# 國立交通大學

## 應用數學系

## 碩 士 論 文

在 B 型代數結構下之 N 相黎曼空間的單擺運動之確切理論與數值計算

The Exact Theory and Numerical Computations of Pendulum Motions on Riemann Surfaces of Genus N with Cut-Structure of Type B

研 究 生：張竣富

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摘 要

Sine-Gordon 方程 $u_{xx} - u_{yy} + \sin u = 0$是被廣泛應用的偏微分方程式，而其某些特殊解滿足非線性二階微分方程 $\frac{d^2u}{dt^2} + \sin u = 0$，此為單擺運動方程式。當求解 $\frac{d^2u}{dt^2} + \sin u = 0$ 我們首先利用 $\sin u$ 的 Maclaurin 級數來替代 $\sin u$ 使得原微分方程變為 $\frac{d^2u}{dt^2} + P(u) = 0$，其中 P(u) 為多項式。此方程的解存在於 N 相黎曼空間。我們利用正確的代數結構來建構這些黎曼空間，使我們可以在黎曼空間中執行路徑積分進而得到方程數值解。之後我們研究古典橢圓函數，利用 Jacobian 橢圓函數來分析單擺運動方程。最後，我們利用 Jacobian 橢圓函數來導出此方程式的確切解與週期。

中 華 民 國 一 百 零 一 年 六 月

# The Exact Theory and Numerical Computations of Pendulum Motions on Riemann Surfaces of Genus N with Cut-Structure of Type B

**Student**：Chun-Fu Chang **Advisor**：Jong-Eao Lee

**Department of Applied Mathematics**

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## Abstract

The Sine-Gordon equation $u_{xx} - u_{yy} + \sin u = 0$ is a well-known Partial differential equation, and there are some special solutions satisfy the nonlinear second-order differential equation $\frac{d^2u}{dt^2} + \sin u = 0$ which is the Pendulum motion. As we solving the differential equation $\frac{d^2u}{dt^2} + \sin u = 0$. We first replace $\sin u$ by the Maclaurin Series of $\sin u$ to get the differential equation of the form $\frac{d^2u}{dt^2} + P(u) = 0$, where P(u) is a polynomial. Solutions of such equations reside in Riemann Surfaces of genus N. We construct these Riemann Surfaces with the correct algebraic structures. So we can perform path integrals on the Riemann Surfaces to get the numerical solution of the equation. Next, we investigate the classical Elliptic functions, and use the Jacobian Elliptic function to analyze this nonlinear pendulum motion. Finally, we derive the exact solutions and the periods of those solutions by the Jacobian Elliptic functions.

June 2012

# 誌　　謝

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# 1 Introduction of the Riemann Surface.

When we want to solve the differential equation $u'' + \sin u = 0$.

$$\begin{aligned}
&\Rightarrow \quad u'' + \sin u = 0 \\
&\Rightarrow \quad (1/2)(u')^2 - \cos u = k \\
&\Rightarrow \quad (1/2)(u')^2 = \cos u + k \\
&\Rightarrow \quad (u')^2 = 2\cos u + 2k \\
&\Rightarrow \quad u' = \frac{du}{dt} = \sqrt{2\cos u + 2k} \\
&\Rightarrow \quad \int \frac{1}{\sqrt{2\cos u + 2k}} du = \int 1 dt = t
\end{aligned}$$

where $k$ is a constant.
There is difficult to integrate

$$\int \frac{1}{\sqrt{2\cos u + 2k}} du$$

into a normal function.
By the Maclaurin Series

$$\sin u = \sum_{n=0}^{\infty} (-1)^n \frac{u^{2n+1}}{(2n+1)!} \sim \frac{u^1}{1!} - \frac{u^3}{3!} + \frac{u^5}{5!} - \frac{u^7}{7!} + \frac{u^9}{9!} - \frac{u^{11}}{11!}$$

When we replace $\sin u$ by

$$\frac{u^1}{1!} - \frac{u^3}{3!} + \frac{u^5}{5!} - \frac{u^7}{7!} + \frac{u^9}{9!} - \frac{u^{11}}{11!}$$

then the differential equation becomes

$$
\begin{aligned}
&\Rightarrow \quad u'' + \frac{u^1}{1!} - \frac{u^3}{3!} + \frac{u^5}{5!} - \frac{u^7}{7!} + \frac{u^9}{9!} - \frac{u^{11}}{11!} = 0 \\
&\Rightarrow \quad u'u'' + \frac{u'u^1}{1!} - \frac{u'u^3}{3!} + \frac{u'u^5}{5!} - \frac{u'u^7}{7!} + \frac{u'u^9}{9!} - \frac{u'u^{11}}{11!} = 0 \\
&\Rightarrow \quad \frac{1}{2}(u')^2 + \frac{\frac{1}{2}u^2}{1!} - \frac{\frac{1}{4}u^4}{3!} + \frac{\frac{1}{6}u^6}{5!} - \frac{\frac{1}{8}u^8}{7!} + \frac{\frac{1}{10}u^{10}}{9!} - \frac{\frac{1}{12}u^{12}}{11!} = k \\
&\Rightarrow \quad (u')^2 = -\frac{u^2}{1} + \frac{u^4}{12} - \frac{u^6}{360} + \frac{u^8}{20160} - \frac{u^{10}}{1814400} + \frac{u^{12}}{239500800} + 2k \\
&\Rightarrow \quad \frac{du}{dt} = \sqrt{-\frac{u^2}{1} + \frac{u^4}{12} - \frac{u^6}{360} + \frac{u^8}{20160} - \frac{u^{10}}{1814400} + \frac{u^{12}}{239500800} + 2k} \\
&\Rightarrow \quad \int \frac{1}{\sqrt{-\frac{u^2}{1} + \frac{u^4}{12} - \frac{u^6}{360} + \frac{u^8}{20160} - \frac{u^{10}}{1814400} + \frac{u^{12}}{239500800} + 2k}} du = \int 1 dt
\end{aligned}
$$

where $k$ is a constant.

We need to compute the integral

$$
\int \frac{1}{\sqrt{-\frac{u^2}{1} + \frac{u^4}{12} - \frac{u^6}{360} + \frac{u^8}{20160} - \frac{u^{10}}{1814400} + \frac{u^{12}}{239500800} + 2k}} du
$$

Before we compute the integral , we need to investigate the space where $u$ reside.

Because

$$
f(z) = \sqrt{\prod_{k=1}^{n} (z - z_k)}
$$

is a two-valued function of $z$ on complex plane $\mathbb{C}$. We use algebra and analysis to develop a new surface such that $f$ becomes a single-valued function on this surface , namely , a Riemann Surface.

## 1.1 The construction of the corresponding Riemann Surface.

Suppose $w, z \in \mathbb{C}$ and $w^k = z$ , we find the solution of $w^k = z$ in polar form

$$
\begin{aligned}
&\Rightarrow \quad w^k = z = |z|\, e^{i\theta} = |z| e^{i(\theta + 2n\pi)} \\
&\Rightarrow \quad w = |z|^{\frac{1}{k}} e^{\frac{i(\theta+2n\pi)}{k}}
\end{aligned}
$$

where $\theta \in [-\pi, \pi)$ and $n \in \mathbb{Z}$.

We will take $f(z) = \sqrt{z} = (z)^{\frac{1}{2}}$ for example first , where $f(z) : \mathbb{C} \to \mathbb{C}$. We will still use polar form , let

$$z = |z|e^{i\theta} = |z|e^{i(\theta+2n\pi)}, n \in \mathbb{Z}$$

then

$$\begin{aligned} f(z) &= \sqrt{z} = |z|^{\frac{1}{2}} e^{i(\frac{\theta+2n\pi}{2})} \\ &= \begin{cases} |z|^{\frac{1}{2}} e^{i(\frac{\theta}{2}+n\pi)} = |z|^{\frac{1}{2}} e^{i(\frac{\theta}{2})} \text{ , if } n \text{ is even} \\ |z|^{\frac{1}{2}} e^{i(\frac{\theta}{2}+n\pi)} = (-1)|z|^{\frac{1}{2}} e^{i(\frac{\theta}{2})} \text{ , if } n \text{ is odd} \end{cases} \end{aligned}$$

This means that $f(z)$ is a two-valued function. We need to let $f(z)$ becomes a single valued function now , so we modify its domain $\mathbb{C}$ to develop the corresponding Riemann Surface such that $f$ becomes a single-valued and analytic function on this surface.

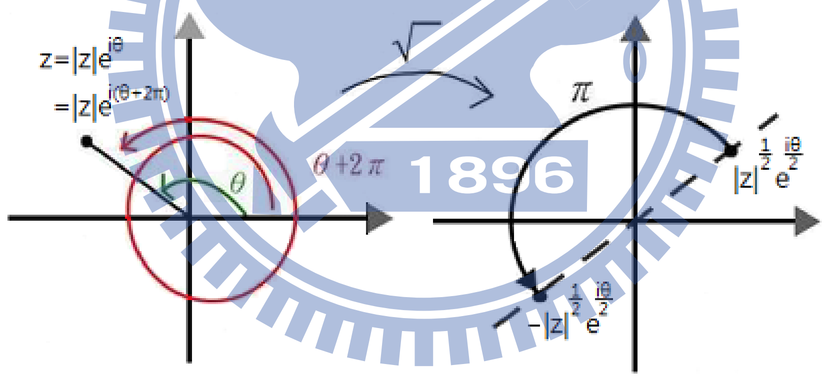



Figure 1. The idea of two sheets.

We start at some $z = |z|e^{i\theta}$ , and then we have $f(z) = \sqrt{z} = \sqrt{|z|}e^{i(\frac{\theta}{2})}, |z| \neq 0$. Fixing $|z|$ and continuing along a closed path once around the origin so that $\theta$ increases by $2\pi$ , $f(z)$ comes to the value

$$\sqrt{|z|}e^{i(\frac{\theta+2\pi}{2})} = -\sqrt{|z|}e^{i(\frac{\theta}{2})}$$

which is just the negative of its original value. When we continuing same way above , we find that as $\theta$ increases by $2\pi$ again , then $f(z)$ comes to oringinal value.

First , we image two sheets lying over the complex plane and cut the plane along negative real axis ( i.e. from 0 to $-\infty$ ) and restrict ourselves such that never to continue $f(z)$ over this cuts , we get single-valued branches of $f(z)$.

Define that

$$
\begin{aligned}
f(z) &= |z|^{\frac{1}{2}} e^{\frac{i\theta}{2}}, -\pi \leq \theta < \pi \\
f(z) &= |z|^{\frac{1}{2}} e^{\frac{i\theta}{2}}, \pi \leq \theta < 3\pi
\end{aligned}
$$

called sheet-I and sheet-II , respectively. There are two edges for every cut on each sheet , we label the starting edge with $(+)$-edge and the terminal edge with $(-)$-edge. (Show in Figure 2)

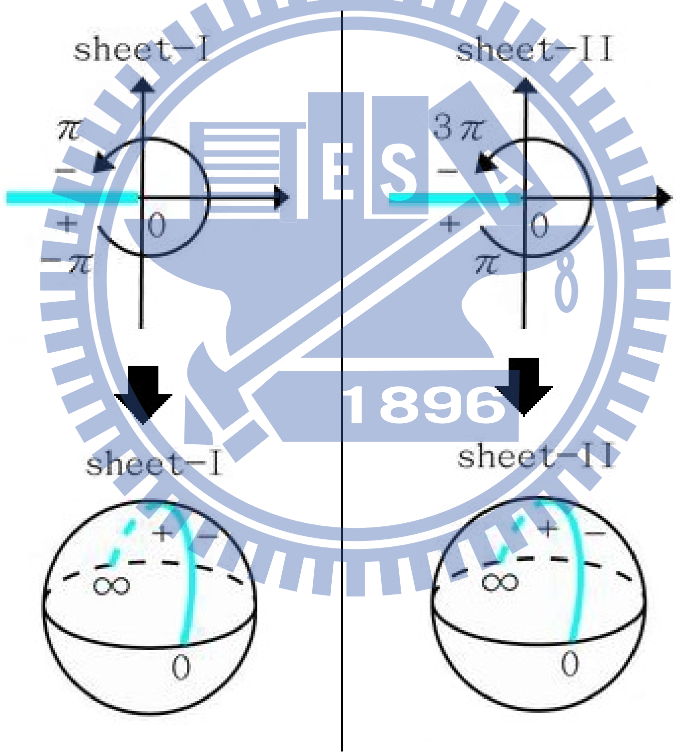



Figure 2. Complex plane and extended complex plane.

Moreover , when crossing the cut , we pass from one sheet to another.

Second , we extend the plane of complex numbers with one additional point at infinity constitute a number system known as the extended complex numbers. Use stereographic projection , we can consider the two sheets to be a spheres. And we image that the spheres are made of rubber and stretch each cut into circular holes.

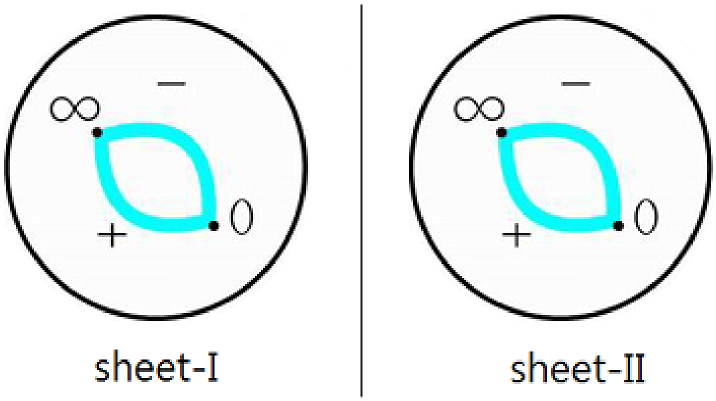



Figure 3. Place the cuts open.

We rotate the spheres to let the holes face each other , and paste two cuts together where (+)-edge of sheet-I with (−)-edge of sheet-II and (−)-edge of sheet-I with (+)-edge of sheet-II. So we can derive a new sphere now. We called this sphere to be "Riemann surface of genus 0" and denoted this sphere by $R_0$. Show in Figure 4.

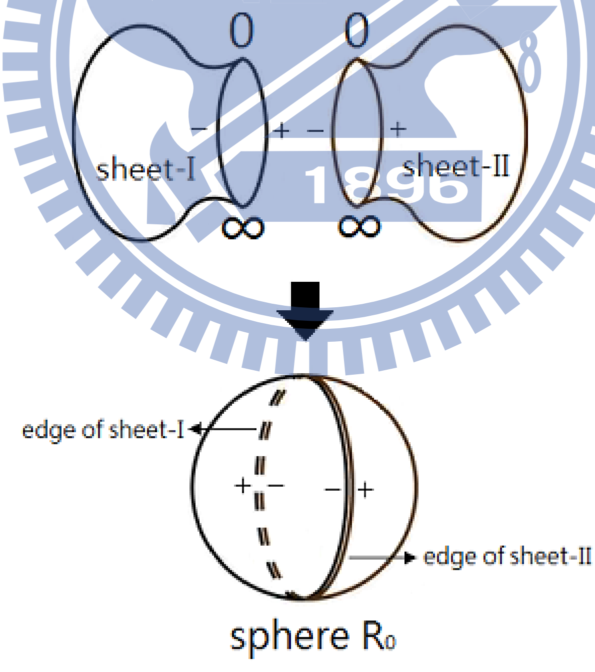



Figure 4. Construct $R_0$.

Notice that in Riemann Surface (+)-edge of sheet-I is equivalent to (−)-edge of sheet-II and (−)-edge of sheet-I is equivalent to (+)-edge of sheet-II.

We could using similar way to develop the corresponding Riemann Surface for

$$f(z) = \sqrt{\prod_{k=1}^{n} (z - z_k)}$$

We use two examples to show this , one have odd roots , and the other have even roots.

**Example 1** *Suppose there are 7 roots where the function* $f(z)$ *have. Construct the Riemann Surface of* $f(z) = \sqrt{\prod_{k=1}^{7} (z - z_k)} = \prod_{k=1}^{7} \sqrt{(z - z_k)}, z_k \in \mathbb{R}$ *where* $z_7 < z_6 < z_5 < z_4 < z_3 < z_2 < z_1$ *and we cut plane starts from* $z_k$ *to* $-\infty$ , $k = 1, 2, 3, 4, 5, 6, 7$.

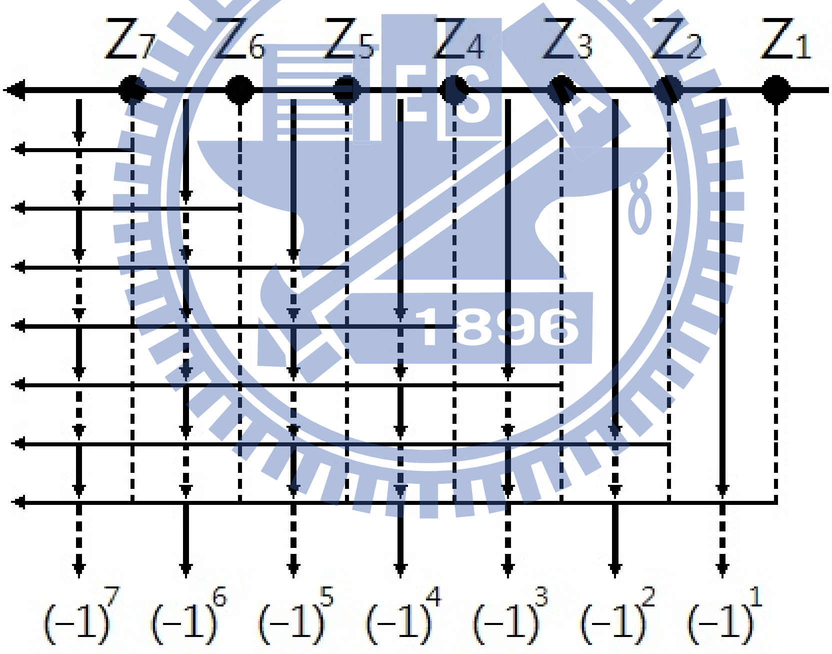



*Figure 5. Cut plane start from* $z_k$ *to* $-\infty$.

*When crossing one cut , we pass from one sheet to another. And at this time the argument of* $z$ *increases by* $2\pi$ *, so the argument of* $f(z)$ *increases by* $\pi$ *which is just the negative of its original value. So when crossing one cut we need to change the sign , using* $(-1)$ *represent that. So when crossing odd times we will change sign and when crossing even times we will not change sign eventually.*


*Figure 6. The cut structure.*

*There are branch cuts in* $(-\infty, z_7]$ , $[z_6, z_5]$ , $[z_4, z_3]$ , $[z_2, z_1]$ *and then using same idea to construct the corresponding Riemann Surface.*

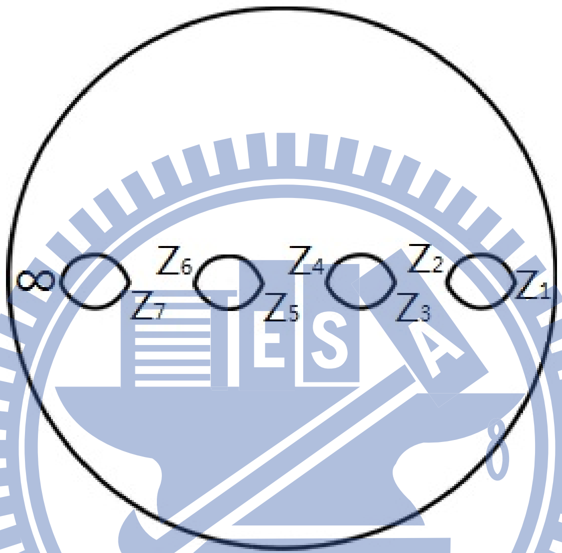



*Figure 7. Placing the cuts open.*

*Finally , we pull the cuts open and paste two sheet together with the rule* $(+)$*-edge with* $(-)$*-edge so we obtain corresponding Riemann Surface of genus 3 eventually.*

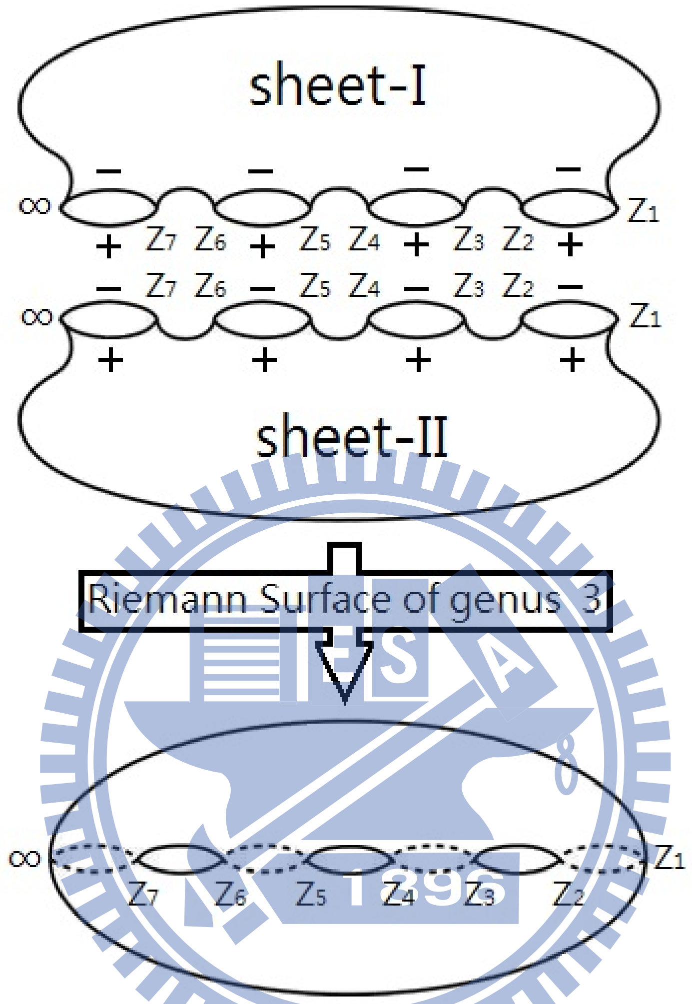



*Figure 8. Geometric graph of $R_3$.*

**Example 2** *Suppose there are 8 roots where the function $f(z)$ have. Construct the Riemann Surface of $f(z) = \sqrt{\prod_{k=1}^{8}(z - z_k)} = \prod_{k=1}^{8}\sqrt{(z - z_k)}, z_k \in \mathbb{R}$ where $z_8 < z_7 < z_6 < z_5 < z_4 < z_3 < z_2 < z_1$ and we cut plane starts from $z_k$ to $-\infty$ , $k = 1, 2, 3, 4, 5, 6, 7, 8$.*

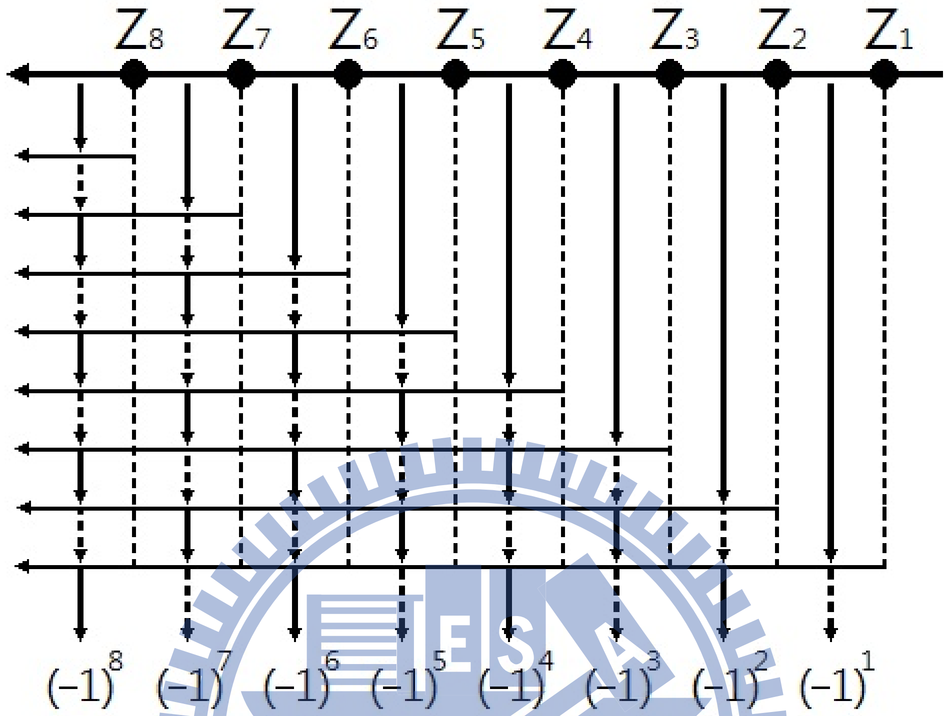



*Figure 9. Cut plane start from $z_k$ to $-\infty$.*

*When crossing one cut , we pass from one sheet to another. And at this time the argument of $z$ increases by $2\pi$ , so the argument of $f(z)$ increases by $\pi$ which is just the negative of its original value. So when crossing one cut we need to change the sign , using $(-1)$ represent that. So when crossing odd times we will change sign and when crossing even times we will not change sign eventually.*


