國立交通大學

應用數學系

碩 士 論 文

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Spatial Entropy of 3- dimensional Face Coloring

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A Thesis
Submitted to Department of Applied Mathematics
College of Science
National Chiao Tung University
in partial Fulfillment of the Requirements
for the Degree of
Master
in

Applied Mathematics

June 2011

Hsinchu, Taiwan, Republic of China

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摘　　要

這個研究主要是要去計算三維度兩個顏色的熵，但首先必須利用有序矩陣以及矩陣自乘的性質所發展出來的遞迴公式去解決三維度兩個顏色下面著色的花樣生成問題。

接下來，給一個限制集則就可以定義出轉移矩陣而且它的遞迴公式也會被表現出來。最後，只需去計算矩陣的最大特徵值即可計算出熵的問題。

Spatial Entropy of 3- dimensional Face Coloring

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ABSTRACT

The work investigates spatial Entropy of 3-dimensional face coloring, but we need to solve three-dimensional pattern generation problem with edge- coloring by using the properties of ordering and self-multiply matrices to establish some recursive formulas, first.

Now, given admissible set of local patterns then the transition matrix is defined and the recursive formulas are presented. Finally the spatial entropy is obtained by computing the maximum eigenvalues of a sequence of transition matrices.

致　　謝

首先，很感謝指導教授— 林松山教授 一路上無論是研究心態、待人之道、人生道理總是耳提面命，好讓我這一路上走得很順遂，這是有錢也買不到的真心對待，所以身為學生的我心裡很感激。

其次，要感謝的是胡文貴學長的幫忙，每次學長都會很有耐心的與我討論問題，有時也會以過來人的角色與我經驗分享，這些都讓我受益良多。

接著，很感激我的家人一路上的支持與鼓勵，因為有了他(她)們的陪伴與鼓勵，這兩年我不孤單。

最後，要謝謝交大一起打排球的朋友們，因為排球所以才有機會在這麼短的時間內認識這們多不同科系的人，也因為打排球這兩年才過的這麼精彩，真的很謝謝他(她)們。

目　　錄

1 Introduction

Here, we consider the problem of 2 symbols, then will get the set of all patterns Σ_2^3, first, give any admissible set $B \subseteq \Sigma_2^3$ and denote $\Sigma_2^3(B)$ be Σ_2^3 which is restricted in B, secondly, denote $\Gamma_{m\times n\times k}(B)$ is the quality of $\Sigma_2^3(B)$, and finally, need to calculate spatial entropy of 3-dimensional face coloring,

$$h(B) = \lim_{m,n,k\to\infty} \frac{\log \Gamma_{m\times n\times k}(B)}{mnk} \tag{1.1}$$

clearly, how to calculate $\Gamma_{m\times n\times k}(B)$ is the first problem we encountered from the equation (1.1). In order to solve the problem we face, we must study the problem of 3-dimensional pattern generation of 2 symbols in section 2 and find a way to control the colors of different directions by the matrix, Y_2.
Now, split section 2 into 3 steps as following:

Step 1 : find the recursive formula, $Y_{2\times n\times 2}$, of y-direction by $Y_{2\times 2\times 2}$, for $n \geq 3$
Step 2 : denote $X_{2\times n\times 2} \equiv Y_{2\times n\times 2}$ and find the recursive formula , $X_{m\times n\times 2}$ by $X_{2\times n\times 2}$, for $n \geq 3$
Step 3 : denote $Z_{m\times n\times 2} \equiv Y_{m\times n\times 2}$ and we will get $Z_{m\times n\times k}$ by $Z_{m\times n\times 2}$ which is self-multiply.

In section 3, we defined $T_{y;2\times 2\times 2;i_y}$ as the transition matrix of $Y_{2\times 2\times 2;i_y}$, for $1 \leq i_y \leq 4$ and find that the main problem will be converted into finding $\Gamma_{z;m\times n\times 2}(B)$ by Peronn Fubini's theorem. Finally, using the result to calculus the entropy of (1.1) where the details will be presented in theorem 1.

2 Three-Dimensional Pattern Generation Problems

This section describes three-dimensional pattern generation problem. Here $m, n, k \geq 2$ are fixed and indices for brevity. Let S be a set of p colors, and $\mathbf{Z}_{m\times n\times k}$ be a fixed finite rectangular sublattice of $\mathbf{Z}^3$, where $\mathbf{Z}^3$ denotes the integer lattice on $\mathbb{R}^3$ and (m, n, k) be a three-tuple of positive integer. Function $U : \mathbf{Z}^3 \longrightarrow S$ and $U_{m\times n\times k} : \mathbf{Z}_{m\times n\times k} \longrightarrow S$ are called global patterns and locally patterns respectively. The set of all patterns U is denoted by $\Sigma_p^3 \equiv S^{\mathbf{Z}^3}$, such that$\Sigma_p^3$ is the set of all patterns with p different colors in a three-dimensional lattice. For clarity, two symbols, $S = \{0, 1\}$ are considered. Let x, y and z coordinate represent 1st-, 2st- and 3st-coordinates respectively as in Fig.1. Six orderings $[w]$ ordering are represented as the following:

$$\begin{aligned}
&[x] : [1] \succ [2] \succ [3] \\
&[y] : [2] \succ [1] \succ [3] \\
&[z] : [3] \succ [1] \succ [2] \\
&[\widehat{x}] : [1] \succ [3] \succ [2] \\
&[\widehat{y}] : [2] \succ [3] \succ [1] \\
&[\widehat{z}] : [3] \succ [2] \succ [1]
\end{aligned} \tag{2.1}$$

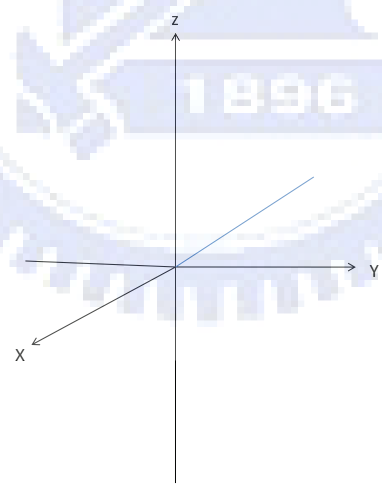

Figure 1. Three-dimension coordinate system.

