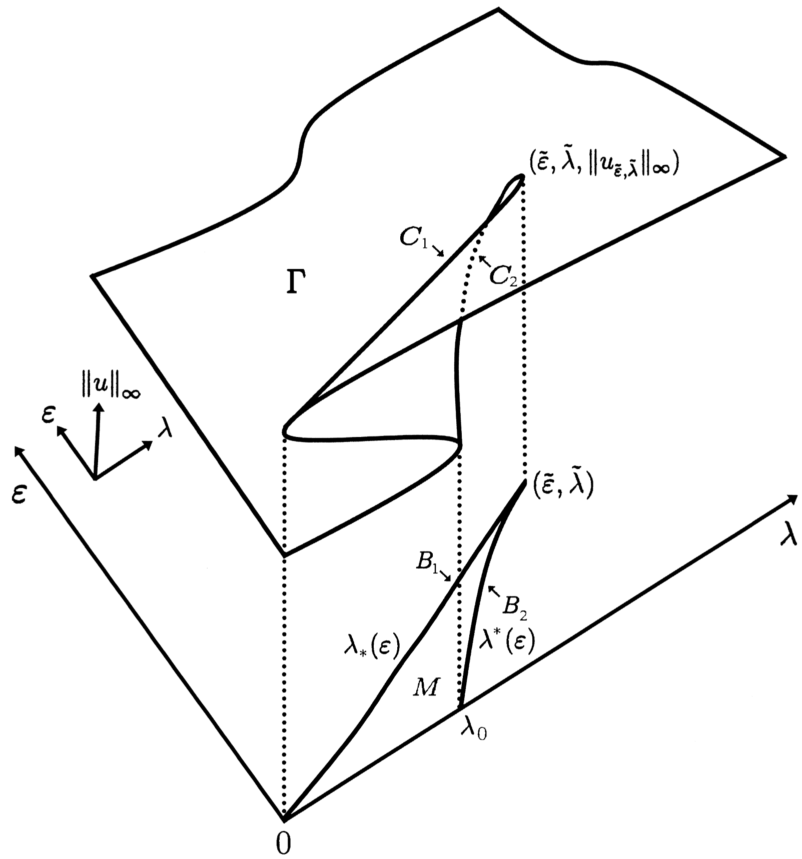

Fig. 4. The bifurcation surface Γ of with the fold curve $C_\Gamma = C_1 \cup C_2$, and the projection of Γ onto F_q. $B_\Gamma = B_1 \cup B_2$ is the bifurcation set and $(\tilde\varepsilon, \tilde\lambda)$ is the cusp point on F_q.

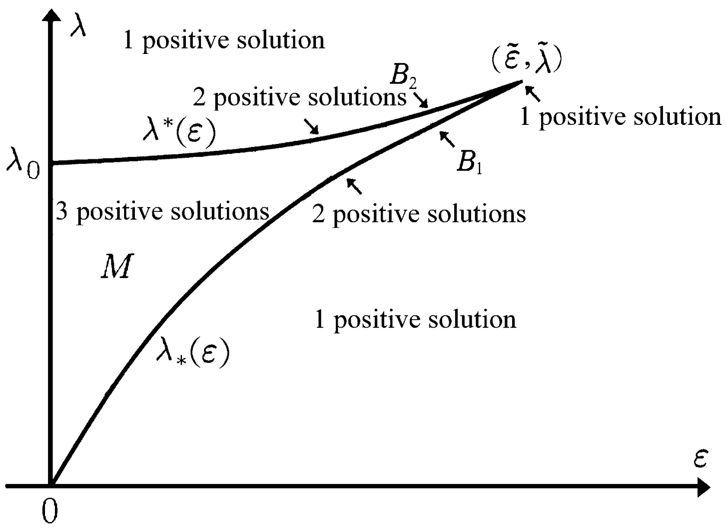

Fig. 5. The projection of the bifurcation surface Γ onto F_q. $B_\Gamma = B_1 \cup B_2$ is the bifurcation set and $(\tilde\varepsilon, \tilde\lambda)$ is the cusp point on F_q.

In Theorem 2.3 for (1.1), (1.2) stated below, we prove the same structure of the bifurcation set B_Γ and the fold curve C_Γ. Hence we are able to determine the exact number of positive solutions of (1.1), (1.2) by the values of ε and λ. See Figs. 4 and 5.

The paper is organized as follows. Section 2 contains statements of the main results: Theorems 2.1–2.3. Section 3 contains several lemmas needed to prove Theorems 2.1–2.3. Section 4 contains the proofs of Theorems 2.1–2.3. Finally, in Section 5, we give some conjectures on shapes of bifurcation curves of problem (1.1) with evolution parameter $\kappa > \sqrt{\sigma\rho}$.

In this section, finally, we note that our main results (Theorems 2.1–2.3) in this paper extend those of Hung and Wang [3, Theorem 2.1, 2.3, and 2.4] from $\kappa \leq 0$ to $\kappa \leq \sqrt{\sigma\rho}$, and the proofs are more complicated. One of the main difficulties is that $f_\varepsilon(u)$ could initially decrease, but then increases to a peak before falling to zero on $(0, \beta_\varepsilon]$, see Fig. 1(i).

2. Main results

Theorem 2.1. *Consider* (1.1), (1.2) *with varying* $\varepsilon > 0$. *There exists a positive number* $\tilde{\varepsilon} = \tilde{\varepsilon}(\sigma, \kappa, \rho)$ *satisfying*

$$\left(\frac{25}{32}\left(\frac{\sigma^3}{27\rho}\right)\right)^{1/2} < \tilde{\varepsilon} < \left(\frac{\sigma^3}{27\rho}\right)^{1/2}$$

such that the following assertions (i)–(iii) hold:

(i) *(See Fig. 2(i).) For* $0 < \varepsilon < \tilde{\varepsilon}$, *the bifurcation curve* S_ε *is S-shaped on the* $(\lambda, \|u\|_\infty)$-*plane. Moreover, there exist two positive numbers* $\lambda_* < \lambda^*$ *such that* (1.1), (1.2) *has exactly one degenerate positive solution* u_{λ_*} *and* u_{λ^*} *for* $\lambda = \lambda_*$ *and* $\lambda = \lambda^*$, *respectively. More precisely,* (1.1), (1.2) *has:*

  (a) *exactly three positive solutions* u_λ, v_λ, w_λ *with* $w_\lambda < u_\lambda < v_\lambda$ *for* $\lambda_* < \lambda < \lambda^*$,

  (b) *exactly two positive solutions* w_λ, u_λ *with* $w_\lambda < u_\lambda$ *for* $\lambda = \lambda_*$, *and exactly two positive solutions* u_λ, v_λ *with* $u_\lambda < v_\lambda$ *for* $\lambda = \lambda^*$,

  (c) *exactly one positive solution* w_λ *for* $0 < \lambda < \lambda_*$, *and exactly one positive solution* v_λ *for* $\lambda > \lambda^*$.

  Furthermore,

  (d) $\lim_{\lambda\to 0^+} \|w_\lambda\|_\infty = 0$ *and* $\lim_{\lambda\to\infty} \|v_\lambda\|_\infty = \beta_\varepsilon$.

(ii) *(See Fig. 2(ii).) For* $\varepsilon = \tilde{\varepsilon}$, *the bifurcation curve* $S_{\tilde{\varepsilon}}$ *is monotone increasing on the* $(\lambda, \|u\|_\infty)$-*plane. Moreover,* (1.1), (1.2) *has exactly one (cusp type) degenerate positive solution* $u_{\tilde{\lambda}}$. *More precisely, for all* $\lambda > 0$, (1.1), (1.2) *has exactly one positive solution* u_λ *satisfying* $\lim_{\lambda\to 0^+} \|u_\lambda\|_\infty = 0$ *and* $\lim_{\lambda\to\infty} \|u_\lambda\|_\infty = \beta_\varepsilon$.

(iii) *(See Fig. 2(iii).) For* $\varepsilon > \tilde{\varepsilon}$, *the bifurcation curve* S_ε *is monotone increasing on the* $(\lambda, \|u\|_\infty)$-*plane. Moreover, all positive solutions* u_λ *of* (1.1), (1.2) *are nondegenerate. More precisely, for all* $\lambda > 0$, (1.1), (1.2) *has exactly one positive solution* u_λ *satisfying* $\lim_{\lambda\to 0^+} \|u_\lambda\|_\infty = 0$ *and* $\lim_{\lambda\to\infty} \|u_\lambda\|_\infty = \beta_\varepsilon$.

Theorem 2.2. *Consider* (1.1), (1.2) *with varying* $\lambda > 0$. *There exist two positive numbers* λ_0 $(= \lambda_0(\sigma, \kappa, \rho)) < \tilde{\lambda}$ $(= \tilde{\lambda}(\sigma, \kappa, \rho))$ *such that the following assertions (i)–(v) hold:*

(i) *(See Fig. 3(i).) For* $0 < \lambda < \lambda_0$, *on the* $(\varepsilon, \|u\|_\infty)$*-plane, the bifurcation curve* Σ_λ *has two disjoint connected components, the upper branch is* $\supset$*-shaped with exactly one turning point, and the lower branch is a monotone decreasing curve. Moreover, there exists a positive number* ε^* *such that* (1.1), (1.2) *has exactly one degenerate positive solution* u_{ε^*} *for* $\varepsilon = \varepsilon^*$. *More precisely, problem* (1.1), (1.2) *has:*

  (a) *exactly three positive solutions* u_ε, v_ε, w_ε *with* $w_\varepsilon < u_\varepsilon < v_\varepsilon$ *for* $0 < \varepsilon < \varepsilon^*$,

  (b) *exactly two positive solutions* w_ε, u_ε *with* $w_\varepsilon < u_\varepsilon$ *for* $\varepsilon = \varepsilon^*$,

  (c) *exactly one positive solution* w_ε *for* $\varepsilon > \varepsilon^*$.
  Furthermore,

  (d) $0 = \lim_{\varepsilon\to\infty} \|w_\varepsilon\|_\infty < \lim_{\varepsilon\to 0^+} \|w_\varepsilon\|_\infty < \lim_{\varepsilon\to 0^+} \|u_\varepsilon\|_\infty < \lim_{\varepsilon\to 0^+} \|v_\varepsilon\|_\infty = \infty$.

(ii) *(See Fig. 3(ii).) For* $\lambda = \lambda_0$, *on the* $(\varepsilon, \|u\|_\infty)$*-plane, the bifurcation curve* Σ_{λ_0} *has two disjoint connected components, the upper branch is* $\supset$*-shaped with exactly one turning point, and the lower branch is a monotone decreasing curve. Moreover, there exists a positive number* ε^* *such that* (1.1), (1.2) *has exactly one degenerate positive solution* u_{ε^*} *for* $\varepsilon = \varepsilon^*$. *More precisely, problem* (1.1), (1.2) *has:*

  (a) *exactly three positive solutions* u_ε, v_ε, w_ε *with* $w_\varepsilon < u_\varepsilon < v_\varepsilon$ *for* $0 < \varepsilon < \varepsilon^*$,

  (b) *exactly two positive solutions* w_ε, u_ε *with* $w_\varepsilon < u_\varepsilon$ *for* $\varepsilon = \varepsilon^*$,

  (c) *exactly one positive solution* w_ε *for* $\varepsilon > \varepsilon^*$.
  Furthermore,

  (d) $0 = \lim_{\varepsilon\to\infty} \|w_\varepsilon\|_\infty < \lim_{\varepsilon\to 0^+} \|w_\varepsilon\|_\infty = \lim_{\varepsilon\to 0^+} \|u_\varepsilon\|_\infty < \lim_{\varepsilon\to 0^+} \|v_\varepsilon\|_\infty = \infty$.

(iii) *(See Fig. 3(iii).) For* $\lambda_0 < \lambda < \tilde{\lambda}$, *the bifurcation curve* Σ_λ *is reversed S-shaped on the* $(\varepsilon, \|u\|_\infty)$*-plane. Moreover, there exist two positive number* $\varepsilon_* < \varepsilon^*$ *such that* (1.1), (1.2) *has exactly one degenerate positive solution* u_{ε_*} *and* u_{ε^*} *for* $\varepsilon = \varepsilon_*$ *and* $\varepsilon = \varepsilon^*$, *respectively. More precisely, problem* (1.1), (1.2) *has:*

  (a) *exactly three positive solutions* u_ε, v_ε, w_ε *with* $w_\varepsilon < u_\varepsilon < v_\varepsilon$ *for* $\varepsilon_* < \varepsilon < \varepsilon^*$,

  (b) *exactly two positive solutions* u_ε, v_ε *with* $u_\varepsilon < v_\varepsilon$ *for* $\varepsilon = \varepsilon_*$, *and exactly two positive solutions* w_ε, u_ε *with* $w_\varepsilon < u_\varepsilon$ *for* $\varepsilon = \varepsilon^*$,

  (c) *exactly one positive solution* v_ε *for* $0 < \varepsilon < \varepsilon_*$, *and exactly one positive solution* w_ε *for* $\varepsilon > \varepsilon^*$.
  Furthermore,

  (d) $\lim_{\varepsilon\to 0^+} \|v_\varepsilon\|_\infty = \infty$ *and* $\lim_{\varepsilon\to\infty} \|w_\varepsilon\|_\infty = 0$.

(iv) *(See Fig. 3(iv).) For $\lambda = \tilde{\lambda}$, the bifurcation curve $\Sigma_{\tilde{\lambda}}$ is monotone decreasing on the $(\varepsilon, \|u\|_\infty)$-plane. Moreover, problem* (1.1), (1.2) *has exactly one (cusp type) degenerate positive solution $u_{\tilde{\varepsilon}}$. More precisely, for all $\varepsilon > 0$, problem* (1.1), (1.2) *has exactly one positive solution u_ε satisfying $\lim_{\varepsilon \to 0^+} \|u_\varepsilon\|_\infty = \infty$ and $\lim_{\varepsilon \to \infty} \|u_\varepsilon\|_\infty = 0$.*

(v) *(See Fig. 3(v).) For $\lambda > \tilde{\lambda}$, the bifurcation curve Σ_λ is monotone decreasing on the $(\varepsilon, \|u\|_\infty)$-plane. Moreover, all positive solutions u_ε of* (1.1), (1.2) *are nondegenerate. More precisely, for all $\varepsilon > 0$, problem* (1.1), (1.2) *has exactly one positive solution u_ε satisfying $\lim_{\varepsilon \to 0^+} \|u_\varepsilon\|_\infty = \infty$ and $\lim_{\varepsilon \to \infty} \|u_\varepsilon\|_\infty = 0$.*

We give next remark to Theorem 2.2.