*Figure 10. The cut structure.*

*There are branch cuts in $[z_8, z_7]$ , $[z_6, z_5]$ , $[z_4, z_3]$ , $[z_2, z_1]$ and then using same idea to construct the corresponding Riemann Surface.*

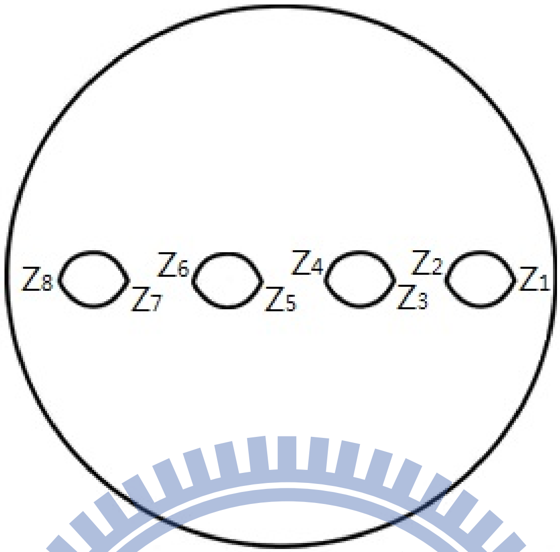



*Figure 11. Placing the cuts open.*

*Finally , we pull the cuts open and paste two sheet together with the rule (+)-edge with (−)-edge so we obtain corresponding Riemann Surface of genus 3 eventually.*

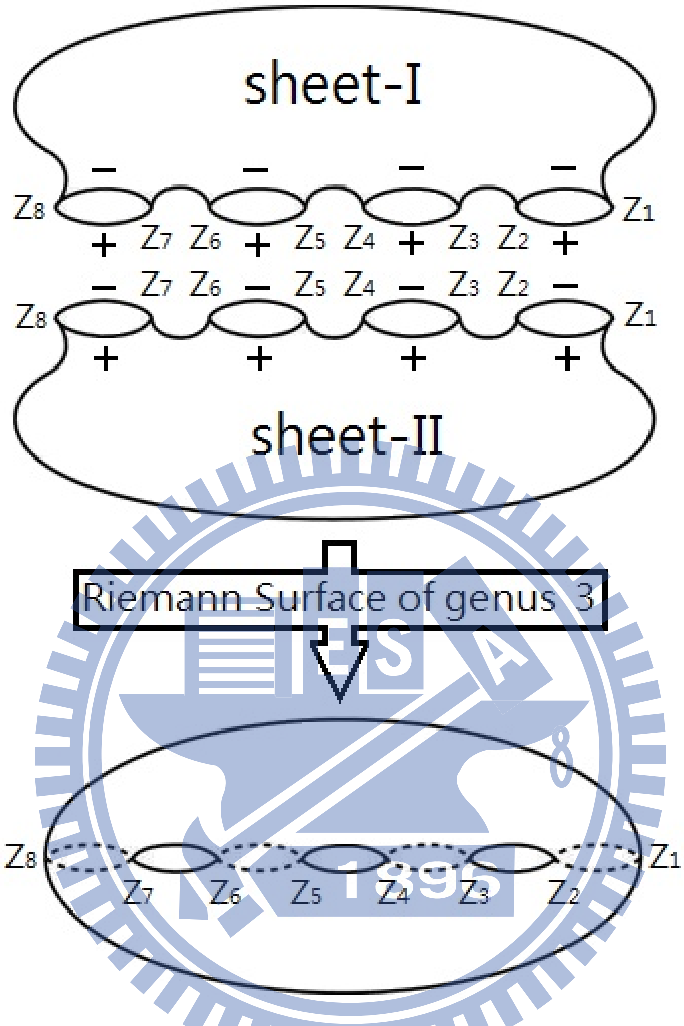


*Figure 12. Geometric graph of $R_3$.*

Although there are different algebraic structures between 7 roots and 8 roots that $f(z)$ have. But they both have the same geometric graph with 3 holes. This means that no matter 7 or 8 roots , we can construct corresponding Riemann Surface of genus 3.

## 1.2 The relationship of curve between algebraic structure and geometric structure.

We will use algebraic to compute the integrals and discuss the integrals later for convenience. We already know the relation of algebraic and geometric structure with $f(z) = \sqrt{\prod_{k=1}^{n} (z - z_k)}$ and how to create the Riemann Surface.

We give some examples to show that the curve in algebraic structure and its corresponding in geometric structure.

We defined something as following:

1. The curve in sheet-I is solid line and the curve in sheet-II is dash line in algebraic structure.

2. The curve in overhead Riemann Surface is solid line and the curve in ventral Riemann Surface is dash line in geometric structure.

**Example 3** *$r_1$ is the curve from a point at $(I,+)$ to $(I,-)$ in sheet-I and $r_2$ is the curve from a point at $(II,-)$ to $(II,+)$ in sheet-II.*

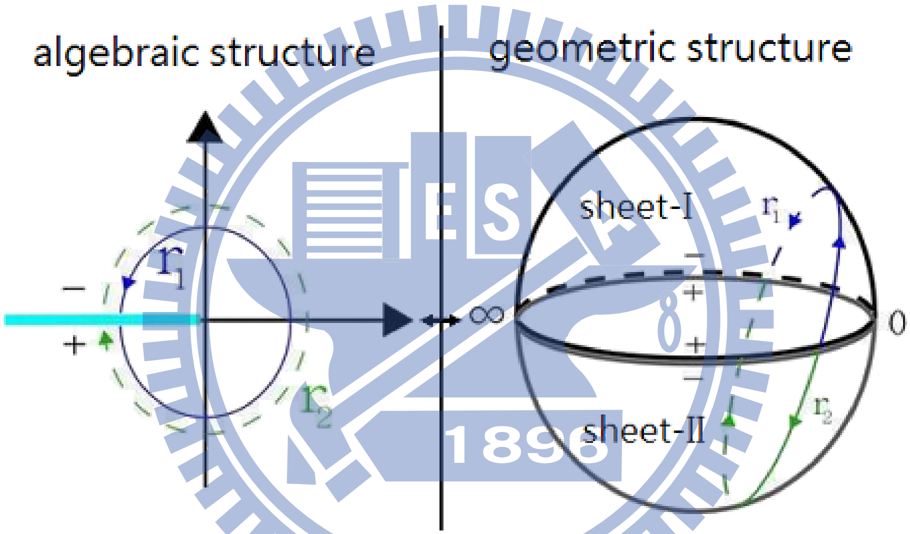



*Figure 13. The figure for Example 3.*

**Example 4** *The curve $r$ is start from point A in sheet-I and cross the cut to point B on sheet-II.*

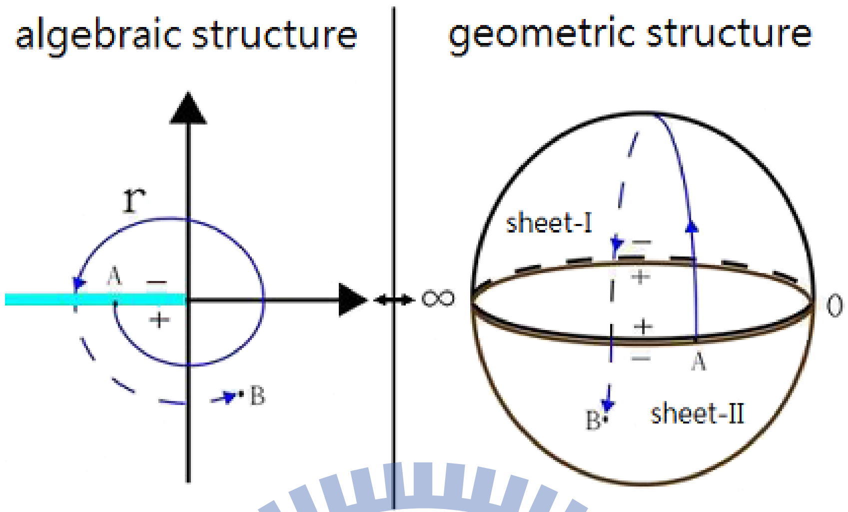



*Figure 14. The figure for Example 4.*

### 1.3 The a,b cycles and its equivalent paths.

We know every closed curve on Riemann Surface $R_k$ can be deformed into an integral combination of the loop-cut $a_i$ and $b_i, i = 1, 2, ..., k$. So in this paper , we will consider the integrals of $f(z)$ over $a$-cycles and $b$-cycles help us to obtain the integrals easier.

**Example 5** *Suppose* $f(z) = \sqrt{(z-0)(z-1)(z-2)(z-3)}$. *Construct the a-cycle , b-cycle and the corresponding geometric structure.*

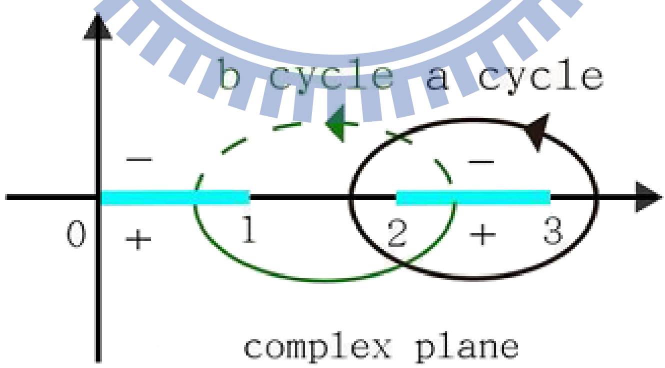



*Figure 15. a,b-cycles of* $f(z) = \sqrt{(z-0)(z-1)(z-2)(z-3)}$ *and the cut plane.*

*Because* $f(z)$ *has four roots , so we can construct two cuts and one a-cycle and one b-cycle. Notice that in this example , the numbers of a-cycle and the numbers of b-cycle are the same.*

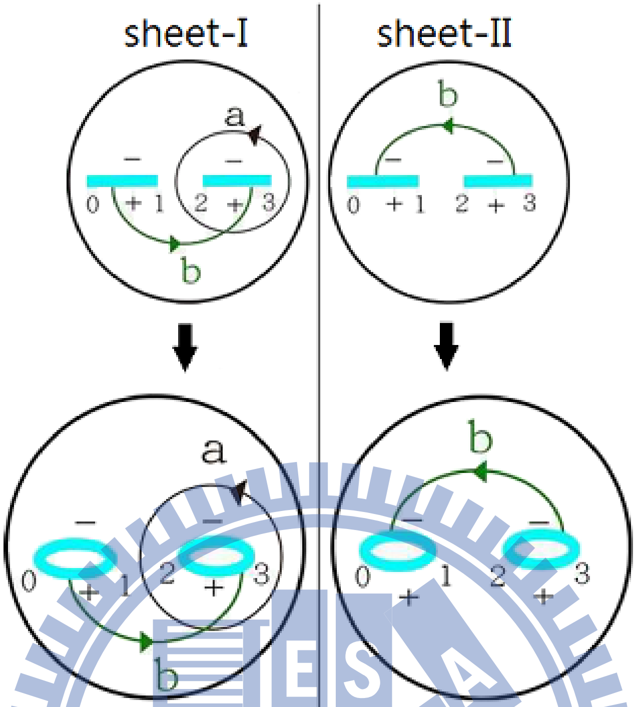



*Figure 16. Draw a,b-cycle in each sheet and then pull the cuts open.*

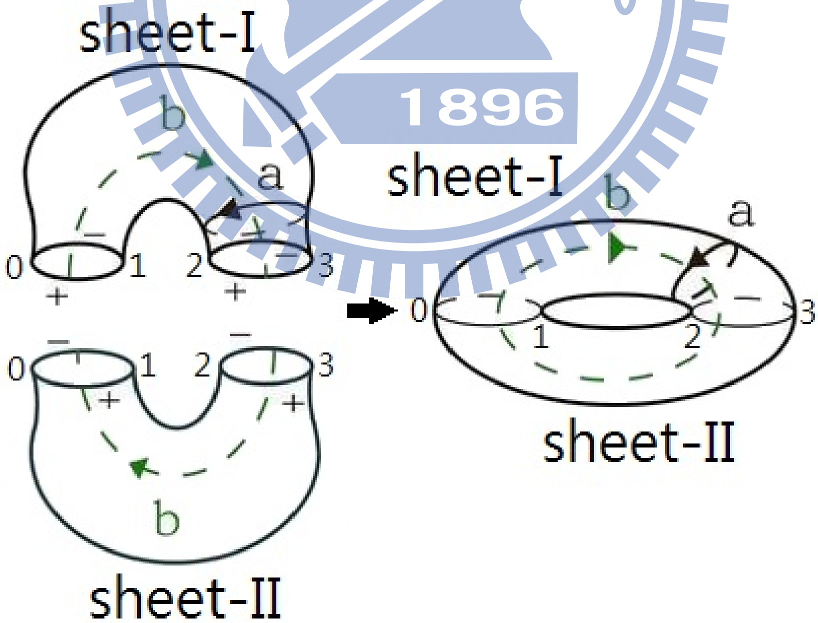



*Figure 17. Corresponding geometric structure and cycles.*

*Finally , we paste two sheets with open cuts and gained corresponding geometric structure and cycles.*

It is difficult to write out the parameters of curves sometimes. But the straight lines are easy to write out their parameters for us. So using homotopic of curves to find the equivalent paths of curves could help us to obtain the integrals over the curves easier and quicker.

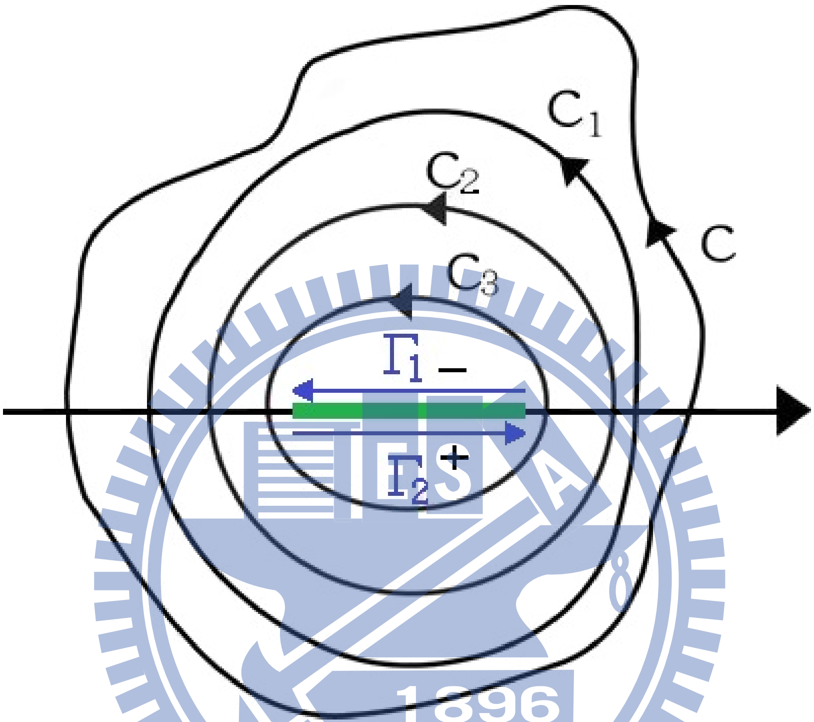



Figure 18. Homotopic curves.

Because the curve $C$ is homotopic to the curve $C_1$ , we denotes $C \approx C_1$. We have $\int_C \frac{1}{f(z)}dz = \int_{C_1} \frac{1}{f(z)}dz$ by Cauchy-Goursat theorem. In Figure 18 , we see that $C \approx C_1 \approx C_2 \approx C_3 \approx \Gamma_1 \cup \Gamma_2$.

So we have

$$
\begin{aligned}
\int_C \frac{1}{f(z)}dz &= \int_{C_1} \frac{1}{f(z)}dz \\
&= \int_{C_2} \frac{1}{f(z)}dz \\
&= \int_{C_3} \frac{1}{f(z)}dz \\
&= \int_{\Gamma_1 \cup \Gamma_2} \frac{1}{f(z)}dz \\
&= \int_{\Gamma_1} \frac{1}{f(z)}dz + \int_{\Gamma_1} \frac{1}{f(z)}dz
\end{aligned}
$$

We will use this method in the whole paper.

## 1.4 Conclusion of Riemann Surface.

Although the result and statement we discuss with above are all in horizontal cut. But the method which handleing other styles of cut is the same as in horizontal cut. We take $\omega^2 = c(z-z_1)(z-z_2)(z-z_3)$ for an example , where $z_1, z_2, z_3 \in \mathbb{C}$ are distinct and $c$ is a constant. Because $\sqrt{c}$ does not influence the cut , so we omit $\sqrt{c}$ and let $f(z) = \sqrt{(z-z_1)(z-z_2)(z-z_3)} = \sqrt{(z-z_1)}\sqrt{(z-z_2)}\sqrt{(z-z_3)}$. Remembering when $\arg(z-z_k)$ changes by $2\pi$ , the factor $\sqrt{(z-z_k)}$ will change the sign. In the figure 19 we label left of cut with (+)-edge and right of cut with (−)-edge.

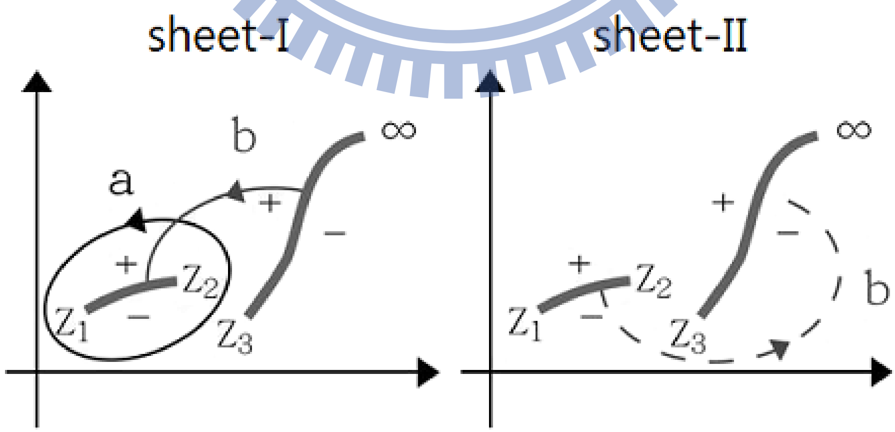



Figure 19. The cut-plane and $a, b$ cycles in each sheets.

We construct the Riemann Surface in similar way before. Imageing both two sheets are made of rubber , and pull cuts to be holes. We rotate the sheets to let the holes face each other , and paste two cuts together where (+)-edge of sheet-I with (−)-edge of sheet-II and (−)-edge of sheet-I with (+)-edge of sheet-II. We will get the corresponding Riemann Surface $R_1$. The $a, b$ curves are corresponding to the meridian curve $a$ and latitude curve $b$ on Rienann Surface $R_1$ , respectively.

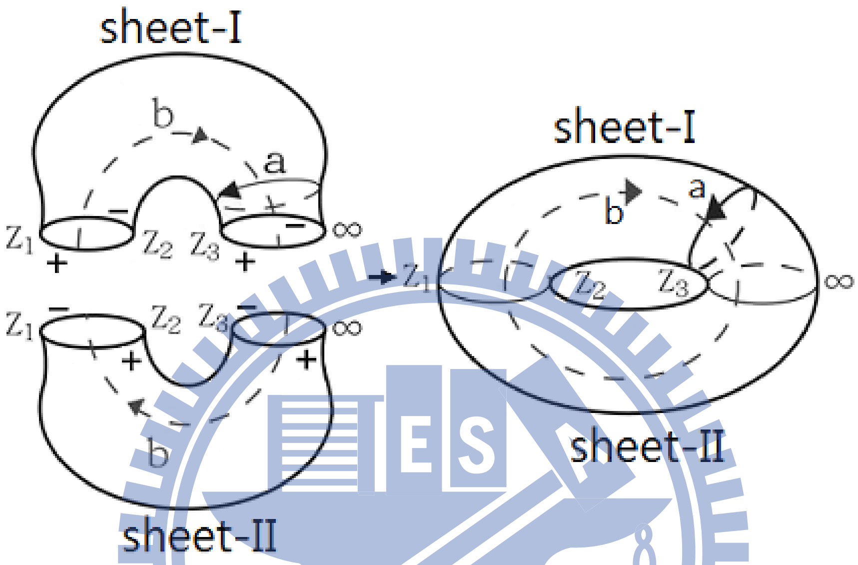



Figure 20. Corresponding Riemann Surface.

# 2 The integrations of $1/f(z)$ over $a, b$ cycles for cuts on Riemann Surface.

When we known the geometric structure of Riemann Surface. It is usefull to know the integration of a function on Riemann Surface. Especially , the $a, b$ cycles for cuts on Riemann Surface.

## 2.1 The integrations of $1/f(z)$ over $a, b$ cycles of the Riemann Surface with horizontal cut-structure.

We will use Mathematica help us to obtain the values of integrations of $1/f(z)$ over $a, b$ cycles. First, We discuss the values in sheet-I, sheet-II and Mathematica for horizontal cuts.

In using polar form

$$\frac{1}{f(z)} = [\prod_{k=1}^{n}(z - z_k)]^{-\frac{1}{2}} = (re^{i\theta})^{-\frac{1}{2}}$$

Let $\theta_1$ denotes $\theta$ in sheet-I and $\theta_2$ denotes $\theta$ in sheet-II. Clearly that $\theta_2 = \theta_1 + 2\pi$ , so we have

$$
\begin{aligned}
(\frac{1}{f(z)})|_{(II)} &= r^{-\frac{1}{2}} e^{i(-\frac{\theta_2}{2})} \\
&= r^{-\frac{1}{2}} e^{i(-\frac{\theta_1+2\pi}{2})} \\
&= r^{-\frac{1}{2}} e^{i(-\frac{\theta_1}{2})} e^{i(-\pi)} \\
&= (-1) r^{-\frac{1}{2}} e^{i(-\frac{\theta_1}{2})} \\
&= (-1)(\frac{1}{f(z)})|_{(I)}
\end{aligned}
$$

where $(\frac{1}{f(z)})|_{(I)}$ denote the value of $\frac{1}{f(z)}$ with $z$ in sheet-I and $(\frac{1}{f(z)})|_{(II)}$ denote the value of $\frac{1}{f(z)}$ with $z$ in sheet-II. Because the difference of argument between $z$ in sheet-I and in sheet-II is $2\pi$, so the difference between $(\frac{1}{f(z)})|_{(I)}$ and $(\frac{1}{f(z)})|_{(II)}$ is $(-\pi)$. Hence ,

$$(\frac{1}{f(z)})|_{(II)} = (-1)(\frac{1}{f(z)})|_{(I)}$$

Now we discuss the difference over sheet-I in theory and in Mathematica.

First , we consider $\sqrt{-1}$. See Figure 21 , in theory we know that $\sqrt{-1} = -i$ by the definition of argument in sheet-I. But in Mathematica , when we compute $\sqrt{-1}$ , we will obtain $\sqrt{-1} \overset{Math}{=} i$. Actually , we found that $re^{i\theta}$ where $\theta \in (-\pi, \pi]$ in Mathematica. This means for any other $\theta$ of $re^{i\theta}$ which does not belong to $(-\pi, \pi]$ , Mathematica will converse $re^{i\theta}$ into $re^{i\theta^*}$ where $\theta^* \in (-\pi, \pi]$ and $re^{i\theta} = re^{i\theta^*}$.

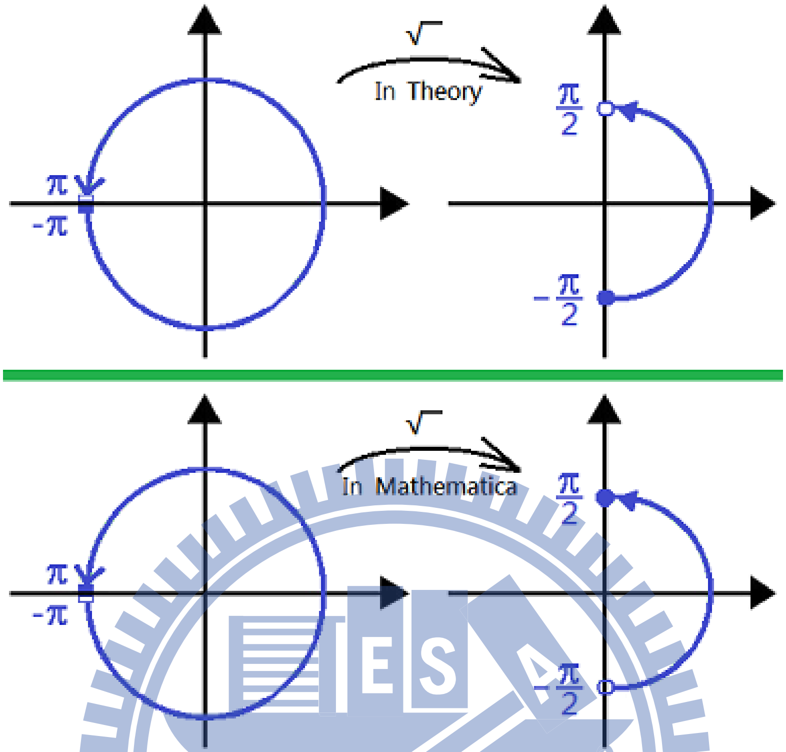



Figure 21. Domain and range of square function in Theory and in Mathematica.