On a fixed lattice $Z_{m\times n\times k}$, an ordering $[w] \succ [j] \succ [k]$ is obtained on $Z_{m\times n\times k}$, which is any one of the above ordering on $Z_{m\times n\times k}$. Therefore, the six ordering of $Z_{2\times 2\times 2}$ are presented as Fig.2, where $\alpha_i = \{0, 1\}$.

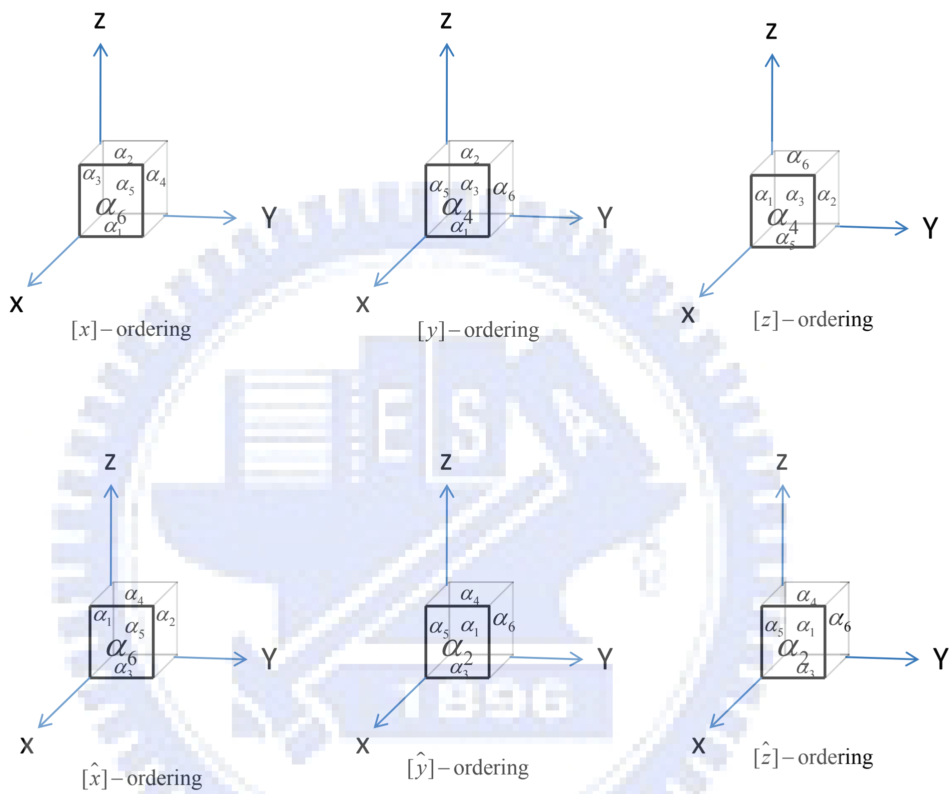

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Figure 2.

2.1 Ordering Matrices

By using the six forms can get six different ordering matrices of $Z_{2\times2\times2}$ and denote the order of matrices as equation (2.2)

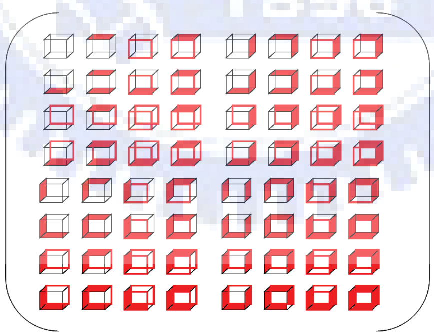
$$i_\alpha = 1 + \sum_{i=1}^{6} \alpha_i 2^{6-i}, \text{ where } \alpha_i \in \{0, 1\}. \tag{2.2}$$

Here, we choose [z] as the order for convenience, and we can denote the order of x, y and z-directions by i_x, i_y and i_z (2.3) respectively, where $1 \leqslant i_x, i_y$ and $i_z \leq 4$.

$$\begin{cases} i_x = & 1 + \alpha_2 + \alpha_1 \times 2 \\ i_y = & 1 + \alpha_4 + \alpha_3 \times 2 \\ i_z = & 1 + \alpha_6 + \alpha_5 \times 2 \end{cases} \tag{2.3}$$

For convenience again, we have to define the matrix $Y_{2\times2\times2}$ (2.4) below which present the relation between colors and each directions.

$\mathrm{Y}_{2\times2\times2} =$



$$= [\mathrm{y}_{2\times2\times2;i;j}]_{2^3\times2^3} = \begin{bmatrix} \mathrm{Y}_{2\times2\times2;1} & \mathrm{Y}_{2\times2\times2;2} \\ \mathrm{Y}_{2\times2\times2;3} & \mathrm{Y}_{2\times2\times2;4} \end{bmatrix}$$

(2.4)

It's not different to discover that the colors of each direct of $Z_{2\times2\times2}$ be controlled in each layer of $Y_{2\times2\times2}$ respectively. It means that $Y_{2\times2\times2}$ is divided into three layers by matrix partitioning as figure 3 and the colors of y-, x-and z-direction are controlled in first, second and the third layer respectively as figure 3 below.

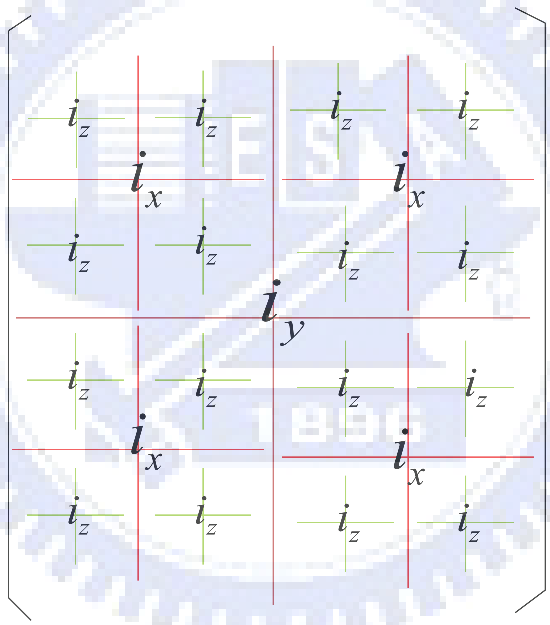


Figure 3. relation between colors and layers.