Remark 1. *Considering* (1.1), (1.2) *with $\varepsilon > 0$ generalized to $\varepsilon \in \mathbb{R}$, we define the bifurcation curve*

$$\tilde{\Sigma}_\lambda \equiv \{(\varepsilon, \|u_\varepsilon\|_\infty) : \varepsilon \in \mathbb{R} \text{ and } u_\varepsilon \text{ is a positive solution of (1.1) with } \kappa \leq \sqrt{\sigma\rho}\}.$$

Actually, it can be easily proved that:

(i) *For $0 < \lambda < \lambda_0$, the bifurcation curve $\tilde{\Sigma}_\lambda$ is reversed S-shaped on the $(\varepsilon, \|u\|_\infty)$-plane. Moreover, there exists $\varepsilon_* < 0$ such that* (1.1), (1.2) *has exactly two positive solutions w_ε, u_ε with $w_\varepsilon < u_\varepsilon$ for $\varepsilon_* < \varepsilon \leq 0$, and exactly one positive solution u_ε for $\varepsilon = \varepsilon_*$, and no positive solution for $\varepsilon < \varepsilon_*$. See Fig. 6(i).*

(ii) *For $\lambda = \lambda_0$, the bifurcation curve $\tilde{\Sigma}_{\lambda_0}$ is reversed S-shaped on the $(\varepsilon, \|u\|_\infty)$-plane. Moreover, problem* (1.1), (1.2) *has exactly one positive solution u_ε for $\varepsilon = 0$, and no positive solution for $\varepsilon < 0$. See Fig. 6(ii).*

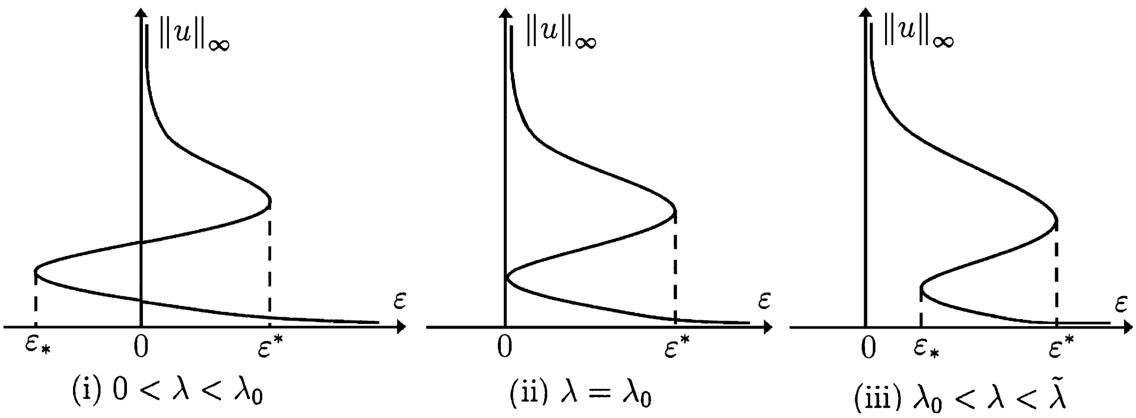

Fig. 6. Global bifurcation of bifurcation curves $\tilde{\Sigma}_\lambda$ of (1.1), (1.2) with $\varepsilon > 0$ generalized to $\varepsilon \in \mathbb{R}$ and with varying $\lambda \in (0, \tilde{\lambda})$.

Notice that, in Theorem 2.1, on the $(\lambda, \|u\|_\infty)$-plane, the bifurcation curve S_ε is S-shaped for $0 < \varepsilon < \tilde{\varepsilon}$, see Fig. 2. While in Theorem 2.2 and Remark 1, on the $(\varepsilon, \|u\|_\infty)$-plane, the bifurcation curve $\tilde{\Sigma}_\lambda$ is *reversed* S-shaped for $0 < \lambda < \tilde{\lambda}$, see Fig. 6.

Let $\tilde{\varepsilon} = \tilde{\varepsilon}(\sigma, \kappa, \rho)$, $\lambda_0 = \lambda_0(\sigma, \kappa, \rho)$, $\tilde{\lambda} = \tilde{\lambda}(\sigma, \kappa, \rho)$, $\lambda_* = \lambda_*(\varepsilon)$, $\lambda^* = \lambda^*(\varepsilon)$, $\varepsilon_* = \varepsilon_*(\lambda)$ and $\varepsilon^* = \varepsilon^*(\lambda)$ be the values in Theorems 2.1 and 2.2 for (1.1), (1.2). We study the structure of the bifurcation set B_Γ in the next theorem.

Theorem 2.3 (See Fig. 5). *Consider* (1.1), (1.2) *with* $(\varepsilon, \lambda) \in F_q$. *Then the bifurcation set* $B_\Gamma = B_1 \cup B_2$ *where*

$$B_1 \equiv \{(\varepsilon, \lambda_*(\varepsilon)) : 0 < \varepsilon \leq \tilde{\varepsilon}\} \quad \text{and} \quad B_2 \equiv \{(\varepsilon, \lambda^*(\varepsilon)) : 0 < \varepsilon \leq \tilde{\varepsilon}\}.$$

Moreover, problem (1.1), (1.2) *has exactly three positive solutions for* $(\varepsilon, \lambda) \in M$, *exactly two positive solutions for* $(\varepsilon, \lambda) \in B_\Gamma \setminus \{(\tilde{\varepsilon}, \tilde{\lambda})\}$, *and exactly one positive solution for* $(\varepsilon, \lambda) \in (F_q \setminus (B_\Gamma \cup M)) \cup \{(\tilde{\varepsilon}, \tilde{\lambda})\}$. *More precisely, the following assertions (i) and (ii) hold:*

(i) Functions $\lambda_*(\varepsilon)$ *and* $\lambda^*(\varepsilon)$ *are both continuous, strictly increasing on* $(0, \tilde{\varepsilon}]$ *and satisfy* $0 = \lim_{\varepsilon \to 0^+} \lambda_*(\varepsilon) < \lim_{\varepsilon \to 0^+} \lambda^*(\varepsilon) = \lambda_0 < \tilde{\lambda} = \lambda_*(\tilde{\varepsilon}) = \lambda^*(\tilde{\varepsilon})$.

(ii) Function $\varepsilon^*(\lambda)$ *is continuous, strictly increasing on* $(0, \tilde{\lambda}]$ *and satisfies* $\lim_{\lambda \to 0^+} \varepsilon^*(\lambda) = 0$ *and* $\varepsilon^*(\tilde{\lambda}) = \tilde{\varepsilon}$. *Function* $\varepsilon_*(\lambda)$ *is continuous, strictly increasing on* $(\lambda_0, \tilde{\lambda}]$ *and satisfies* $\lim_{\lambda \to \lambda_0^+} \varepsilon_*(\lambda) = 0$ *and* $\varepsilon_*(\tilde{\lambda}) = \tilde{\varepsilon}$.

In next remark, we give a precise characterization of the fold curve C_Γ in the $(\varepsilon, \lambda, \|u\|_\infty)$-space.

Remark 2 (See Fig. 4). *Consider* (1.1), (1.2). *Then, by Theorem 2.3(i), the fold curve* $C_\Gamma = C_1 \cup C_2$ *where*

$$C_1 \equiv \left\{(\varepsilon, \lambda_*(\varepsilon), \left\|u_{\varepsilon,\lambda_*(\varepsilon)}\right\|_\infty) : 0 < \varepsilon \leq \tilde{\varepsilon}\right\} \quad \text{and} \quad C_2 \equiv \left\{(\varepsilon, \lambda^*(\varepsilon), \left\|u_{\varepsilon,\lambda^*(\varepsilon)}\right\|_\infty) : 0 < \varepsilon \leq \tilde{\varepsilon}\right\}.$$

Moreover, by applying (4.4)–(4.6) *stated below, we are able to prove that:*

(i) $\left\|u_{\varepsilon,\lambda_*(\varepsilon)}\right\|_\infty > \left\|u_{\varepsilon,\lambda^*(\varepsilon)}\right\|_\infty$ *for* $0 < \varepsilon < \tilde{\varepsilon}$ *and* $\left\|u_{\tilde{\varepsilon},\lambda_*(\tilde{\varepsilon})}\right\|_\infty = \left\|u_{\tilde{\varepsilon},\lambda^*(\tilde{\varepsilon})}\right\|_\infty = \left\|u_{\tilde{\varepsilon},\tilde{\lambda}}\right\|_\infty$.

(ii) $\left\|u_{\varepsilon,\lambda_*(\varepsilon)}\right\|_\infty$ *is a continuous, strictly decreasing function of* $\varepsilon \in (0, \tilde{\varepsilon}]$ *and* $\left\|u_{\varepsilon,\lambda^*(\varepsilon)}\right\|_\infty$ *is a continuous, strictly increasing function of* $\varepsilon \in (0, \tilde{\varepsilon}]$.

(iii) C_Γ *is a continuous curve in the* $(\varepsilon, \lambda, \|u\|_\infty)$*-space.*

Observe that both $\lambda^*(\varepsilon)$ and $\lambda_*(\varepsilon)$ have continuous inverse functions on $(0, \tilde{\varepsilon}]$. Indeed, $\varepsilon_*(\lambda)$ is the inverse function of $\lambda^*(\varepsilon)$ on $(\lambda_0, \tilde{\lambda}]$ and $\varepsilon^*(\lambda)$ is the inverse function of $\lambda_*(\varepsilon)$ on $(0, \tilde{\lambda}]$.

3. Lemmas

To prove our results (Theorems 2.1–2.3), we need the following Lemmas 3.1–3.8 in which we develop new time-map techniques different from those developed in [3]. In particular, Lemma 3.3 is a key lemma in the proofs of Theorems 2.1–2.3. In Lemma 3.3, for any fixed $\varepsilon > 0$, we prove that the bifurcation curve S_ε is either monotone increasing or S-shaped on the $(\lambda, \|u\|_\infty)$-plane. To apply the time-map techniques for (1.1), (1.2), in the following, we consider $\varepsilon \geq 0$. The time map formula which we apply to study (1.1), (1.2) takes the form as follows:

$$\sqrt{\lambda} = \frac{1}{\sqrt{2}} \int_0^{\alpha} [F_\varepsilon(\alpha) - F_\varepsilon(u)]^{-1/2} \, du \equiv T_\varepsilon(\alpha) \quad \text{for} \quad 0 < \alpha < \beta_\varepsilon \quad \text{and } \varepsilon \geq 0, \tag{3.1}$$

where $F_\varepsilon(u) \equiv \int_0^u f_\varepsilon(t)dt$ and β_ε the unique positive zero of cubic polynomial $f_\varepsilon(u)$ for $\varepsilon > 0$, and we let $\beta_{\varepsilon=0} \equiv \infty$. Observe that positive solutions $u_{\varepsilon,\lambda}$ for (1.1), (1.2) correspond to

$$\|u_{\varepsilon,\lambda}\|_\infty = \alpha \quad \text{and} \quad T_\varepsilon(\alpha) = \sqrt{\lambda}. \tag{3.2}$$

Thus, studying of the exact number of positive solutions of (1.1), (1.2) for fixed $\varepsilon \geq 0$ is equivalent to studying the shape of the time map $T_\varepsilon(\alpha)$ on $(0, \beta_\varepsilon)$; and studying the exact number of positive solutions of (1.1), (1.2) for fixed $\lambda > 0$ is equivalent to studying the number of roots of the equation $T_\varepsilon(\alpha) = \sqrt{\lambda}$ on $(0, \beta_\varepsilon)$ for varying $\varepsilon > 0$. Note that it can be proved that $T_\varepsilon(\alpha)$ is a thrice differentiable function of $\alpha \in (0, \beta_\varepsilon)$ for $\varepsilon \geq 0$. The proof is easy but tedious and we omit it.

We call a positive solution $u_{\varepsilon,\lambda}$ of (1.1), (1.2) is *degenerate* if $T_\varepsilon'(\|u_{\varepsilon,\lambda}\|_\infty) = 0$ and is *nondegenerate* if $T_\varepsilon'(\|u_{\varepsilon,\lambda}\|_\infty) \neq 0$. So to find the degenerate positive solutions of (1.1), (1.2), we only need to find the critical points of $T_\varepsilon(\alpha)$ on $(0, \beta_\varepsilon)$. It is known that a *degenerate* positive solution $u_{\varepsilon,\lambda}$ of (1.1), (1.2) is of *cusp type* if $T_\varepsilon''(\|u_{\varepsilon,\lambda}\|_\infty) = 0$ and $T_\varepsilon'''(\|u_{\varepsilon,\lambda}\|_\infty) \neq 0$, see Shi [9, p. 497] and [10, p. 214].

The main difficulty in proving our main results is to determine the *exact* number of critical points of the time map $T_\varepsilon(\alpha)$ on $(0, \beta_\varepsilon)$ for all $\varepsilon > 0$. This question is partially answered in the following Lemmas 3.1 and 3.2. Lemma 3.1 follows from [8, Theorems 2.6, 2.9 and 3.2] and Lemma 3.2 mainly follows by applying [4, Theorem 2.1]. We omit the proofs.

Lemma 3.1. *Consider* (1.1), (1.2). *For any fixed* $\varepsilon > 0$, *the following assertions (i) and (ii) hold:*

(i) $\lim_{\alpha \to 0^+} T_\varepsilon(\alpha) = 0$ *and* $\lim_{\alpha \to \beta_\varepsilon^-} T_\varepsilon(\alpha) = \infty$.

(ii) If $T_\varepsilon(\alpha)$ *is not strictly increasing on* $(0, \gamma_\varepsilon)$, *then* $T_\varepsilon(\alpha)$ *is strictly increasing on* $(0, \tilde{\gamma}_\varepsilon)$ *and strictly decreasing on* $(\tilde{\gamma}_\varepsilon, \gamma_\varepsilon)$ *for some* $\tilde{\gamma}_\varepsilon \in (0, \gamma_\varepsilon)$.

Lemma 3.2. *Consider* (1.1), (1.2). *Then the following assertions (i) and (ii) hold:*

(i) For any fixed $\varepsilon \geq (\frac{\sigma^3}{27\rho})^{1/2}$, $T_\varepsilon(\alpha)$ *is a strictly increasing function on* $(0, \beta_\varepsilon)$.

(ii) For any fixed positive $\varepsilon \leq (\frac{7}{10}(\frac{\sigma^3}{27\rho}))^{1/2}$, $T_\varepsilon(\alpha)$ *has exactly one local maximum and one local minimum on* $(0, \beta_\varepsilon)$.

However, there is a gap, what about the case where ε is between $(\frac{7}{10}(\frac{\sigma^3}{27\rho}))^{1/2}$ and $(\frac{\sigma^3}{27\rho})^{1/2}$? First, in the next Lemma 3.3, we prove

Lemma 3.3. *Consider* (1.1), (1.2). *For any fixed* $\varepsilon > 0$, $T_\varepsilon(\alpha)$ *is either a strictly increasing function or has exactly two critical points, a local maximum and a local minimum, on* $(0, \beta_\varepsilon)$.

To prove Lemma 3.3, we develop some new time-map techniques. First, for time-map function $T_\varepsilon(\alpha)$ with $\alpha \in (0, \beta_\varepsilon)$ in (3.1), letting $u = \alpha v$, we have

$$T_\varepsilon(\alpha) = \frac{\alpha}{\sqrt{2}} \int_0^1 \frac{1}{[F_\varepsilon(\alpha) - F_\varepsilon(\alpha v)]^{1/2}} dv.$$

For any fixed $\varepsilon > 0$, we compute that

$$T'_\varepsilon(\alpha) = \frac{1}{\sqrt{2}} \int_0^1 \frac{1}{[F_\varepsilon(\alpha) - F_\varepsilon(\alpha v)]^{1/2}} dv - \frac{\alpha}{2\sqrt{2}} \int_0^1 \frac{f_\varepsilon(\alpha) - f_\varepsilon(\alpha v)v}{[F_\varepsilon(\alpha) - F_\varepsilon(\alpha v)]^{3/2}} dv \tag{3.3}$$

and

$$\begin{aligned} T''_\varepsilon(\alpha) &= -\frac{1}{\sqrt{2}} \int_0^1 \frac{f_\varepsilon(\alpha) - f_\varepsilon(\alpha v)v}{[F_\varepsilon(\alpha) - F_\varepsilon(\alpha v)]^{3/2}} dv - \frac{\alpha}{2\sqrt{2}} \int_0^1 \frac{f'_\varepsilon(\alpha) - f'_\varepsilon(\alpha v)v^2}{[F_\varepsilon(\alpha) - F_\varepsilon(\alpha v)]^{3/2}} dv \\ &\quad + \frac{3\alpha}{4\sqrt{2}} \int_0^1 \frac{[f_\varepsilon(\alpha) - f_\varepsilon(\alpha v)v]^2}{[F_\varepsilon(\alpha) - F_\varepsilon(\alpha v)]^{5/2}} dv. \end{aligned} \tag{3.4}$$

We define the auxiliary function

$$G_\varepsilon(\alpha) = 8\sqrt{2}\alpha^{\frac{5}{2}} T''_\varepsilon(\alpha). \tag{3.5}$$

Then we have the following lemma. The proof of Lemma 3.4 is rather long and technical, therefore we postpone it to the Appendix.