Compare the value of $(1/f(z))$ with $z$ in sheet-I and in Mathematica , we discover that

**Lemma 6** *If* $(1/f(z)) = [\prod_{k=1}^{n}(z - z_k)]^{-\frac{1}{2}} = (re^{i\theta})^{-\frac{1}{2}}$ *in sheet-I for horizontal cut , then*

$$\left(\frac{1}{f(z)}\right)\Big|_{(I)} = \begin{cases} (\frac{1}{f(z)})|_{Mathematica} \ , \ if\ \theta \in (-\pi, \pi) \\ (-1)(\frac{1}{f(z)})|_{Mathematica} \ , \ if\ \theta = -\pi \end{cases}$$

***Proof.*** *Since* $(-\pi)$ *does not in* $(-\pi, \pi]$ *, then Mathematica will converse* $re^{i(-\pi)}$ *into* $re^{i\pi}$ *and* $re^{i(-\pi)} = re^{i\pi}$*. We compute* $(1/f(z))$ *in theory and in Mathematica.*

*In theory :* $\quad -r = re^{i(-\pi)} \quad \overset{\frac{1}{\sqrt{*}}}{\longrightarrow} (re^{i(-\pi)})^{-\frac{1}{2}} = ir^{-\frac{1}{2}}$

*In Mathematica :* $\quad -r = re^{i(-\pi)} \overset{Math.}{=} re^{i\pi} \overset{\frac{1}{\sqrt{*}}}{\longrightarrow} (re^{i\pi})^{-\frac{1}{2}} \overset{Math.}{=} (-1)ir^{-\frac{1}{2}}$

*Hence ,* $(1/f(z))|_{(I)} \overset{Math.}{=} (-1)(1/f(z))|_{Mathematica}$ *where* $\theta = -\pi$*.* ∎

In this whole paper , $(1/f(z)) \overset{Math.}{=} (-1)(1/f(z))$ denotes the function $(1/f(z))$ in front of $\overset{Math.}{=}$ is the value of $(1/f(z))$ in theory and the function $(1/f(z))$ behind the $\overset{Math.}{=}$ is the value of $(1/f(z))$ in Mathematica. After we known the state above, we must modify the computation when we want to use Mathematica to calculate the value. Take examples to explain.

**Example 7** *Evaluate* $\int_r \frac{1}{f(z)}dz$ *where* $f(z) = \sqrt{(z-1)(z-2)(z-3)}, z \in \mathbb{R}$ *and* $r = r_1 \cup r_2$ *where* $r_1$ *is the path on a horizontal cut from* 2 *to* 3 *with* $(+)$*-edge of sheet-I and* $r_2$ *is the path on a horizontal cut from* 3 *to* 2 *with* $(-)$*-edge of sheet-I.*

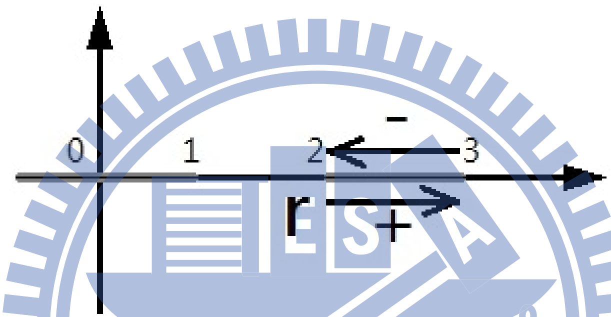



*Figure 22. Cuts in complex plane of* $f(z) = \sqrt{(z-1)(z-2)(z-3)}$ .

***Solution 8*** *Since* $f(z) = \sqrt{(z-1)(z-2)(z-3)} = \sqrt{z-1}\sqrt{z-2}\sqrt{z-3}$

1. *If* $z \in r_1$ *:*

   *(1) In theory :*

$$
\begin{aligned}
z-1 &\geq 0 \Rightarrow \frac{1}{\sqrt{z-1}} = |z-1|^{-\frac{1}{2}} \\
z-2 &\geq 0 \Rightarrow \frac{1}{\sqrt{z-2}} = |z-2|^{-\frac{1}{2}} \\
z-3 &< 0 \Rightarrow z-3 = |z-3|e^{-i\pi} \Rightarrow \frac{1}{\sqrt{z-3}} = |z-3|^{-\frac{1}{2}}e^{i\frac{\pi}{2}} = i|z-3|^{-\frac{1}{2}}
\end{aligned}
$$

*we have*

$$\int_{r_1} \frac{1}{f(z)}dz = i\int_2^3 |z-1|^{-\frac{1}{2}}|z-2|^{-\frac{1}{2}}|z-3|^{-\frac{1}{2}}dz$$

*(2) In Mathematica :*

$$\begin{aligned} z-1 &\geq 0 \Rightarrow \frac{1}{\sqrt{z-1}} = |z-1|^{-\frac{1}{2}} \\ z-2 &\geq 0 \Rightarrow \frac{1}{\sqrt{z-2}} = |z-2|^{-\frac{1}{2}} \\ z-3 &< 0 \Rightarrow z-3 = |z-3|e^{i\pi} \Rightarrow \frac{1}{\sqrt{z-3}} = |z-3|^{-\frac{1}{2}}e^{-i\frac{\pi}{2}} = -i|z-3|^{-\frac{1}{2}} \end{aligned}$$

*we have*

$$\int_{r_1} \frac{1}{f(z)}dz = -i\int_2^3 |z-1|^{-\frac{1}{2}}|z-2|^{-\frac{1}{2}}|z-3|^{-\frac{1}{2}}dz$$

*Compare* (1) *and* (2) *, we found there is a difference of minus sign with the value in sheet-I and in Mathematica.*

*2. If* $z \in r_2$ *:*

*(1) In theory :*

$$\begin{aligned} z-1 &\geq 0 \Rightarrow \frac{1}{\sqrt{z-1}} = |z-1|^{-\frac{1}{2}} \\ z-2 &\geq 0 \Rightarrow \frac{1}{\sqrt{z-2}} = |z-2|^{-\frac{1}{2}} \\ z-3 &< 0 \Rightarrow z-3 = |z-3|e^{i\pi} \Rightarrow \frac{1}{\sqrt{z-3}} = |z-3|^{-\frac{1}{2}}e^{-i\frac{\pi}{2}} = -i|z-3|^{-\frac{1}{2}} \end{aligned}$$

*we have*

$$\int_{r_2} \frac{1}{f(z)}dz = -i\int_3^2 |z-1|^{-\frac{1}{2}}|z-2|^{-\frac{1}{2}}|z-3|^{-\frac{1}{2}}dz$$

*(2) In Mathematica :*

$$\begin{aligned} z-1 &\geq 0 \Rightarrow \frac{1}{\sqrt{z-1}} = |z-1|^{-\frac{1}{2}} \\ z-2 &\geq 0 \Rightarrow \frac{1}{\sqrt{z-2}} = |z-2|^{-\frac{1}{2}} \\ z-3 &< 0 \Rightarrow z-3 = |z-3|e^{i\pi} \Rightarrow \frac{1}{\sqrt{z-3}} = |z-3|^{-\frac{1}{2}}e^{-i\frac{\pi}{2}} = -i|z-3|^{-\frac{1}{2}} \end{aligned}$$

*we have*

$$\int_{r_2} \frac{1}{f(z)} dz = -i \int_3^2 |z-1|^{-\frac{1}{2}} |z-2|^{-\frac{1}{2}} |z-3|^{-\frac{1}{2}} dz$$

*Compare* (1) *and* (2) *, the value is the same.*

*By* 1, 2 *we have*

$$\begin{aligned} \int_r \frac{1}{f(z)} dz &= \begin{cases} 2i \int_2^3 |z-1|^{-\frac{1}{2}} |z-2|^{-\frac{1}{2}} |z-3|^{-\frac{1}{2}} dz & \text{in sheet-I .} \\ 0 & \text{in Mathematica .} \end{cases} \\ &= \begin{cases} 0. + 5.24412i & \text{in sheet-I .} \\ 0 & \text{in Mathematica .} \end{cases} \end{aligned}$$

Clearly, there is a mistake when $\theta = -\pi$. When we use Mathematica to get the value of integration we want , we need modify some range where the value will wrong. Determine the difference of $sign(f)$ (same or negative) and then modify the computation of Mathematica to get right value. Because sometimes the form of integration is complex , if we could simplify the way about modify the difference of $sign(f)$ , it will help us to get right value easier.

**Example 9** *Same $f(z)$ as the example before , using Lemma in this section to modify.*

***Solution 10*** *.*

1. *If $z \in r_1, z : 2 \to 3$*

$$\begin{aligned} z-1 &\geq 0 \Rightarrow \arg(z-1) = 0 \Rightarrow \frac{1}{\sqrt{z-1}} \overset{Math.}{=} \frac{1}{\sqrt{z-1}} \\ z-2 &\geq 0 \Rightarrow \arg(z-2) = 0 \Rightarrow \frac{1}{\sqrt{z-2}} \overset{Math.}{=} \frac{1}{\sqrt{z-2}} \\ z-3 &< 0 \Rightarrow \arg(z-3) = -\pi \Rightarrow \frac{1}{\sqrt{z-3}} \overset{Math.}{=} -\frac{1}{\sqrt{z-3}} \end{aligned}$$

*we have*

$$\int_{r_1} \frac{1}{f(z)} dz \overset{Math.}{=} -\int_2^3 \frac{1}{\sqrt{z-1}} \frac{1}{\sqrt{z-2}} \frac{1}{\sqrt{z-3}} dz$$

*2. If* $z \in r_2, z : 3 \to 2$

$$\begin{aligned} z-1 &\geq 0 \Rightarrow \arg(z-1) = 0 \Rightarrow \frac{1}{\sqrt{z-1}} \overset{Math.}{=} \frac{1}{\sqrt{z-1}} \\ z-2 &\geq 0 \Rightarrow \arg(z-2) = 0 \Rightarrow \frac{1}{\sqrt{z-2}} \overset{Math.}{=} \frac{1}{\sqrt{z-2}} \\ z-3 &< 0 \Rightarrow \arg(z-3) = \pi \Rightarrow \frac{1}{\sqrt{z-3}} \overset{Math.}{=} \frac{1}{\sqrt{z-3}} \end{aligned}$$

*we have*

$$\int_{r_2} \frac{1}{f(z)} dz \overset{Math.}{=} \int_3^2 \frac{1}{\sqrt{z-1}} \frac{1}{\sqrt{z-2}} \frac{1}{\sqrt{z-3}} dz$$

*By* $1, 2$ *we have*

$$\int_r \frac{1}{f(z)} dz \overset{Math.}{=} -2 \int_2^3 \frac{1}{\sqrt{z-1}} \frac{1}{\sqrt{z-2}} \frac{1}{\sqrt{z-3}} dz = 0. + 5.24412i$$

**Example 11** *Evaluate* $\int \frac{1}{f(z)} dz$ *over* $a_1$ , $a_2$ *and* $a_3$ *cycles where* $f(z) = \sqrt{(z+4)(z+2)(z-2)(z-4)(z-5)(z-7)(z-8)}$*. We analysis the integral in Mathematica and in theory to compare the result and using the result of angle to modify the computation to get value. Let* $z_1 = 8$ , $z_2 = 7$ , $z_3 = 5$ , $z_4 = 4$ , $z_5 = 2$ , $z_6 = -2$ , $z_7 = -4$.

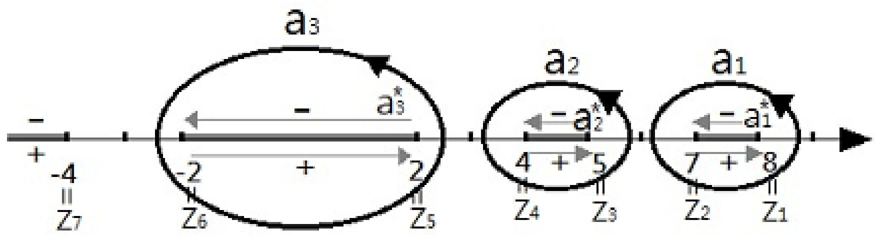



*Figure 23. a-cycles and their equivalent path* $a^*$*.*

***Solution 12*** *The detail is in appendix. And we just only give numerical solution here.*

$$\int_{a_1} \frac{1}{f(z)} dz \stackrel{Math.}{=} 0. + 0.0890282i$$
$$\int_{a_2} \frac{1}{f(z)} dz \stackrel{Math.}{=} 0. + 0.1832730i$$
$$\int_{a_3} \frac{1}{f(z)} dz \stackrel{Math.}{=} 0. + 0.1115720i$$

**Example 13** *Evaluate* $\int \frac{1}{f(z)}dz$ *over* $b_1$ , $b_2$ *and* $b_3$ *cycles where* $f(z) = \sqrt{(z+4)(z+2)(z-2)(z-4)(z-5)(z-7)(z-8)}$. *We analysis the integral in Mathematica and in theory to compare the result and using the result of angle to modify the computation to get value. Let* $z_1 = 8$ , $z_2 = 7$ , $z_3 = 5$ , $z_4 = 4$ , $z_5 = 2$ , $z_6 = -2$ , $z_7 = -4$.

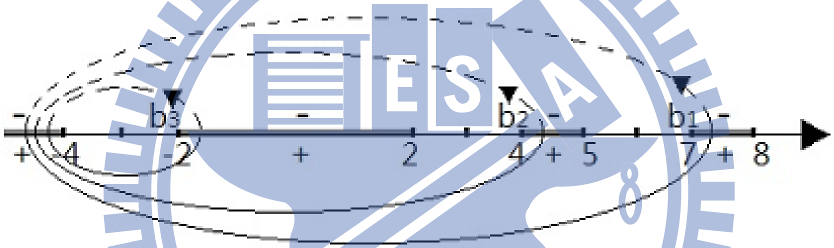



*Figure 24.* $b_1$ , $b_2$ *and* $b_3$ *cycles.*

***Solution 14*** *The detail is in appendix. And we just only give numerical solution here.*

$$\int_{b_1} \frac{1}{f(z)} dz \stackrel{Math.}{=} 0.4132335$$
$$\int_{b_2} \frac{1}{f(z)} dz \stackrel{Math.}{=} 0.2196815$$
$$\int_{b_3} \frac{1}{f(z)} dz \stackrel{Math.}{=} 0.0372385$$

## 2.2 The integrations of $1/f(z)$ over $a, b$ cycles of the Riemann Surface with vertical cut-structure.

After knowing the integrations in horizontal cut , we will discuss the integrations for vertical cuts. In this case , we define that

$$z - z_k = \begin{cases} re^{i\theta}, \theta \in [-\frac{3\pi}{2}, \frac{\pi}{2}) & \text{iff } z \text{ in sheet-I} \\ re^{i\theta}, \theta \in [\frac{\pi}{2}, \frac{5\pi}{2}) & \text{iff } z \text{ in sheet-II} \end{cases}$$

the cut in each sheet has two edges , label the starting edge with $(+)$-edge and the terminal edge with $(-)$-edge and $z_k$ is the end point of the vertical cut.

First , we will analysis the value of $1/f(z)$ on sheet-I and sheet-II in theory.

Second , we will discuss the difference between the value of $1/f(z)$ in theory and in Mathematica and find out how to modify the computation.

1. Analysis the value of $1/f(z)$ on sheet-I and sheet-II in theory :

   For a simple case $f(z) = \sqrt{z}$ , by the figure below , we know that

   $$\text{if} \begin{cases} z = |z|\, e^{i\theta} \text{ where } \theta \in [-\frac{3\pi}{2}, \frac{\pi}{2}) & \text{i.e. } z = |z|\, e^{i\theta} \in \text{ sheet-I} \\ z = |z|\, e^{i\theta} \text{ where } \theta \in [\frac{\pi}{2}, \frac{5\pi}{2}) & \text{i.e. } z = |z|\, e^{i\theta} \in \text{ sheet-II} \end{cases}$$

   $$\text{, then} \begin{cases} \sqrt{z} = |z|^{\frac{1}{2}}\, e^{\frac{i\theta}{2}}, \frac{\theta}{2} \in [-\frac{3\pi}{4}, \frac{\pi}{4}) \\ \sqrt{z} = |z|^{\frac{1}{2}}\, e^{\frac{i\theta}{2}}, \frac{\theta}{2} \in [\frac{\pi}{4}, \frac{5\pi}{4}) \end{cases}$$

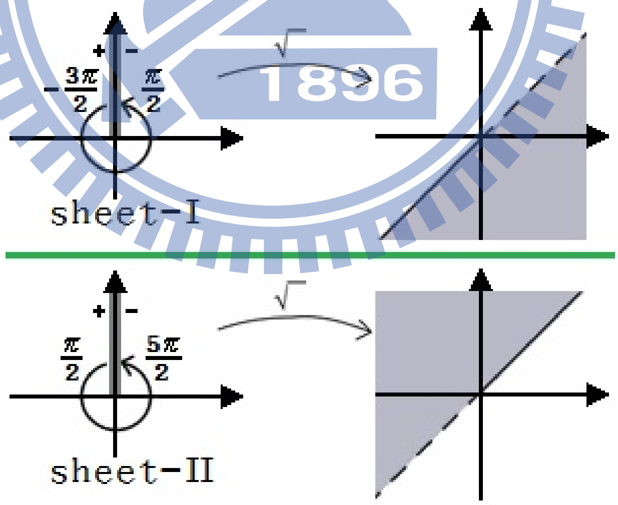



Figure 25. Case of $f(z) = \sqrt{z}$.

$$\text{and} \begin{cases} 1/\sqrt{z} = |z|^{-\frac{1}{2}}\, e^{i(-\frac{\theta}{2})}, -\frac{\theta}{2} \in (-\frac{\pi}{4}, \frac{3\pi}{4}] \\ 1/\sqrt{z} = |z|^{-\frac{1}{2}}\, e^{i(-\frac{\theta}{2})}, -\frac{\theta}{2} \in (-\frac{5\pi}{4}, -\frac{\pi}{4}] \end{cases}$$

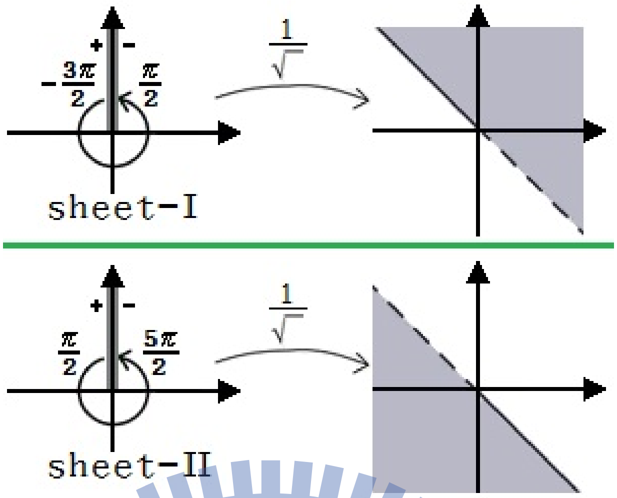



Figure 26. Case of $f(z) = 1/\sqrt{z}$.

For the general case , suppose $f(z) = \sqrt{\prod_{k=1}^{n}(z - z_k)}$

, then $\begin{cases} \prod_{k=1}^{n}(z - z_k) = re^{i\theta_1}, \theta_1 \in [-\frac{3\pi}{2}, \frac{\pi}{2}) \text{ in sheet-I} \\ \prod_{k=1}^{n}(z - z_k) = re^{i\theta_2}, \theta_2 \in [\frac{\pi}{2}, \frac{5\pi}{2}) \text{ in sheet-II} \end{cases}$

From the idea of definition , $re^{i\theta_1} = re^{i\theta_2}$ and $\theta_2 = \theta_1 + 2\pi$ , we have

$$\begin{aligned} (\frac{1}{f(z)})|_{(II)} &= r^{-\frac{1}{2}}e^{i(-\frac{\theta_2}{2})} \\ &= r^{-\frac{1}{2}}e^{i(-\frac{\theta_1+2\pi}{2})} \\ &= r^{-\frac{1}{2}}e^{i(-\frac{\theta_1}{2})}e^{i(-\pi)} \\ &= (-1)r^{-\frac{1}{2}}e^{i(-\frac{\theta_1}{2})} \\ &= (-1)(\frac{1}{f(z)})|_{(I)} \end{aligned}$$

2. Discuss the difference between the value of $1/f(z)$ in theory and in Mathematica and find out how to modify the computation :

   In the Figure below , we see the value of $f(z) = \sqrt{z}$ in sheet-I and the value of $f(z) = \sqrt{z}$ in Mathematica. So we need to modify the computation in Mathematica such that the numerical result of Mathematica is identical to the numerical result of theory when $\theta \in [-\frac{3\pi}{2}, -\pi]$.

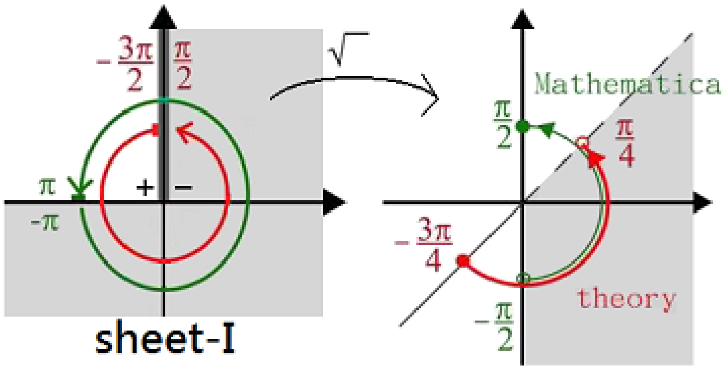



Figure 27. The value of $f(z) = \sqrt{z}$ in sheet-I and in Mathematica.

**Lemma 15** *When $z$ in sheet-I for vertical cut whose one of the end points is $z_k$ , we have*

$$\frac{1}{\sqrt{z-z_k}} \overset{Math.}{=} \begin{cases} (-1)\frac{1}{\sqrt{z-z_k}} & \text{if } \arg(z-z_k) \in [-\frac{3\pi}{2}, -\pi] \ , \\ \frac{1}{\sqrt{z-z_k}} & \text{if } \arg(z-z_k) \in (-\pi, \frac{\pi}{2}) \end{cases}$$

***Proof.*** *Let $z$ in sheet-I and using polar form $z - z_k = re^{i\theta}$. When $\theta \in (-\pi, \frac{\pi}{2})$ , the argument in theory or Mathematica is the same. When $\theta \in [-\frac{3\pi}{2}, -\pi]$ , Mathematica will conversion $\theta \in [-\frac{3\pi}{2}, -\pi]$ into $\theta + 2\pi \in [\frac{\pi}{2}, \pi]$ and $re^{i\theta} = re^{i(\theta+2\pi)} = re^{i\theta + i(2\pi)}$ , but*

$$\begin{cases} \text{In theory :} & (z-z_k)^{-\frac{1}{2}} = (re^{i\theta})^{-\frac{1}{2}} = r^{-\frac{1}{2}} e^{i(-\frac{\theta}{2})} \\ \text{In Mathematica :} & (z-z_k)^{-\frac{1}{2}} = (re^{i\theta + i(2\pi)})^{-\frac{1}{2}} = (-1) r^{-\frac{1}{2}} e^{i(-\frac{\theta}{2})} \end{cases}$$

*So if $\theta \in [-\frac{3\pi}{2}, -\pi]$ , we have*

$$\frac{1}{\sqrt{z-z_k}} \overset{Math.}{=} (-1)\frac{1}{\sqrt{z-z_k}}$$

■

As same as horizontal cut , we first discuss the difference of values of $1/f(z)$ between sheet-I and sheet-II in theory. And discuss the value of $1/f(z)$ in theory and in Mathematica , compare their $sign(f)$ is different or not? Using statement before and modify to get the value. The result is similar to horizontal cut.

**Example 16** *Evaluate the integrals of* $1/f(z)$ *over* $a_1$ *cycle for vertical cut where* $f(z) = \sqrt{(z-i)(z-2i)(z-3i)(z-5i)(z-6i)(z-8i)}$.