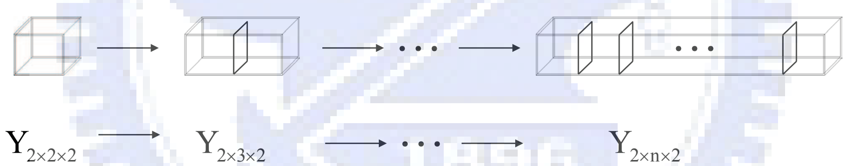
The process of investigating the pattern generation problem should be broken down in the following steps:

Step 1. find the recursive formula $Y_{2\times2\times2} \rightarrow Y_{2\times3\times2} \rightarrow \ldots \rightarrow Y_{2\times n\times2}$, that is extend on the y-direction.

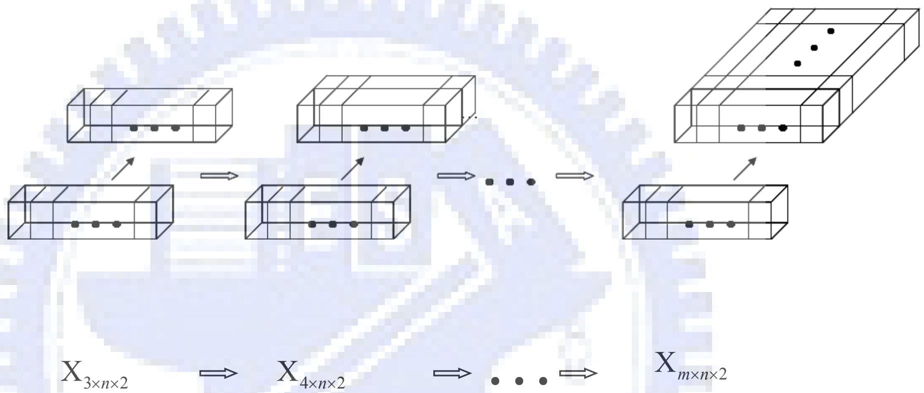
Step 2. here replaces $X_{2\times n\times2}$ with $Y_{2\times n\times2}$, it means that $X_{2\times n\times2}$ is really to extend the x-direction and we get the recursive formula like $X_{2\times n\times2} \rightarrow X_{3\times n\times2} \rightarrow \ldots \rightarrow X_{m\times n\times2}$ that is extend the x-direction by $Y_{2\times n\times2}$.

Step 3. here replaces $Z_{m\times n\times2}$ with $X_{m\times n\times2}$, it means that $X_{m\times n\times2}$ is really to extend the z – direction. By using the matrix to self-multiply, we can generate $Z_{m\times n\times2} \rightarrow Z_{m\times n\times3} \rightarrow \ldots \rightarrow Z_{m\times n\times k}$, that is extend the z-direction by $X_{m\times n\times2}$.

STEP 1.

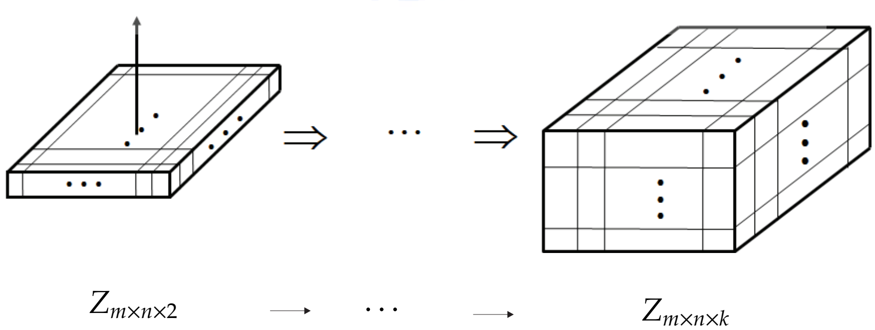


STEP 2.



$X_{3\times n\times 2}$
$X_{4\times n\times 2}$
$X_{m\times n\times 2}$

STEP 3.



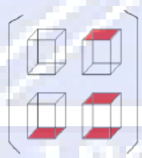
$Z_{m\times n\times 2}$
$Z_{m\times n\times k}$

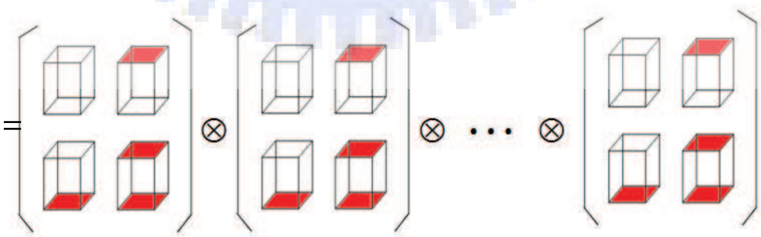
Proposition 2.1.

(1) If we only consider the colers of the z-direction of those matrices, $Z_{2\times2\times2;1}$, $Z_{2\times2\times2;2}$, $Z_{2\times2\times2;3}$ *and* $Z_{2\times2\times2;4}$ *with ignoring x- and y- directions, we can represent this scenario as* $\overline{Z}_2$. *Moreover, we denote* $\overline{Z}_n = \overline{Z}_2 \otimes \overline{Z}_2 \otimes \ldots \otimes \overline{Z}_2$ *and discover that* $\overline{Z}_n$ *is self-multiply, where*

$$Z_{2\times2\times2} = \begin{pmatrix} Z_{2\times2\times2;1} & Z_{2\times2\times2;2} \\ Z_{2\times2\times2;3} & Z_{2\times2\times2;4} \end{pmatrix},$$

$$Z_{2\times2\times2;l} = \begin{bmatrix} \left(Y^{(re_z)}_{2\times2\times2;1;l} \right)_{4\times4} & \left(Y^{(re_z)}_{2\times2\times2;2;l} \right)_{4\times4} \\ \left(Y^{(re_z)}_{2\times2\times2;3;l} \right)_{4\times4} & \left(Y^{(re_z)}_{2\times2\times2;4;l} \right)_{4\times4} \end{bmatrix}_{2\times2}, 1 \le l \le 4,$$