Lemma 3.4. *Consider* (1.1), (1.2). *For any fixed* $\varepsilon \in [(\frac{7}{10}(\frac{\sigma^3}{27\rho}))^{1/2}, (\frac{\sigma^3}{27\rho})^{1/2}]$, $G'_\varepsilon(\alpha) > 0$ *for* $\alpha \in [\gamma_\varepsilon, \beta_\varepsilon)$.

For any fixed $\alpha > 0$, let

$$I_\alpha = \{\varepsilon > 0 : \alpha \in (0, \beta_\varepsilon)\}.$$

Since β_ε is a continuous, strictly decreasing function of $\varepsilon > 0$, and $\lim_{\varepsilon \to 0^+} \beta_\varepsilon = \infty$ and $\lim_{\varepsilon \to \infty} \beta_\varepsilon = 0$, we obtain that $I_\alpha = (0, \varepsilon(\alpha))$ where $\alpha = \beta_{\varepsilon(\alpha)}$, and $\varepsilon(\alpha)$ is strictly decreasing in α.

Lemma 3.5. *Consider* (1.1), (1.2). *For any fixed* $\alpha > 0$, $T'_\varepsilon(\alpha)$ *is a continuously differentiable, strictly increasing function of* $\varepsilon \in I_\alpha \cup \{0\}$.

Proof of Lemma 3.5. First, for any fixed $\alpha > 0$, it can be proved that $T'_\varepsilon(\alpha)$ is a continuously differentiable function of $\varepsilon \in I_\alpha \cup \{0\}$. The proof is easy but tedious and we omit it.

Secondly, since $f_\varepsilon(u) = -\varepsilon u^3 + \sigma u^2 - \kappa u + \rho$, $F_\varepsilon(u) = \int_0^u f_\varepsilon(t)dt$ and by (3.3), we compute that

$$
\begin{aligned}
T'_\varepsilon(\alpha) &= \frac{1}{\sqrt{2}}\int_0^1 \frac{1}{[F_\varepsilon(\alpha) - F_\varepsilon(\alpha v)]^{1/2}}dv - \frac{\alpha}{2\sqrt{2}}\int_0^1 \frac{f_\varepsilon(\alpha) - f_\varepsilon(\alpha v)v}{[F_\varepsilon(\alpha) - F_\varepsilon(\alpha v)]^{3/2}}dv \\
&= \frac{1}{2\sqrt{2}\alpha}\int_0^\alpha \frac{2\,[F_\varepsilon(\alpha) - F_\varepsilon(u)] - [\alpha f_\varepsilon(\alpha) - u f_\varepsilon(u)]}{[F_\varepsilon(\alpha) - F_\varepsilon(u)]^{3/2}}du \\
&= \frac{1}{2\sqrt{2}\alpha}\int_0^\alpha \frac{\varepsilon\frac{(\alpha^4-u^4)}{2} - \sigma\frac{(\alpha^3-u^3)}{3} + \rho(\alpha-u)}{\left[-\varepsilon\frac{(\alpha^4-u^4)}{4} + \sigma\frac{(\alpha^3-u^3)}{3} - \kappa\frac{(\alpha^2-u^2)}{2} + \rho(\alpha-u)\right]^{3/2}}du
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{\partial}{\partial\varepsilon}T'_\varepsilon(\alpha) \\
= {} & \frac{1}{96\sqrt{2}\alpha}\int_0^\alpha \frac{(\alpha^4-u^4)\,[3\varepsilon(\alpha^4-u^4) + 2\sigma(\alpha^3-u^3) - 12\kappa(\alpha^2-u^2) + 42\rho(\alpha-u)]}{\left[-\varepsilon\frac{(\alpha^4-u^4)}{4} + \sigma\frac{(\alpha^3-u^3)}{3} - \kappa\frac{(\alpha^2-u^2)}{2} + \rho(\alpha-u)\right]^{5/2}}du \\
> {} & \frac{1}{48\sqrt{2}\alpha}\int_0^\alpha \frac{(\alpha^4-u^4)(\alpha-u)\,[\sigma(\alpha^2+\alpha u+u^2) - 6\kappa(\alpha+u) + 21\rho]}{\left[-\varepsilon\frac{(\alpha^4-u^4)}{4} + \sigma\frac{(\alpha^3-u^3)}{3} - \kappa\frac{(\alpha^2-u^2)}{2} + \rho(\alpha-u)\right]^{5/2}}du. \qquad (3.6)
\end{aligned}
$$

Let

$$
\begin{aligned}
H(u) &\equiv \sigma(\alpha^2 + \alpha u + u^2) - 6\kappa(\alpha+u) + 21\rho \\
&= \sigma u^2 + (\sigma\alpha - 6\kappa)u + (\sigma\alpha^2 - 6\kappa\alpha + 21\rho).
\end{aligned}
$$

Therefore, the proof is complete if we can prove that

$$
H(u) > 0 \;\text{ for any given numbers } \sigma, \rho, \alpha > 0,\; 0 < \kappa \le \sqrt{\sigma\rho}. \qquad (3.7)
$$

Note that the discriminant of quadratic polynomial $H(u)$ is $-3\sigma^2\alpha^2 + 12\sigma\kappa\alpha + (36\kappa^2 - 84\sigma\rho) \equiv \tilde{H}(\alpha)$. By the assumption that $\kappa \le \sqrt{\sigma\rho}$, the discriminant of quadratic polynomial $\tilde{H}(\alpha)$ is $144\sigma^2(4\kappa^2 - 7\sigma\rho) < 0$. So $\tilde{H}(\alpha) < 0$ for any given numbers $\sigma, \rho > 0$, $0 < \kappa \le \sqrt{\sigma\rho}$. This implies that (3.7) holds. By (3.6) and (3.7), for any fixed $\alpha > 0$, $T'_\varepsilon(\alpha)$ is a strictly increasing function of $\varepsilon \in I_\alpha \cup \{0\}$.

This completes the proof of Lemma 3.5. ■

We are now in a position to prove Lemma 3.3.

Proof of Lemma 3.3. First, we prove that for any fixed $\varepsilon > 0$, $T_\varepsilon(\alpha)$ is either a strictly increasing function or has a local maximum and a local minimum, on $(0, \beta_\varepsilon)$. By Lemma 3.2, we only need to consider the case $(\frac{7}{10}(\frac{\sigma^3}{27\rho}))^{1/2} < \varepsilon < (\frac{\sigma^3}{27\rho})^{1/2}$.

For any fixed $(\frac{7}{10}(\frac{\sigma^3}{27\rho}))^{1/2} < \varepsilon < (\frac{\sigma^3}{27\rho})^{1/2}$, by Lemma 3.1(ii) (resp. Lemma 3.4), we know that all (possible) critical points of $T_\varepsilon(\alpha)$ on $(0, \gamma_\varepsilon]$ (resp. on $[\gamma_\varepsilon, \beta_\varepsilon)$) are discrete. Moreover, since $\lim_{\alpha\to 0^+} T_\varepsilon(\alpha) = 0$ and $\lim_{\alpha\to\beta_\varepsilon^-} T_\varepsilon(\alpha) = \infty$ and by Lemma 3.1(i), we obtain that $T'_\varepsilon(\alpha)$ changes sign an even number of times or infinitely times. Assume that $T_\varepsilon(\alpha)$ is neither a strictly increasing function nor does it have exactly one local maximum and one local minimum on $(0, \beta_\varepsilon)$. Then there exist numbers $\alpha_1, \alpha_2, \alpha_3 \in (0, \beta_\varepsilon)$ such that $\alpha_1 < \alpha_2 < \alpha_3$ are critical points of $T_\varepsilon(\alpha)$, α_1, α_3 are local maxima, and α_2 is a local minimum. Thus $T''_\varepsilon(\alpha_1), T''_\varepsilon(\alpha_3) \le 0$ and $T''_\varepsilon(\alpha_2) \ge 0$.

By Lemma 3.4, for any fixed $(\frac{7}{10}(\frac{\sigma^3}{27\rho}))^{1/2} < \varepsilon < (\frac{\sigma^3}{27\rho})^{1/2}$, $G_\varepsilon(\alpha) = 8\sqrt{2}\alpha^{\frac{5}{2}} T''_\varepsilon(\alpha)$ is a strictly increasing function on $[\gamma_\varepsilon, \beta_\varepsilon)$. Since $\alpha_2 \ge \gamma_\varepsilon$ by Lemma 3.1(ii), we obtain that

$$8\sqrt{2}\alpha_3^{\frac{5}{2}} T''_\varepsilon(\alpha_3) = G_\varepsilon(\alpha_3) > G_\varepsilon(\alpha_2) = 8\sqrt{2}\alpha_2^{\frac{5}{2}} T''_\varepsilon(\alpha_2) \ge 0.$$

Therefore $T''_\varepsilon(\alpha_3) > 0$. This contradicts to that $T''_\varepsilon(\alpha_3) \le 0$. So $T_\varepsilon(\alpha)$ is either a strictly increasing function or has exactly one local maximum and one local minimum on $(0, \beta_\varepsilon)$.

Next, suppose that $T_\varepsilon(\alpha)$ has exactly a local maximum α_M and a local minimum α_m for some fixed $\varepsilon > 0$, then $0 < \alpha_M < \alpha_m < \beta_\varepsilon$ by Lemma 3.1(i). We can prove that $T_\varepsilon(\alpha)$ has exactly two critical points α_M, α_m on $(0, \beta_\varepsilon)$ by applying similar arguments used in the proof of [3, Lemma 3.3]; we omit it. (Note that Lemma 3.5 was used in the skipped part.)

This completes the proof of Lemma 3.3. ■

Let

$$E = \left\{ \begin{array}{l} \varepsilon > 0 : T_\varepsilon(\alpha) \text{ has exactly two critical points,} \\ \text{a local maximum and a local minimum, on } (0, \beta_\varepsilon) \end{array} \right\}.$$

By Lemma 3.3, for any $\varepsilon > 0$, $T_\varepsilon(\alpha)$ is either a strictly increasing function or has exactly two critical points, a local maximum and a local minimum, on $(0, \beta_\varepsilon)$. Thus

$$\begin{aligned} E &= \left\{ \begin{array}{l} \varepsilon > 0 : T_\varepsilon(\alpha) \text{ has exactly two critical points,} \\ \text{a local maximum and a local minimum, on } (0, \beta_\varepsilon) \end{array} \right\} \\ &= \{\varepsilon > 0 : T'_\varepsilon(\alpha) < 0 \text{ for some } \alpha \in (0, \beta_\varepsilon)\}. \end{aligned} \tag{3.8}$$

We obtain the following two lemmas by modifying the same arguments used in the proof of [3, Lemmas 3.7–3.8]; we omit the proofs.

Lemma 3.6. *The set E is open and connected.*

Lemma 3.7. $(0, (\frac{25}{32}(\frac{\sigma^3}{27\rho}))^{1/2}] \subset E$.

The following Lemma 3.8(i) determine the shape of $T_{\varepsilon=0}(\alpha)$ on $(0, \infty)$, and Lemma 3.8(ii) is a basic comparison theorem for the time map formula. Lemma 3.8(i) follows from [8, Theorem 3.2] and Lemma 3.8(ii) by modifying [8, Theorems 2.3 and 2.4]. We omit the proofs.

Lemma 3.8. *Consider* (1.1), (1.2). *The following assertions (i) and (ii) hold:*

(i) *$T_{\varepsilon=0}(\alpha)$ has exactly one critical point at some α_0, a maximum, on $(0, \infty)$. Moreover,* $\lim_{\alpha \to 0^+} T_{\varepsilon=0}(\alpha) = \lim_{\alpha \to \infty} T_{\varepsilon=0}(\alpha) = 0$.

(ii) *For any fixed $\alpha > 0$, $T_\varepsilon(\alpha)$ is a continuous, strictly increasing function of $\varepsilon \in I_\alpha \cup \{0\}$.*

4. Proofs of the main results

We first recall that a positive solution $u_{\varepsilon,\lambda}$ of (1.1) is *degenerate* if $T'_\varepsilon(\|u_{\varepsilon,\lambda}\|_\infty) = 0$ and is *nondegenerate* if $T'_\varepsilon(\|u_{\varepsilon,\lambda}\|_\infty) \neq 0$. Also, a *degenerate* positive solution $u_{\varepsilon,\lambda}$ of (1.1) is of *cusp type* if $T''_\varepsilon(\|u_{\varepsilon,\lambda}\|_\infty) = 0$ and $T'''_\varepsilon(\|u_{\varepsilon,\lambda}\|_\infty) \neq 0$.

Proof of Theorem 2.1. To prove Theorem 2.1, by (3.1) and Lemma 3.1(i), it suffices to prove that there exists a positive number $\tilde{\varepsilon} = \tilde{\varepsilon}(\sigma, \kappa, \rho)$ such that the following parts (I)–(III) hold:

(I) For $0 < \varepsilon < \tilde{\varepsilon}$, on $(0, \beta_\varepsilon)$, $T_\varepsilon(\alpha)$ has exactly two critical points, a local maximum at some α_ε^- and a local minimum at some α_ε^+ $(> \alpha_\varepsilon^-)$, satisfying $\lambda^* = (T_\varepsilon(\alpha_\varepsilon^-))^2$ and $\lambda_* = (T_\varepsilon(\alpha_\varepsilon^+))^2$.

(II) For $\varepsilon = \tilde{\varepsilon}$, $T_{\tilde{\varepsilon}}(\alpha)$ is a strictly increasing function and has exactly one critical point, at some $\tilde{\alpha}$, on $(0, \beta_{\tilde{\varepsilon}})$. Moreover, $T'_{\tilde{\varepsilon}}(\tilde{\alpha}) = 0$, $T'_{\tilde{\varepsilon}}(\alpha) > 0$ for $\alpha \in (0, \beta_{\tilde{\varepsilon}}) \setminus \{\tilde{\alpha}\}$, $T''_{\tilde{\varepsilon}}(\tilde{\alpha}) = 0$ and $T'''_{\tilde{\varepsilon}}(\tilde{\alpha}) \neq 0$ (So (1.1), (1.2) has exactly one (cusp type) degenerate positive solution $u_{\tilde{\lambda}}$ with $\tilde{\lambda} \equiv (T_{\tilde{\varepsilon}}(\tilde{\alpha}))^2$ and $\tilde{\alpha} = \|u_{\tilde{\lambda}}\|_\infty$.)

(III) For $\varepsilon > \tilde{\varepsilon}$, $T_\varepsilon(\alpha)$ is a strictly increasing function and has no critical point on $(0, \beta_\varepsilon)$. Moreover, $T'_\varepsilon(\alpha) > 0$ for $\alpha \in (0, \beta_\varepsilon)$.

Note that, by (3.2) and the above parts (I)–(III), we obtain immediately the exact multiplicity result of positive solutions of (1.1), (1.2) for $0 < \varepsilon < \tilde{\varepsilon}$ and the uniqueness result of positive solution of (1.1), (1.2) for $\varepsilon \geq \tilde{\varepsilon}$. Moreover, ordering properties and asymptotic behaviors of positive solutions of (1.1), (1.2) in parts (I)–(III) can be obtained easily. We then prove parts (I)–(III) as follows.