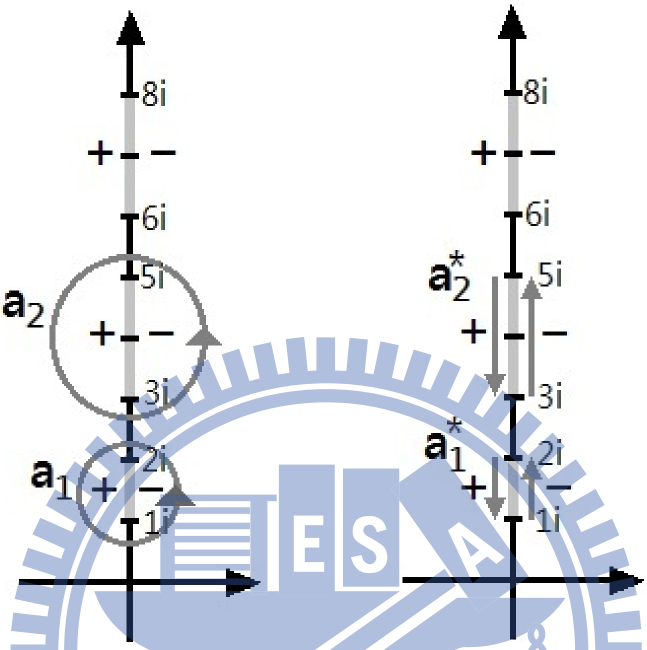



*Figure 28. path a and its equivalent path* $a^*$.

***Solution 17*** *In the Figure 28. , we know that* $a_1^*$ *is an equivalent path for* $a_1$ *and* $a_1^*$ *is the path along vertical cut from* $2i$ *to* $i$ *on* $(+)$*-edge of sheet-I (called* $a_{11}^*$*) and then back from* $i$ *to* $2i$ *on* $(-)$*-edge of sheet-I (called* $a_{12}^*$*). So we compute* $\int_{a_1^*} \frac{1}{f(z)} dz$.

1. $a_{11}^*$ *: Let* $z = ri$ *where* $r : 2 \xrightarrow{+} 1$ *and* $dz = idr$

   (1) *Analysis in theory :*
   *Since* $z - ki = |z-ki|\, e^{i \arg(z-ki)}$ *, so we consider* $\arg(z-ki)$.

$$\begin{aligned} \arg(z-i) &= -\frac{3}{2}\pi \Rightarrow \arg(\frac{1}{\sqrt{z-i}}) = \frac{3\pi}{4} \\ \arg(z-ki) &= -\frac{1}{2}\pi \Rightarrow \arg(\frac{1}{\sqrt{z-ki}}) = \frac{\pi}{4}, k = 2,3,5,6,8 \end{aligned}$$

   *we have*

$$
\begin{aligned}
\frac{1}{f(z)} &= \Big(\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}}\Big)(e^{\frac{3\pi}{4}})(e^{\frac{\pi}{4}})^5 \\
&= \Big(\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}}\Big)(e^{2\pi}) \\
&= \Big(\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}}\Big) \\
&= R
\end{aligned}
$$

*(2) Analysis in Mathematica (no matter in which sheet) :*
*Since* $z-ki = |z-ki|\, e^{i\arg(z-ki)}$ *, so we consider* $\arg(z-ki)$.

$$
\begin{aligned}
\arg(z-i) &= \frac{1}{2}\pi \Rightarrow \arg\Big(\frac{1}{\sqrt{z-i}}\Big) = -\frac{\pi}{4} \\
\arg(z-ki) &= -\frac{1}{2}\pi \Rightarrow \arg\Big(\frac{1}{\sqrt{z-i}}\Big) = \frac{\pi}{4}, k = 2,3,5,6,8
\end{aligned}
$$

*we have*

$$
\begin{aligned}
\frac{1}{f(z)} &= \Big(\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}}\Big)(e^{-\frac{\pi}{4}})(e^{\frac{\pi}{4}})^5 \\
&= \Big(\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}}\Big)(e^{\pi}) \\
&= \Big(\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}}\Big)(-1) \\
&= -R
\end{aligned}
$$

*Compare with* (1) *and* (2) *we find that when we want to obtain true value, the value which we have from Mathematica should multiply* $(-1)$ *, i.e.* $sign(f(z)|_{(I)}) = (-1)sign(f(z)|_{Mathematica})$.

*(3) Using the Lemma 15 to modify :*

$$
\begin{aligned}
\arg(z-i) &= -\frac{3}{2}\pi \Rightarrow \frac{1}{\sqrt{z-i}} \overset{Math.}{=} (-1)\frac{1}{\sqrt{z-i}} \\
\arg(z-ki) &= -\frac{1}{2}\pi \Rightarrow \frac{1}{\sqrt{z-i}} \overset{Math.}{=} \frac{1}{\sqrt{z-ki}}, k = 2,3,5,6,8
\end{aligned}
$$

*we have*

$$\frac{1}{f(z)} \overset{Math.}{=} (-1)\frac{1}{f(z)}$$

*The same result as above difference between in theory and in Mathematica , the difference is a minus sign.*

2. $a^*_{12}$ *: Let* $z = ri$ *where* $r : 1 \xrightarrow{-} 2$ *and* $dz = idr$

   (1) *Analysis in theory :*
   *Since* $z - ki = |z - ki|\, e^{i\arg(z-ki)}$ *, so we consider* $\arg(z - ki)$.

$$\begin{aligned}\arg(z-i) &= \frac{1}{2}\pi \Rightarrow \arg(\frac{1}{\sqrt{z-i}}) = -\frac{\pi}{4} \\ \arg(z-ki) &= -\frac{1}{2}\pi \Rightarrow \arg(\frac{1}{\sqrt{z-ki}}) = \frac{\pi}{4}, k = 2,3,5,6,8\end{aligned}$$

   *we have*

$$\begin{aligned}\frac{1}{f(z)} &= (\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}})(e^{-\frac{\pi}{4}})(e^{\frac{\pi}{4}})^5 \\ &= (\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}})(e^{\pi}) \\ &= (\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}})(-1) \\ &= -R\end{aligned}$$

   (2) *Analysis in Mathematica (no matter in which sheet) :*
   *Since* $z - ki = |z - ki|\, e^{i\arg(z-ki)}$ *, so we consider* $\arg(z - ki)$.

$$\begin{aligned}\arg(z-i) &= \frac{1}{2}\pi \Rightarrow \arg(\frac{1}{\sqrt{z-i}}) = -\frac{\pi}{4} \\ \arg(z-ki) &= -\frac{1}{2}\pi \Rightarrow \arg(\frac{1}{\sqrt{z-ki}}) = \frac{\pi}{4}, k = 2,3,5,6,8\end{aligned}$$

   *we have*

$$\begin{aligned}\frac{1}{f(z)} &= (\prod_{k=1,k\neq 4,7}^{8}|z-ki|^{-\frac{1}{2}})(e^{-\frac{\pi}{4}})(e^{\frac{\pi}{4}})^5\\ &= (\prod_{k=1,k\neq 4,7}^{8}|z-ki|^{-\frac{1}{2}})(e^{\pi})\\ &= (\prod_{k=1,k\neq 4,7}^{8}|z-ki|^{-\frac{1}{2}})(-1)\\ &= -R\end{aligned}$$

*Compare with* (1) *and* (2) *we find the value is same.*

*(3) Using the Lemma 15 to modify :*

$$\begin{aligned}\arg(z-i) &= \frac{1}{2}\pi \Rightarrow \frac{1}{\sqrt{z-i}} \overset{Math.}{=} \frac{1}{\sqrt{z-i}}\\ \arg(z-ki) &= -\frac{1}{2}\pi \Rightarrow \frac{1}{\sqrt{z-i}} \overset{Math.}{=} \frac{1}{\sqrt{z-ki}}, k=2,3,5,6,8\end{aligned}$$

*we have*

$$\frac{1}{f(z)} \overset{Math.}{=} \frac{1}{f(z)}$$

*The same result as above.*

*By* 1. *and* 2. *above , we have*

$$\begin{aligned}\int_{a_1}\frac{1}{f(z)}dz &= \int_{a_1^*}\frac{1}{f(z)}dz\\ &= \int_{a_{11}^*}\frac{1}{f(z)}dz + \int_{a_{12}^*}\frac{1}{f(z)}dz\\ &= -2\int_2^1(\prod_{k=1,k\neq 4,7}^{8}|ri-ki|^{-\frac{1}{2}})idr\\ &= 0.-0.531987i\end{aligned}$$

**Example 18** *Evaluate the integrals of* $1/f(z)$ *over* $a, b$ *cycles for vertical cut where* $f(z)=\sqrt{(z-i)(z-2i)(z-3i)(z-5i)(z-6i)(z-8i)}$.

***Solution 19*** *The detail is in appendix.*

We can integrate $1/f(z)$ over $a, b$ cycles of the Riemann Surface with horizontal cut-structure and with vertical cut-structure. We give some more examples here , and the solution could see in appendix.

**Example 20** *Compute the integrals of* $1/f(z)$ *over every cycles in the Figure below where*

$$f(z) = \sqrt{(z - z_1)(z - z_2)(z - z_3)(z - z_4)(z - z_5)(z - z_6)(z - z_7)(z - z_8)}$$

*for* $z_1 = -2 - i, z_2 = -2 + i, z_3 = -1 - i, z_4 = -1 + i, z_5 = 0 + 0i, z_6 = 0 + i, z_7 = 1 + i, z_8 = 1 + 2i.$

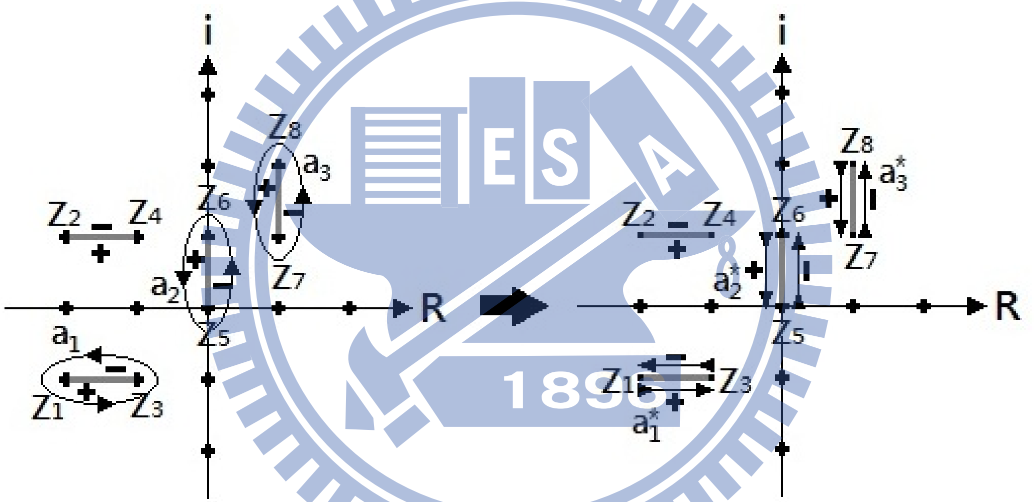



*Figure 29. cycles $a_1, a_2, a_3$ and equivalent pathes $a_1^*, a_2^*, a_3^*$.*

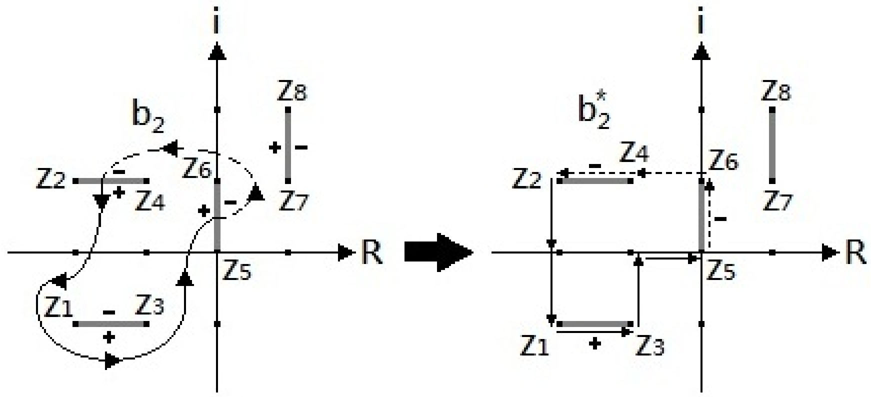



*Figure 30. Cycle $b_2$ and equivalent path $b_2^*$.*

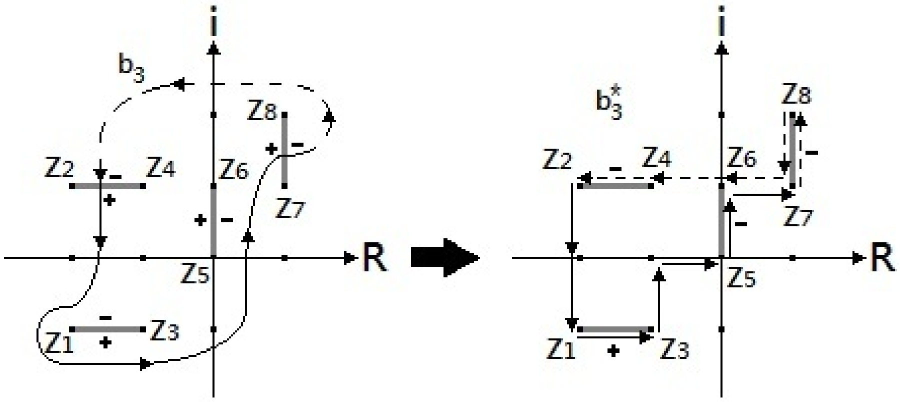



*Figure 31. Cycle $b_3$ and equivalent path $b_3^*$.*

## 2.3 The integrations of the Sine-Gordon Equation over $a, b$ cycles.

Now we want to compute the integral

$$\int \frac{1}{\sqrt{(-1)\frac{u^2}{1} + \frac{u^4}{12} - \frac{u^6}{360} + \frac{u^8}{20160} - \frac{u^{10}}{1814400} + \frac{u^{12}}{239500800} + 2k}} du$$

over $a, b$ cycles.

Let $k = 1$ , and compute the roots of the equation

$$(-1)\frac{u^2}{1} + \frac{u^4}{12} - \frac{u^6}{360} + \frac{u^8}{20160} - \frac{u^{10}}{1814400} + \frac{u^{12}}{239500800} + 2k = 0$$

We have the roots of the equation

$$(-1)\frac{u^2}{1} + \frac{u^4}{12} - \frac{u^6}{360} + \frac{u^8}{20160} - \frac{u^{10}}{1814400} + \frac{u^{12}}{239500800} + 2 = 0$$

are similar to $Z_1 = -6.58948 + 5.23118i$ , $Z_2 = -6.58948 - 5.23118i$ , $Z_3 = -6.31381 + 1.46139i$ , $Z_4 = -6.31381 - 1.46139i$ , $Z_5 = -4.68652 + 0.0i$ , $Z_6 = -1.57080 + 0.0i$ , $Z_7 = 1.57080 + 0.0i$ , $Z_8 = 4.68652 + 0.0i$ , $Z_9 = 6.31381 + 1.46139i$ , $Z_{10} = 6.31381 - 1.46139i$ , $Z_{11} = 6.58948 + 5.23118i$ , $Z_{12} = 6.58948 - 5.23118i$.

So we draw the roots and its cuts in the Firure 32 below

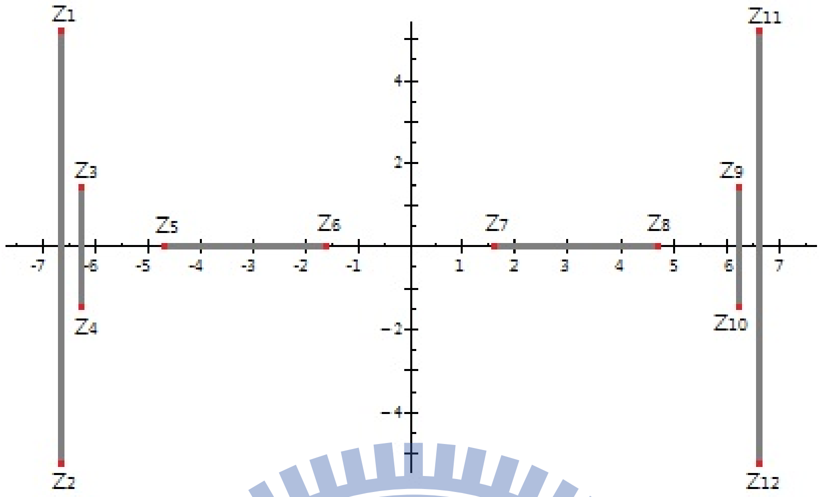



Figure 32. $Z_i, i \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ and its cuts.

First , we will compute the integral of $1/f(z)$ over $a_1, a_2, a_3, a_4, a_5$ cycles in the Figure 33 below where

$$f(z) = \prod_{k=1}^{12} \sqrt{(z - z_k)}$$

and $Z_1 = -6.58948 + 5.23118i$ , $Z_2 = -6.58948 - 5.23118i$ , $Z_3 = -6.31381 + 1.46139i$ , $Z_4 = -6.31381 - 1.46139i$ , $Z_5 = -4.68652 + 0.0i$ , $Z_6 = -1.57080 + 0.0i$ , $Z_7 = 1.57080 + 0.0i$ , $Z_8 = 4.68652 + 0.0i$ , $Z_9 = 6.31381 + 1.46139i$ , $Z_{10} = 6.31381 - 1.46139i$ , $Z_{11} = 6.58948 + 5.23118i$ , $Z_{12} = 6.58948 - 5.23118i$.

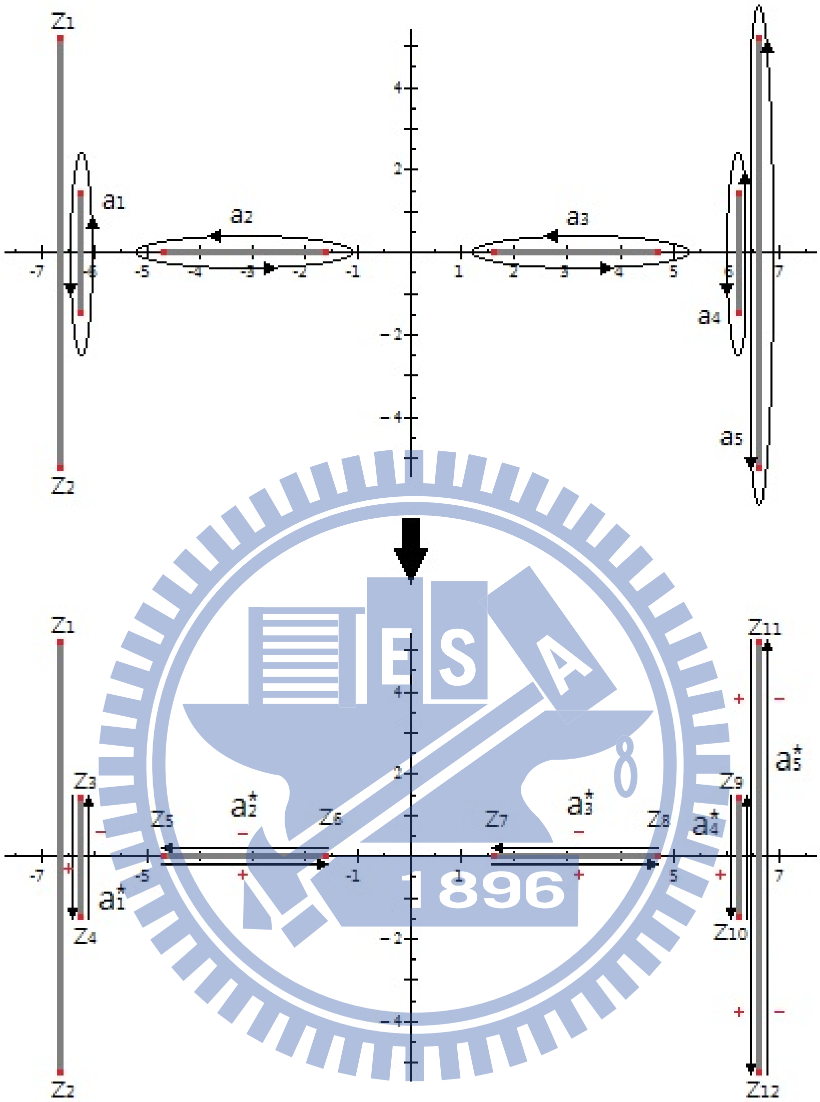



Figure 33. $a_1, a_2, a_3, a_4, a_5$ cycles and its equivalent path $a_1^*, a_2^*, a_3^*, a_4^*, a_5^*$.

We will just write solution here , and the calculation is putted in appendix.

$$\int_{a_1} \frac{1}{f(z)} dz = 9.52646 \times 10^{-18} + 0.000197837i$$
$$\int_{a_2} \frac{1}{f(z)} dz = 2.31913 \times 10^{-14} - 0.000472233i$$
$$\int_{a_3} \frac{1}{f(z)} dz = -2.23575 \times 10^{-14} + 0.000472233i$$
$$\int_{a_4} \frac{1}{f(z)} dz = 9.52151 \times 10^{-18} - 0.000197837i$$
$$\int_{a_5} \frac{1}{f(z)} dz = -1.54107 \times 10^{-17} + 0.000262034i$$

Second , we will compute the integral of $1/f(z)$ over $b_1, b_2, b_3, b_4, b_5$ cycles in the Figure 34 below where

$$f(z) = \prod_{k=1}^{12} \sqrt{(z - z_k)}$$

and $Z_1 = -6.58948 + 5.23118i$ , $Z_2 = -6.58948 - 5.23118i$ , $Z_3 = -6.31381 + 1.46139i$ , $Z_4 = -6.31381 - 1.46139i$ , $Z_5 = -4.68652 + 0.0i$ , $Z_6 = -1.57080 + 0.0i$ , $Z_7 = 1.57080 + 0.0i$ , $Z_8 = 4.68652 + 0.0i$ , $Z_9 = 6.31381 + 1.46139i$ , $Z_{10} = 6.31381 - 1.46139i$ , $Z_{11} = 6.58948 + 5.23118i$ , $Z_{12} = 6.58948 - 5.23118i$.

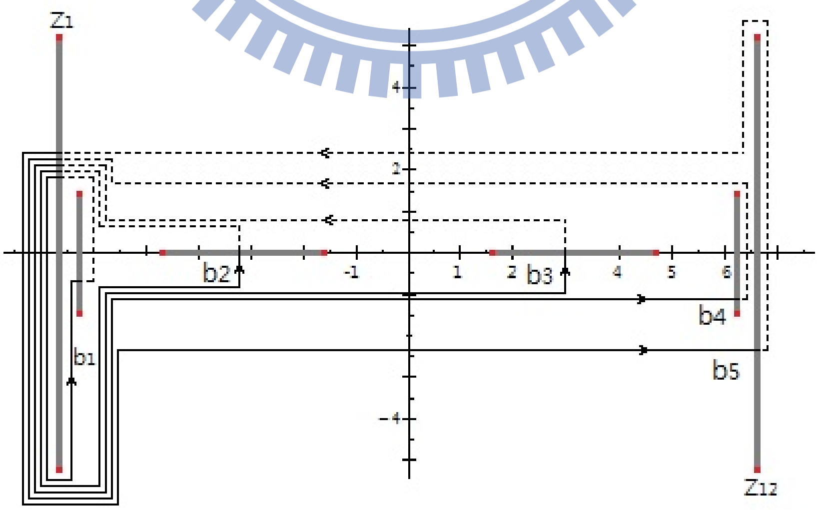



Figure 34. $b_1, b_2, b_3, b_4, b_5$ cycles.

Similarly , we will just write solution here , and the calculation is putted in appendix.

$$\begin{aligned}
\int_{b_1} \frac{1}{f(z)} dz &= -0.0000106043 - 0.0000764721i \\
\int_{b_2} \frac{1}{f(z)} dz &= 0.00025277 + 0.0000169501i \\
\int_{b_3} \frac{1}{f(z)} dz &= -0.000226449 + 0.0000169501i \\
\int_{b_4} \frac{1}{f(z)} dz &= 0.0000523241 + 0.000115868i \\
\int_{b_5} \frac{1}{f(z)} dz &=
\end{aligned}$$

The integrations of the equation over $a, b$ cycles above are all numerical approximation. Can we get exact solution of the Sine-Gordon equation? We need more tools in Mathematics.

# 3 The Elliptic functions , the Theta functions , and the Jacobian Elliptic functions.

When we handle the problem before. It is useful for us to know about the Elliptic functions , the Theta functions and the Jacobian Elliptic functions.

## 3.1 The Elliptic Functions.

**Definition 21** *A function $f(z)$ is called a doubly-periodic function of $z$ with periods $2\omega_1$ , $2\omega_2$ , if function $f(z)$ satisfies the equations below for all values of $z$ for which $f(z)$ exists.*

$$\begin{cases} f(z + 2\omega_1) = f(z) \\ f(z + 2\omega_2) = f(z) \end{cases}$$

*Where $\omega_1$ , $\omega_2$ are any two numbers (complex or real) whose ratio is not purely real.*

**Definition 22** *A doubly-periodic function which is analytic (except at poles) , and which has no singularities other than poles in the finite part of the plane , is called an elliptic function.*

In mathematics , a singularity is in general a point at which a given mathematical object is not defined , or a point of an exceptional set where it fails to be well-behaved in some particular way , such as differentiability.