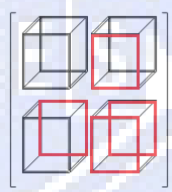
$\bar{Z}_2 =$  ,

and $\bar{Z}_n =$ 

(2) As the ideal of (1) above, we consider the colors of the x-direction of those matrices, $X_{2\times2\times2;1}$, $X_{2\times2\times2;2}$, $X_{2\times2\times2;3}$ and $X_{2\times2\times2;4}$ with ignoring y- and z- directions, then we can represent this scenario as $\overline{X}_2$. Moreover, we define $\overline{X}_n = \overline{X}_2 \otimes \overline{X}_2 \otimes \ldots \otimes \overline{X}_2$ and discover that $\overline{X}_n$ is a matrix which control the colors of the front and rear faces by the columns and rows respectively, where

$$X_{2\times2\times2} = \begin{pmatrix} X_{2\times2\times2;1} & X_{2\times2\times2;2} \\ X_{2\times2\times2;3} & X_{2\times2\times2;4} \end{pmatrix};$$

$$X_{2\times2\times2;l} = \begin{bmatrix} \left(Y^{(re_z)}_{2\times2\times2;1;l} \right)_{4\times4} & \left(Y^{(re_z)}_{2\times2\times2;2;l} \right)_{4\times4} \\ \left(Y^{(re_z)}_{2\times2\times2;3;l} \right)_{4\times4} & \left(Y^{(re_z)}_{2\times2\times2;4;l} \right)_{4\times4} \end{bmatrix}_{2\times2}, 1 \le l \le 4$$

$\overline{X}_2 =$ 

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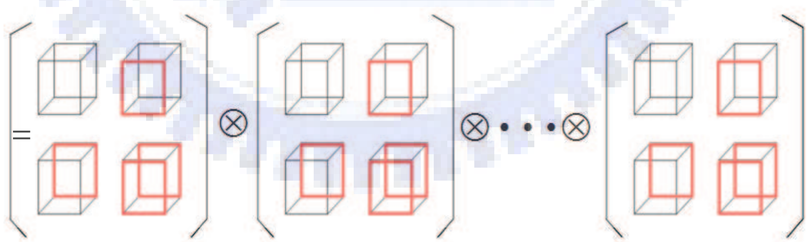
and $\overline{X}_{2;n} =$ 

Example 2.2. As $n = 3$

(1) We focus on the colors of z-direction, and the order be presented as $\mathbf{Z}_2$:

$$\overline{Z}_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ & \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$$

Now, consider

$$\overline{Z}_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ & \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ & \\ 0 & 1 \\ 1 & 1 \end{pmatrix} = \left(\begin{array}{cc|cc|cc|cc} 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right)$$

Here $\overline{Z}_3$ is a self-multiply matrix.

(2) We focus on the colors of x-direction, and the order be presented as $\mathbf{X}_2$:

$$\overline{X}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Now, consider

$$\overline{X}_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} = \left(\begin{array}{cc|cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ \hline 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ \hline 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{array}\right)$$

Here $\overline{X}_3$ is a matrix which controls the the colors of front and rear faces by columns and rows respectively.

Now, we talk about the details of those steps. First, the recursive formula can be developed by using the properties of $Y_{2\times2\times2}$ as the following:

$$Y_{2\times3\times2} = \begin{pmatrix} Y_{2\times2\times2;1}\otimes Y_{2\times2\times2;1} & Y_{2\times2\times2;1}\otimes Y_{2\times2\times2;2} \\ +Y_{2\times2\times2;2}\otimes Y_{2\times2\times2;3} & +Y_{2\times2\times2;2}\otimes Y_{2\times2\times2;4} \\ & \\ Y_{2\times2\times2;3}\otimes Y_{2\times2\times2;1} & Y_{2\times2\times2;3}\otimes Y_{2\times2\times2;2} \\ +Y_{2\times2\times2;4}\otimes Y_{2\times2\times2;3} & +Y_{2\times2\times2;4}\otimes Y_{2\times2\times2;4} \end{pmatrix}_{2\times2}$$

$$= \begin{pmatrix} Y_{2\times3\times2;1} & Y_{2\times3\times2;2} \\ Y_{2\times3\times2;3} & Y_{2\times3\times2;4} \end{pmatrix}_{2\times2},$$

where $Y_{2\times3\times2;l}$ is $4^2\times4^2$ matrix, $1\le l\le4$.

$$Y_{2\times n\times2} = \begin{pmatrix} Y_{2\times2\times2;1}\otimes Y_{2\times n-1\times2;1} & Y_{2\times2\times2;1}\otimes Y_{2\times n-1\times2;2} \\ +Y_{2\times2\times2;2}\otimes Y_{2\times n-1\times2;3} & +Y_{2\times2\times2;2}\otimes Y_{2\times n-1\times2;4} \\ & \\ Y_{2\times2\times2;3}\otimes Y_{2\times n-1\times2;1} & Y_{2\times2\times2;3}\otimes Y_{2\times n-1\times2;2} \\ +Y_{2\times2\times2;4}\otimes Y_{2\times n-1\times2;3} & +Y_{2\times2\times2;4}\otimes Y_{2\times n-1\times2;4} \end{pmatrix}_{2\times2} \tag{2.5}$$

$$= \begin{pmatrix} Y_{2\times n\times2;1} & Y_{2\times n\times2;2} \\ Y_{2\times n\times2;3} & Y_{2\times n\times2;4} \end{pmatrix}_{2\times2},$$

where $Y_{2\times n\times2;l}$ is a $4^{n-1}\times4^{n-1}$ matrix, $1\le l\le4$.