By Lemma 3.2, we obtain that the set E is nonempty and bounded above by $(\frac{\sigma^3}{27\rho})^{1/2}$. By Lemmas 3.6 and 3.7, $E = (0, \tilde{\varepsilon})$ where $\tilde{\varepsilon} = \sup E$ satisfies $(\frac{25}{32}(\frac{\sigma^3}{27\rho}))^{1/2} < \tilde{\varepsilon} < (\frac{\sigma^3}{27\rho})^{1/2}$. So, for $0 < \varepsilon < \tilde{\varepsilon}$, on $(0, \beta_\varepsilon)$, $T_\varepsilon(\alpha)$ has exactly two critical points, a local maximum at some α_ε^- and a local minimum at some α_ε^+ $(> \alpha_\varepsilon^-)$, satisfying $\lambda^* = (T_\varepsilon(\alpha_\varepsilon^-))^2$ and $\lambda_* = (T_\varepsilon(\alpha_\varepsilon^+))^2$. So part (I) holds.

For $\varepsilon > \tilde{\varepsilon}$, by Lemma 3.5 and (3.8), we obtain that

$$T'_\varepsilon(\alpha) > T'_{\tilde{\varepsilon}}(\alpha) \geq 0 \quad \text{for } \alpha \in (0, \beta_\varepsilon) \subset (0, \beta_{\tilde{\varepsilon}}),$$

and hence $T_\varepsilon(\alpha)$ has no critical point on $(0, \beta_\varepsilon)$. So part (III) holds.

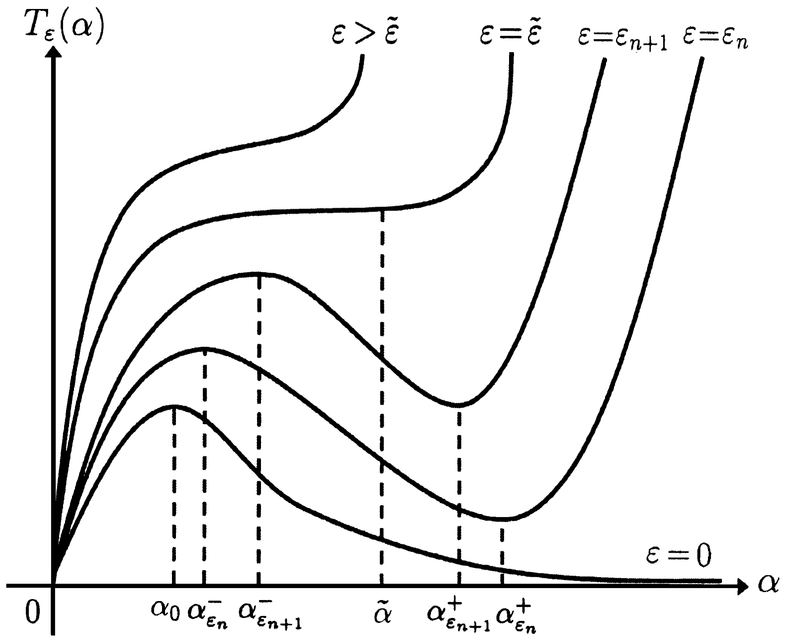

Fig. 7. Graphs of $T_\varepsilon(\alpha)$ for $\alpha \in (0, \beta_\varepsilon)$ with varying $\varepsilon \geq 0$.

We prove the remaining part (II). For $\varepsilon = \tilde{\varepsilon}$, we know that

$$T'_{\tilde{\varepsilon}}(\alpha) \geq 0 \text{ on } (0, \beta_{\tilde{\varepsilon}}). \tag{4.1}$$

We first prove the existence of a critical point of $T_{\tilde{\varepsilon}}(\alpha)$ on $(0, \beta_{\tilde{\varepsilon}})$. Choose a sequence $\{\varepsilon_n\} \subset E = (0, \tilde{\varepsilon})$ such that $\varepsilon_n \nearrow \tilde{\varepsilon}$ as $n \to \infty$. Let $\alpha^-_{\varepsilon_n} < \alpha^+_{\varepsilon_n}$ be two critical points of $T_{\varepsilon_n}(\alpha)$ on $(0, \beta_{\varepsilon_n})$ for each $n \in \mathbb{N}$ (see Fig. 7). Then by Lemma 3.5 again, we obtain that

$$T'_{\varepsilon_n}(\alpha^-_{\varepsilon_{n+1}}) < T'_{\varepsilon_{n+1}}(\alpha^-_{\varepsilon_{n+1}}) = 0 \text{ and } T'_{\varepsilon_n}(\alpha^+_{\varepsilon_{n+1}}) < T'_{\varepsilon_{n+1}}(\alpha^+_{\varepsilon_{n+1}}) = 0.$$

Hence $\alpha^-_{\varepsilon_n} < \alpha^-_{\varepsilon_{n+1}} < \alpha^+_{\varepsilon_{n+1}} < \alpha^+_{\varepsilon_n}$ and

$$\alpha^-_{\varepsilon_n} < \tilde{\alpha}^- \equiv \lim_{n\to\infty} \alpha^-_{\varepsilon_n} \leq \tilde{\alpha}^+ \equiv \lim_{n\to\infty} \alpha^+_{\varepsilon_n} < \alpha^+_{\varepsilon_n} \text{ for all } n \in \mathbb{N}.$$

These imply that

$$T'_{\varepsilon_n}(\tilde{\alpha}^-), T'_{\varepsilon_n}(\tilde{\alpha}^+) < 0 \text{ for all } n \in \mathbb{N}.$$

By Lemma 3.5, we obtain that $T'_\varepsilon(\alpha)$ is a continuous function of $\varepsilon \in I_\alpha$. Thus

$$T'_{\tilde{\varepsilon}}(\tilde{\alpha}^-) = \lim_{n\to\infty} T'_{\varepsilon_n}(\tilde{\alpha}^-) \leq 0 \text{ and } T'_{\tilde{\varepsilon}}(\tilde{\alpha}^+) = \lim_{n\to\infty} T'_{\varepsilon_n}(\tilde{\alpha}^+) \leq 0. \tag{4.2}$$

So $T'_{\tilde{\varepsilon}}(\tilde{\alpha}^-) = T'_{\tilde{\varepsilon}}(\tilde{\alpha}^+) = 0$ by (4.1) and (4.2), and hence $T_{\tilde{\varepsilon}}(\alpha)$ has critical points at $\tilde{\alpha}^-, \tilde{\alpha}^+$ on $(0, \beta_{\tilde{\varepsilon}})$.

We then prove the uniqueness of critical point of $T_{\tilde{\varepsilon}}(\alpha)$ on $(0, \beta_{\tilde{\varepsilon}})$. That is, we prove that $\tilde{\alpha} \equiv \tilde{\alpha}^- = \tilde{\alpha}^+$ is the unique critical point of $T_{\tilde{\varepsilon}}(\alpha)$ on $(0, \beta_{\tilde{\varepsilon}})$. Suppose that $\hat{\alpha} < \bar{\alpha}$ are two critical points of $T_{\tilde{\varepsilon}}(\alpha)$ on $(0, \beta_{\tilde{\varepsilon}})$. We know that all (possible) critical points of $T_\varepsilon(\alpha)$ on $(0, \beta_\varepsilon)$ are discrete as in the proof of Lemma 3.3. Hence there exist positive numbers $\alpha_1 < \hat{\alpha} < \alpha_2 < \bar{\alpha}$ such that

$$T'_{\tilde{\varepsilon}}(\alpha_1), T'_{\tilde{\varepsilon}}(\alpha_2) > 0.$$

By Lemma 3.5, we obtain that $T'_{\varepsilon}(\alpha)$ is a continuous, strictly increasing function of $\varepsilon \in I_{\alpha}$. Hence there exists a positive $\hat{\varepsilon} < \tilde{\varepsilon}$ such that

$$T'_{\hat{\varepsilon}}(\alpha_1) > 0,\, T'_{\hat{\varepsilon}}(\hat{\alpha}) < 0,\, T'_{\hat{\varepsilon}}(\alpha_2) > 0,\, T'_{\hat{\varepsilon}}(\bar{\alpha}) < 0.$$

Thus $T_{\hat{\varepsilon}}(\alpha)$ has at least two local maxima on $(0, \beta_{\hat{\varepsilon}})$, which contradicts to the facts that $\hat{\varepsilon} \in E$ and $T_{\hat{\varepsilon}}(\alpha)$ has exactly one local maximum on $(0, \beta_{\hat{\varepsilon}})$. So $T_{\tilde{\varepsilon}}(\alpha)$ has at most one critical point on $(0, \beta_{\tilde{\varepsilon}})$. By the above analysis,

$$T'_{\tilde{\varepsilon}}(\tilde{\alpha}) = 0 \;\text{ and }\; T'_{\tilde{\varepsilon}}(\alpha) > 0 \;\text{ for } \alpha \in (0, \beta_{\tilde{\varepsilon}}) \setminus \{\tilde{\alpha}\}\,. \tag{4.3}$$

Next, if $T''_{\tilde{\varepsilon}}(\tilde{\alpha}) > 0$ (resp. $T''_{\tilde{\varepsilon}}(\tilde{\alpha}) < 0$), then $T_{\tilde{\varepsilon}}(\alpha)$ has a local minimum (resp. a local maximum) at $\tilde{\alpha}$, which contradicts to (4.3). So $T''_{\tilde{\varepsilon}}(\tilde{\alpha}) = 0$. By Lemma 3.1(ii), we have

$$\alpha^+_{\varepsilon_n} \geq \gamma_{\varepsilon_n} > \gamma_{\tilde{\varepsilon}} \;\text{ for all } n \in \mathbb{N},$$

and hence $\tilde{\alpha} = \lim_{n\to\infty} \alpha^+_{\varepsilon_n} \geq \gamma_{\tilde{\varepsilon}}$. By Lemma 3.4, $G'_{\tilde{\varepsilon}}(\alpha) > 0$ for all $\alpha \in [\gamma_{\tilde{\varepsilon}}, \beta_{\tilde{\varepsilon}})$. So

$$G'_{\tilde{\varepsilon}}(\tilde{\alpha}) = \tilde{\alpha}^{\frac{3}{2}} \left[20\sqrt{2}T''_{\tilde{\varepsilon}}(\tilde{\alpha}) + 8\sqrt{2}\tilde{\alpha}T'''_{\tilde{\varepsilon}}(\tilde{\alpha})\right] > 0.$$

Therefore $T'''_{\tilde{\varepsilon}}(\tilde{\alpha}) > 0$ since $T''_{\tilde{\varepsilon}}(\tilde{\alpha}) = 0$. This completes the proof of part (II).

The proof of Theorem 2.1 is complete. ■

Proof of Theorem 2.2. Recall (3.1) with $\varepsilon \geq 0$,

$$\sqrt{\lambda} = \frac{1}{\sqrt{2}} \int_0^{\alpha} [F_{\varepsilon}(\alpha) - F_{\varepsilon}(u)]^{-1/2}\, du \equiv T_{\varepsilon}(\alpha) \;\text{ for }\; 0 < \alpha < \beta_{\varepsilon},$$

where β_{ε} the unique positive zero of cubic polynomial $f_{\varepsilon}(u)$ for $\varepsilon > 0$ and $\beta_{\varepsilon=0} = \infty$. Thus, studying the exact number of positive solutions of (1.1), (1.2) for fixed $\lambda > 0$ is equivalent to studying the number of roots of the equation $T_{\varepsilon}(\alpha) = \sqrt{\lambda}$ on $(0, \beta_{\varepsilon})$ for varying $\varepsilon > 0$. Since we have studied the behaviors of $T_{\varepsilon}(\alpha)$ for all varying $\varepsilon \geq 0$ (see the proofs of Theorem 2.1 and Lemma 3.8(i) and Fig. 7), there exist two positive numbers λ_0 $(= \lambda_0(\sigma, \kappa, \rho)) < \tilde{\lambda}$ $(= \tilde{\lambda}(\sigma, \kappa, \rho))$ such that the following parts (I)–(III) hold:

(I) For $0 < \lambda \leq \lambda_0$, there exists a positive number $\varepsilon^* = \varepsilon^*(\lambda)$ such that the equation $T_{\varepsilon}(\alpha) = \sqrt{\lambda}$ has exactly three roots on $(0, \beta_{\varepsilon})$ for $0 < \varepsilon < \varepsilon^*$, exactly two roots on $(0, \beta_{\varepsilon})$ for $\varepsilon = \varepsilon^*$, and exactly one root on $(0, \beta_{\varepsilon})$ for $\varepsilon > \varepsilon^*$.

(II) For $\lambda_0 < \lambda < \tilde{\lambda}$, there exist two positive number ε_* $(= \varepsilon_*(\lambda)) < \varepsilon^*$ $(= \varepsilon^*(\lambda))$ such that the equation $T_{\varepsilon}(\alpha) = \sqrt{\lambda}$ has exactly three roots on $(0, \beta_{\varepsilon})$ for $\varepsilon_* < \varepsilon < \varepsilon^*$, exactly two roots on $(0, \beta_{\varepsilon})$ for $\varepsilon = \varepsilon_*$ and $\varepsilon = \varepsilon^*$, and exactly one root on $(0, \beta_{\varepsilon})$ for $0 < \varepsilon < \varepsilon_*$ and $\varepsilon > \varepsilon^*$.

(III) For $\lambda \geq \tilde{\lambda}$, the equation $T_{\varepsilon}(\alpha) = \sqrt{\lambda}$ has exactly one root on $(0, \beta_{\varepsilon})$ for all $\varepsilon > 0$.

Notice that $\lambda_0 = (T_{\varepsilon=0}(\alpha_0))^2$ and $\tilde{\lambda} = (T_{\tilde{\varepsilon}}(\tilde{\alpha}))^2$, where α_0 is the unique critical point of $T_{\varepsilon=0}(\alpha)$ and $\tilde{\alpha}$ be the unique critical point of $T_{\tilde{\varepsilon}}(\alpha)$. Hence (3.2) and the above parts (I)–(III) imply immediately the exact multiplicity result of positive solutions of (1.1), (1.2) for $\lambda \in (0, \tilde{\lambda})$ and the uniqueness result of positive solution of (1.1), (1.2) for $\lambda \geq \tilde{\lambda}$. Moreover, ordering properties and asymptotic behaviors of positive solutions of (1.1), (1.2) in parts (I)–(III) can be obtained easily.

The proof of Theorem 2.2 is complete. ■

Proof of Theorem 2.3. By Theorem 2.1, for any $\varepsilon \geq \tilde{\varepsilon}$, we obtain that (1.1), (1.2) has exactly one positive solution for all $\lambda > 0$. In addition, for any $\varepsilon \in (0, \tilde{\varepsilon})$, there exist two positive numbers $\lambda_*(\varepsilon) < \lambda^*(\varepsilon)$ such that (1.1), (1.2) has exactly three positive solutions for $\lambda_*(\varepsilon) < \lambda < \lambda^*(\varepsilon)$, exactly two positive solutions for $\lambda = \lambda_*(\varepsilon)$ and $\lambda^*(\varepsilon)$, and exactly one positive solution for $0 < \lambda < \lambda_*(\varepsilon)$ and $\lambda > \lambda^*(\varepsilon)$, where $\lambda_*(\varepsilon) = (T_\varepsilon(\alpha_\varepsilon^+))^2$ and $\lambda^*(\varepsilon) = (T_\varepsilon(\alpha_\varepsilon^-))^2$ in which $\alpha_\varepsilon^- < \alpha_\varepsilon^+$ are two critical points of $T_\varepsilon(\alpha)$ on $(0, \beta_\varepsilon)$.