**Remark 23** *Suppose $f$ is a complex differentiable function defined on some neighborhood $N_p$ around point $p$ , excluding point $p$ i.e. $N_p - \{p\}$ , where $N_p$ is an open subset of the complex numbers $\mathbb{C}$ , and the point $p$ is an element of $N_p$. There are four classes of singularities in complex analysis.*

1. *Isolated singularities : Suppose the function $f$ is not defined at $p$ , although it does have values defined on $N_p - \{p\}$.*

   (1) *The point $p$ is a removable singularity of $f$ if there exists a holomorphic function $g$ defined on all of $N_p$ such that $f(z) = g(z)$ for all $z$ in $N_p - \{p\}$. The function $g$ is a continuous replacement for the function $f$.*

   (2) *The point $p$ is a pole or non-essential singularity of $f$ if there exists a holomorphic function $g$ defined on $N_p$ and a natural number $n$ such that $f(z) = \frac{g(z)}{(z-p)^n}$ for all $z$ in $N_p - \{p\}$. The derivative at a non-essential singularity may or may not exist. If $g(p)$ is nonzero , then we say that $p$ is a pole of order $n$.*

   (3) *The point $p$ is an essential singularity of $f$ if is neither a removable singularity nor a pole. The point $p$ is an essential singularity if and only if the Laurent series has infinitely many powers of negative degree.*

2. *Branch points are generally the result of a multi-valued function, such as $\sqrt{z}$ or $\log(z)$ being defined within a certain limited domain so that the function can be made single-valued within the domain. The cut is a line or curve excluded from the domain to introduce a technical separation between discontinuous values of the function. When the cut is genuinely required , the function will have distinctly different values on each side of the branch cut. The location and shape of most of the branch cut is usually a matter of choice , with perhaps only one point (like $z = 0$ for $\log(z)$) which is fixed in place.*

**Definition 24** *A period-parallelogram is called a cell if there are none of the poles of the integrands considered on the sides of the parallelogram.*

**Definition 25** *A set is called an irreducible set if it is a set of poles ( or zeros ) of an elliptic function in any given cell.*

Remember that all other poles ( or zeros ) of the elliptic function outside the irreducible set are congruent to one or other of them.

There are some simple properties of elliptic functions.

1. The number of poles of an elliptic function in any cell is finite.

   **Proof.** Suppose that the number of poles of an elliptic function $f(z)$ in some cell is not finite , then the poles must have a limit point $p$. Clearly , this point $p$ is a singularity but not a pole. So by definition of elliptic function , the function $f(z)$ is not an elliptic function. $(\rightarrow\leftarrow)$ ■

2. The number of zeros of an elliptic function in any cell is finite.

   **Proof.** Suppose the number of zeros of an elliptic function $f(z)$ in some cell is not finite , then $1/f(z)$ is an elliptic function which have infinite poles in this cell. But this is a contradiction by elliptic function simple property 1. ■

3. The sum of the residues of an elliptic function , $f(z)$ , at its poles in any cell is zero.

   **Proof.** Suppose the corners of the cell are $t, t+2\omega_1, t+2\omega_1+2\omega_2, t+2\omega_2$. Let $C$ be the contour formed by the edges of the cell. The sum of the residues of $f(z)$ at its poles inside $C$ is

$$\frac{1}{2\pi i}\int_C f(z)dz = \frac{1}{2\pi i}\left\{\int_t^{t+2\omega_1} + \int_{t+2\omega_1}^{t+2\omega_1+2\omega_2} + \int_{t+2\omega_1+2\omega_2}^{t+2\omega_2} + \int_{t+2\omega_2}^{t}\right\} f(z)dz$$

Let $x = z - 2\omega_1$ , then $dz = dx$ , thus

$$\begin{aligned}
\frac{1}{2\pi i}\int_{t+2\omega_1}^{t+2\omega_1+2\omega_2} f(z)dz &= \frac{1}{2\pi i}\int_t^{t+2\omega_2} f(x+2\omega_1)dx \\
&= \frac{1}{2\pi i}\int_t^{t+2\omega_2} f(z+2\omega_1)dz \\
&= \frac{1}{2\pi i}\int_t^{t+2\omega_2} f(z)dz
\end{aligned}$$

Let $y = z - 2\omega_2$ , then $dz = dy$ , thus

$$
\begin{aligned}
\frac{1}{2\pi i}\int_{t+2\omega_1+2\omega_2}^{t+2\omega_2} f(z)dz &= \frac{1}{2\pi i}\int_{t+2\omega_1}^{t} f(y+2\omega_2)dy \\
&= \frac{1}{2\pi i}\int_{t+2\omega_1}^{t} f(z+2\omega_2)dz \\
&= \frac{1}{2\pi i}\int_{t+2\omega_1}^{t} f(z)dz
\end{aligned}
$$

So we have

$$
\begin{aligned}
\frac{1}{2\pi i}\int_C f(z)dz &= \frac{1}{2\pi i}\left\{\int_t^{t+2\omega_1} + \int_t^{t+2\omega_2} + \int_{t+2\omega_1}^{t} + \int_{t+2\omega_2}^{t}\right\} f(z)dz \\
&= 0
\end{aligned}
$$

■

4. (Liouville's Theorem) An elliptic function , $f(z)$ , with no poles in a cell is merely a constant.

   **Proof.** Suppose $f(z)$ has no poles inside the cell. $\because f(z)$ is analytic inside and on the boundary of the cell. $\therefore f(z)$ is bounded inside and on the boundary of the cell. $\therefore$there exists a number $K \in \mathbb{R}$ such that $|f(z)| < K$ when $z$ is inside or on the boundary of the cell. $\because f(z)$ is a doubly-periodic function and $f(z)$ is analytic and $|f(z)| < K$ for all values of $z$. $\therefore f(z)$ is a constant. ■

Let $f(z)$ be an elliptic function and $C$ be any cell with corners $t, t+2\omega_1, t+2\omega_1+2\omega_2, t+2\omega_2$ and $a$ be any constant. Because the difference between the number of zeros of $f(z)-a$ and the number of poles of $f(z)-a$ which lie in the cell $C$ is

$$
\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)-a}dz
$$

When we try to compute its value. Since $f(z)$ is an elliptic function , thus $f'(z+2\omega_1) = f'(z+2\omega_2) = f'(z)$.

Let $x = z - 2\omega_1$ , then $dz = dx$ , so we have

$$\begin{aligned}\frac{1}{2\pi i}\int_{t+2\omega_1}^{t+2\omega_1+2\omega_2}\frac{f'(z)}{f(z)-a}dz &= \frac{1}{2\pi i}\int_{t}^{t+2\omega_2}\frac{f'(x+2\omega_1)}{f(x+2\omega_1)-a}dx \\ &= \frac{1}{2\pi i}\int_{t}^{t+2\omega_2}\frac{f'(z+2\omega_1)}{f(z+2\omega_1)-a}dz \\ &= \frac{1}{2\pi i}\int_{t}^{t+2\omega_2}\frac{f'(z)}{f(z)-a}dz\end{aligned}$$

Let $y = z - 2\omega_2$ , then $dz = dy$ , so we have

$$\begin{aligned}\frac{1}{2\pi i}\int_{t+2\omega_1+2\omega_2}^{t+2\omega_2}\frac{f'(z)}{f(z)-a}dz &= \frac{1}{2\pi i}\int_{t+2\omega_1}^{t}\frac{f'(y+2\omega_2)}{f(y+2\omega_2)-a}dy \\ &= \frac{1}{2\pi i}\int_{t+2\omega_1}^{t}\frac{f'(z+2\omega_2)}{f(z+2\omega_2)-a}dz \\ &= \frac{1}{2\pi i}\int_{t+2\omega_1}^{t}\frac{f'(z)}{f(z)-a}dz\end{aligned}$$

Hence , we have

$$\frac{1}{2\pi i}\int_{C}\frac{f'(z)}{f(z)-a}dz = 0$$

Therefore the number of zeros of $f(z)-a$ is equal to the number of poles of $f(z)-a$. Because any pole of $f(z)-a$ is also a pole of $f(z)$ and conversely. Hence the number of zeros of $f(z)-a$ is equal to the number of poles of $f(z)$ , which is independent of $a$. So we have the following definition.

**Definition 26** *The order of an elliptic function $f(z)$ is the number $n$ of roots of the equation*

$$f(z) = a$$

*which lie in any cell depends only on $f(z)$ , but not on $a$. And this number $n$ is also equal to the number of poles of $f(z)$ in the cell.*

**Remark 27** *The order of an elliptic function is never less than 2.*
***Proof.*** *Suppose an elliptic function $f(z)$ of order 1 would have a single irreducible pole ; and if this point actually were a pole ( and not an ordinary point ) the residue there would not be zero , which is contrary to the simple property 3 of elliptic functions.* ■

**Remark 28** *The simplest elliptic functions are those of order 2. There are two classes of such functions :*

1. *Function have a single irreducible double pole with residue is* 0*.*
2. *Function have two simple poles with the residues are numerically equal but opposite in sign.*

**Lemma 29** *The sum of the affixes of the zeros minus the sum of the affixes of the poles is a period.*
***Proof.*** *With the notation previously employed. Because the difference between the sums of the zeros and the sums of the poles is*

$$
\begin{aligned}
\frac{1}{2\pi i}\int_C \frac{zf'(z)}{f(z)}dz &= \frac{1}{2\pi i}\int_t^{t+2\omega_1}\left\{\frac{zf'(z)}{f(z)} - \frac{(z+2\omega_2)f'(z+2\omega_2)}{f(z+2\omega_2)}\right\}dz \\
&\quad -\frac{1}{2\pi i}\int_t^{t+2\omega_2}\left\{\frac{zf'(z)}{f(z)} - \frac{(z+2\omega_1)f'(z+2\omega_1)}{f(z+2\omega_1)}\right\}dz \\
&= \frac{1}{2\pi i}\left\{-2\omega_2\int_t^{t+2\omega_1}\frac{f'(z)}{f(z)}dz + 2\omega_1\int_t^{t+2\omega_2}\frac{f'(z)}{f(z)}dz\right\} \\
&= \frac{1}{2\pi i}\left\{-2\omega_2\left[\log f(z)|_t^{t+2\omega_1}\right] + 2\omega_1\left[\log f(z)|_t^{t+2\omega_2}\right]\right\} \\
&= \frac{1}{2\pi i}\left\{-2\omega_2\left[\log\frac{f(t+2\omega_1)}{f(t)}\right] + 2\omega_1\left[\log\frac{f(t+2\omega_2)}{f(t)}\right]\right\} \\
&= \frac{1}{2\pi i}\left\{-2\omega_2\left[\log\frac{f(t)}{f(t)}\right] + 2\omega_1\left[\log\frac{f(t)}{f(t)}\right]\right\} \\
&= \frac{1}{2\pi i}\left\{-2\omega_2\left[\log(1)\right] + 2\omega_1\left[\log(1)\right]\right\}
\end{aligned}
$$

*on making use of the substitutions used in simple property* 3 *of elliptic functions and of the periodic properties of* $f(z)$ *and* $f'(z)$*.*

$\because$

$$
\begin{aligned}
e^{2k\pi i} &= \cos(2k\pi) + i\sin(2k\pi) \\
&= 1
\end{aligned}
$$

$\therefore$

$$
\begin{aligned}
\log(1) &= \log(e^{2k\pi i}) \\
&= 2k\pi i
\end{aligned}
$$

*for all* $k \in \mathbb{Z}$. *Thus* ,

$$\begin{aligned}\frac{1}{2\pi i}\int_C \frac{zf'(z)}{f(z)}dz &= \frac{1}{2\pi i}\left\{-2\omega_2\left[\log(1)\right]+2\omega_1\left[\log(1)\right]\right\}\\ &= \frac{1}{2\pi i}\left\{-2\omega_2(-2n\pi i)+2\omega_1(2m\pi i)\right\}\\ &= 2m\omega_1+2n\omega_2\end{aligned}$$

*where* $m, n \in \mathbb{Z}$. ■

## 3.2 Weierstrass Elliptic function.

After knowing some basic properties of elliptic function. We will introduce the Weierstrass elliptic function.

**Definition 30** *The Weierstrass elliptic function* $\wp(z)$ *is defined by the equation*

$$\wp(z) = \frac{1}{z^2} + \sum_{m,n}{}' \left\{ \frac{1}{(z-2m\omega_1-2n\omega_2)^2} - \frac{1}{(2m\omega_1+2n\omega_2)^2} \right\}$$

*The summation extends over all integer values ( positive , negative , and zero ) of* $m$ *and* $n$ *, but simultaneous zero values of* $m$ *and* $n$ *excepted.*

Throughout this paper we will use the notation $\sum_{m,n}$ to denote a summation over all integer values of $m$ and $n$ , and using $\sum_{m,n}{}'$ when the term for which $m=n=0$ has to be omitted from the summation. Sometimes , for brevity , we write $\Omega_{m,n}$ in place of $2m\omega_1+2n\omega_2$ , so that

$$\begin{aligned}\wp(z) &= \frac{1}{z^2} + \sum_{m,n}{}' \left\{ \frac{1}{(z-2m\omega_1-2n\omega_2)^2} - \frac{1}{(2m\omega_1+2n\omega_2)^2} \right\}\\ &= z^{-2} + \sum_{m,n}{}' \left\{ (z-\Omega_{m,n})^{-2} - \Omega_{m,n}^{-2} \right\}\end{aligned}$$

**Remark 31** *When* $m, n$ *such that* $|\Omega_{m,n}|$ *is large , the general terms of the series defining* $\wp(z)$ *is* $O(|\Omega_{m,n}|^{-3})$. *Hence* $\wp(z)$ *converges absolutely and uniformly.*

**Remark 32** *$\wp(z)$ is analytic except the poles , namely the points $\Omega_{m,n}$ and the points $\Omega_{m,n}$ are all double poles.*

We now proceed to discuss properties of $\wp(z)$ and properties of $\wp'(z)$.

1. Periodicity and other properties of $\wp(z)$.

   Since $\wp(z)$ is a uniformly convergent series of analytic functions , term-by-term differentiation is legitimate , hence

$$\begin{aligned} \wp'(z) &= \frac{d}{dz}\wp(z) \\ &= -2\frac{1}{z^3} + {\sum_{m,n}}' -2\frac{1}{(z-\Omega_{m,n})^3} \\ &= -2\sum_{m,n}\frac{1}{(z-\Omega_{m,n})^3} \end{aligned}$$

   Since the set of points $-\Omega_{m,n}$ is the same as the set $\Omega_{m,n}$ , and the series for $\wp'(z)$ being absolutely convergent. The derangement of the terms does not affect its sum , thus

$$\begin{aligned} \wp'(-z) &= -2\sum_{m,n}\frac{1}{(-z-\Omega_{m,n})^3} \\ &= -\left[-2\sum_{m,n}\frac{1}{(z+\Omega_{m,n})^3}\right] \\ &= -\left[-2\sum_{m,n}\frac{1}{(z-\Omega_{m,n})^3}\right] \\ &= -\wp'(z) \end{aligned}$$

   Hence the function $\wp'(z)$ is an odd function of $z$.

   In similar manner , the series for $\wp(z)$ being absolutely convergent. The derangement of the terms does not affect its sum , thus

$$\begin{aligned}
\wp(-z) &= \frac{1}{(-z)^2} + \sum_{m,n}{}' \left\{ \frac{1}{(-z-\Omega_{m,n})^2} - \frac{1}{(\Omega_{m,n})^2} \right\} \\
&= \frac{1}{(z)^2} + \sum_{m,n}{}' \left\{ \frac{1}{(z+\Omega_{m,n})^2} - \frac{1}{(\Omega_{m,n})^2} \right\} \\
&= \frac{1}{(z)^2} + \sum_{m,n}{}' \left\{ \frac{1}{(z-\Omega_{m,n})^2} - \frac{1}{(\Omega_{m,n})^2} \right\} \\
&= \wp(z)
\end{aligned}$$

Hence the function $\wp(z)$ is an even function of $z$.

In similar manner , the series for $\wp'(z)$ being absolutely convergent. The derangement of the terms does not affect its sum , thus

$$\begin{aligned}
\wp'(z+2\omega_1) &= -2\sum_{m,n} \frac{1}{(z-\Omega_{m,n}+2\omega_1)^3} \\
&= -2\sum_{m,n} \frac{1}{(z-2m\omega_1-2n\omega_2+2\omega_1)^3} \\
&= -2\sum_{m,n} \frac{1}{(z-2(m-1)\omega_1-2n\omega_2)^3} \\
&= -2\sum_{m,n} \frac{1}{(z-\Omega_{m-1,n})^3} \\
&= -2\sum_{m,n} \frac{1}{(z-\Omega_{m,n})^3} \\
&= \wp'(z)
\end{aligned}$$

Hence the function $\wp'(z)$ has the period $2\omega_1$ , in similar manner the function $\wp'(z)$ has the period $2\omega_2$.

Since $\wp'(z)$ is analytic except at its poles , and $\wp'(z)$ is a doubly-periodic function. Hence $\wp'(z)$ is an elliptic function.

If we integrate the equation $\wp'(z+2\omega_1) = \wp'(z)$ , we get

$$\wp(z+2\omega_1) = \wp(z) + K$$

where $K$ is a constant. Putting $z = -\omega_1$ into the equation $\wp(z+2\omega_1) = \wp(z) + K$ , we have

$$\wp(-\omega_1 + 2\omega_1) = \wp(-\omega_1) + K$$

Since $\wp(z)$ is an even function , we have

$$\begin{aligned} \wp(\omega_1) &= \wp(-\omega_1) + K \\ &= \wp(\omega_1) + K \end{aligned}$$

$\therefore K = 0$ , this shows that $\wp(z + 2\omega_1) = \wp(z)$. In similar manner $\wp(z + 2\omega_2) = \wp(z)$. Since $\wp(z)$ is a doubly-periodic function , and $\wp(z)$ has no singularities but poles , it follows that $\wp(z)$ is an elliptic function.

We give the following table as conclusion.

| Function | Definition | Periods | Parity | Poles |
|---|---|---|---|---|
| $\wp(z)$ | $\frac{1}{z^2} + \sum\limits_{m,n}{}' \left\{ \frac{1}{(z-\Omega_{m,n})^2} - \frac{1}{(\Omega_{m,n})^2} \right\}$ | $2\omega_1, 2\omega_2$ | even | $\Omega_{m,n}$ |
| $\wp'(z)$ | $-2\sum\limits_{m,n} \frac{1}{(z-\Omega_{m,n})^3}$ | $2\omega_1, 2\omega_2$ | odd | $\Omega_{m,n}$ |

2. The differential equation satisfied by $\wp(z)$.

Let $f(z) = \wp(z) - z^{-2} = \sum\limits_{m,n}{}' \{(z - \Omega_{m,n})^{-2} - \Omega_{m,n}^{-2}\}$ is analytic in a region of which the origin is an internal point , and it is an even function of $z$. By Taylor's theorem , we have

$$
\begin{aligned}
f(z) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k \\
&= f(0) + f'(0)z + \frac{f^{(2)}(0)}{2!} z^2 + \frac{f^{(3)}(0)}{3!} z^3 + \frac{f^{(4)}(0)}{4!} z^4 + \cdots \\
&= \sum_{m,n}{}' \left\{-2(-\Omega_{m,n})^{-3}\right\} z + \sum_{m,n}{}' \frac{\{6(-\Omega_{m,n})^{-4}\}}{2} z^2 \\
&\quad + \sum_{m,n}{}' \frac{\{-24(-\Omega_{m,n})^{-5}\}}{6} z^3 + \sum_{m,n}{}' \frac{\{120(-\Omega_{m,n})^{-6}\}}{24} z^4 + \cdots \\
&= \sum_{m,n}{}' \left\{-2(-\Omega_{m,n})^{-3}\right\} z + \sum_{m,n}{}' \left\{3(-\Omega_{m,n})^{-4}\right\} z^2 \\
&\quad + \sum_{m,n}{}' \left\{-4(-\Omega_{m,n})^{-5}\right\} z^3 + \sum_{m,n}{}' \left\{5(-\Omega_{m,n})^{-6}\right\} z^4 + \cdots
\end{aligned}
$$

Since $f(z)$ is an even function of $z$ , $\therefore$

$$
\begin{aligned}
f(z) &= f(-z) \\
&= \sum_{m,n}{}' \left\{+2(-\Omega_{m,n})^{-3}\right\} z + \sum_{m,n}{}' \left\{3(-\Omega_{m,n})^{-4}\right\} z^2 \\
&\quad + \sum_{m,n}{}' \left\{+4(-\Omega_{m,n})^{-5}\right\} z^3 + \sum_{m,n}{}' \left\{5(-\Omega_{m,n})^{-6}\right\} z^4 + \cdots
\end{aligned}
$$

When we compute $f(z) + f(-z)$ , we have

$$
\begin{aligned}
2f(z) &= f(z)+f(-z) \\
&= \sum_{m,n}{}' \left\{-2(-\Omega_{m,n})^{-3}\right\} z + \sum_{m,n}{}' \left\{3(-\Omega_{m,n})^{-4}\right\} z^2 \\
&\quad + \sum_{m,n}{}' \left\{-4(-\Omega_{m,n})^{-5}\right\} z^3 + \sum_{m,n}{}' \left\{5(-\Omega_{m,n})^{-6}\right\} z^4 \\
&\quad + \sum_{m,n}{}' \left\{+2(-\Omega_{m,n})^{-3}\right\} z + \sum_{m,n}{}' \left\{3(-\Omega_{m,n})^{-4}\right\} z^2 \\
&\quad + \sum_{m,n}{}' \left\{+4(-\Omega_{m,n})^{-5}\right\} z^3 + \sum_{m,n}{}' \left\{5(-\Omega_{m,n})^{-6}\right\} z^4 + \cdots \\
&= 2\left\{ \sum_{m,n}{}' \left\{3(-\Omega_{m,n})^{-4}\right\} z^2 + \sum_{m,n}{}' \left\{5(-\Omega_{m,n})^{-6}\right\} z^4 + \cdots \right\}
\end{aligned}
$$

So we have $\wp(z) - z^{-2} = \frac{1}{20}g_2 z^2 + \frac{1}{28}g_3 z^4 + O(z^6)$ for sufficiently small values of $|z|$ where

$$
g_2 = 60\sum_{m,n}{}'(\Omega_{m,n})^{-4}, \; g_3 = 140\sum_{m,n}{}'(\Omega_{m,n})^{-6}
$$

and

$$
\wp(z) = z^{-2} + \frac{1}{20}g_2 z^2 + \frac{1}{28}g_3 z^4 + O(z^6)
$$

differentiating the equation , we have

$$
\wp'(z) = -2z^{-3} + \frac{1}{10}g_2 z + \frac{1}{7}g_3 z^3 + O(z^5)
$$

Cubing $\wp(z)$ and squaring $\wp'(z)$ respectively , we have

$$
\begin{aligned}
(\wp(z))^3 &= z^{-6} + \frac{3}{20}g_2 z^{-2} + \frac{3}{28}g_3 + O(z^2) \\
(\wp'(z))^2 &= 4z^{-6} - \frac{2}{5}g_2 z^{-2} - \frac{4}{7}g_3 + O(z^2)
\end{aligned}
$$

Hence

$$(\wp'(z))^2 - 4(\wp(z))^3 = -g_2 z^{-2} - g_3 + O(z^2)$$

and so

$$(\wp'(z))^2 - 4(\wp(z))^3 + g_2\wp(z) + g_3 = O(z^2)$$

$\therefore$ the function $(\wp'(z))^2 - 4(\wp(z))^3 + g_2\wp(z) + g_3$ is analytic at the origin. Since the function $(\wp'(z))^2 - 4(\wp(z))^3 + g_2\wp(z) + g_3$ is an elliptic function , so it is also analytic at all congruent points about the origin point. Since such points are the only possible singularities , so the function $(\wp'(z))^2 - 4(\wp(z))^3 + g_2\wp(z) + g_3$ is an elliptic function with no singularities. Hence $(\wp'(z))^2 - 4(\wp(z))^3 + g_2\wp(z) + g_3$ is a constant by the simple property 4 of elliptic functions. And we can make $z \to 0$ to gain the differential equation

$$(\wp'(z))^2 = 4(\wp(z))^3 - g_2\wp(z) - g_3$$

Hence the function $\wp(z)$ satisfies the differential equation

$$(y')^2 = 4y^3 - g_2 y - g_3$$

where

$$g_2 = 60\sum_{m,n}{}' (\Omega_{m,n})^{-4}, g_3 = 140\sum_{m,n}{}' (\Omega_{m,n})^{-6}$$

Conversely , if numbers $\omega_1, \omega_2$ can be determined such that

$$g_2 = 60\sum_{m,n}{}' (\Omega_{m,n})^{-4}, g_3 = 140\sum_{m,n}{}' (\Omega_{m,n})^{-6}$$

then the general solution of the differential equation $(\frac{dy}{dz})^2 = 4y^3 - g_2 y - g_3$ is

$$y = \wp(\pm z + \alpha)$$

where $\alpha$ is the constant of integration. Since $\wp(z)$ is an even function of $z$ , we can write $y = \wp(z + \alpha)$ without loss of generality.

3. The integral formula for $\wp(z)$.

   When we consider the equation

$$z = \int_{\zeta}^{\infty} (4t^3 - g_2 t - g_3)^{-\frac{1}{2}} dt$$

   where the path of integration may be any curve which does not pass through a zero of $4t^3 - g_2 t - g_3$.