Secondly, we denote $X_{2\times n\times2}=\sum_{i_y=1}^{4}Y_{2\times n\times2;i_y}$ and find that the order of the colors of each layers which be controlled by $X_{2\times n\times2}$ as $i_{z_1}\to i_{x_1}\to i_{z_2}\to i_{x_2}\to\ldots\to i_{z_{n-2}}\to i_{x_{n-2}}\to i_{z_{n-1}}\to i_{x_{n-1}}$, for $n\ge3$, so need to rearrange the order as $i_{z_1}\to i_{z_2}\to\cdots\to i_{z_{n-2}}\to i_{x_1}\to i_{x_2}\to\cdots\to i_{x_{n-2}}\to i_{z_{n-1}}\to i_{x_{n-1}}$. For fixing any z and it's the fact that change the view of x- into y-direction by property 1 ,where the method of the detail will be presented by items below:

- (1.) Rearrange i_{z_1}

$$(X_{2\times n\times 2}^{(r_{i_{z_2}})})_{i_{z_1}} = (X_{2\times n\times 2})_{i x_1}; i_{z_1}$$

- (2.) Rearrange i_{z_2}, for i_{z_1}

$$(X_{2\times n\times 2}^{(r_{i_{z_2}})})_{i_{z_1};i_{z_2}} = (X_{2\times n\times 2})_{i_{z_1};i_{x_1};i_{z_2}}$$

- (3.) Rearrange i_{z_3}, for i_{z_1}, i_{z_2}

$$(X_{2\times n\times 2}^{(r_{i_{z_2};i_{z_3}})})_{i_{z_1};i_{z_2};i_{z_3}} = (X_{2\times n\times 2}^{(r_{i_{z_2}})})_{i_{z_1};i_{z_2};i_{x_1};i_{x_2};i_{z_3}}$$

$$\vdots$$

- (n-1.) Rearrange $i_{z_{n-1}}$, for $i_{z_1}, i_{z_2}, \cdots, i_{z_{n-2}}$

$$(X_{2\times n\times 2}^{r_{i_{z_1};i_{z_2};\cdots;i_{z_{n-1}}}})_{i_{z_1};i_{z_2};\cdots;i_{z_{n-1}}}$$

$$= (X_{2\times n\times 2}^{(re_{i_{z_2};i_{z_3};\cdots;i_{z_{n-2}}})})_{i_{z_1};i_{z_2};\cdots;i_{z_{n-2}};i_{x_1};i_{x_2};\cdots;i_{x_{n-1}};i_{z_{n-1}}}$$

here $(X_{2\times n\times 2}^{(re_{i_{z_2};i_{z_3};\ldots;i_{z_{n-2}}})})_{i_{z_1};i_{z_2};\ldots;i_{z_{n-2}}}$ is abbreviated as $(X_{2\times n\times 2}^{r})_{z}$, for $n \geq 4$.
For convenience, we need to define some forms of matrices as the orders below, before entering the second step:

1. For any z, $(X^r_{2\times n\times 2})_z$ controls the orders i_1, i_2 by rows and columns respectively, where i_2 and i_1 are the orders of the front and rear faces respectively, $1 \leq i_1, i_2 \leq 2^{n-1}$.

$$[(X^r_{2\times n\times 2})_z]_{2^{n-1}\times 2^{n-1}} = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,2^{n-1}} \\ a_{2,1} & & & a_{2,2^{n-1}} \\ \vdots & & \ddots & \vdots \\ a_{2^{n-1},1} & \cdots & & a_{2^{n-1},2^{n-1}} \end{pmatrix}_{2^{n-1}\times 2^{n-1}}$$

Now, the order becomes $z_1 \to z_2 \to \cdots z_{n-2} \to z_{n-1} \to i_1 \to i_2$ from the previous discussion, but is not well enough to control the order, so we still need to complete the following items as below before entering second step.

2. For any z and i_1, we denote $[(X^r_{2,n})_{z;i_1}]^{(r_{i_2})}$.

$$[(X^r_{2\times n\times 2})_{z;i_1}]^{(r_{i_2})} = \begin{pmatrix} a_{i_1,1} & \cdots & a_{i_1,2^{k_2}} \\ \vdots & \ddots & \vdots \\ a_{i_1,(2^{k_1}-1)(2^{k_2})+1} & \cdots & a_{i_1,2^{k_1+k_2}} \end{pmatrix}_{2^{k_1}\times 2^{k_2}},$$

$$\text{where} \begin{cases} k_1 = \frac{n-1}{2}, k_2 = \frac{n-1}{2} & \text{, n is odd.} \\ k_1 = \frac{n}{2}, k_2 = \frac{n-2}{2} & \text{, n is even.} \end{cases}$$

3. To assign the seats of i_1 is the goal in this item.

$[(X^r_{2\times n\times 2})_z]^{(r_{i_2})} =$

$$\begin{pmatrix} [(X^r_{2\times n\times 2})_{z;1}]^{(r_{i_2})}[(X^r_{2\times n\times 2})_{z;2}]^{(r_{i_2})}\cdots & [(X^r_{2\times n\times 2})_{z;2^{2k_2}}]^{(r_{i_2})} \\ \vdots & \vdots \\ \left[(X^r_{2\times n\times 2})_{z;(2^{k_1}-1)(2^{k_2})+1}\right]^{(r_{i_2})}\cdots & \left[(X^r_{2\times n\times 2})_{z;2^{k_1+k_2}}\right]^{(r_{i_2})} \end{pmatrix}_{2^{k_1}\times 2^{k_2}},$$

$$\forall z \text{ ,where } \begin{cases} k_1 = \frac{n-1}{2}, k_2 = \frac{n-1}{2} & \text{n is odd.} \\ k_1 = \frac{n-2}{2}, k_2 = \frac{n}{2} & \text{n is even.} \end{cases}$$

Here, we denote the process from (1) to (3) as $(X^r_{2;n})^{(r_{i_2})}$ which controls the color of each direction in each layer respectively, $i_{z_1} \to i_{z_2} \to \cdots \to i_{z_{n-2}} \to i_{z_{n-1}} \to i_1 \to i_2$. For convenience again, we need to rearrange i_1 and i_2 as the define as following:

- $[(X^r_{2\times n\times 2})^{(r_{i_2})}]^{(r_{i_1})}$ is called rearrangement matrix of i_1 of $(X^r_{2\times n\times 2})^{(r_{i_2})}$, if $([(X^r_{2\times n\times 2})^{(r_{i_2})}]^{(r_{i_1})})_{i_1;z} = [(X^r_{2\times n\times 2})^{(r_{i_2})}]_{z;i_1}$, for $1 \le i_1, i_{z_l} \le 4, 1 \le l \le n-2$.