First, letting $\alpha_{\tilde{\varepsilon}}^- = \alpha_{\tilde{\varepsilon}}^+ \equiv \tilde{\alpha}$, we prove that α_ε^- (resp. α_ε^+) is a continuous, strictly increasing (resp. strictly decreasing) function on $(0, \tilde{\varepsilon}]$ and $\lim_{\varepsilon \to 0^+} \alpha_\varepsilon^- = \alpha_0$ (resp. $\lim_{\varepsilon \to 0^+} \alpha_\varepsilon^+ = \infty$) as follows. By similar arguments in the proof of Theorem 2.1, we obtain that α_ε^- (resp. α_ε^+) is a strictly increasing (resp. strictly decreasing) function on $(0, \tilde{\varepsilon}]$. For any fixed $\alpha \in (\alpha_0, \tilde{\alpha})$, by Theorem 2.1(ii) and Lemma 3.8(i), we obtain that

$$T'_{\varepsilon=0}(\alpha) < 0 \text{ and } T'_{\tilde{\varepsilon}}(\alpha) > 0.$$

Then by Lemma 3.5, $T'_\varepsilon(\alpha)$ is a continuously differentiable, strictly increasing function of $\varepsilon \in [0, \tilde{\varepsilon}]$. This implies that there exists a unique $\varepsilon \in (0, \tilde{\varepsilon})$ such that $T'_\varepsilon(\alpha) = 0$. So

$$\alpha_\varepsilon^- : (0, \tilde{\varepsilon}] \to (\alpha_0, \tilde{\alpha}] \text{ is a strictly increasing, surjective function,} \tag{4.4}$$

and hence α_ε^- is a continuous function on $(0, \tilde{\varepsilon}]$ and $\lim_{\varepsilon \to 0^+} \alpha_\varepsilon^- = \alpha_0$. Similarly, we can prove that

$$\alpha_\varepsilon^+ : (0, \tilde{\varepsilon}] \to [\tilde{\alpha}, \infty) \text{ is a strictly decreasing, surjective function,}$$

and hence α_ε^+ is also a continuous function on $(0, \tilde{\varepsilon}]$ and $\lim_{\varepsilon \to 0^+} \alpha_\varepsilon^+ = \infty$.

Secondly, letting

$$\lambda_*(0) \equiv 0,\ \lambda^*(0) \equiv \lambda_0 = (T_{\varepsilon=0}(\alpha_0))^2, \text{ and } \lambda_*(\tilde{\varepsilon}) = \lambda^*(\tilde{\varepsilon}) \equiv \tilde{\lambda} = (T_{\tilde{\varepsilon}}(\tilde{\alpha}))^2,$$

we then show $\lambda_*(\varepsilon)$ and $\lambda^*(\varepsilon)$ are both strictly increasing functions on $[0, \tilde{\varepsilon}]$. By Lemma 3.8(ii), $T_\varepsilon(\alpha)$ is a continuous, strictly increasing function of $\varepsilon \in I_\alpha \cup \{0\}$. So for any $\varepsilon_1, \varepsilon_2$ satisfying $0 \leq \varepsilon_1 < \varepsilon_2 \leq \tilde{\varepsilon}$, we obtain that

$$\sqrt{\lambda^*(\varepsilon_2)} = T_{\varepsilon_2}(\alpha_{\varepsilon_2}^-) > T_{\varepsilon_2}(\alpha_{\varepsilon_1}^-) > T_{\varepsilon_1}(\alpha_{\varepsilon_1}^-) = \sqrt{\lambda^*(\varepsilon_1)}$$

and

$$\sqrt{\lambda_*(\varepsilon_2)} = T_{\varepsilon_2}(\alpha_{\varepsilon_2}^+) > T_{\varepsilon_1}(\alpha_{\varepsilon 2}^+) > T_{\varepsilon_1}(\alpha_{\varepsilon_1}^+) = \sqrt{\lambda_*(\varepsilon_1)};$$

cf. Fig. 7. Hence $\lambda_*(\varepsilon)$ and $\lambda^*(\varepsilon)$ are both strictly increasing functions on $[0, \tilde{\varepsilon}]$. Note that $T_{\varepsilon=0}(\alpha_0^-) \equiv \sqrt{\lambda^*(0)} = T_{\varepsilon=0}(\alpha_0)$ and $T_{\varepsilon=0}(\alpha_0^+) \equiv \sqrt{\lambda_*(0)} = 0$.

Thirdly, we show $\lambda_*(\varepsilon)$ and $\lambda^*(\varepsilon)$ are both continuous functions on $[0, \tilde{\varepsilon}]$. By Lemma 3.5, for any fixed $\alpha > 0$, $T'_\varepsilon(\alpha)$ is a continuously differentiable, strictly increasing function of $\varepsilon \in I_\alpha \cup \{0\}$. So for any $\bar{\varepsilon} \in (0, \tilde{\varepsilon}]$ and for any given $\eta > 0$, there exists $\delta > 0$ such that

$$T_{\bar{\varepsilon}}(\alpha_{\bar{\varepsilon}}^-) - \eta < T_\varepsilon(\alpha_{\bar{\varepsilon}}^-) < T_{\bar{\varepsilon}}(\alpha_{\bar{\varepsilon}}^-) \text{ for all } \varepsilon \in (\bar{\varepsilon} - \delta, \bar{\varepsilon}).$$

By Theorem 2.1(i) and the fact that $\lambda^*(\varepsilon)$ is a strictly increasing function on $[0,\tilde{\varepsilon}]$, we have that

$$T_{\bar{\varepsilon}}(\alpha_{\bar{\varepsilon}}^-) - \eta < T_{\varepsilon}(\alpha_{\bar{\varepsilon}}^-) < T_{\varepsilon}(\alpha_{\varepsilon}^-) = \sqrt{\lambda^*(\varepsilon)} < \sqrt{\lambda^*(\bar{\varepsilon})} = T_{\bar{\varepsilon}}(\alpha_{\bar{\varepsilon}}^-) \quad \text{for all } \varepsilon \in (\bar{\varepsilon}-\delta, \bar{\varepsilon}).$$

Hence

$$\left| \sqrt{\lambda^*(\varepsilon)} - \sqrt{\lambda^*(\bar{\varepsilon})} \right| < \eta \quad \text{for all } \varepsilon \in (\bar{\varepsilon}-\delta, \bar{\varepsilon}).$$

This implies that $\sqrt{\lambda^*(\varepsilon)}$ is left continuous on $(0,\tilde{\varepsilon}]$, and so is $\lambda^*(\varepsilon)$. By similar arguments, we can prove that $\lambda_*(\varepsilon)$ is right continuous on $[0,\tilde{\varepsilon})$.

Now, assume that $\lambda^*(\varepsilon)$ is *not* right continuous at some point $\hat{\varepsilon} \in [0,\tilde{\varepsilon})$, then there exist $\eta_1 > 0$ and a sequence $\{\varepsilon_n\} \subset (\hat{\varepsilon},\tilde{\varepsilon})$ such that $\varepsilon_n \searrow \hat{\varepsilon}$ as $n \to \infty$, and

$$\left|T_{\varepsilon_n}(\alpha_{\varepsilon_n}^-) - T_{\hat{\varepsilon}}(\alpha_{\hat{\varepsilon}}^-)\right| = \left| \sqrt{\lambda^*(\varepsilon_n)} - \sqrt{\lambda^*(\hat{\varepsilon})} \right| \geq \eta_1 \quad \text{for all } n \in \mathbb{N}.$$

Then for any positive integers $m > n$, by Lemma 3.9(ii),

$$T_{\varepsilon_n}(\alpha_{\varepsilon_m}^-) > T_{\varepsilon_m}(\alpha_{\varepsilon_m}^-) \geq T_{\hat{\varepsilon}}(\alpha_{\hat{\varepsilon}}^-) + \eta_1.$$

By (4.4), α_{ε}^- is a continuous function on $(0,\tilde{\varepsilon}]$, so

$$T_{\varepsilon_n}(\alpha_{\hat{\varepsilon}}^-) = \lim_{m\to\infty} T_{\varepsilon_n}(\alpha_{\varepsilon_m}^-) \geq T_{\hat{\varepsilon}}(\alpha_{\hat{\varepsilon}}^-) + \eta_1 \quad \text{for all } n \in \mathbb{N}.$$

Note that $\alpha_0^- \equiv \alpha_0 = \lim_{\varepsilon\to 0^+} \alpha_{\varepsilon}^-$. Then by Lemma 3.8(ii) again, we have that

$$T_{\hat{\varepsilon}}(\alpha_{\hat{\varepsilon}}^-) = \lim_{n\to\infty} T_{\varepsilon_n}(\alpha_{\hat{\varepsilon}}^-) \geq T_{\hat{\varepsilon}}(\alpha_{\hat{\varepsilon}}^-) + \eta_1 > T_{\hat{\varepsilon}}(\alpha_{\hat{\varepsilon}}^-),$$

and we obtain a contradiction. This implies that $\sqrt{\lambda^*(\varepsilon)}$ is right continuous on $[0,\tilde{\varepsilon})$, and so is $\lambda^*(\varepsilon)$. By similar arguments, we can prove that $\lambda_*(\varepsilon)$ is left continuous on $(0,\tilde{\varepsilon}]$.

By the above analysis, we conclude that

$$\lambda^*(\varepsilon) : [0,\tilde{\varepsilon}] \to [\lambda_0,\tilde{\lambda}] \quad \text{is a continuous, strictly increasing function} \tag{4.5}$$

and

$$\lambda_*(\varepsilon) : [0,\tilde{\varepsilon}] \to [0,\tilde{\lambda}] \quad \text{is a continuous, strictly increasing function.} \tag{4.6}$$

Moreover,

$$\lim_{\varepsilon\to 0^+} \lambda^*(\varepsilon) = \lambda_0, \quad \lim_{\varepsilon\to 0^+} \lambda_*(\varepsilon) = 0, \quad \text{and} \quad \lambda_*(\tilde{\varepsilon}) = \lambda^*(\tilde{\varepsilon}) = \tilde{\lambda}. \tag{4.7}$$

Finally, by (4.5)–(4.7), $\lambda^*(\varepsilon)$ and $\lambda_*(\varepsilon)$ both have continuous inverse functions on $(0,\tilde{\varepsilon}]$. Indeed, by Theorem 2.2 and (3.1), $\varepsilon_*(\lambda) = (\lambda^*)^{-1}(\varepsilon)$ on $(\lambda_0,\tilde{\lambda}]$ and $\varepsilon^*(\lambda) = (\lambda_*)^{-1}(\varepsilon)$ on $(0,\tilde{\lambda}]$ where $\varepsilon_*(\tilde{\lambda}) = \varepsilon^*(\tilde{\lambda}) \equiv \tilde{\varepsilon}$. So we obtain that

$$\varepsilon^*(\lambda) : (0,\tilde{\lambda}] \to (0,\tilde{\varepsilon}] \quad \text{is a continuous, strictly increasing function}$$

and

$$\varepsilon_*(\lambda) : (\lambda_0,\tilde{\lambda}] \to (0,\tilde{\varepsilon}] \quad \text{is a continuous, strictly increasing function.}$$

Moreover,

$$\lim_{\lambda\to 0^+} \varepsilon^*(\lambda) = \lim_{\lambda\to\lambda_0^+} \varepsilon_*(\lambda) = 0.$$

The proof of Theorem 2.3 is complete. ■

5. Conclusions and conjectures

We consider the problem

$$\begin{cases} u''(x) + \lambda f_\varepsilon(u) = 0, \quad -1 < x < 1, \ u(-1) = u(1) = 0, \\ f_\varepsilon(u) = -\varepsilon u^3 + \sigma u^2 - \kappa u + \rho, \ \ \lambda, \varepsilon > 0. \end{cases} \tag{5.1}$$

Problem (5.1) was first systematically studied by a celebrated paper by Smoller and Wasserman [11]. In particular, they consider (5.1) with $\varepsilon = 1$ and that cubic nonlinearity $f_{\varepsilon=1}(u)$ has *three real zeros* $a < b < c$. In this section we discuss the general case with $\varepsilon > 0$ and $\sigma, \rho, \kappa \in \mathbb{R}$, so that $f_\varepsilon(u)$ may have exactly one positive zero, two distinct positive zeros or three distinct positive zeros. First, note that, if $\sigma \le 0$ or $\rho \le 0$, we can show that the structure of bifurcation curve S_ε of positive solutions for (5.1) is one of the following cases:

(i) The bifurcation curve S_ε of (5.1) is an empty set (that is, (5.1) has no positive solution for all $\lambda > 0$).

(ii) The bifurcation curve S_ε of (5.1) is a monotone curve on the $(\lambda, \|u\|_\infty)$-plane.

(iii) The bifurcation curve S_ε of (5.1) has exactly one turning point where the curve turns to the right on the $(\lambda, \|u\|_\infty)$-plane.

Thus problem (5.1) has at most two positive solutions if $\sigma \le 0$ or $\rho \le 0$. See [5] for the details of the above results.

If $\sigma > 0$ and $\rho > 0$, then (5.1) reduces to (1.1). It is more difficult to determine precisely the exact multiplicity of (1.1) since problem (1.1) may have three positive solutions for some positive numbers $\varepsilon, \kappa, \lambda$. We analyze (1.1) more precisely in this section. First, if

$$\kappa \le \sqrt{\sigma\rho},$$

the exact multiplicity results of positive solutions for problem (1.1) was determine precisely by Theorem 2.1 and [3, Theorem 2.1]. By some numerical simulations, we give next three conjectures on the shape of bifurcation curves $\hat{S}_\varepsilon$ of positive solutions of (1.1) with $\kappa > \sqrt{\sigma\rho}$, defined by

$$\hat{S}_\varepsilon \equiv \{(\lambda, \|u_\lambda\|_\infty) : \lambda > 0 \text{ and } u_\lambda \text{ is a positive solution of (1.1) with } \kappa > \sqrt{\sigma\rho}\}.$$

Conjecture 5.1. *Consider* (1.1) *where*

$$\sqrt{\sigma\rho} < \kappa \le \sqrt{3\sigma\rho}.$$

Then there exists a positive number $\tilde{\varepsilon} = \tilde{\varepsilon}(\sigma, \kappa, \rho)$ *satisfying satisfying*

$$\left(\frac{25}{32}\left(\frac{\sigma^3}{27\rho}\right)\right)^{1/2} < \tilde{\varepsilon} < \left(\frac{\sigma^3}{27\rho}\right)^{1/2}$$

such that all results in Theorem 2.1(i)–(iii) hold.

While

$$\kappa > \sqrt{3\sigma\rho}, \tag{5.2}$$

we remark that there exists some $\check{\varepsilon} > 0$ such that cubic nonlinearity $f_{\check{\varepsilon}}(u)$ has three positive zeros $0 < a < b < c$ and $\int_a^c f_{\check{\varepsilon}}(t)dt > 0$ (see Fig. 8(i).) For these $f_{\check{\varepsilon}}(u)$, it is easy to check that $a + c > 2b$ and there exists $\mu \in (b, c)$ such that $\int_a^\mu f_{\check{\varepsilon}}(t)dt = 0$. So problem (1.1), (5.2) can be written as

$$\begin{cases} u''(x) + \lambda\check{\varepsilon}(u-a)(u-b)(c-u) = 0, \quad -1 < x < 1, \ u(-1) = u(1) = 0, \\ \lambda, \check{\varepsilon} > 0, \ \ 0 < a < b < c, \ \ a + c > 2b. \end{cases} \tag{5.3}$$

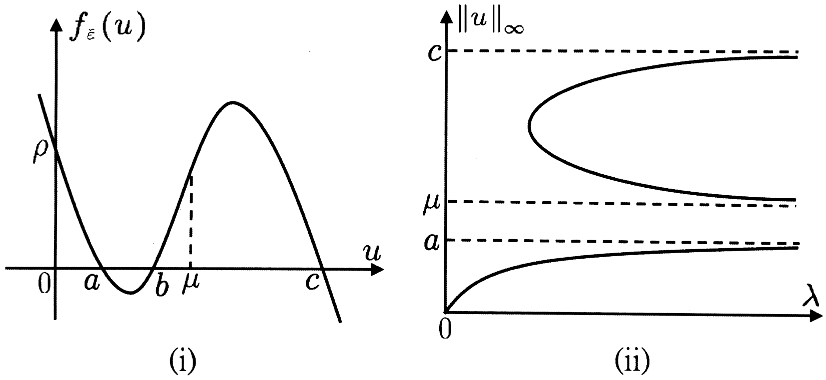

Fig. 8. (i) The graph of $f_{\check{\varepsilon}}(u)$ satisfying (5.3). (ii) The conjectured bifurcation curve of problem (5.3).