   By The Fundamental Theorem Of Calculus , when we differentiate the equation

$$z = \int_{\zeta}^{\infty} (4t^3 - g_2 t - g_3)^{-\frac{1}{2}} dt$$

   we get the equation

$$(\frac{d\zeta}{dz})^2 = 4\zeta^3 - g_2\zeta - g_3$$

   and so

$$\zeta = \wp(z + \alpha)$$

   where $\alpha$ is a constant. Since $z = \int_{\zeta}^{\infty} (4t^3 - g_2 t - g_3)^{-\frac{1}{2}} dt \to 0$ as $\zeta \to \infty$. So $\alpha$ is a pole of the function $\wp(z)$ i.e. $\alpha$ is of the form $\Omega_{m,n}$ , thus

$$\begin{aligned} \zeta &= \wp(z + \alpha) \\ &= \wp(z + \Omega_{m,n}) \\ &= \wp(z) \end{aligned}$$

So the equation $z = \int_{\zeta}^{\infty}(4t^3 - g_2 t - g_3)^{-\frac{1}{2}}dt$ is called the integral formula for $\wp(z)$ , and it is sometimes written as

$$z = \int_{\wp(z)}^{\infty}(4t^3 - g_2 t - g_3)^{-\frac{1}{2}}dt$$

4. The addition-theorem for the function $\wp(z)$.

   We want to express $\wp(z+y)$ as an algrbraic function of $\wp(z)$ and $\wp(y)$ for general values of $z$ and $y$. Consider the equations

$$\wp'(z) = A\wp(z) + B, \wp'(y) = A\wp(y) + B$$

which determine $A$ and $B$ in terms of $z$ and $y$ unless $\wp(z) = \wp(y)$ , i.e. unless $z \equiv \pm y(\text{mod}\,.2\omega_1, 2\omega_2)$. Since

$$\wp(z) = \frac{1}{z^2} + {\sum_{m,n}}' \left\{ \frac{1}{(z - \Omega_{m,n})^2} - \frac{1}{(\Omega_{m,n})^2} \right\}, \wp'(z) = -2\sum_{m,n} \frac{1}{(z - \Omega_{m,n})^3}$$

, so the function

$$\wp'(x) - A\wp(x) - B$$

has a triple pole at $x = 0$ and it has three , and only three , irreducible zero.( the number of roots of $f(z)$ is equal to the number of poles of $f(z)$ ).

Since the sum of the zeros minus the sum of the poles is a period. So if $x = z, x = y$ are two zeros , the third irreducible zero must be congruent to $-z - y$ , i.e. $-z - y$ is a zero of $\wp'(x) - A\wp(x) - B$. Thus

$$\wp'(-z - y) = A\wp(-z - y) + B$$

Eliminating $A$ and $B$ from the equations

$$\begin{aligned}
\wp'(z) &= A\wp(z) + B \\
\wp'(y) &= A\wp(y) + B \\
\wp'(-z-y) &= A\wp(-z-y) + B
\end{aligned}$$

we have

$$\begin{vmatrix} \wp(z) & \wp'(z) & 1 \\ \wp(y) & \wp'(y) & 1 \\ \wp(z+y) & -\wp'(z+y) & 1 \end{vmatrix} = 0$$

It is called an addition-theorem for the function $\wp(z)$.

**Remark 33** *The following equation is another form of the addition-theorem*

$$\wp(z+y) = \frac{1}{4}\left\{\frac{\wp'(z) - \wp'(y)}{\wp(z) - \wp(y)}\right\}^2 - \wp(z) - \wp(y)$$

*This equation expresses $\wp(z+y)$ explicitly in terms of functions of $z$ and $y$.*

5. The duplication formula for $\wp(z)$.

   Taking the limiting form of the equation $\wp(z+y) = \frac{1}{4}\left\{\frac{\wp'(z)-\wp'(y)}{\wp(z)-\wp(y)}\right\}^2 - \wp(z) - \wp(y)$ when $y$ approaches $z$ , we have

$$\lim_{y\to z}\wp(z+y) = \frac{1}{4}\lim_{y\to z}\left\{\frac{\wp'(z) - \wp'(y)}{\wp(z) - \wp(y)}\right\}^2 - \wp(z) - \lim_{y\to z}\wp(y)$$

   If $2z$ is not a period , we have

$$
\begin{aligned}
\wp(2z) &= \frac{1}{4}\lim_{h\to 0}\left\{\frac{\wp'(z)-\wp'(z+h)}{\wp(z)-\wp(z+h)}\right\}^2 - 2\wp(z) \\
&= \frac{1}{4}\lim_{h\to 0}\left\{\frac{-h\wp''(z)}{-h\wp'(z)}\right\}^2 - 2\wp(z) \\
&= \frac{1}{4}\left\{\frac{\wp''(z)}{\wp'(z)}\right\}^2 - 2\wp(z)
\end{aligned}
$$

by the definition of derivative.

When $2z$ is not a period , the equation

$$\wp(2z) = \frac{1}{4}\left\{\frac{\wp''(z)}{\wp'(z)}\right\}^2 - 2\wp(z)$$

is called the duplication formula.

6. The constants $e_1, e_2, e_3$.

We want to claim that if $\wp(\omega_1) = e_1, \wp(\omega_2) = e_2, \wp(\omega_3) = e_3$ are all unequal where $\omega_3 = -\omega_1 - \omega_2$ , then $e_1, e_2, e_3$ are the roots of the equation

$$4y^3 - g_2 y - g_3 = 0$$

(1) Since $\wp'(z)$ is an odd periodic function , we have

$$
\begin{aligned}
\wp'(\omega_1) &= \wp'(-(-\omega_1)) \\
&= -\wp'(-\omega_1) \\
&= -\wp'(-\omega_1 + 2\omega_1) \\
&= -\wp'(\omega_1)
\end{aligned}
$$

and so

$$\wp'(\omega_1) = 0$$

Similarly ,

$$\wp'(\omega_2) = \wp'(\omega_3) = 0$$

This means that $\wp'(z)$ has three zeros. Since $\wp'(z) = -2\sum_{m,n} \frac{1}{(z-\Omega_{m,n})^3}$ is an elliptic function whose only singularities are triple poles at points congruent to the origin. Therefore that $\wp'(z)$ has three , and only three irreducible zeros which are points congruent to $\omega_1, \omega_2, \omega_3$.

(2) Since $\wp'(z)$ has three irreducible zeros , thus $\wp(z)$ has only two irreducible poles. Clearly , $\wp(\omega_1) - e_1 = e_1 - e_1 = 0$. It follows that the only zero of $\wp(z) - e_1$ is a double zero at point congruent to $\omega_1$. Similarly , the only zeros of $\wp(z) - e_2, \wp(z) - e_3$ are double zeros at points congruent to $\omega_2, \omega_3$ respectively.

(3) Suppose $e_1 = e_2$ , then $\wp(z) - e_1$ has zero at $\omega_2$ which is a point not congruent to $\omega_1$. Thus , $e_1 \neq e_2$. Similarly , $e_2 \neq e_3$ and $e_3 \neq e_1$. Hence $e_1 \neq e_2 \neq e_3$.
Clearly $(\wp'(z))^2 = 4\wp^3(z) - g_2\wp(z) - g_3$ , By (1) , we have

$$\begin{aligned} 4\wp^3(\omega_1) - g_2\wp(\omega_1) - g_3 &= (\wp'(\omega_1))^2 = 0 \\ 4\wp^3(\omega_2) - g_2\wp(\omega_2) - g_3 &= (\wp'(\omega_2))^2 = 0 \\ 4\wp^3(\omega_3) - g_2\wp(\omega_3) - g_3 &= (\wp'(\omega_3))^2 = 0 \end{aligned}$$

This is to say , by (2) , (3) , $e_1, e_2, e_3$ are the roots of the equation

$$4y^3 - g_2 y - g_3 = 0$$

Use the formula connecting roots of equations with their coefficients , we have

$$\begin{aligned} e_1 + e_2 + e_3 &= 0 \\ e_2e_3 + e_3e_1 + e_1e_2 &= -\frac{1}{4}g_2 \\ e_1e_2e_3 &= \frac{1}{4}g_3 \end{aligned}$$

7. The addition of a half-period to the argument of $\wp(z)$.

The form of the addition-theorem is $\wp(z+y) = \frac{1}{4}\left\{\frac{\wp'(z)-\wp'(y)}{\wp(z)-\wp(y)}\right\}^2 - \wp(z) - \wp(y)$ , let $y = \omega_1$ , we have

$$\wp(z+\omega_1) = \frac{1}{4}\left\{\frac{\wp'(z)-\wp'(\omega_1)}{\wp(z)-\wp(\omega_1)}\right\}^2 - \wp(z) - \wp(\omega_1)$$

Since $[\wp'(z)]^2 = 4\wp^3(z) - g_2\wp(z) - g_3 = 4[\wp(z)-e_1][\wp(z)-e_2][\wp(z)-e_3]$ and $e_1+e_2+e_3 = 0$ and $\wp'(\omega_1) = \wp'(\omega_2) = \wp'(\omega_3) = 0$ , we have

$$\begin{aligned}
\wp(z+\omega_1) &= \frac{1}{4}\left\{\frac{\wp'(z)-\wp'(\omega_1)}{\wp(z)-\wp(\omega_1)}\right\}^2 - \wp(z) - \wp(\omega_1) \\
&= \frac{1}{4}\frac{[\wp'(z)]^2}{[\wp(z)-e_1]^2} - \wp(z) - e_1 \\
&= \frac{[\wp(z)-e_2][\wp(z)-e_3]}{\wp(z)-e_1} - \wp(z) - e_1 \\
&= e_1 + \frac{-(e_1+e_2+e_3)\wp(z) + e_2e_3 + e_1(-e_2-e_3) + e_1^2}{\wp(z)-e_1} \\
&= e_1 + \frac{(e_1-e_2)(e_1-e_3)}{\wp(z)-e_1}
\end{aligned}$$

Using similar method , we have

$$\begin{aligned}
\wp(z+\omega_2) &= e_2 + \frac{(e_2-e_3)(e_2-e_1)}{\wp(z)-e_2} \\
\wp(z+\omega_3) &= e_3 + \frac{(e_3-e_1)(e_3-e_2)}{\wp(z)-e_3}
\end{aligned}$$

We collect the addition of a half-period to the argument of $\wp(z)$ as following

$$\begin{aligned}
\wp(z+\omega_1) &= e_1 + \frac{(e_1-e_2)(e_1-e_3)}{\wp(z)-e_1} \\
\wp(z+\omega_2) &= e_2 + \frac{(e_2-e_3)(e_2-e_1)}{\wp(z)-e_2} \\
\wp(z+\omega_3) &= e_3 + \frac{(e_3-e_1)(e_3-e_2)}{\wp(z)-e_3}
\end{aligned}$$

Afetr we introduce some properties about the Weierstrass elliptic function , we will introduce the Jacobian elliptic function. The Weierstrass elliptic function $\wp(z)$ is one of the simplest example for the elliptic function with single double pole.

## 3.3 Jacobian Elliptic Functions.

We will first discuss the Theta-functions before discussing the Jacobian elliptic functions.

**Definition 34** *The theta function is defined by the series*

$$\begin{aligned}
\vartheta(z,q) &= \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2niz} \\
&= 1 + 2\sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos(2nz)
\end{aligned}$$

*where $q = e^{\pi i \tau}$ with $|q| < 1$ , and $\tau$ is a constant complex number whose imaginary part is positive. It is customary to write $\vartheta_4(z,q)$ (Tannery's and Molk's notation) in place of $\vartheta(z,q)$ (Jacobi's notation).*

Before we define the other three types of Theta-functions , we give some properties of $\vartheta_4(z,q)$.

**Remark 35** *By simple computation for $\vartheta_4(z,q) = \sum\limits_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2niz}$ , we have*

$$\begin{aligned}
\vartheta_4(z+\pi,q) &= \vartheta_4(z,q) \\
\vartheta_4(z+\pi\tau,q) &= (-q^{-1}e^{-2iz})\vartheta_4(z,q)
\end{aligned}$$

By the Remark , we say that $\vartheta_4(z,q)$ is a quasi doubly-periodic function of $z$. The numbers 1 and $-q^{-1}e^{-2iz}$ are called the multipliers or periodicity factors associated with the periods $\pi$ and $\tau\pi$ respectively.

**Definition 36** *The other three types of Theta-functions are defined as follows :*

$$\begin{aligned}
\vartheta_1(z,q) &= -ie^{iz+\frac{1}{4}\pi i\tau}\vartheta_4(z+\frac{1}{2}\pi\tau,q)\\
\vartheta_2(z,q) &= \vartheta_1(z+\frac{1}{2}\pi,q)\\
\vartheta_3(z,q) &= \vartheta_4(z+\frac{1}{2}\pi,q)\\
\vartheta_4(z,q) &= \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2}e^{2niz}
\end{aligned}$$

*and in series form*

$$\begin{aligned}
\vartheta_1(z,q) &= 2\sum_{n=0}^{\infty}(-1)^n q^{(n+\frac{1}{2})^2}\sin(2n+1)z\\
\vartheta_2(z,q) &= 2\sum_{n=0}^{\infty}q^{(n+\frac{1}{2})^2}\cos(2n+1)z\\
\vartheta_3(z,q) &= 1+2\sum_{n=1}^{\infty}q^{n^2}\cos 2nz\\
\vartheta_4(z,q) &= 1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos(2nz)
\end{aligned}$$

**Remark 37** .

1. *By Definition above and the parity of trigonometric functions , we have*

| *function* | $\vartheta_1(z,q)$ | $\vartheta_2(z,q)$ | $\vartheta_3(z,q)$ | $\vartheta_4(z,q)$ |
|---|---|---|---|---|
| *parity* | *odd* | *even* | *even* | *even* |

2. *The parameter $q$ will not usually be specified , so we will write $\vartheta_1(z)$ , $\vartheta_2(z)$ , $\vartheta_3(z)$ , $\vartheta_4(z)$ for $\vartheta_1(z,q)$ , $\vartheta_2(z,q)$ , $\vartheta_3(z,q)$ , $\vartheta_4(z,q)$ respectively.*

3. *By Definition in series form above and simple computation , we have*

$$\begin{aligned}
\vartheta_3(z,q) &= \vartheta_3(2z,q^4)+\vartheta_2(2z,q^4)\\
\vartheta_4(z,q) &= \vartheta_3(2z,q^4)-\vartheta_2(2z,q^4)
\end{aligned}$$

4. *By Definition in series form above and simple computation , we have relations between four types of Theta-functions*

$$\begin{aligned}
\vartheta_1(z) &= -\vartheta_2(z+\frac{1}{2}\pi) = -iM\vartheta_3(z+\frac{1}{2}\pi+\frac{1}{2}\pi\tau) = -iM\vartheta_4(z+\frac{1}{2}\pi\tau)\\
\vartheta_2(z) &= M\vartheta_3(z+\frac{1}{2}\pi\tau) = M\vartheta_4(z+\frac{1}{2}\pi+\frac{1}{2}\pi\tau) = \vartheta_1(z+\frac{1}{2}\pi)\\
\vartheta_3(z) &= \vartheta_4(z+\frac{1}{2}\pi) = M\vartheta_1(z+\frac{1}{2}\pi+\frac{1}{2}\pi\tau) = M\vartheta_2(z+\frac{1}{2}\pi\tau)\\
\vartheta_4(z) &= -iM\vartheta_1(z+\frac{1}{2}\pi\tau) = iM\vartheta_2(z+\frac{1}{2}\pi+\frac{1}{2}\pi\tau) = \vartheta_3(z+\frac{1}{2}\pi)
\end{aligned}$$

*where* $M = q^{\frac{1}{4}}e^{iz}$.

5. *The periodicity factors of the four types of Theta-functions associated with the periods* $\pi, \pi\tau$ *are made by the table below*

| | $\vartheta_1(z)$ | $\vartheta_2(z)$ | $\vartheta_3(z)$ | $\vartheta_4(z)$ |
|---|---|---|---|---|
| $\pi$ | -1 | -1 | 1 | 1 |
| $\pi\tau$ | -N | N | N | -N |

*where* $N = q^{-1}e^{-2iz}$.

6. *The Theta-function* $\vartheta(z)$ *satisfy the following equations*

$$\begin{aligned}
\frac{\vartheta'(z+\pi)}{\vartheta(z+\pi)} &= \frac{\vartheta'(z)}{\vartheta(z)}\\
\frac{\vartheta'(z+\pi\tau)}{\vartheta(z+\pi\tau)} &= -2i+\frac{\vartheta'(z)}{\vartheta(z)}
\end{aligned}$$

*where* $\vartheta(z)$ *is any one of the four Theta-functions and* $\vartheta'(z)$ *its derivate with respect to* $z$.

In this paper we will interpret to mean $e^{\lambda\pi i\tau}$ for the many-valued function $q^{\lambda}$.

Suppose $\vartheta(z_0) = 0$ where $\vartheta(z)$ is any one of the four types of Theta-functions. Then

$$\vartheta(z_0 + m\pi + n\pi\tau) = 0$$

from the quasi-periodic properties of the Theta-functions for all $m, n \in \mathbb{Z}$.

**Theorem 38** *If $C$ be a cell with corners $t, t+\pi, t+\pi+\pi\tau, t+\pi\tau$ , then $\vartheta(z)$ has one and only one zero inside $C$ , where $\vartheta(z)$ is any one of the four Theta-functions.*

***Proof.*** *Since $\vartheta(z)$ is analytic throughout the finite part of the $z$-plane , thus the number of its zeros inside $C$ is*

$$\frac{1}{2\pi i}\int_C \frac{\vartheta'(z)}{\vartheta(z)}dz = \frac{1}{2\pi i}\left\{\int_t^{t+\pi} + \int_{t+\pi}^{t+\pi+\pi\tau} + \int_{t+\pi+\pi\tau}^{t+\pi\tau} + \int_{t+\pi\tau}^{t}\right\}\frac{\vartheta'(z)}{\vartheta(z)}dz$$

1. *Let $x = z - \pi$ , then $dx = dz$*

   *and since*

$$\frac{\vartheta'(z+\pi)}{\vartheta(z+\pi)} = \frac{\vartheta'(z)}{\vartheta(z)}$$

   *we have*

$$\begin{aligned}\frac{1}{2\pi i}\int_{t+\pi}^{t+\pi+\pi\tau}\frac{\vartheta'(z)}{\vartheta(z)}dz &= \frac{1}{2\pi i}\int_t^{t+\pi\tau}\frac{\vartheta'(x+\pi)}{\vartheta(x+\pi)}dx \\ &= \frac{1}{2\pi i}\int_t^{t+\pi\tau}\frac{\vartheta'(x)}{\vartheta(x)}dx \\ &= \frac{1}{2\pi i}\int_t^{t+\pi\tau}\frac{\vartheta'(z)}{\vartheta(z)}dz\end{aligned}$$

2. *Let $y = z - \pi\tau$ , then $dy = dz$*

   *and since*

$$\frac{\vartheta'(z+\pi\tau)}{\vartheta(z+\pi\tau)} = -2i + \frac{\vartheta'(z)}{\vartheta(z)}$$

*we have*

$$\begin{aligned}
\frac{1}{2\pi i}\int_{t+\pi+\pi\tau}^{t+\pi\tau} \frac{\vartheta'(z)}{\vartheta(z)} dz &= \frac{1}{2\pi i}\int_{t+\pi}^{t} \frac{\vartheta'(y+\pi\tau)}{\vartheta(y+\pi\tau)} dy \\
&= -\frac{1}{2\pi i}\int_{t}^{t+\pi} \frac{\vartheta'(y+\pi\tau)}{\vartheta(y+\pi\tau)} dy \\
&= -\frac{1}{2\pi i}\int_{t}^{t+\pi} -2i + \frac{\vartheta'(y)}{\vartheta(y)} dy \\
&= -\frac{1}{2\pi i}\int_{t}^{t+\pi} -2i + \frac{\vartheta'(z)}{\vartheta(z)} dz \\
&= 1 - \frac{1}{2\pi i}\int_{t}^{t+\pi} \frac{\vartheta'(z)}{\vartheta(z)} dz
\end{aligned}$$

*By* 1. , 2. , *we have*

$$\begin{aligned}
\frac{1}{2\pi i}\int_C \frac{\vartheta'(z)}{\vartheta(z)} dz &= \frac{1}{2\pi i}\left\{\int_t^{t+\pi} + \int_{t+\pi}^{t+\pi+\pi\tau} + \int_{t+\pi+\pi\tau}^{t+\pi\tau} + \int_{t+\pi\tau}^{t}\right\} \frac{\vartheta'(z)}{\vartheta(z)} dz \\
&= 1
\end{aligned}$$

*Hence* $\vartheta(z)$ *has one and only one zero inside* $C$. ■

**Remark 39 (The zeros of the Theta-functions.)** *The zeros of* $\vartheta_1(z)$ , $\vartheta_2(z)$ , $\vartheta_3(z)$ , $\vartheta_4(z)$ *are the points congruent to* $0$ , $\frac{1}{2}\pi$ , $\frac{1}{2}\pi + \frac{1}{2}\pi\tau$ , $\frac{1}{2}\pi\tau$ *respectively.*

***Proof.*** *Clearly ,* $z = 0$ *is a zero of* $\vartheta_1(z,q) = 2\sum_{n=0}^{\infty}(-1)^n q^{(n+\frac{1}{2})^2}\sin(2n+1)z$ *by definition. Hence* $z = 0 + m\pi + n\pi\tau$ *is also a zero of* $\vartheta_1(z,q)$ *for all* $m, n \in \mathbb{Z}$. *This means the zeros of* $\vartheta_1(z)$ *are the points congruent to* 0. *By the relations between four types of Theta-functions*

$$\vartheta_1(z) = -\vartheta_2(z+\frac{1}{2}\pi) = -iM\vartheta_3(z+\frac{1}{2}\pi+\frac{1}{2}\pi\tau) = -iM\vartheta_4(z+\frac{1}{2}\pi\tau)$$

*where* $M = q^{\frac{1}{4}} e^{iz}$. *We find that*

$$\vartheta_1(0) = -\vartheta_2(\frac{1}{2}\pi) = -iM\vartheta_3(\frac{1}{2}\pi + \frac{1}{2}\pi\tau) = -iM\vartheta_4(\frac{1}{2}\pi\tau) = 0$$

*where* $M = q^{\frac{1}{4}} e^{iz}$ , *thus*

$$\vartheta_1(0) = \vartheta_2(\frac{1}{2}\pi) = \vartheta_3(\frac{1}{2}\pi + \frac{1}{2}\pi\tau) = \vartheta_4(\frac{1}{2}\pi\tau) = 0$$

*So* $z = \frac{1}{2}\pi, \frac{1}{2}\pi + \frac{1}{2}\pi\tau, \frac{1}{2}\pi\tau$ *are zeros of* $\vartheta_2(z), \vartheta_3(z), \vartheta_4(z)$ *respectively. Hence , the zeros of* $\vartheta_1(z), \vartheta_2(z), \vartheta_3(z), \vartheta_4(z)$ *are the points congruent to* $0, \frac{1}{2}\pi, \frac{1}{2}\pi + \frac{1}{2}\pi\tau, \frac{1}{2}\pi\tau$ *respectively.* ∎

We summarized the result as the following table.

| | Zeros | Relations |
|---|---|---|
| $\vartheta_1(z,q)$ | $z = 0 \bmod (\pi, \pi\tau)$ | $2\sum_{n=0}^{\infty}(-1)^n q^{(n+\frac{1}{2})^2} \sin(2n+1)z$ |
| $\vartheta_2(z,q)$ | $z = \frac{1}{2}\pi \bmod (\pi, \pi\tau)$ | $\vartheta_1(z,q) = -\vartheta_2(z + \frac{1}{2}\pi, q)$ |
| $\vartheta_3(z,q)$ | $z = \frac{1}{2}\pi + \frac{1}{2}\pi\tau \bmod (\pi, \pi\tau)$ | $\vartheta_1(z,q) = -iq^{\frac{1}{4}} e^{iz}\vartheta_3(z + \frac{1}{2}\pi + \frac{1}{2}\pi\tau, q)$ |
| $\vartheta_4(z,q)$ | $z = \frac{1}{2}\pi\tau \bmod (\pi, \pi\tau)$ | $\vartheta_1(z,q) = -iq^{\frac{1}{4}} e^{iz}\vartheta_4(z + \frac{1}{2}\pi\tau, q)$ |

**Theorem 40** *It is possible to express any Theta-function in terms of any other pair of Theta-functions by the following equations.*

$$\begin{aligned}
\vartheta_4^2(0)\vartheta_1^2(z) &= \vartheta_2^2(0)\vartheta_3^2(z) - \vartheta_3^2(0)\vartheta_2^2(z) \\
\vartheta_4^2(0)\vartheta_2^2(z) &= \vartheta_2^2(0)\vartheta_4^2(z) - \vartheta_3^2(0)\vartheta_1^2(z) \\
\vartheta_4^2(0)\vartheta_3^2(z) &= \vartheta_3^2(0)\vartheta_4^2(z) - \vartheta_2^2(0)\vartheta_1^2(z) \\
\vartheta_4^2(0)\vartheta_4^2(z) &= \vartheta_3^2(0)\vartheta_3^2(z) - \vartheta_2^2(0)\vartheta_2^2(z)
\end{aligned}$$

***Proof.*** *Since each of the four functions* $\vartheta_1^2(z), \vartheta_2^2(z), \vartheta_3^2(z), \vartheta_4^2(z)$ *is analytic and has periodicity factors* $1, q^{-2}e^{-4iz}$ *associated with the periods* $\pi, \pi\tau$. *Thus each of the functions*

$$\frac{a\vartheta_1^2(z) + b\vartheta_4^2(z)}{\vartheta_2^2(z)}, \frac{a'\vartheta_1^2(z) + b'\vartheta_4^2(z)}{\vartheta_3^2(z)}$$

*is a doubly-periodic function ( with periods* $\pi, \pi\tau$ *) where the constants* $a, b, a', b'$ *are suitably chosen.*

*Since each of the four functions* $\vartheta_1^2(z), \vartheta_2^2(z), \vartheta_3^2(z), \vartheta_4^2(z)$ *has a double zero ( and no other zeros ) in any cell. Thus each of the functions*

$$\frac{a\vartheta_1^2(z) + b\vartheta_4^2(z)}{\vartheta_2^2(z)}, \frac{a'\vartheta_1^2(z) + b'\vartheta_4^2(z)}{\vartheta_3^2(z)}$$

*have at most only a simple pole in each cell where the constants* $a, b, a', b'$ *are suitably chosen. But the order of an elliptic function is never less than 2 otherwise such a function is merely a constant.*

*So we assume that there exist* $a, b, a', b'$ *such that*

$$\frac{a\vartheta_1^2(z) + b\vartheta_4^2(z)}{\vartheta_2^2(z)} = 1, \frac{a'\vartheta_1^2(z) + b'\vartheta_4^2(z)}{\vartheta_3^2(z)} = 1$$

*i.e.*

$$\vartheta_2^2(z) = a\vartheta_1^2(z) + b\vartheta_4^2(z), \vartheta_3^2(z) = a'\vartheta_1^2(z) + b'\vartheta_4^2(z)$$

*Using the relations between four types of Theta-functions*

$$\begin{aligned}
\vartheta_1(z) &= -\vartheta_2(z + \frac{1}{2}\pi) = -iM\vartheta_3(z + \frac{1}{2}\pi + \frac{1}{2}\pi\tau) = -iM\vartheta_4(z + \frac{1}{2}\pi\tau) \\
\vartheta_2(z) &= M\vartheta_3(z + \frac{1}{2}\pi\tau) = M\vartheta_4(z + \frac{1}{2}\pi + \frac{1}{2}\pi\tau) = \vartheta_1(z + \frac{1}{2}\pi) \\
\vartheta_3(z) &= \vartheta_4(z + \frac{1}{2}\pi) = M\vartheta_1(z + \frac{1}{2}\pi + \frac{1}{2}\pi\tau) = M\vartheta_2(z + \frac{1}{2}\pi\tau) \\
\vartheta_4(z) &= -iM\vartheta_1(z + \frac{1}{2}\pi\tau) = iM\vartheta_2(z + \frac{1}{2}\pi + \frac{1}{2}\pi\tau) = \vartheta_3(z + \frac{1}{2}\pi)
\end{aligned}$$

*where* $M = q^{\frac{1}{4}}e^{iz}$.