- $([(X^r_{2\times n\times 2})^{(r_{i_2})}]^{(r_{i_1})})^{(r_{i_2})}$ is called rearrangement matrix of i_2 of $[(X^r_{2\times n\times 2})^{(r_{i_2})}]^{r(i_1)}$ if $([(X^r_{2\times n\times 2})^{(r_{i_2})}]^{(r_{i_1})})^{(r_{i_2})}_{i_1;i_2;z} = [(X^r_{2\times n\times 2})^{(r_{i_2})}]^{(re_{i_1})}_{i_1;z;i_2}$ for $1 \le i_1, i_2, i_{z_l} \le 4, 1 \le l \le n-2, n \ge 3$.

By doing step 2, we need to denote $X^{(a)}_{2\times n\times 2} = ([(X^r_{2\times n\times 2})^{(r_{i_2})}]^{(r_{i_1})})^{(r_{i_2})}$ and $X^{(b)}_{2\times n\times 2} = \sum_{i_2=1}^{2^{n-1}} X^{(a)}_{2\times n\times 2;i_1;i_2}$ to develop the recursive formula of $X^{(b)}_{m,n}$ as following (2.6), (2.7):

$$X^{(2)}_{m\times n\times 2;i_1;i_2} = X^{(a)}_{2\times n\times 2;i_1;i_2} \otimes X^{(b)}_{m-1\times n\times 2;i_2} \tag{2.6}$$

$$X^{(b)}_{m\times n\times 2;i_1} = \sum_{i_2=1}^{2^{n-1}} X^{(2)}_{m\times n\times 2;i_1;i_2}, \text{ for all } m \ge 3. \tag{2.7}$$

Finally, denote $Z_{m\times n\times 2} = \sum_{i_1=1}^{2^{n-1}} X^{(b)}_{m\times n\times 2;i_1}$ which records the colors of the z-direction as $i_{z_1} \rightarrow i_{z_2} \rightarrow \ldots \rightarrow i_{z_{n-1}}$ and we know that $Z_{m\times n\times 2}$ is a self-multiply matrix by property 1, therefore $Z_{m\times n\times k} = Z^{k-1}_{m\times n\times 2}$. Here, we give a method to solve the problem of 3-d pattern generation.

3 Transition Matrices and Spatial Entropy

3.1 Transition Matrices

Based on the process of the ordering matrix, we have to define transition matrix as the following :

1. Given an admissible set $B \subseteq \Sigma^{z_3}_{2\times 2\times 2}$.

2.Define

$$\begin{cases} t_{y;2\times 2\times 2;i} = 1 & , y_{2\times 2\times 2;i;j;k} \in B \\ t_{y;2\times 2\times 2;i} = 0 & , y_{2\times 2\times 2;i;j;k} \notin B \end{cases}$$

3. Define $T^r_{y_2;i_y;i_z;i_x} = T_{y_2;i_y;i_x;i_z}$,where $1 \leq i_y; i_x; i_z \leq 4$.

4. The recursive formula for y-direction is as following:

$$T^r_{y;2\times 3\times 2} = \begin{pmatrix} \begin{matrix} T_{y;2\times 2\times 2;1} \otimes T_{y;2\times 2\times 2;1} \\ +T_{y;2\times 2\times 2;2} \otimes T_{y;2\times 2\times 2;3} \end{matrix} & \begin{matrix} T_{y;2\times 2\times 2;1} \otimes T_{y;2\times 2\times 2;2} \\ +T_{y;2\times 2\times 2;2} \otimes T_{y;2\times 2\times 2;4} \end{matrix} \\ & \\ \begin{matrix} T_{y;2\times 2\times 2;3} \otimes T_{y;2\times 2\times 2;1} \\ +T_{y;2\times 2\times 2;4} \otimes T_{y;2\times 2\times 2;3} \end{matrix} & \begin{matrix} T_{y;2\times 2\times 2;3} \otimes T_{y;2\times 2\times 2;2} \\ +T_{y;2\times 2\times 2;4} \otimes T_{y;2\times 2\times 2;4} \end{matrix} \end{pmatrix}_{2\times 2}$$

$$= \begin{pmatrix} T_{y;2\times 3\times 2;1} & T_{y;2\times 3\times 2;2} \\ T_{y;2\times 3\times 2;3} & T_{y;2\times 3\times 2;4} \end{pmatrix}_{2\times 2},$$

where $T_{y;2\times3\times2;l}$ is $4^2\times4^2 matrix, 1\le l\le 4.$

$$T_{y;2\times n\times 2} = \begin{pmatrix} T_{y;2\times2\times2;1}\otimes T_{y;2\times n-1\times2;1} & T_{y;2\times2\times2;1}\otimes T_{y;2\times n-1\times2;2} \\ +T_{y;2\times2\times2;2}\otimes T_{y;2\times n-1\times2;3} & +T_{y;2\times2\times2;2}\otimes T_{y;2\times n-1\times2;4} \\ & \\ T_{y;2\times2\times2;3}\otimes T_{y;2\times n-1\times2;1} & T_{y;2\times2\times2;3}\otimes T_{y;2\times n-1\times2;2} \\ +T_{y;2\times2\times2;4}\otimes T_{y;2\times n-1\times2;3} & +T_{y;2\times2\times2;4}\otimes T_{y;2\times n-1\times2;4} \end{pmatrix}_{2\times2}$$

$$= \begin{pmatrix} T_{y;2\times n\times2;1} & T_{y;2\times n\times2;2} \\ T_{y;2\times n\times2;3} & T_{y;2\times n\times2;4} \end{pmatrix}_{2\times2},$$

where $Y_{y;2\times n\times2;l}$ is $4^{n-1}\times4^{n-1} matrix, 1\le l\le 4.$

5. Rearrangement.

- (1.) Rearrange i_{z_1}

$$(T^{(r_{i_{z_2}})}_{x;2\times n\times2})_{i_{z_1}} = (T_{x;2\times n\times2})_{ix_1}; i_{z_1}$$