It was conjectured that the bifurcation curve of positive solution of problem (5.3) is broken S-shaped (see Fig. 8(ii)) on the $(\lambda, \|u\|_\infty)$-plane. A proof was claimed by Smoller and Wasserman [11, Theorem 2.1], but their proof has a gap. Assuming different conditions on constants a, b and c, Wang [12] and Korman, Li and Ouyang [6] gave partial proof of the above conjecture independently. For this conjecture, Korman, Li and Ouyang [7] gave a computer-assisted proof. Further investigation on this conjecture is needed. We give two conjectures about problem (1.1), (5.2) which is more general than problem (5.3).

Conjecture 5.2. *Consider* (1.1) *where*

$$\sqrt{3\sigma\rho} < \kappa < 2\sqrt{\sigma\rho}. \tag{5.4}$$

Then there exist two positive numbers $\tilde{\varepsilon}_0 = \tilde{\varepsilon}_0(\sigma, \kappa, \rho) < \varepsilon_0 = \varepsilon_0(\sigma, \kappa, \rho)$ such that the following assertions (i)–(iii) hold:

(i) (See Fig. 2(i).) If $0 < \varepsilon < \tilde{\varepsilon}_0$, then the bifurcation curve $\hat{S}_\varepsilon$ is S-shaped on the $(\lambda, \|u\|_\infty)$-plane. Moreover, the exact multiplicity results of positive solutions in Theorem 2.1(i) hold.

(ii) (See Fig. 8(ii).) If $\tilde{\varepsilon}_0 \le \varepsilon < \varepsilon_0$, then the bifurcation curve $\hat{S}_\varepsilon$ is broken S-shaped on the $(\lambda, \|u\|_\infty)$-plane. Moreover, there exist $\lambda^* > 0$ such that problem (1.1), (5.4) *has exactly three positive solutions for $\lambda > \lambda^*$, exactly two positive solutions for $\lambda = \lambda^*$, and exactly one positive solution for $0 < \lambda < \lambda^*$.*

(iii) *(See Fig. 2(iii).) If $\varepsilon \geq \varepsilon_0$, then the bifurcation curve $\hat{S}_\varepsilon$ is a monotone curve on the $(\lambda, ||u||_\infty)$-plane. Moreover, problem* (1.1), (5.4) *has exactly one positive solution for all $\lambda > 0$.*

Conjecture 5.3. *Consider* (1.1) *where*

$$\kappa \geq 2\sqrt{\sigma\rho}. \tag{5.5}$$

Then there exists a positive number $\varepsilon_0 = \varepsilon_0(\sigma, \kappa, \rho)$ such that the following assertions (i) and (ii) hold:

(i) *(See Fig. 8(ii).) If $0 < \varepsilon < \varepsilon_0$, then the bifurcation curve $\hat{S}_\varepsilon$ is broken S-shaped on the $(\lambda, \|u\|_\infty)$-plane. Moreover, there exist $\lambda^* > 0$ such that problem* (1.1), (5.5) *has exactly three positive solutions for $\lambda > \lambda^*$, exactly two positive solutions for $\lambda = \lambda^*$, and exactly one positive solution for $0 < \lambda < \lambda^*$.*

(ii) *(See Fig. 2(iii).) If $\varepsilon \geq \varepsilon_0$, then the bifurcation curve $\hat{S}_\varepsilon$ is a monotone curve on the $(\lambda, ||u||_\infty)$-plane. Moreover, problem* (1.1), (5.5) *has exactly one positive solution for all $\lambda > 0$.*

Acknowledgments. Most of the computation in this paper has been checked using the symbolic manipulator *Mathematica 7.0*.

References

[1] K.J. Brown, M.M.A. Ibrahim, R. Shivaji, S-shaped bifurcation curves, Nonlinear Anal. 5 (1981) 475–486.

[2] M.G. Crandall, P.H. Rabinowitz, Bifurcation, perturbation of simple eigenvalues and linearized stability, Arch. Rational Mech. Anal. 52 (1973) 161–180.

[3] K.-C. Hung, S.-H. Wang, Global bifurcation and exact multiplicity of positive solutions for a positone problem with cubic nonlinearity and their applications, Trans. Amer. Math. Soc., accepted to appear under minor revision.

[4] K.-C. Hung, S.-H. Wang, A theorem on S-shaped bifurcation curve for a positone problem with convex-concave nonlinearity and its applications to the perturbed Gelfand problem, J. Differential Equations 251 (2011) 223–237.

[5] K.-C. Hung, S.-H. Wang, Exact multiplicity of positive solutions for a Dirichlet problem with cubic nonlinearity, preprint.

[6] P. Korman, Y. Li, T. Ouyang, Exact multiplicity results for boundary value problems with nonlinearities generalising cubic. Proc. Roy. Soc. Edinburgh Sect. A 126 (1996) 599–616.

[7] P. Korman, Y. Li, T. Ouyang, Computing the location and direction of bifurcation, Math. Res. Lett. 12 (2005) 933–944.

[8] T. Laetsch, The number of solutions of a nonlinear two point boundary value problem, Indiana Univ. Math. J. 20 (1970) 1–13.

[9] J. Shi, Persistence and bifurcation of degenerate solutions. J. Funct. Anal. 169 (1999) 494–531.

[10] J. Shi, Multi-parameter bifurcation and applications, in: H. Brezis, K.C. Chang, S.J. Li, P. Rabinowitz (Eds.), ICM2002 Satellite Conference on Nonlinear Functional Analysis: Topological Methods, Variational Methods and Their Applications, World Scientific, Singapore, 2003, pp. 211–222.

[11] J. Smoller, A. Wasserman, Global bifurcation of steady-state solutions, J. Differential Equations 39 (1981) 269–290.

[12] S.-H. Wang, A correction for a paper by J. Smoller and A. Wasserman, J. Differential Equations 77 (1989) 199-202.

6. Appendix

Proof of Lemma 3.4.

The proof of Lemma 3.4 is rather long and technical, we divide the proof into next Steps 1–5.

Step 1. We compute $G'_\varepsilon(\alpha)$.

By (3.3)–(3.5), we compute that

$$\begin{aligned} G_\varepsilon(\alpha) &= 8\sqrt{2}\alpha^{\frac{5}{2}}T''_\varepsilon(\alpha) \\ &= -8\alpha^{\frac{5}{2}}\int_0^1 \frac{f_\varepsilon(\alpha)-f_\varepsilon(\alpha v)v}{[F_\varepsilon(\alpha)-F_\varepsilon(\alpha v)]^{3/2}}dv - 4\alpha^{\frac{7}{2}}\int_0^1 \frac{f'_\varepsilon(\alpha)-f'_\varepsilon(\alpha v)v^2}{[F_\varepsilon(\alpha)-F_\varepsilon(\alpha v)]^{3/2}}dv \\ &\quad +6\alpha^{\frac{7}{2}}\int_0^1 \frac{[f_\varepsilon(\alpha)-f_\varepsilon(\alpha v)v]^2}{[F_\varepsilon(\alpha)-F_\varepsilon(\alpha v)]^{5/2}}dv \end{aligned}$$

and

$$\begin{aligned} G'_\varepsilon(\alpha) &= -20\alpha^{\frac{3}{2}}\int_0^1 \frac{f_\varepsilon(\alpha)-f_\varepsilon(\alpha v)v}{[F_\varepsilon(\alpha)-F_\varepsilon(\alpha v)]^{3/2}}dv - 22\alpha^{\frac{5}{2}}\int_0^1 \frac{f'_\varepsilon(\alpha)-f'_\varepsilon(\alpha v)v^2}{[F_\varepsilon(\alpha)-F_\varepsilon(\alpha v)]^{3/2}}dv \\ &\quad -4\alpha^{\frac{7}{2}}\int_0^1 \frac{f''_\varepsilon(\alpha)-f''_\varepsilon(\alpha v)v^3}{[F_\varepsilon(\alpha)-F_\varepsilon(\alpha v)]^{3/2}}dv + 33\alpha^{\frac{5}{2}}\int_0^1 \frac{[f_\varepsilon(\alpha)-f_\varepsilon(\alpha v)v]^2}{[F_\varepsilon(\alpha)-F_\varepsilon(\alpha v)]^{5/2}}dv \\ &\quad +18\alpha^{\frac{7}{2}}\int_0^1 \frac{[f_\varepsilon(\alpha)-f_\varepsilon(\alpha v)v]\,[f'_\varepsilon(\alpha)-f'_\varepsilon(\alpha v)v^2]}{[F_\varepsilon(\alpha)-F_\varepsilon(\alpha v)]^{5/2}}dv \\ &\quad -15\alpha^{\frac{7}{2}}\int_0^1 \frac{[f_\varepsilon(\alpha)-f_\varepsilon(\alpha v)v]^3}{[F_\varepsilon(\alpha)-F_\varepsilon(\alpha v)]^{7/2}}dv \\ &= \frac{1}{\sqrt{\alpha}}\int_0^\alpha \frac{K_\varepsilon(\alpha,u)}{[\triangle F_\varepsilon]^{7/2}}du, \end{aligned} \tag{6.1}$$

where

$$K_\varepsilon(\alpha,u) = -20(\triangle F_\varepsilon)^2(\triangle f_\varepsilon) - 22(\triangle F_\varepsilon)^2(\triangle \tilde{f}_\varepsilon) - 4(\triangle F_\varepsilon)^2(\triangle \hat{f}_\varepsilon) + 33(\triangle F_\varepsilon)(\triangle f_\varepsilon)^2 + 18(\triangle F_\varepsilon)(\triangle f_\varepsilon)(\triangle \tilde{f}_\varepsilon) - 15(\triangle f_\varepsilon)^3, \tag{6.2}$$

$$\triangle F_\varepsilon = F_\varepsilon(\alpha) - F_\varepsilon(u), \tag{6.3}$$

$$\triangle f_\varepsilon = \alpha f_\varepsilon(\alpha) - u f_\varepsilon(u), \tag{6.4}$$

$$\triangle \tilde{f}_\varepsilon = \alpha^2 f'_\varepsilon(\alpha) - u^2 f'_\varepsilon(u), \tag{6.5}$$

$$\triangle \hat{f}_\varepsilon = \alpha^3 f''_\varepsilon(\alpha) - u^3 f''_\varepsilon(u). \tag{6.6}$$

Since $f_\varepsilon(u) = -\varepsilon u^3 + \sigma u^2 - \kappa u + \rho$, we have that

$$F_\varepsilon(u) = -\varepsilon u^4/4 + \sigma u^3/3 - \kappa u^2/2 + \rho u, \tag{6.7}$$

$$u f_\varepsilon(u) = -\varepsilon u^4 + \sigma u^3 - \kappa u^2 + \rho u, \tag{6.8}$$

$$u^2 f'_\varepsilon(u) = -3\varepsilon u^4 + 2\sigma u^3 - \kappa u^2, \tag{6.9}$$

$$u^3 f''_\varepsilon(u) = -6\varepsilon u^4 + 2\sigma u^3. \tag{6.10}$$

For $0 < u < \alpha$, we let $A \equiv \varepsilon(\alpha^4 - u^4)$, $B \equiv \sigma(\alpha^3 - u^3)$, $C \equiv \kappa(\alpha^2 - u^2)$, $D \equiv \rho(\alpha - u)$. Then $A, B, C, D > 0$. By (6.3)–(6.10), we obtain that

$$\triangle F_\varepsilon = -A/4 + B/3 - C/2 + D, \tag{6.11}$$

$$\triangle f_\varepsilon = -A + B - C + D, \tag{6.12}$$

$$\triangle \tilde{f}_\varepsilon = -3A + 2B - C, \tag{6.13}$$

$$\triangle \hat{f}_\varepsilon = -6A + 2B. \tag{6.14}$$

Substitute (6.11)–(6.14) into (6.2), we have

$$K_\varepsilon(\alpha,u) = \frac{1}{72}(168ABC - 1356ABD - 504ACD - 168BCD + 9A^3 - 144D^3 - 2AB^2 + 12A^2B + 90AC^2 - 207A^2C - 60B^2C + 2646AD^2 + 1134A^2D - 1248BD^2 + 560B^2D + 468CD^2 + 72C^2D). \tag{6.15}$$

So Lemma 3.4 holds if we can prove that $K_\varepsilon(\alpha,u) > 0$ for any fixed $\varepsilon \in [(\frac{7}{10}(\frac{\sigma^3}{27\rho}))^{1/2}, (\frac{\sigma^3}{27\rho})^{1/2}]$, $\alpha \in [\gamma_\varepsilon, \beta_\varepsilon)$ and $0 < u < \alpha$.

Step 2. We make a transformation for $K_\varepsilon(\alpha,u)$.

Although both $T_\varepsilon(\alpha)$ and $G_\varepsilon(\alpha)$ are only defined for $\alpha \in (0, \beta_\varepsilon)$, $K_\varepsilon(\alpha,u)$ is well defined for all $\alpha \in \mathbb{R}$. So Lemma 3.4 holds if we can prove $K_\varepsilon(\alpha,u) > 0$ for any fixed $\varepsilon \in [(\frac{7}{10}(\frac{\sigma^3}{27\rho}))^{1/2}, (\frac{\sigma^3}{27\rho})^{1/2}]$, $\alpha \geq \gamma_\varepsilon$, $0 \leq \kappa \leq \sqrt{\sigma\rho}$ and $0 < u < \alpha$. Since $\gamma_\varepsilon = \frac{\sigma}{3\varepsilon}$, we consider $K_\varepsilon(\alpha,u)$ when $\alpha \geq \gamma_\varepsilon$, $0 \leq \kappa \leq \sqrt{3}\varepsilon\gamma_\varepsilon^2$, $\frac{7}{10}\varepsilon\gamma_\varepsilon^3 \leq \rho \leq \varepsilon\gamma_\varepsilon^3$, and $0 < u < \alpha$. Let

$$\alpha = (r+1)\gamma_\varepsilon, \quad r \in [0, \infty),$$

$$\kappa = s\varepsilon\gamma_\varepsilon^2, \quad s \in [0, \sqrt{3}],$$
$$\rho = t\varepsilon\gamma_\varepsilon^3, \quad t \in [\frac{7}{10}, 1],$$
$$u = y\gamma_\varepsilon, \quad y \in (0, r+1).$$

Thus

$$A = \varepsilon(\alpha^4 - u^4) = \varepsilon\gamma_\varepsilon^4[(r+1)^4 - y^4], \tag{6.16}$$
$$B = \sigma(\alpha^3 - u^3) = 3\varepsilon\gamma_\varepsilon^4[(r+1)^3 - y^3], \tag{6.17}$$
$$C = \kappa(\alpha^2 - u^2) = s\varepsilon\gamma_\varepsilon^4[(r+1)^2 - y^2], \tag{6.18}$$
$$D = \rho(\alpha - u) = t\varepsilon\gamma_\varepsilon^4(r+1-y). \tag{6.19}$$