*We have*

$$\begin{aligned}
\vartheta_1(\frac{1}{2}\pi\tau) &= iq^{-\frac{1}{4}}\vartheta_4(0) \\
\vartheta_2(\frac{1}{2}\pi\tau) &= q^{-\frac{1}{4}}\vartheta_3(0) \\
\vartheta_3(\frac{1}{2}\pi\tau) &= q^{-\frac{1}{4}}\vartheta_2(0) \\
\vartheta_4(\frac{1}{2}\pi\tau) &= 0
\end{aligned}$$

To determine $a, b, a', b'$ , let $z = \frac{1}{2}\pi\tau$ and $z = 0$ into the equations $\vartheta_2^2(z) = a\vartheta_1^2(z) + b\vartheta_4^2(z)$ , $\vartheta_3^2(z) = a'\vartheta_1^2(z) + b'\vartheta_4^2(z)$ respectively. Then we have

$$\vartheta_3^2 = -a\vartheta_4^2, \vartheta_2^2 = b\vartheta_4^2(z); \vartheta_2^2(z) = -a'\vartheta_4^2, \vartheta_3^2 = b'\vartheta_4^2$$

So we obtained the relations

$$\begin{aligned} \vartheta_4^2(0)\vartheta_2^2(z) &= \vartheta_2^2(0)\vartheta_4^2(z) - \vartheta_3^2(0)\vartheta_1^2(z) \\ \vartheta_4^2(0)\vartheta_3^2(z) &= \vartheta_3^2(0)\vartheta_4^2(z) - \vartheta_2^2(0)\vartheta_1^2(z) \end{aligned}$$

Replace $z$ by $z + \frac{1}{2}\pi$ , we have

$$\begin{aligned} \vartheta_4^2(0)\vartheta_1^2(z) &= \vartheta_2^2(0)\vartheta_3^2(z) - \vartheta_3^2(0)\vartheta_2^2(z) \\ \vartheta_4^2(0)\vartheta_4^2(z) &= \vartheta_3^2(0)\vartheta_3^2(z) - \vartheta_2^2(0)\vartheta_2^2(z) \end{aligned}$$

■

**Corollary 41** *In the last relation , we write $z = 0$ to get the equation*

$$\vartheta_2^4(0) + \vartheta_4^4(0) = \vartheta_3^4(0)$$

**Remark 42** *The addition-formula for the Theta-functions is in the following equation.*

$$\vartheta_3(z+y)\vartheta_3(z-y)\vartheta_3^2 = \vartheta_3^2(y)\vartheta_3^2(z) + \vartheta_1^2(y)\vartheta_1^2(z)$$

***Proof.*** *Clearly the function $\vartheta_3(z+y)\vartheta_3(z-y)$ of $z$ has periodicity factors associated with the periods $\pi$ and $\pi\tau$ are 1 and $q^{-1}e^{-2i(z+y)} \cdot q^{-1}e^{-2i(z-y)} = q^{-2}e^{-4iz}$ , and the function $a\vartheta_3^2(z) + b\vartheta_1^2(z)$ has the same periodicity factors where $a, b \in \mathbb{C}$ are constants. So we can choose the ratio $a : b$ such that the doubly-periodic function*

$$\frac{a\vartheta_3^2(z) + b\vartheta_1^2(z)}{\vartheta_3(z+y)\vartheta_3(z-y)}$$

*has no poles at the zeros of $\vartheta_3(z-y)$ , then it at most has a single simple pole in any cell. And this simple is the zero of $\vartheta_3(z+y)$ in that cell. Since* *the order of an elliptic function is never less than 2 otherwise such a function is merely a constant. So we may choose suitable $a, b$ for the ratio $a : b$ such that*

$$\frac{a\vartheta_3^2(z) + b\vartheta_1^2(z)}{\vartheta_3(z+y)\vartheta_3(z-y)} = 1$$

*i.e.*

$$a\vartheta_3^2(z) + b\vartheta_1^2(z) = \vartheta_3(z+y)\vartheta_3(z-y)$$

*To determine $a$ and $b$ , we put $z = 0$ and $z = \frac{1}{2}\pi + \frac{1}{2}\pi\tau$ into the equation $a\vartheta_3^2(z) + b\vartheta_1^2(z) = \vartheta_3(z+y)\vartheta_3(z-y)$ respectively. Since $\vartheta_3(z)$ is an even function , so we get*

$$a\vartheta_3^2 = \vartheta_3^2(y), b\vartheta_1^2(\frac{1}{2}\pi + \frac{1}{2}\pi\tau) = \vartheta_3(\frac{1}{2}\pi + \frac{1}{2}\pi\tau + y)\vartheta_3(\frac{1}{2}\pi + \frac{1}{2}\pi\tau - y)$$

*Using the relations between Theta-functions*

$$\vartheta_3(z) = q^{\frac{1}{4}}e^{iz}\vartheta_1(z + \frac{1}{2}\pi + \frac{1}{2}\pi\tau), \vartheta_1(z) = -iq^{\frac{1}{4}}e^{iz}\vartheta_3(z + \frac{1}{2}\pi + \frac{1}{2}\pi\tau)$$

*we have*

$$\vartheta_1^2(\frac{1}{2}\pi + \frac{1}{2}\pi\tau) = q^{-\frac{1}{2}}\vartheta_3^2, \vartheta_3(\frac{1}{2}\pi + \frac{1}{2}\pi\tau + y)\vartheta_3(\frac{1}{2}\pi + \frac{1}{2}\pi\tau - y) = q^{-\frac{1}{2}}\vartheta_1^2(y)$$

*So we get*

$$a = \frac{\vartheta_3^2(y)}{\vartheta_3^2}, b = \frac{\vartheta_1^2(y)}{\vartheta_3^2}$$

*Hence*

$$\vartheta_3(z+y)\vartheta_3(z-y)\vartheta_3^2 = \vartheta_3^2(y)\vartheta_3^2(z) + \vartheta_1^2(y)\vartheta_1^2(z)$$

■

**Remark 43 (Jacobi's fundamental formula.)** *Suppose*

$$\begin{aligned} 2w' &= -w + x + y + z \\ 2x' &= w - x + y + z \\ 2y' &= w + x - y + z \\ 2z' &= w + x + y - z \end{aligned}$$

*and let*

$$[r] = \vartheta_r(w)\vartheta_r(x)\vartheta_r(y)\vartheta_r(z), [r]' = \vartheta_r(w')\vartheta_r(x')\vartheta_r(y')\vartheta_r(z')$$

*where we will consider* $[r], [r]'$ *qua functions of* $z$*. Then we have*

$$\begin{aligned} 2[3] &= -[1]' + [2]' + [3]' + [4]' \\ 2[4] &= [1]' - [2]' + [3]' + [4]' \\ 2[2] &= [1]' + [2]' + [3]' - [4]' \\ 2[1] &= [1]' + [2]' - [3]' + [4]' \end{aligned}$$

***Proof.*** *By the simple computation , the effect of increasing* $z$ *by* $\pi$ *or* $\pi\tau$ *is to transform the functions in the first row of the following table into those in the second row respectively.*

| | $[3]$ | $[1]'$ | $[2]'$ | $[3]'$ | $[4]'$ |
|---|---|---|---|---|---|
| $\pi$ | $[3]$ | $-[2]'$ | $-[1]'$ | $[4]'$ | $[3]'$ |
| $\pi\tau$ | $N[3]$ | $-N[4]'$ | $N[3]'$ | $N[2]'$ | $-N[1]'$ |

*where* $N = q^{-\frac{1}{2}}e^{-2iz}$.

*In the table , we know that both* $-[1]' + [2]' + [3]' + [4]'$ *and* $[3]$ *have periodicity factors* $1$ *and* $N$*. Thus the quotient*

$$\frac{-[1]' + [2]' + [3]' + [4]'}{[3]}$$

*is a doubly-periodic function. And in any cell this function have at most a single simple pole namely the zero of* $\vartheta_3(z)$ *in that cell. Since the order of an elliptic function is never less than 2 otherwise such a function is merely a constant. So this quotient is merely a constant ; i.e. independent of* $z$*. And by considerations of symmetry , we know that this quotient is also independent of* $w, x, y$*. So we let*

$$A[3] = -[1]' + [2]' + [3]' + [4]'$$

*where $A$ is independent of $w, x, y, z$. Let $w = x = y = z = 0$ , then we get*

$$A\vartheta_3^4 = \vartheta_2^4 + \vartheta_3^4 + \vartheta_4^4$$

*By the corollary before , because $\vartheta_2^4 + \vartheta_4^4 = \vartheta_3^4$ ,we get $A = 2$. Therefore*

$$2[3] = -[1]' + [2]' + [3]' + [4]'$$

*Increasing $w, x, y, z$ by $\frac{1}{2}\pi$ and therefore $w', x', y', z'$ will also increase by $\frac{1}{2}\pi$ , then we get*

$$2[4] = [1]' - [2]' + [3]' + [4]'$$

*Increasing $w, x, y, z$ by $\frac{1}{2}\pi\tau$ in*

$$\begin{aligned} 2[3] &= -[1]' + [2]' + [3]' + [4]' \\ 2[4] &= [1]' - [2]' + [3]' + [4]' \end{aligned}$$

*we will get*

$$\begin{aligned} 2[2] &= [1]' + [2]' + [3]' - [4]' \\ 2[1] &= [1]' + [2]' - [3]' + [4]' \end{aligned}$$

■

**Remark 44** *We can express Theta-functions as infinite products in the following*

$$\begin{aligned} \vartheta_1(z) &= 2Gq^{\frac{1}{4}} \sin z \prod_{n=1}^{\infty} (1 - 2q^{2n} \cos 2z + q^{4n}) \\ \vartheta_2(z) &= 2Gq^{\frac{1}{4}} \cos z \prod_{n=1}^{\infty} (1 + 2q^{2n} \cos 2z + q^{4n}) \\ \vartheta_3(z) &= G \prod_{n=1}^{\infty} (1 + 2q^{2n-1} \cos 2z + q^{4n-2}) \\ \vartheta_4(z) &= G \prod_{n=1}^{\infty} (1 - 2q^{2n-1} \cos 2z + q^{4n-2}) \end{aligned}$$

*where $G$ is independent of $z$.*

***Proof.*** *Let*

$$\begin{aligned} f(z) &= \prod_{n=1}^{\infty}(1-q^{2n-1}e^{2iz})\prod_{n=1}^{\infty}(1-q^{2n-1}e^{-2iz}) \\ &= \prod_{n=1}^{\infty}(1-2q^{2n-1}\cos 2z+q^{4n-2}) \end{aligned}$$

*Since $\sum_{n=1}^{\infty} q^{2n-1}$ is convergent absolutely , thus each of the two products converges absolutely and uniformly in any bounded domain of values of $z$. Hence $f(z)$ is analytic throughout the finite part of the $z$-plane , and so it is an integral function.*

*The zeros of $f(z)$ are simple zeros at the points where*

$$e^{2iz} = e^{(2n+1)\pi i\tau}, (n \in \mathbb{Z})$$

*i.e. where*

$$2iz = (2n+1)\pi i\tau + 2m\pi i$$

*, thus $f(z)$ and $\vartheta_4(z)$ have the same zeros. Hence the quotient $\vartheta_4(z)/f(z)$ has neither zeros nor poles in the finite part of the plane.*

*Clearly , $f(z+\pi) = f(z)$ and*

$$\begin{aligned} f(z+\pi\tau) &= \prod_{n=1}^{\infty}(1-q^{2n+1}e^{2iz})\prod_{n=1}^{\infty}(1-q^{2n-3}e^{-2iz}) \\ &= f(z)\frac{1-q^{-1}e^{-2iz}}{1-qe^{2iz}} \\ &= -q^{-1}e^{-2iz}f(z) \end{aligned}$$

*, thus $f(z)$ and $\vartheta_4(z)$ have the same periodicity factors. Therefore $\vartheta_4(z)/f(z)$ is a doubly-periodic function with no zeros or poles , so it is a constant $G$. Hence*

$$\begin{aligned} \vartheta_4(z) &= Gf(z) \\ &= G\prod_{n=1}^{\infty}(1-2q^{2n-1}\cos 2z+q^{4n-2}) \end{aligned}$$

*Write $z + \frac{1}{2}\pi$ for $z$ , we get*

$$\vartheta_3(z) = G\prod_{n=1}^{\infty}(1 + 2q^{2n-1}\cos 2z + q^{4n-2})$$

*And*

$$\begin{aligned}
\vartheta_1(z) &= -iq^{\frac{1}{4}}e^{iz}\vartheta_4(z + \frac{1}{2}\pi\tau) \\
&= -iq^{\frac{1}{4}}e^{iz}G\prod_{n=1}^{\infty}(1 - q^{2n}e^{2iz})\prod_{n=1}^{\infty}(1 - q^{2n-2}e^{-2iz}) \\
&= 2Gq^{\frac{1}{4}}\sin z\prod_{n=1}^{\infty}(1 - q^{2n}e^{2iz})\prod_{n=1}^{\infty}(1 - q^{2n}e^{-2iz}) \\
&= 2Gq^{\frac{1}{4}}\sin z\prod_{n=1}^{\infty}(1 - 2q^{2n}\cos 2z + q^{4n})
\end{aligned}$$

*so*

$$\begin{aligned}
\vartheta_2(z) &= \vartheta_1(z + \frac{1}{2}\pi) \\
&= 2Gq^{\frac{1}{4}}\cos z\prod_{n=1}^{\infty}(1 + 2q^{2n}\cos 2z + q^{4n})
\end{aligned}$$

■

**Remark 45** *We may regard any one of four type of Theta-functions as a function $\vartheta(z|\tau)$ of two independent variables $z$ and $\tau$. By compute directly we have*

$$\frac{\partial^2\vartheta(z|\tau)}{\partial z^2} = -\frac{4}{\pi i}\frac{\partial\vartheta(z|\tau)}{\partial\tau}$$

*Therefore the Theta-functions $\vartheta(z|\tau)$ satisfies the partial differential equation*

$$\frac{1}{4}\pi i\frac{\partial^2 y}{\partial z^2} + \frac{\partial y}{\partial\tau} = 0$$

**Remark 46 (A relation between Theta-functions of zero argument)** *For Theta-functions , we have the following relation*

$$\vartheta_1'(0) = \vartheta_2(0)\cdot\vartheta_3(0)\cdot\vartheta_4(0)$$

**Remark 47 (The value of the constant $G$.)** *The constant $G$ in the following equations*

$$\begin{aligned}
\vartheta_1(z) &= 2Gq^{\frac{1}{4}}\sin z\prod_{n=1}^{\infty}(1-2q^{2n}\cos 2z+q^{4n}) \\
\vartheta_2(z) &= 2Gq^{\frac{1}{4}}\cos z\prod_{n=1}^{\infty}(1+2q^{2n}\cos 2z+q^{4n}) \\
\vartheta_3(z) &= G\prod_{n=1}^{\infty}(1+2q^{2n-1}\cos 2z+q^{4n-2}) \\
\vartheta_4(z) &= G\prod_{n=1}^{\infty}(1-2q^{2n-1}\cos 2z+q^{4n-2})
\end{aligned}$$

*have the value*

$$G=\prod_{n=1}^{\infty}(1-q^{2n})$$

**Remark 48** *The quotient of Theta-functions $\vartheta_1(z)/\vartheta_4(z)$ is a solution of the following equation*

$$(\frac{d\xi}{dz})^2=(\vartheta_2^2-\vartheta_3^2\cdot\xi^2)(\vartheta_3^2-\vartheta_2^2\cdot\xi^2)$$

***Proof.*** *By the table below*

| | $\vartheta_1(z)$ | $\vartheta_2(z)$ | $\vartheta_3(z)$ | $\vartheta_4(z)$ |
|---|---|---|---|---|
| $\pi$ | -1 | -1 | 1 | 1 |
| $\pi\tau$ | -N | N | N | -N |

*we know that the function $\vartheta_1(z)/\vartheta_4(z)$ has periodicity factors $(-1),(+1)$ associated with the periods $(\pi),(\pi\tau)$ respectively. So its derivative*

$$\frac{d}{dz}(\frac{\vartheta_1(z)}{\vartheta_4(z)})=\frac{\vartheta_1'(z)\vartheta_4(z)-\vartheta_4'(z)\vartheta_1(z)}{\vartheta_4^2(z)}$$

*also has periodicity factors $(-1),(+1)$ associated with the periods $(\pi),(\pi\tau)$ respectively.*

*By the same table we verify that*

$$\frac{\vartheta_2(z)\cdot\vartheta_3(z)}{\vartheta_4^2(z)}$$

*has periodicity factors* $(-1), (+1)$ *associated with the periods* $(\pi), (\pi\tau)$ *respectively. So the function*

$$\phi(z) = \frac{\vartheta_1'(z)\vartheta_4(z) - \vartheta_4'(z)\vartheta_1(z)}{\vartheta_2(z) \cdot \vartheta_3(z)}$$

*is doubly-periodic with periods* $\pi$ *and* $\pi\tau$ *; and the only possible poles of* $\phi(z)$ *are simple poles at points congruent to* $(1/2)\pi$ *and* $(1/2)\pi + (1/2)\pi\tau$*.*

*By the relations between four types of Theta-functions*

$$\begin{aligned} \vartheta_1(z + \frac{1}{2}\pi\tau) &= iq^{-\frac{1}{4}}e^{-iz}\vartheta_4(z) \\ \vartheta_4(z + \frac{1}{2}\pi\tau) &= iq^{-\frac{1}{4}}e^{-iz}\vartheta_1(z) \\ \vartheta_2(z + \frac{1}{2}\pi\tau) &= q^{-\frac{1}{4}}e^{-iz}\vartheta_3(z) \\ \vartheta_3(z + \frac{1}{2}\pi\tau) &= q^{-\frac{1}{4}}e^{-iz}\vartheta_2(z) \end{aligned}$$

*we can see that*

$$\phi(z + \frac{1}{2}\pi\tau) = \frac{-\vartheta_4'(z)\vartheta_1(z) + \vartheta_1'(z)\vartheta_4(z)}{\vartheta_3(z) \cdot \vartheta_2(z)} = \phi(z)$$

*Hence* $\phi(z)$ *is doubly-periodic with periods* $\pi$ *and* $(1/2)\pi\tau$ *, and the only possible poles of* $\phi(z)$ *are simple poles at points congruent to* $(1/2)\pi$ *relative to these periods. Because the order of an elliptic function is never less than 2 , otherwise a constant. So* $\phi(z) = A$ *is a constant , and making* $z \to 0$ *, we get that*

$$\phi(z) = A = \frac{\vartheta_1' \cdot \vartheta_4}{\vartheta_2 \cdot \vartheta_3} = \vartheta_4^2$$

*by the relation* $\vartheta_1' = \vartheta_2 \cdot \vartheta_3 \cdot \vartheta_4$*. So we have*

$$\frac{\vartheta_1'(z)\vartheta_4(z) - \vartheta_4'(z)\vartheta_1(z)}{\vartheta_2(z) \cdot \vartheta_3(z)} = \vartheta_4^2$$

*i.e.*

$$\vartheta_1'(z)\vartheta_4(z) - \vartheta_4'(z)\vartheta_1(z) = \vartheta_4^2\vartheta_2(z)\vartheta_3(z)$$

*thus we get*

$$\frac{d}{dz}(\frac{\vartheta_1(z)}{\vartheta_4(z)}) = \frac{\vartheta_1'(z)\vartheta_4(z) - \vartheta_4'(z)\vartheta_1(z)}{\vartheta_4^2(z)} = \vartheta_4^2\frac{\vartheta_2(z)}{\vartheta_4(z)} \cdot \frac{\vartheta_3(z)}{\vartheta_4(z)}$$

*Let* $\xi \equiv \vartheta_1(z)/\vartheta_4(z)$ *, and by the relation* $\vartheta_1' = \vartheta_2 \cdot \vartheta_3 \cdot \vartheta_4$ *again , we eventually get the differential equation*

$$(\frac{d\xi}{dz})^2 = (\vartheta_2^2 - \vartheta_3^2 \cdot \xi^2)(\vartheta_3^2 - \vartheta_2^2 \cdot \xi^2)$$

■

Now , we could introduce the Jacobian elliptic functioms.