- (2.) Rearrange i_{z_2}, for i_{z_1}

$$(T^{(r_{i_{z_2}})}_{x;2\times n\times2})_{i_{z_1};i_{z_2}} = (T_{x;2\times n\times2})_{i_{z_1};i_{x_1};i_{z_2}}$$

- (3.) Rearrange i_{z_3}, for i_{z_1}, i_{z_2}

$$(T^{(r_{i_{z_2};i_{z_3}})}_{x;2\times n\times2})_{i_{z_1};i_{z_2};i_{z_3}} = (T^{(r_{i_{z_2}})}_{x;2\times n\times2})_{i_{z_1};i_{z_2};i_{x_1};i_{x_2};i_{z_3}}$$

$$\vdots$$

- (n-1.) Rearrange $i_{z_{n-1}}$, for $i_{z_1}, i_{z_2}, \cdots, i_{z_{n-2}}$

$$(T^{r_{i_{z_1};i_{z_2};\cdots;i_{z_{n-1}}}}_{x;2\times n\times2})_{i_{z_1};i_{z_2};\cdots;i_{z_{n-1}}}$$

$$= (T^{(re_{i_{z_2};i_{z_3};\cdots;i_{z_{n-2}}})}_{x;2\times n\times2})_{i_{z_1};i_{z_2};\cdots;i_{z_{n-2}};i_{x_1};i_{x_2};\cdots;i_{x_{n-1}};i_{z_{n-1}}}$$

Here $(T^{(r_{i_{z_2};i_{z_3};\ldots;i_{z_{n-2}}})}_{x;2\times n\times2})_{i_{z_1};i_{z_2};\ldots;i_{z_{n-2}}}$ is abbreviated as $(T^{r}_{x;2\times n\times2})_{z}$.

6. For convenience, we need to define some forms of matrices as the following:

(1)

$$[(T^r_{x;2\times n\times 2})_z]_{2^{n-1}\times 2^{n-1}} = \begin{pmatrix} t_{1,1} & t_{1,2} & \cdots & t_{1,2^{n-1}} \\ t_{2,1} & & & t_{2,2^{n-1}} \\ \vdots & & \ddots & \vdots \\ t_{2^{n-1},1} & \cdots & & t_{2^{n-1},2^{n-1}} \end{pmatrix}_{2^{n-1}\times 2^{n-1}}$$

$\forall z$, here the colors of the front and rear faces are represent as i_2 and i_1 respectively,

where $1 \leq i_1, i_2 \leq 2^{n-1}$.

(2)

$$[(T^r_{x;2\times n\times 2})_{z;i_1}]^{(r_{i_2})} = \begin{pmatrix} t_{i_1,1} & \cdots & t_{i_1,2^{k_2}} \\ \vdots & \ddots & \vdots \\ t_{i_{1,(2^{k_1}-1)(2^{k_2})+1}} & \cdots & t_{i_1,2^{k_1+k_2}} \end{pmatrix}_{2^{k_1}\times 2^{k_2}},$$

$$\forall z, i_1 \text{ ,where } \begin{cases} k_1 = \frac{n-1}{2}, k_2 = \frac{n-1}{2} & \text{and n is odd.} \\ k_1 = \frac{n}{2}, k_2 = \frac{n-2}{2} & \text{and n is even.} \end{cases}$$

(3)

$[(T^r_{x;2\times n\times 2})_z]^{(r_{i_2})} =$

$$\begin{pmatrix} [(T^r_{x;2\times n\times 2})_{z;1}]^{(r_{i_2})} & [(T^r_{x;2\times n\times 2})_{x;2\times n\times 2}]^{(r_{i_2})} & \cdots & [(T^r_{x;2\times n\times 2})_{z;2^{2k_2}}]^{(r)} \\ \vdots & & \ddots & \vdots \\ \left[(T^r_{x;2\times n\times 2})_{z;(2^{k_1}-1)(2^{k_2})+1}\right]^{(r_{i_2})} & & \cdots & \left[(T^r_{x;2\times n\times 2})_{z;2^{k_1+k_2}}\right]^{(r_{i_2})} \end{pmatrix}_{2^{k_1}\times 2^{k_2}},$$

$$\forall z \text{ ,where } \begin{cases} k_1 = \frac{n-1}{2}, k_2 = \frac{n-1}{2} & \text{and n is odd.} \\ k_1 = \frac{n-2}{2}, k_2 = \frac{n}{2} & \text{and n is even.} \end{cases}$$

7.Rearrange i_1, and i_2 as the define as following:

$[(T^r_{x;2\times n\times 2})^{(r_{i_2})}]^{(r_{i_1})}$ is called rearrangement matrix of i_1 of $(T^r_{2;n})^{(r_{i_2})}$,
if $([(T^r_{x;2\times n\times 2})^{(r_{i_2})}]^{(re_{i_1})})_{i_1;z} = [(T^r_{x;2\times n\times 2})^{(r_{i_2})}]_{z;i_1}$, for $1 \le i_1, i_{z_l} \le 4, 1 \le l \le n-2$.

8. Rearrange i_2 as the define as following:

$([(T^r_{x;2\times n\times 2})^{(r_{i_2})}]^{(re_{i_1})})^{(re_{i_2})}$ is called rearrangement matrix of i_2 of $[(T^r_{x;2\times n\times 2})^{(r_{i_2})}]^{re_{(i_1)}}$
if $([(T^r_{x;2\times n\times 2})^{r}]^{(re_{i_1})}))^{(re_{i_2})}_{i_1;i_2;z} = [(T^r_{x;2\times n\times 2})^{(r_{i_2})}]^{re_{i_1}}_{i_1;z;i_2}$
for $1 \le i_1, i_2, i_{z_l} \le 4, 1 \le l \le n-2, n \ge 3$.

9.