Substitute (6.16)–(6.19) into (6.15), we obtain

$$K_\varepsilon(\alpha, u) = \frac{1}{8}\varepsilon^3\gamma_\varepsilon^{12}(r+1-y)^3\widetilde{K}_\varepsilon(r,s,t,y), \tag{6.20}$$

where

$$\widetilde{K}_\varepsilon(r,s,t,y) = \sum_{j=0}^{9} k_j(r,s,t)y^j, \tag{6.21}$$

$$\begin{aligned}
k_0(r,s,t) \;=\; & (3 - 122t^2 + 10s^2 - 16t^3 + 8s^2t + 234t - 27s + 52st^2 - 112st) \\
& +(-392st + 16s^2t + 50t^2 + 736t + 50s^2 + 27 - 125s + 52st^2)r \\
& +(100s^2 + 466t^2 + 8s^2t - 243s + 730t - 504st + 106)r^2 + (-280st \\
& +238 + 294t^2 + 240t + 100s^2 - 285s)r^3 + (-265s - 56st + 50s^2 \\
& +336 + 190t)r^4 + (-207s + 304t + 308 + 10s^2)r^5 + (126t - 105s \\
& +182)r^6 + (66 - 23s)r^7 + 13r^8 + r^9,
\end{aligned}$$

$$\begin{aligned}
k_1(r,s,t) \;=\; & (9 + 52st^2 + 468t + 30s^2 - 81s - 224st - 122t^2 + 16s^2t) + (16s^2t \\
& +172t^2 - 560st + 72 - 294s + 1004t + 120s^2)r + (294t^2 + 180s^2 \\
& -435s - 448st + 456t + 246)r^2 + (-112st + 24t + 468 + 120s^2 \\
& -420s)r^3 + (30s^2 + 356t + 540 - 375s)r^4 + (252t + 384 - 246s)r^5 \\
& +(162 - 69s)r^6 + 36r^7 + 3r^8,
\end{aligned}$$

$$\begin{aligned}
k_2(r,s,t) \;=\; & (18 + 40s^2 - 135s + 702t - 224st + 8s^2t - 122t^2) + (126 + 294t^2 \\
& +804t + 120s^2 - 355s - 336st)r + (-370s - 120t + 366 - 112st \\
& +120s^2)r^2 + (40s^2 + 156t - 330s + 570)r^3 + (-295s + 378t + 510)r^4 \\
& +(258 - 115s)r^5 + 66r^6 + 6r^7,
\end{aligned}$$

$$\begin{aligned}
k_3(r,s,t) \;=\; & (-125s + 40s^2 + 268t - 168st + 30 + 294t^2) + (-236s - 80t + 80s^2 \\
& +176 - 112st)r + (-258s + 156t + 40s^2 + 414)r^2 + (496 - 308s \\
& +504t)r^3 + (314 - 161s)r^4 + 96r^5 + 10r^6,
\end{aligned}$$

$$\begin{aligned} k_4(r,s,t) &= (36-61s-56st+34t+30s^2)+(172-148t-103s+30s^2)r \\ &\quad +(-203s+378t+312)r^2+(264-161s)r^3+100r^4+12r^5, \end{aligned}$$

$$\begin{aligned} k_5(r,s,t) &= (-7s+10s^2+36-200t)+(252t+132-62s)r+(168-115s)r^2 \\ &\quad +84r^3+12r^4, \end{aligned}$$

$$k_6(r,s,t) = (-13s+28+126t)+(72-69s)r+54r^2+10r^3,$$

$$k_7(r,s,t) = 16-23s+24r+6r^2,$$

$$k_8(r,s,t) = 7+3r,$$

$$k_9(r,s,t) = 1.$$

So Lemma 3.4 holds if we can prove $\widetilde{K}_\varepsilon(r,s,t,y) > 0$ for any fixed $y \in (0, r+1)$, $(r,t) \in \Omega \equiv [0,\infty) \times [\frac{7}{10}, 1]$ and $s \in [0, \sqrt{3}]$.

Step 3. For any fixed $y \in (0, r+1)$ and $(r,t) \in \Omega \equiv [0,\infty)\times[\frac{7}{10},1]$, we show $\widetilde{K}_\varepsilon(r,s,t,y)$ is strictly decreasing with respect to s on $[0,\sqrt{3}]$.

From (6.15), we have

$$72\frac{\partial K_\varepsilon}{\partial C} = -207A^2-60B^2-504AD-168BD+180AC+468D^2+144CD+168AB \quad (6.22)$$

and

$$72\frac{\partial^2 K_\varepsilon}{\partial C^2} = 180A+144D > 0.$$

By (6.16)–(6.20), we compute that

$$\begin{aligned} \frac{\partial^2 \widetilde{K}_\varepsilon}{\partial s^2} &= \frac{8}{\varepsilon^3\gamma_\varepsilon^{12}(r+1-y)^3}\frac{\partial^2 K_\varepsilon}{\partial s^2} \\ &= \frac{8}{\varepsilon^3\gamma_\varepsilon^{12}(r+1-y)^3}\left[\frac{\partial^2 K_\varepsilon}{\partial C^2}\left(\frac{\partial C}{\partial s}\right)^2 + \frac{\partial K_\varepsilon}{\partial C}\frac{\partial^2 C}{\partial s^2}\right] \\ &= \frac{1}{9\varepsilon^3\gamma_\varepsilon^{12}(r+1-y)^3}(180A+144D)\left[\varepsilon\gamma_\varepsilon^4((r+1)^2-y^2)\right]^2 \\ &= \left\{20\left[(r+1)^3+(r+1)^2y+(r+1)y^2+y^3\right]+18t\right\}(r+1+y)^2 > 0. \end{aligned}$$

This implies that for any fixed $y \in (0, r+1)$ and $(r,t) \in \Omega$, $\widetilde{K}_\varepsilon(r,s,t,y)$ is concave up as a function of $s \in [0,\sqrt{3}]$, hence $\widetilde{K}_\varepsilon(r,s,t,y)$ is strictly decreasing with respect to s on $[0,\sqrt{3}]$ if we can prove

$$\frac{\partial \widetilde{K}_\varepsilon}{\partial s}(r,\sqrt{3},t,y) < 0 \quad \text{for any } y \in (0,r+1),\ (r,t) \in \Omega. \quad (6.23)$$

By (6.16)–(6.19) and (6.22), we compute that

$$\begin{aligned}
&\frac{\partial \widetilde{K}_\varepsilon}{\partial s}\\
= \;& \frac{8}{\varepsilon^3\gamma_\varepsilon^{12}(r+1-y)^3}\frac{\partial K_\varepsilon}{\partial C}\frac{\partial C}{\partial s}\\
= \;& \frac{[-207A^2-60B^2-504AD-168BD+180AC+468D^2+144CD+168AB]\,(r+1+y)}{9\varepsilon^2\gamma_\varepsilon^8(r+1-y)^2},
\end{aligned}$$

and

$$-\frac{\partial \widetilde{K}_\varepsilon}{\partial s}(r,\sqrt{3},t,y) = (r+1+y)\sum_{j=0}^{6} g_j(r,t)y^j, \tag{6.24}$$

where

$$\begin{aligned}
g_0(r,t) \;=\;& 23r^6+82r^5+(125-20\sqrt{3})r^4+(140-80\sqrt{3}+56t)r^3\\
&+(145-120\sqrt{3}+224t)r^2+(98-80\sqrt{3}+280t-16\sqrt{3}t)r\\
&+(27-20\sqrt{3}+112t-16\sqrt{3}t-52t^2),
\end{aligned}$$

$$\begin{aligned}
g_1(r,t) \;=\;& 46r^5+118r^4+(132-40\sqrt{3})r^3+(148-120\sqrt{3}+56t)r^2\\
&+(142-120\sqrt{3}+168t)r+(54-40\sqrt{3}+112t-16\sqrt{3}t),
\end{aligned}$$

$$g_2(r,t) = 69r^4+108r^3+(90-40\sqrt{3})r^2+(132-80\sqrt{3}+56t)r+(81-40\sqrt{3}+112t),$$

$$g_3(r,t) = 92r^3+108r^2+(60-40\sqrt{3})r+(44-40\sqrt{3}+56t),$$

$$g_4(r,t) = 69r^2+26r+(17-20\sqrt{3}),$$

$$g_5(r,t) = 46r-10,$$

$$g_6(r,t) = 23.$$

In order to prove (6.23), we claim that for any fixed $y\in(0,r+1)$ and $(r,t)\in\Omega$,

$$\sum_{j=0}^{n} g_j(r,t)y^j \geq \left(\frac{y}{r+1}\right)^n \tilde{g}_n(r,t) > 0, \quad n=0,1,2,3,4,5,6, \tag{6.25}$$

where

$$\tilde{g}_n(r,t) = \sum_{j=0}^{n}(r+1)^j g_j(r,t), \quad n=0,1,2,3,4,5,6.$$

First, we compute $\tilde{g}_0(r,t)=g_0(r,t)$ and

$$\begin{aligned}
\tilde{g}_1(r,t) \;=\;& 69r^6+246r^5+(375-60\sqrt{3})r^4+(420-240\sqrt{3}+112t)r^3\\
&+(435-360\sqrt{3}+448t)r^2+(294-240\sqrt{3}+560t-32\sqrt{3}t)r\\
&+(81-60\sqrt{3}+224t-32\sqrt{3}t-52t^2),
\end{aligned}$$

$$\begin{aligned}
\tilde{g}_2(r,t) \;=\;& 138r^6+492r^5+(750-100\sqrt{3})r^4+(840-400\sqrt{3}+168t)r^3\\
&+(870-600\sqrt{3}+672t)r^2+(588-400\sqrt{3}+840t-32\sqrt{3}t)r\\
&+(162-100\sqrt{3}+336t-32\sqrt{3}t-52t^2),
\end{aligned}$$

$$\begin{aligned}\tilde{g}_3(r,t) &= 230r^6 + 876r^5 + (1410 - 140\sqrt{3})r^4 + (1480 - 560\sqrt{3} + 224t)r^3 \\ &\quad + (1290 - 840\sqrt{3} + 840t)r^2 + (780 - 560\sqrt{3} + 1008t - 32\sqrt{3}t)r \\ &\quad + (206 - 140\sqrt{3} + 392t - 32\sqrt{3}t - 52t^2),\end{aligned}$$

$$\begin{aligned}\tilde{g}_4(r,t) &= 299r^6 + 1178r^5 + (1945 - 160\sqrt{3})r^4 + (1980 - 640\sqrt{3} + 224t)r^3 \\ &\quad + (1565 - 960\sqrt{3} + 840t)r^2 + (874 - 640\sqrt{3} + 1008t - 32\sqrt{3}t)r \\ &\quad + (223 - 160\sqrt{3} + 392t - 32\sqrt{3}t - 52t^2),\end{aligned}$$

$$\begin{aligned}\tilde{g}_5(r,t) &= 345r^6 + 1398r^5 + (2355 - 160\sqrt{3})r^4 + (2340 - 640\sqrt{3} + 224t)r^3 \\ &\quad + (1695 - 960\sqrt{3} + 840t)r^2 + (870 - 640\sqrt{3} + 1008t - 32\sqrt{3}t)r \\ &\quad + (213 - 160\sqrt{3} + 392t - 32\sqrt{3}t - 52t^2),\end{aligned}$$

$$\begin{aligned}\tilde{g}_6(r,t) &= 368r^6 + 1536r^5 + (2700 - 160\sqrt{3})r^4 + (2800 - 640\sqrt{3} + 224t)r^3 \\ &\quad + (2040 - 960\sqrt{3} + 840t)r^2 + (1008 - 640\sqrt{3} + 1008t - 32\sqrt{3}t)r \\ &\quad + (236 - 160\sqrt{3} + 392t - 32\sqrt{3}t - 52t^2).\end{aligned}$$

As a polynomial of r, it is easy to check that the coefficients of $\tilde{g}_n(r,t)$ are all positive for $n \in \{0,1,2,3,4,5,6\}$, where $t \in [\frac{7}{10}, 1]$. So for any fixed $y \in (0, r+1)$ and $(r,t) \in \Omega$,

$$\tilde{g}_n(r,t) > 0, \quad n = 0,1,2,3,4,5,6. \tag{6.26}$$

Suppose (6.25) holds for $n = l$ where $l \in \{0,1,2,3,4,5\}$, by (6.26) and $0 < y < r+1$, we have

$$\begin{aligned}\sum_{j=0}^{l+1} g_j(r,t)y^j &= \sum_{j=0}^{l} g_j(r,t)y^j + g_{l+1}(r,t)y^{l+1} \\ &\geq \left[\left(\frac{y}{r+1}\right)^l \tilde{g}_l(r,t)\right]\left(\frac{y}{r+1}\right) + g_{l+1}(r,t)y^{l+1} \\ &= \left(\frac{y}{r+1}\right)^{l+1}\left[\sum_{j=0}^{l}(r+1)^j g_j(r,t)\right] + \left(\frac{y}{r+1}\right)^{l+1}(r+1)^{l+1}g_{l+1}(r,t) \\ &= \left(\frac{y}{r+1}\right)^{l+1}\sum_{j=0}^{l+1}(r+1)^j g_j(r,t) \\ &= \left(\frac{y}{r+1}\right)^{l+1}\tilde{g}_{l+1}(r,t).\end{aligned} \tag{6.27}$$

So (6.25) holds for $n = l+1$ where $l \in \{0,1,2,3,4,5\}$. By (6.24)–(6.27), we obtain (6.23)

$$\frac{\partial \widetilde{K}_\varepsilon}{\partial s}(r, \sqrt{3}, t, y) = -(r+1+y)\sum_{j=0}^{6} g_j(r,t)y^j < 0 \ \text{ for any } y \in (0, r+1),\ (r,t) \in \Omega.$$

So for any fixed $y \in (0, r+1)$ and $(r,t) \in \Omega \equiv [0,\infty) \times [\frac{7}{10}, 1]$, $\widetilde{K}_\varepsilon(r,s,t,y)$ is strictly decreasing with respect to s on $[0, \sqrt{3}]$.

Step 4. We show $\widetilde{K}_\varepsilon(r,s,t,y) > 0$ for any fixed $y \in (0, r+1)$, $(r,t) \in \Omega \equiv [0,\infty) \times [\frac{7}{10}, 1]$ and $s \in [0, \sqrt{3}]$.

By Step 3, Step 4 holds if we can prove

$$\widetilde{K}_\varepsilon(r,\sqrt{3},t,y) > 0 \quad \text{for any } y \in (0,r+1),\ (r,t) \in \Omega. \tag{6.28}$$

By (6.21) in Step 2,

$$\widetilde{K}_\varepsilon(r,\sqrt{3},t,y) = \sum_{j=0}^{9} h_j(r,t) y^j, \tag{6.29}$$

where

$$\begin{aligned} h_0(r,t) \ = \ & r^9 + 13r^8 + (66 - 23\sqrt{3})r^7 + (182 - 105\sqrt{3} + 126t)r^6 + (338 - 207\sqrt{3} + 304t)r^5 \\ & + (486 - 265\sqrt{3} + 190t - 56\sqrt{3}t)r^4 + (538 - 285\sqrt{3} + 240t - 280\sqrt{3}t + 294t^2)r^3 \\ & + (406 - 243\sqrt{3} + 754t - 504\sqrt{3}t + 466t^2)r^2 + (177 - 125\sqrt{3} + 784t - 392\sqrt{3}t \\ & + 50t^2 + 52\sqrt{3}t^2)r + (33 - 27\sqrt{3} + 258t - 112\sqrt{3}t + 52\sqrt{3}t^2 - 122t^2 - 16t^3), \end{aligned}$$

$$\begin{aligned} h_1(r,t) \ = \ & 3r^8 + 36r^7 + (162 - 69\sqrt{3})r^6 + (384 - 246\sqrt{3} + 252t)r^5 \\ & + (630 - 375\sqrt{3} + 356t)r^4 + (828 - 420\sqrt{3} + 24t - 112\sqrt{3}t)r^3 \\ & + (786 - 435\sqrt{3} + 456t - 448\sqrt{3}t + 294t^2)r^2 + (432 - 294\sqrt{3} + 1052t \\ & - 560\sqrt{3}t + 172t^2)r + (99 - 81\sqrt{3} + 516t - 224\sqrt{3}t + 52\sqrt{3}t^2 - 122t^2), \end{aligned}$$

$$\begin{aligned} h_2(r,t) \ = \ & 6r^7 + 66r^6 + (258 - 115\sqrt{3})r^5 + (510 - 295\sqrt{3} + 378t)r^4 + (690 - 330\sqrt{3} \\ & + 156t)r^3 + (726 - 370\sqrt{3} - 120t - 112\sqrt{3}t)r^2 + (486 - 355\sqrt{3} + 804t \\ & - 336\sqrt{3}t + 294t^2)r + (138 - 135\sqrt{3} + 726t - 224\sqrt{3}t - 122t^2), \end{aligned}$$

$$\begin{aligned} h_3(r,t) \ = \ & 10r^6 + 96r^5 + (314 - 161\sqrt{3})r^4 + (496 - 308\sqrt{3} + 504t)r^3 \\ & + (534 - 258\sqrt{3} + 156t)r^2 + (416 - 236\sqrt{3} - 112\sqrt{3}t - 80t)r \\ & + (150 - 125\sqrt{3} + 268t - 168\sqrt{3}t + 294t^2), \end{aligned}$$

$$\begin{aligned} h_4(r,t) \ = \ & 12r^5 + 100r^4 + (264 - 161\sqrt{3})r^3 + (312 - 203\sqrt{3} + 378t)r^2 \\ & + (262 - 103\sqrt{3} - 148t)r + (126 - 61\sqrt{3} + 34t - 56\sqrt{3}t), \end{aligned}$$

$$h_5(r,t) = 12r^4 + 84r^3 + (168 - 115\sqrt{3})r^2 + (132 - 62\sqrt{3} + 252t)r + (66 - 7\sqrt{3} - 200t),$$

$$h_6(r,t) = 10r^3 + 54r^2 + (72 - 69\sqrt{3})r + (28 - 13\sqrt{3} + 126t),$$

$$h_7(r,t) = 6r^2 + 24r + (16 - 23\sqrt{3}),$$

$$h_8(r,t) = 3r + 7,$$

$$h_9(r,t) = 1.$$

In order to prove (6.28), we claim that for any fixed $y \in (0, r+1)$ and $(r,t) \in \Omega$,

$$\sum_{j=0}^{n} h_j(r,t) y^j \geq \left(\frac{y}{r+1}\right)^n \tilde{h}_n(r,t) > 0, \quad n = 0, 1, 2, \cdots, 9, \tag{6.30}$$

where

$$\tilde{h}_n(r,t) = \sum_{j=0}^{n} (r+1)^j h_j(r,t), \quad n = 0, 1, 2, \cdots, 9.$$

First, we compute $\widetilde{h}_0(r,t) = h_0(r,t)$ and

$$\begin{aligned}
\tilde{h}_1(r,t) \;=\;& 4r^9 + 52r^8 + (264 - 92\sqrt{3})r^7 + (728 - 420\sqrt{3} + 378t)r^6 + (1352 - 828\sqrt{3} \\
& +912t)r^5 + (1944 - 1060\sqrt{3} + 570t - 168\sqrt{3}t)r^4 + (2152 - 1140\sqrt{3} \\
& +720t - 840\sqrt{3}t + 588t^2)r^3 + (1624 - 972\sqrt{3} + 2262t - 1512\sqrt{3}t \\
& +932t^2)r^2 + (708 - 500\sqrt{3} + 2352t - 1176\sqrt{3}t + 100t^2 + 104\sqrt{3}t^2)r \\
& +(132 - 108\sqrt{3} + 774t - 336\sqrt{3}t + 104\sqrt{3}t^2 - 244t^2 - 16t^3),
\end{aligned} \tag{6.31}$$

$$\begin{aligned}
\tilde{h}_2(r,t) \;=\;& 10r^9 + 130r^8 + (660 - 207\sqrt{3})r^7 + (1820 - 945\sqrt{3} + 756t)r^6 + (3320 \\
& -1863\sqrt{3} + 1824t)r^5 + (4560 - 2385\sqrt{3} + 1140t - 280\sqrt{3}t)r^4 + (4780 \\
& -2565\sqrt{3} + 1440t - 1400\sqrt{3}t + 882t^2)r^3 + (3460 - 2187\sqrt{3} + 4476t \\
& -2520\sqrt{3}t + 1398t^2)r^2 + (1470 - 1125\sqrt{3} + 4608t - 1960\sqrt{3}t + 150t^2 \\
& +104\sqrt{3}t^2)r + (270 - 243\sqrt{3} + 1500t - 560\sqrt{3}t + 104\sqrt{3}t^2 - 366t^2 - 16t^3),
\end{aligned}$$

$$\begin{aligned}
\widetilde{h}_3(r,t) \;=\;& 20r^9 + 256r^8 + (1292 - 368\sqrt{3})r^7 + (3556 - 1736\sqrt{3} + 1260t)r^6 + (6380 \\
& -3528\sqrt{3} + 3492t)r^5 + (8380 - 4480\sqrt{3} + 3040t - 392\sqrt{3}t)r^4 + (8276 \\
& -4480\sqrt{3} + 2440t - 1904\sqrt{3}t + 1176t^2)r^3 + (5692 - 3528\sqrt{3} + 5196t \\
& -3360\sqrt{3}t + 2280t^2)r^2 + (2336 - 1736\sqrt{3} + 5332t - 2576\sqrt{3}t + 1032t^2 \\
& +104\sqrt{3}t^2)r + (420 - 368\sqrt{3} + 1768t - 728\sqrt{3}t + 104\sqrt{3}t^2 - 72t^2 - 16t^3),
\end{aligned}$$

$$\begin{aligned}
\widetilde{h}_4(r,t) \;=\;& 32r^9 + 404r^8 + (2028 - 529\sqrt{3})r^7 + (5572 - 2583\sqrt{3} + 1638t)r^6 + (9886 \\
& -5409\sqrt{3} + 4856t)r^5 + (12582 - 6815\sqrt{3} + 4750t - 448\sqrt{3}t)r^4 + (11864 \\
& -6315\sqrt{3} + 3200t - 2128\sqrt{3}t + 1176t^2)r^3 + (7808 - 4509\sqrt{3} + 5186t \\
& -3696\sqrt{3}t + 2280t^2)r^2 + (3102 - 2083\sqrt{3} + 5320t - 2800\sqrt{3}t + 1032t^2 \\
& +104\sqrt{3}t^2)r + (546 - 429\sqrt{3} + 1802t - 784\sqrt{3}t + 104\sqrt{3}t^2 - 72t^2 - 16t^3),
\end{aligned}$$

$$\begin{aligned}
\widetilde{h}_5(r,t) \;=\;& 44r^9 + 548r^8 + (2736 - 644\sqrt{3})r^7 + (7504 - 3220\sqrt{3} + 1890t)r^6 + (13192 \\
& -6876\sqrt{3} + 5916t)r^5 + (16344 - 8620\sqrt{3} + 6270t - 448\sqrt{3}t)r^4 + (14768 \\
& -7580\sqrt{3} + 3720t - 2128\sqrt{3}t + 1176t^2)r^3 + (9296 - 5004\sqrt{3} + 4446t \\
& -3696\sqrt{3}t + 2280t^2)r^2 + (3564 - 2180\sqrt{3} + 4572t - 2800\sqrt{3}t + 1032t^2 \\
& +104\sqrt{3}t^2)r + (612 - 436\sqrt{3} + 1602t - 784\sqrt{3}t + 104\sqrt{3}t^2 - 72t^2 - 16t^3),
\end{aligned}$$

$$\begin{aligned}\widetilde{h}_6(r,t) &= 54r^9+662r^8+(3282-713\sqrt{3})r^7+(8974-3647\sqrt{3}+2016t)r^6+(15670\\&-7989\sqrt{3}+6672t)r^5+(19074-10195\sqrt{3}+8160t-448\sqrt{3}t)r^4+(16742\\&-8875\sqrt{3}+6240t-2128\sqrt{3}t+1176t^2)r^3+(10202-5613\sqrt{3}+6336t\\&-3696\sqrt{3}t+2280t^2)r^2+(3804-2327\sqrt{3}+5328t-2800\sqrt{3}t+1032t^2\\&+104\sqrt{3}t^2)r+(640-449\sqrt{3}+1728t-784\sqrt{3}t+104\sqrt{3}t^2-72t^2-16t^3),\end{aligned}$$

$$\begin{aligned}\widetilde{h}_7(r,t) &= 60r^9+728r^8+(3592-736\sqrt{3})r^7+(9800-3808\sqrt{3}+2016t)r^6+(17056\\&-8472\sqrt{3}+6672t)r^5+(20600-11000\sqrt{3}+8160t-448\sqrt{3}t)r^4+(17848\\&-9680\sqrt{3}+6240t-2128\sqrt{3}t+1176t^2)r^3+(10712-6096\sqrt{3}+6336t\\&-3696\sqrt{3}t+2280t^2)r^2+(3940-2488\sqrt{3}+5328t-2800\sqrt{3}t+1032t^2\\&+104\sqrt{3}t^2)r+(656-472\sqrt{3}+1728t-784\sqrt{3}t+104\sqrt{3}t^2-72t^2-16t^3),\end{aligned}$$

$$\begin{aligned}\widetilde{h}_8(r,t) &= 63r^9+759r^8+(3732-736\sqrt{3})r^7+(10164-3808\sqrt{3}+2016t)r^6+(17658\\&-8472\sqrt{3}+6672t)r^5+(21258-11000\sqrt{3}+8160t-448\sqrt{3}t)r^4+(18324\\&-9680\sqrt{3}+6240t-2128\sqrt{3}t+1176t^2)r^3+(10932-6096\sqrt{3}+6336t\\&-3696\sqrt{3}t+2280t^2)r^2+(3999-2488\sqrt{3}+5328t-2800\sqrt{3}t+1032t^2\\&+104\sqrt{3}t^2)r+(663-472\sqrt{3}+1728t-784\sqrt{3}t+104\sqrt{3}t^2-72t^2-16t^3),\end{aligned}$$

$$\begin{aligned}\widetilde{h}_9(r,t) &= 64r^9+768r^8+(3768-736\sqrt{3})r^7+(10248-3808\sqrt{3}+2016t)r^6+(17784\\&-8472\sqrt{3}+6672t)r^5+(21384-11000\sqrt{3}+8160t-448\sqrt{3}t)r^4+(18408\\&-9680\sqrt{3}+6240t-2128\sqrt{3}t+1176t^2)r^3+(10968-6096\sqrt{3}+6336t\\&-3696\sqrt{3}t+2280t^2)r^2+(4008-2488\sqrt{3}+5328t-2800\sqrt{3}t+1032t^2\\&+104\sqrt{3}t^2)r+(664-472\sqrt{3}+1728t-784\sqrt{3}t+104\sqrt{3}t^2-72t^2-16t^3).\end{aligned}$$

As polynomials of r, it is easy to check that the coefficients of $\widetilde{h}_n(r,t)$ are all positive for $n\in\{0,1,2,\cdots,9\}\setminus\{1\}$, where $t\in[\frac{7}{10},1]$. So for any fixed $y\in(0,r+1)$ and $(r,t)\in\Omega$,

$$\widetilde{h}_n(r,t)>0,\quad n\in\{0,1,2,\cdots,9\}\setminus\{1\}.$$

Note that for $n=1$ and $t\in[\frac{7}{10},1]$, it is easy to check that the coefficients of $\widetilde{h}_1(r,t)$ are all positive besides that of r^3. By (6.31), we obtain that

$$\begin{aligned}&\widetilde{h}_1(r,t)\\ >\ & 52r^8+(2152-1140\sqrt{3}+720t-840\sqrt{3}t+588t^2)r^3\\ \geq\ & 52r^3+(2152-1140\sqrt{3}+720t-840\sqrt{3}t+588t^2)r^3\\ =\ & (2204-1140\sqrt{3}+720t-840\sqrt{3}t+588t^2)r^3>0\ \text{ for } t\in[\frac{7}{10},1] \text{ and } r\geq 1,\end{aligned}$$

and

$$\begin{aligned}
&\widetilde{h}_1(r,t)\\
> \ &(2152-1140\sqrt{3}+720t-840\sqrt{3}t+588t^2)r^3+(1624-972\sqrt{3}\\
&+2262t-1512\sqrt{3}t+932t^2)r^2\\
\geq \ &(2152-1140\sqrt{3}+720t-840\sqrt{3}t+588t^2)r^3+(1624-972\sqrt{3}\\
&+2262t-1512\sqrt{3}t+932t^2)r^3\\
= \ &(3776-2112\sqrt{3}+2982t-2352\sqrt{3}t+1520t^2)r^3 \geq 0 \ \text{ for } t \in [\frac{7}{10},1] \text{ and } 0 \leq r < 1.
\end{aligned}$$

So, for any fixed $y \in (0, r+1)$ and $(r,t) \in \Omega$,

$$\widetilde{h}_n(r,t) > 0, \quad n = 0,1,2...,9. \tag{6.32}$$

Suppose (6.30) holds for $n = l$ where $l \in \{0,1,2,\cdots,8\}$, by (6.32) and since $0 < y < r+1$, we have that

$$\begin{aligned}
\sum_{j=0}^{l+1} h_j(r,t)y^j &= \sum_{j=0}^{l} h_j(r,t)y^j + h_{l+1}(r,t)y^{l+1}\\
&\geq \left[\left(\frac{y}{r+1}\right)^l \tilde{h}_l(r,t)\right]\left(\frac{y}{r+1}\right) + h_{l+1}(r,t)y^{l+1}\\
&= \left(\frac{y}{r+1}\right)^{l+1}\left[\sum_{j=0}^{l}(r+1)^j h_j(r,t)\right] + \left(\frac{y}{r+1}\right)^{l+1}(r+1)^{l+1}h_{l+1}(r,t)\\
&= \left(\frac{y}{r+1}\right)^{l+1}\sum_{j=0}^{l+1}(r+1)^j h_j(r,t)\\
&= \left(\frac{y}{r+1}\right)^{l+1}\widetilde{h}_{l+1}(r,t).
\end{aligned}$$

So (6.30) holds for $n = l+1$ where $l \in \{0,1,2,\cdots,8\}$. By (6.29) and (6.30), we obtain (6.28)

$$\widetilde{K}_\varepsilon(r,\sqrt{3},t,y) = \sum_{j=0}^{9} h_j(r,t)y^j > 0 \ \text{ for any } y \in (0,r+1),\ (r,t) \in \Omega.$$

By Step 3 and (6.28), $\widetilde{K}_\varepsilon(r,s,t,y) > 0$ for any fixed $y \in (0, r+1)$, $(r,t) \in \Omega \equiv [0,\infty) \times [\frac{7}{10},1]$ and $s \in [0,\sqrt{3}]$.

Step 5. Finally, we prove the lemma by the above analyses.

By Step 4 and (6.20) in Step 2, we have $K_\varepsilon(\alpha,u) > 0$ for any fixed $\varepsilon \in [(\frac{7}{10}(\frac{\sigma^3}{27\rho}))^{1/2}, (\frac{\sigma^3}{27\rho})^{1/2}]$, $\alpha \in [\gamma_\varepsilon, \beta_\varepsilon)$ and $0 < u < \alpha$. In addition to (6.1) in Step 1, we obtain that for any fixed $\varepsilon \in [(\frac{7}{10}(\frac{\sigma^3}{27\rho}))^{1/2}, (\frac{\sigma^3}{27\rho})^{1/2}]$,

$$G'_\varepsilon(\alpha) = \frac{1}{\sqrt{\alpha}}\int_0^\alpha \frac{K_\varepsilon(\alpha,u)}{[\triangle F_\varepsilon]^{7/2}}du > 0 \ \text{ on } \ [\gamma_\varepsilon, \beta_\varepsilon).$$

So Steps 1–5 complete the proof of Lemma 3.4. ■