$T^{(a)}_{x;2\times n\times 2} = ([(T^r_{2\times n\times 2})^{(r_{i_2})}]^{re_{(i_1)}})^{re_{(i_2)}}$ and $T^{(b)}_{x;2\times n\times 2} = \sum_{i_2=1}^{2^{n-1}} T^{(a)}_{x;2,n;i_1;i_2}$
to develop the recursive formula of $T^{(b)}_{x;m,n}$ as following:

$T^{(2)}_{x;m\times n\times 2;i_1;i_2} = T^{(a)}_{x;2\times n\times 2;i_1;i_2} \otimes T^{(b)}_{x;m-1\times n\times 2;i_2}$ and $T^{(b)}_{x;m\times n\times 2;i_1} = \sum_{i_2=1}^{2^{n-1}} T^{(2)}_{x;m\times n\times 2;i_1;i_2}, \forall m \ge 3$.

10.

$T_{z;m\times n\times 2} = \sum_{i_1=1}^{2^{n-1}} T^{(b)}_{x;m\times n\times 2;i_1}$ which record the colors of the z-direction as $i_{z_1} \to i_{z_2} \to \ldots \to i_{z_{n-1}}$, by property 1. $T_{z;m\times n\times 2}$ is self-multiply, therefore $T_{z;m\times n\times k} = T^{k-1}_{z;m\times n\times 2}$.

Theorem 3.1. *Let $\lambda_{T_{z;m\times n\times 2}}$ be the maximum eigenvalue of $T_{z;m\times n\times 2}$, then*

$$h(B) = \lim_{m,n\to\infty} \frac{\log \lambda_{T_{z;m\times n\times 2}}}{mn}$$

Proof. By the same arguments as in [Chow et al., 1996a], the limit Eq. (1) is well-defined and exists.
From

$$T_{z;m\times n\times k}(B) = \sum_{1\le i,j\le 2^{(m-1)(n-1)}} (T_{z;m\times n\times k})_{i,j} = |(T^{k-1}_{z;m\times n\times 2}B)|$$

As in the one-dimensional case,

$$\lim_{m\to\infty} \frac{\log |(T^{k-1}_{z;m\times n\times 2})|}{m} = \log \lambda_{T_{z;m,n,k}},$$

as for example[Ban Lin,2005]. Hence,

$$\begin{aligned} h(B) = & \lim_{m,n,k\to\infty} \frac{\log |T^{k-1}_{z;m\times n\times 2}|}{mnk} \\ = & \lim_{m,n\to\infty} \frac{1}{mn} \lim_{k\to\infty} \frac{\log |T^{k-1}_{z;m\times n\times 2}|}{k} \\ = & \lim_{m,n\to\infty} \frac{\log \lambda_{T_{z;m\times n\times 2}}}{mn} \end{aligned}$$

3.2 Computation of $\lambda_{Tz;m\times n\times 2}$ and entropy

Let

$$G = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \;,\; E = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Here, $\rho(E) = 2$, $\rho(G) = g$, where $g = \frac{1+\sqrt{5}}{2}$.

Let $T_{i_y;i} = E \otimes G$ and $T^r_{i_y;i} = G \otimes E$, then we will get the following formula:

1. $T^r_{y;2\times n\times 2;i_y} = 2^{(n-2)}(G \otimes E)^{(n-2)} \otimes (E \otimes G)$.

2. $T^r_{x;2\times n\times 2} = \sum_{i_y=1}^{4} T_{y;n;i_y} = 2^n(G \otimes E)^{(n-2)} \otimes (E \otimes G)$.

3. $T^r_{x;2\times n\times 2} = 2^n(G)^{n-2} \otimes (E)^{n-1} \otimes G$.

4. $(T^r_{x;2\times n\times 2})^{(r_{i_2})} = 2^n[(G)^{n-2} \otimes (E)^{n-1} \otimes G]$.

5.

$$[(T^r_{x;2\times n\times 2})^{(r_{i_2})}]^{(re_{i_1})} = \begin{cases} 2^n[(E)^{\frac{n-1}{2}} \otimes (G)^{n-2} \otimes (E)^{\frac{n-1}{2}} \otimes (G)^{n-2}] & ,n \text{ is odd.} \\ 2^n[E_{2^{\frac{n-2}{2}}\times 2^{\frac{n}{2}}} \otimes (G)^{n-2} \otimes E_{2^{\frac{n}{2}}\times 2^{\frac{n-2}{2}}} \otimes G] & ,\text{n is even.} \end{cases}$$

6. $([(T^r_{x;2\times n\times 2})^{(r_{i_2})}]^{(re_{i_1})})^{(re_{i_2})} = 2^n[(E)^{n-1} \otimes (G)^{n-1}]$, $n \geq 3$.

7. $(T^{(a)}_{x;2\times n\times 2})_{i_1;i_2} = 2^n(G)^{n-1}$.

8. $(T^{(b)}_{x;m\times n\times 2})_{i_1} = \sum_{i_2=1}^{2^{n-1}} (T^{(a)}_{x;2\times n\times 2})_{i_1;i_2} = 2^{(m-1)(2n-1)}(G)^{(m-1)(n-1)}$.

9. $T_{z;m\times n\times 2} = \sum_{i_1=1}^{2^{n-1}} (T^{(b)}_{x;m\times n\times 2})_{i_1} = 2^{(m-1)n+m(n-1)}(G)^{(m-1)(n-1)}$.

10. $T_{z;m\times n\times k} = T^{k-1}_{z;m\times n\times 2} = 2^{(k-1)(m-1)n+m(n-1)}G^{(m-1)(n-1)(k-1)}$.

Now, we calculate the entropy ,

$$
\begin{aligned}
h(B) = & \lim_{m,n\to\infty}\frac{1}{mn}\log(\rho(T_{z;m\times n\times 2})) \\
= & \lim_{m,n\to\infty}\frac{1}{mn}log(\rho(2^{(2mn-n-m)}G^{(m-1)(n-1)})) \\
= & \lim_{m,n\to\infty}\frac{1}{mn}\log(2^{(2mn-n-m)}(\frac{1+\sqrt{5}}{2})^{(m-1)(n-1)})) \\
= & 2\log 2 + \log g \geq 0
\end{aligned}
$$

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