**Remark 49 (The genesis of the Jacobian Elliptic function.)** *Let* $y = (\vartheta_3/\vartheta_2)\xi$ *,* $u = \vartheta_3^2 z$ *,* $\kappa = (\vartheta_2/\vartheta_3)^2$ *, then the equation*

$$(\frac{d\xi}{dz})^2 = (\vartheta_2^2 - \vartheta_3^2 \cdot \xi^2)(\vartheta_3^2 - \vartheta_2^2 \cdot \xi^2)$$

*would be written as*

$$(\frac{dy}{du})^2 = (1 - y^2)(1 - \kappa^2 y^2)$$

*It is customary to regard the solution* $y$ *as a function of* $u$ *and* $\kappa$ *, so we denote* $y = sn(u, \kappa)$ *or simply* $y = sn(u)$*. Evidently ,* $sn(u, \kappa)$ *is an elliptic function and* $sn(u, \kappa) \to \sin u$ *as* $\kappa \to 0$*. The constant* $\kappa = (\vartheta_2/\vartheta_3)^2$ *is called the modulus , and the constant* $\kappa' = (\vartheta_4/\vartheta_3)^2$ *is called the complementary modulus. Because* $\vartheta_2^4 + \vartheta_4^4 = \vartheta_3^4$ *,we get* $(\kappa)^2 + (\kappa')^2 = 1$*.*

*The function*

$$y = \frac{\vartheta_3}{\vartheta_2}\xi = \frac{\vartheta_3}{\vartheta_2}\frac{\vartheta_1(\vartheta_3^{-2}u)}{\vartheta_4(\vartheta_3^{-2}u)}$$

*has periodicity factors* $(-1), (+1)$ *associated with the periods* $(\pi\vartheta_3^2), (\pi\tau\vartheta_3^2)$ *, therefore it is a doubly-periodic function with periods* $(2\pi\vartheta_3^2), (\pi\tau\vartheta_3^2)$*. It has* *two simple poles at the points congruent to* $(\frac{1}{2}\pi\tau\vartheta_3^2)$ *and* $(\pi\vartheta_3^2 + \frac{1}{2}\pi\tau\vartheta_3^2)$ *in any cell. The zeros of the function are the points congruent to* $0$ *and* $\pi\vartheta_3^2$ *, and on account of the nature of the quasi-periodicity of* $y$ *, the residues at these points are equal and opposite in sign. The quasi-periods* $(\pi\vartheta_3^2), (\pi\tau\vartheta_3^2)$ *are usually written* $(2K), (2iK')$ *, so that* $sn(u,k)$ *has periods* $(4K), (2iK')$*.*

*On the other version , the equation*

$$(\frac{dy}{du})^2 = (1-y^2)(1-\kappa^2 y^2)$$

*can be written as the integral form*

$$u = \int_0^y (1-t^2)^{-\frac{1}{2}}(1-\kappa^2 t^2)^{-\frac{1}{2}} dt$$

*where* $y = sn(u,\kappa)$ *satisfies it.*

**Definition 50** *The Jacobian elliptic functions* $sn(u), cn(u), dn(u)$ *are defined as following*

$$\begin{aligned} sn(u) &= \frac{\vartheta_3}{\vartheta_2}\frac{\vartheta_1(\vartheta_3^{-2}u)}{\vartheta_4(\vartheta_3^{-2}u)} \\ cn(u) &= \frac{\vartheta_4}{\vartheta_2}\frac{\vartheta_2(\vartheta_3^{-2}u)}{\vartheta_4(\vartheta_3^{-2}u)} \\ dn(u) &= \frac{\vartheta_4}{\vartheta_3}\frac{\vartheta_3(\vartheta_3^{-2}u)}{\vartheta_4(\vartheta_3^{-2}u)} \end{aligned}$$

*where* $u = \vartheta_3^2 z$ *,* $\kappa = (\vartheta_2/\vartheta_3)^2$*.*

**Remark 51 (Relations between the Jacobian elliptic functions.)** *We have some relations between the Jacobian elliptic functions as following*

$$\begin{aligned} cn(u)\cdot dn(u) &= \frac{d}{du}sn(u) \\ sn^2(u) + cn^2(u) &= 1 \\ \kappa^2 sn^2(u) + dn^2(u) &= 1 \\ cn(0) &= 1 \\ dn(0) &= 1 \end{aligned}$$

***Proof.*** *By*

$$\frac{d}{dz}(\frac{\vartheta_1(z)}{\vartheta_4(z)}) = \vartheta_4^2 \frac{\vartheta_2(z)}{\vartheta_4(z)} \cdot \frac{\vartheta_3(z)}{\vartheta_4(z)}$$

*we have*

$$\frac{d}{du} sn(u) = cn(u) \cdot dn(u)$$

*By*

$$\begin{aligned} \vartheta_4^2(0)\vartheta_2^2(z) &= \vartheta_2^2(0)\vartheta_4^2(z) - \vartheta_3^2(0)\vartheta_1^2(z) \\ \vartheta_4^2(0)\vartheta_3^2(z) &= \vartheta_3^2(0)\vartheta_4^2(z) - \vartheta_2^2(0)\vartheta_1^2(z) \end{aligned}$$

*we have*

$$\begin{aligned} sn^2(u) + cn^2(u) &= 1 \\ \kappa^2 sn^2(u) + dn^2(u) &= 1 \end{aligned}$$

*By compute directly and definition , $cn(0) = dn(0) = 1$ is obvious.* ■

**Remark 52 (Simple properties of sn(u),cn(u),dn(n).)** *We summarize simple properties of $sn(u)$ , $cn(u)$ , $dn(u)$ by the following table.*

| | $sn(u)$ | $cn(u)$ | $dn(u)$ |
|---|---|---|---|
| *Periods* | $4K, 2iK'$ | $4K, 4iK'$ | $2K, 4iK'$ |
| *Poles* | $iK', 2K + iK'$ $\mathrm{mod}(4K, 2iK')$ | $iK', 2K + iK'$ $\mathrm{mod}(4K, 4iK')$ | $iK', K + iK'$ $\mathrm{mod}(2K, 4iK')$ |
| *Zeros* | $0$ $\mathrm{mod}(2K, 2iK')$ | $K$ $\mathrm{mod}(2K, 2iK')$ | $K + iK'$ $\mathrm{mod}(2K, 2iK')$ |
| *Parity* | *odd* | *even* | *even* |
| *Derivative* | $cn(u) \cdot dn(u)$ | $-sn(u) \cdot dn(u)$ | $-\kappa^2 sn(u) \cdot cn(u)$ |

# 4 The Exact Theory of the Sine-Gordon Equation.

When we discussed with the simple properties of the Jacobian elliptic functions. We will discuss the exact theory of the Sine-Gordon equation.

## 4.1 The Exact Theory.

The form of partial differential equation

$$u_{xx} - u_{yy} + \sin u = 0$$

is called Sine-Gordon equation. We want to find the traveling wave solution of $u_{xx} - u_{yy} + \sin u = 0$. Let $t = mx - ny$ , then we have

$$\begin{aligned} u_x &= \frac{du}{dt} \cdot \frac{dt}{dx} = \frac{du}{dt} \cdot m \\ u_y &= \frac{du}{dt} \cdot \frac{dt}{dy} = \frac{du}{dt} \cdot (-n) \end{aligned}$$

Using the same method again , we have

$$\begin{aligned} u_{xx} &= u_{tt} \cdot m^2 \\ u_{yy} &= u_{tt} \cdot n^2 \end{aligned}$$

So the equation $u_{xx} - u_{yy} + \sin u = 0$ becomes

$$(m^2 - n^2) \cdot u_{tt} + \sin u = 0$$

Assume $m^2 - n^2 = 1$ , then we get

$$u_{tt} + \sin u = 0$$

Multiplying $\frac{du}{dt}$ to $u_{tt} + \sin u = 0$ , then we get

$$\frac{du}{dt} \cdot u_{tt} + \frac{du}{dt} \cdot \sin u = 0$$

Integrating it with respect to $t$ , the equation will become

$$\frac{1}{2}(\frac{du}{dt})^2 - \cos u = E$$

where $E$ is a constant. The square roots of $\frac{du}{dt}$ are $\pm\sqrt{2(E + \cos u)}$. We will focus on the equation of positive sign

$$\frac{du}{dt} = \sqrt{2(E + \cos u)}$$

Using the relation of trigonometric function that $\cos u = 1 - 2\sin^2(u/2)$. We get the equation

$$\frac{du}{dt} = \sqrt{2(E + 1 - 2\sin^2(\frac{u}{2}))}$$

Since the equation is a separable equation , we could get

$$t = \int_0^{U(t)} \frac{1}{\sqrt{2(E+1) - 4\sin^2(\frac{u}{2})}} du$$

Our goal is to find its solution i.e. we want to find the representation of $U(t)$ in terms of $t$. We will discuss it in three different cases according to different $E$.

Case 1. $-1 < E < 1$

Suppose $E \in (-1, 1)$ , then

$$\begin{aligned} t &= \int_0^{U(t)} \frac{1}{\sqrt{2(E+1) - 4\sin^2(\frac{u}{2})}} du \\ &= \sqrt{\frac{2}{E+1}} \int_0^{\frac{U(t)}{2}} \frac{1}{\sqrt{1 - \frac{2}{E+1}\sin^2(\frac{u}{2})}} d(\frac{u}{2}) \end{aligned}$$

Let $s = \sqrt{\frac{2}{E+1}} \sin(\frac{u}{2})$ , then

$$d(\frac{u}{2}) = \frac{1}{\sqrt{\frac{2}{E+1} - s^2}} ds$$

Thus ,

$$\begin{aligned} t &= \sqrt{\frac{2}{E+1}} \int_0^{\frac{U(t)}{2}} \frac{1}{\sqrt{1-\frac{2}{E+1}\sin^2(\frac{u}{2})}} d(\frac{u}{2}) \\ &= \int_0^{\sqrt{\frac{2}{E+1}}\sin(\frac{U(t)}{2})} \frac{1}{\sqrt{1-s^2}} \frac{1}{\sqrt{1-(\frac{E+1}{2})\cdot s^2}} ds \end{aligned}$$

Let $\kappa = \sqrt{\frac{E+1}{2}}$ , then

$$t = \int_0^{\kappa^{-1}\sin(\frac{U(t)}{2})} \frac{1}{\sqrt{1-s^2}} \frac{1}{\sqrt{1-\kappa^2\cdot s^2}} ds$$

By Jacobian elliptic function $sn(u,\kappa)$ , the equation implies that

$$sn(t,\kappa) = \kappa^{-1}\sin(\frac{U(t)}{2})$$

So we have

$$U(t) = 2\sin^{-1}(\kappa \cdot sn(t,\kappa))$$

where $\kappa = \sqrt{\frac{E+1}{2}}$.

**Remark 53** *Since* $E \in (-1,1)$ *, thus* $\sqrt{\frac{E+1}{2}} \in (0,1)$ *i.e.* $0 < \kappa < 1$*. Furthermore ,* $\kappa \propto E$*.*

Case 2. $E = 1$

Suppose $E = 1$ , then

$$\begin{aligned} t &= \int_0^{U(t)} \frac{1}{\sqrt{2(E+1) - 4\sin^2(\frac{u}{2})}} du \\ &= \int_0^{U(t)} \frac{1}{\sqrt{4 - 4\sin^2(\frac{u}{2})}} du \\ &= \int_0^{\frac{U(t)}{2}} \frac{1}{\sqrt{1 - \sin^2(\frac{u}{2})}} d(\frac{u}{2}) \end{aligned}$$

Let $s = \sqrt{\frac{2}{E+1}} \sin(\frac{u}{2}) = \sin(\frac{u}{2})$ , then

$$d(\frac{u}{2}) = \frac{1}{\sqrt{1-s^2}} ds$$

Thus ,

$$\begin{aligned} t &= \int_0^{\frac{U(t)}{2}} \frac{1}{\sqrt{1 - \sin^2(\frac{u}{2})}} d(\frac{u}{2}) \\ &= \int_0^{\sin(\frac{U(t)}{2})} \frac{1}{\sqrt{1-s^2}} \cdot \frac{1}{\sqrt{1-s^2}} ds \end{aligned}$$

Let $\kappa = \sqrt{\frac{E+1}{2}} = 1$ , then

$$\begin{aligned} t &= \int_0^{\sin(\frac{U(t)}{2})} \frac{1}{\sqrt{1-s^2}} \cdot \frac{1}{\sqrt{1-s^2}} ds \\ &= \int_0^{\kappa^{-1}\sin(\frac{U(t)}{2})} \frac{1}{\sqrt{1-s^2}} \frac{1}{\sqrt{1-\kappa^2 \cdot s^2}} ds \end{aligned}$$

By Jacobian elliptic function $sn(u, \kappa)$ , the equation implies that

$$sn(t,1) = \sin(\frac{U(t)}{2})$$

So we have

$$U(t) = 2\sin^{-1}(sn(t,1))$$

where $\kappa = 1$.

**Remark 54** *If we do not use Jacobian elliptic function , we also can get the solution by Calculus in this case. And the solution would be*

$$U(t) = 2\sin^{-1}(\tanh t)$$

Case 3. $E > 1$

Suppose $E \in (1,\infty)$ , then

$$\begin{aligned} t &= \int_0^{U(t)} \frac{1}{\sqrt{2(E+1) - 4\sin^2(\frac{u}{2})}} du \\ &= \sqrt{\frac{2}{E+1}} \int_0^{\frac{U(t)}{2}} \frac{1}{\sqrt{1 - \frac{2}{E+1}\sin^2(\frac{u}{2})}} d(\frac{u}{2}) \\ &= \kappa \int_0^{\frac{U(t)}{2}} \frac{1}{\sqrt{1 - \kappa^2 \sin^2(\frac{u}{2})}} d(\frac{u}{2}) \end{aligned}$$

where $\kappa = \sqrt{\frac{2}{E+1}}$. Let $s = \sin(\frac{u}{2})$ , then

$$d(\frac{u}{2}) = \frac{1}{\sqrt{1-s^2}} ds$$

Thus ,

$$
\begin{aligned}
t &= \kappa \int_0^{\frac{U(t)}{2}} \frac{1}{\sqrt{1-\kappa^2\sin^2(\frac{u}{2})}} d(\frac{u}{2}) \\
&= \kappa \int_0^{\sin(\frac{U(t)}{2})} \frac{1}{\sqrt{1-s^2}\sqrt{1-\kappa^2 s^2}} ds
\end{aligned}
$$

i.e.

$$
\frac{t}{\kappa} = \int_0^{\sin(\frac{U(t)}{2})} \frac{1}{\sqrt{1-s^2}\sqrt{1-\kappa^2 s^2}} ds
$$

By Jacobian elliptic function $sn(u,\kappa)$ , the equation implies that

$$
sn(\frac{t}{\kappa},\kappa) = \sin(\frac{U(t)}{2})
$$

So we have

$$
U(t) = 2\sin^{-1}(sn(\frac{t}{\kappa},\kappa))
$$

where $\kappa = \sqrt{\frac{2}{E+1}}$.

**Remark 55** *Since* $E \in (1,\infty)$ *, thus* $\sqrt{\frac{2}{E+1}} \in (0,1)$ *i.e.* $0 < \kappa < 1$*. Furthermore ,* $\kappa \propto \frac{1}{E}$*.*

## 4.2 The Periods.

We had found the solutions for the ordinary differential equation in the form of Jacobian elliptic function with different constant $E$. In this section , we want to find the period of solution. The idea is to find the rest position $U(t_0)$ and the period is the four times time of the particle moves from $U(0)$ to $U(t_0)$.

Case 1. $-1 < E < 1$

In this case , the solution is

$$U(t) = 2\sin^{-1}(\kappa \cdot sn(t,\kappa))$$

where $\kappa = \sqrt{\frac{E+1}{2}}$.

The velocity of the particle is

$$\begin{aligned} U_t &= \sqrt{2(E + 1 - 2\sin^2(\frac{U}{2}))} \\ &= \sqrt{2(E+1) - 4\sin^2(\frac{U}{2})} \end{aligned}$$

Let $U_t = 0$ , we have

$$U(t) = \pm 2\sin^{-1}(\kappa)$$

Hence , by the equation

$$t = \int_0^{\kappa^{-1}\sin(\frac{U(t)}{2})} \frac{1}{\sqrt{1-s^2}} \frac{1}{\sqrt{1-\kappa^2 \cdot s^2}} ds$$

The period for this case is

$$\begin{aligned} T &= 4t \\ &= 4\int_0^{\kappa^{-1}\sin(\frac{U(t)}{2})} \frac{1}{\sqrt{1-s^2}} \frac{1}{\sqrt{1-\kappa^2 \cdot s^2}} ds \\ &= 4\int_0^{\kappa^{-1}\sin(\sin^{-1}(\kappa))} \frac{1}{\sqrt{1-s^2}} \frac{1}{\sqrt{1-\kappa^2 \cdot s^2}} ds \\ &= 4\int_0^{1} \frac{1}{\sqrt{1-s^2}} \frac{1}{\sqrt{1-\kappa^2 \cdot s^2}} ds \\ &= 4K \end{aligned}$$

**Remark 56** *The constant $K$ here is defined as $K = \int_0^1 (1-s^2)^{-\frac{1}{2}}(1-\kappa^2 s^2)^{-\frac{1}{2}} ds$ , where $\kappa = ((E+1)/2)^{\frac{1}{2}}$ is the modulus.*

**Remark 57** *The constant $K \propto \kappa$ implies that the period $T \propto \kappa$. It is not difficult to calculate that if $\kappa = 0$ , then $T = 2\pi$. This means that the period $T > 2\pi$ , $\forall \kappa \in (0,1)$.*

**Remark 58** $U(t) = \pm 2\sin^{-1}(\kappa) < 2\sin^{-1}(1) = \pi$ , $\forall \kappa \in (0,1)$.

Case 2. $E = 1$

In this case , the solution is

$$U(t) = 2\sin^{-1}(sn(t,1))$$

where $\kappa = 1$.

The velocity of the particle is

$$\begin{aligned} U_t &= \sqrt{2(E+1-2\sin^2(\frac{U}{2}))} \\ &= \sqrt{2(1+1-2\sin^2(\frac{U}{2}))} \\ &= \sqrt{4-4\sin^2(\frac{U}{2})} \end{aligned}$$

Let $U_t = 0$ , we have

$$U(t) = \pm\pi$$

Hence , by the equation

$$t = \int_0^{\sin(\frac{U(t)}{2})} \frac{1}{\sqrt{1-s^2}} \cdot \frac{1}{\sqrt{1-s^2}} ds$$

The period for this case is

$$
\begin{aligned}
T &= 4t \\
&= 4\int_0^{\sin(\frac{U(t)}{2})} \frac{1}{\sqrt{1-s^2}} \cdot \frac{1}{\sqrt{1-s^2}} ds \\
&= 4\int_0^{\sin(\frac{\pi}{2})} \frac{1}{\sqrt{1-s^2}} \cdot \frac{1}{\sqrt{1-s^2}} ds \\
&= 4\int_0^{1} \frac{1}{1-s^2} ds \\
&= \infty
\end{aligned}
$$

Although it is not a periodic solution , the period of this case could be regarded as $\infty$. This means that if we release the particle at the position $-\pi$ , it needs infinity time to approach the position $\pi$.

Case 3. $E > 1$

The velocity of the particle is

$$
\begin{aligned}
U_t &= \sqrt{2(E + 1 - 2\sin^2(\frac{U}{2}))} \\
&> \sqrt{4 - 4\sin^2(\frac{U}{2})} \\
&\geq 0
\end{aligned}
$$

This means that each position $U(t)$ , the pendulum always has velocity , so the pendulum will never stop. This implies that it has no periodicity.

We construct a table to collect the results we had gotten in the end of this section.

| Energy $(E)$ | $-1 < E < 1$ | $E = 1$ | $E > 1$ |
|---|---|---|---|
| Solution $U(t)$ | $2\sin^{-1}(\kappa sn(t,\kappa))$ | $2\sin^{-1}(sn(t,1))$ | $2\sin^{-1}(sn((t/\kappa),\kappa))$ |
| Modulus $(\kappa)$ | $((E+1)/2)^{\frac{1}{2}}$ | 1 | $(2/(E+1))^{\frac{1}{2}}$ |
| Period $(T)$ | $4K$ | $\infty$ | No periodicity |

## 4.3 The Phase Portraits.

The oridinary differential equation we had discussed is the mathematical model of ideal pendulum. Now we try to plot the relation between $U$ and $U_t$ and the graph is called phase portrait. Before drawing the phase portrait , we see back to the equation

$$\frac{1}{2}(\frac{du}{dt})^2 - \cos u = E$$

where $E$ is a constant first. It shows that

$$\frac{1}{2}(\frac{du}{dt})^2 - \cos u$$

is a constant. It can be regarded as a conservation law in the view point of mathematics since $-\cos(u)$ is not always larger than 0. (But this case can be transferred to the conservation law in the view point of physics by plus a constant $a \geq 1$ for equation $\frac{1}{2}(\frac{du}{dt})^2 - \cos u = E$.).This means that its total energy is a constant and the former part $\frac{1}{2}(\frac{du}{dt})^2$ can be regarded as kinetic energy and the latter part $-\cos(u)$ can be regarded as potential energy. The following we discuss the potential energy and phase portrait with different cases.

Case 1. $-1 < E < 1$

We set $E = 0$ to analyze the case. By the equation

$$\frac{1}{2}(\frac{du}{dt})^2 - \cos u = E$$

we have the equation

$$\frac{du}{dt} = \pm\sqrt{2\cos u}$$

The following graphs are potential energy and phase portrait , respectively. This means that they are the relation between $u$ and $\cos u$ and the relation between $u$ and $\frac{du}{dt}$.

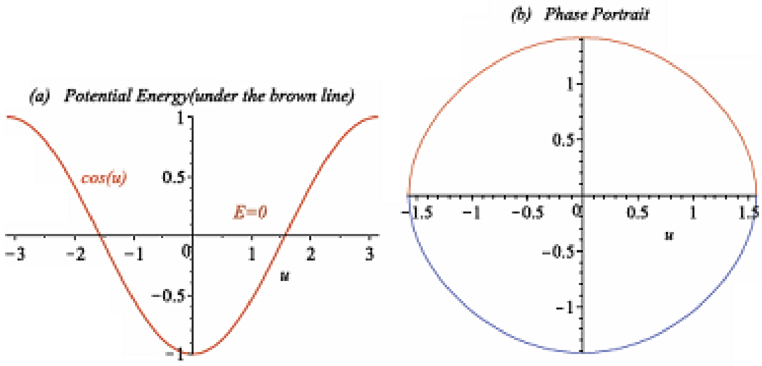



Figure 35. The potential energy and phase portrait for $E = 0$.

**Remark 59** *From the graph of the phase portrait , the red curve means that the velocity at those position are positive and the blue curve means that the velocity at those position are negative. The positive velocity is defined by rotating counterclockwise and the negative velocity is defined by rotating clockwise.*

**Remark 60** *By the graph of potential energy , we can find out that the maximum of amplitude , $u(t)$ , for the pendulum is $\frac{\pi}{2}$ and it oscillates forth and back.*

Case 2. $E = 1$

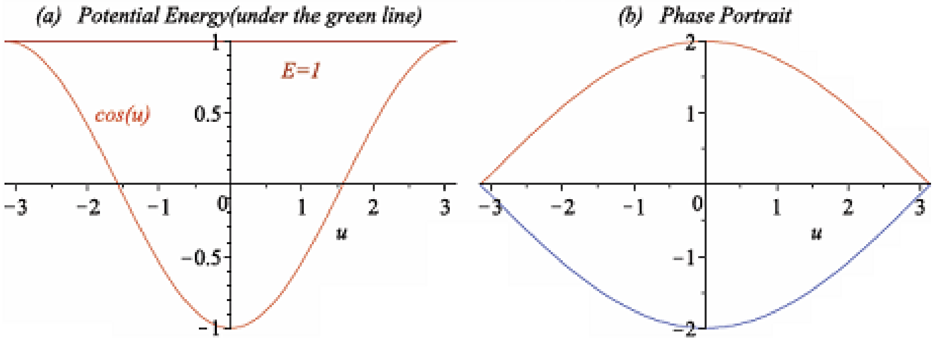
Now we focus on the case with $E = 1$. By the equation

$$\frac{1}{2}(\frac{du}{dt})^2 - \cos u = E$$

we have the equation

$$\frac{du}{dt} = \pm\sqrt{2(1 + \cos u)}$$

We see the potential energy and phase portrait as following.




The potential energy and phase portrait for E = 1

Figure 36. The potential energy and phase portrait for $E = 1$.

**Notation 61** *By the graph of potential energy , we can find out that the maximum of amplitude ,* $u(t)$ *, for the pendulum is* $\pi$*. If we release the pendulum at position* $\pi$ *, the particle will approach to the position* $-\pi$ *after infinite time.*

Case 3. $E > 1$

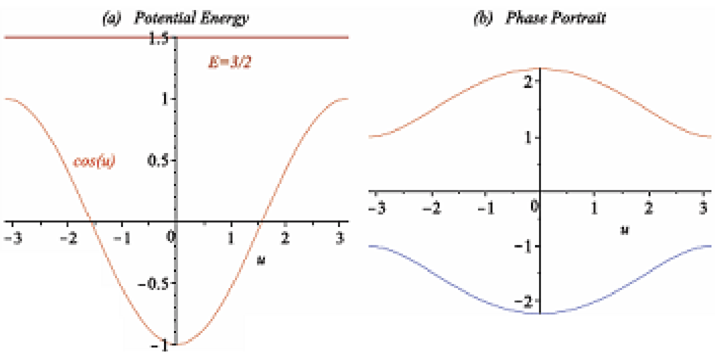
Last , we see the case $E > 1$ with $E = \frac{3}{2}$. By the equation

$$\frac{1}{2}(\frac{du}{dt})^2 - \cos u = E$$

we have the equation

$$\frac{du}{dt} = \pm\sqrt{2(\frac{3}{2} + \cos u)}$$

We see the potential energy and phase portrait as following.




The potential energy and phase portrait for E = 3/2

Figure 37. The potential energy and phase portrait for $E = \frac{3}{2}$.

**Remark 62** *From the graph of the phase portrait , we know that the pendulum of this case will never stop since the phase portrait has no intersection with the u-axis.*

**Remark 63** *By the graph of potential energy , we observe that the kinetic energy is never equal 0. This implies that the case has no periodic solution and the result is corresponded to the property which we had discussed.*

By our discussion , there are three kinds of the phase portraits. Before fnishing the section , we combine the three phase portraits and the vector field together.

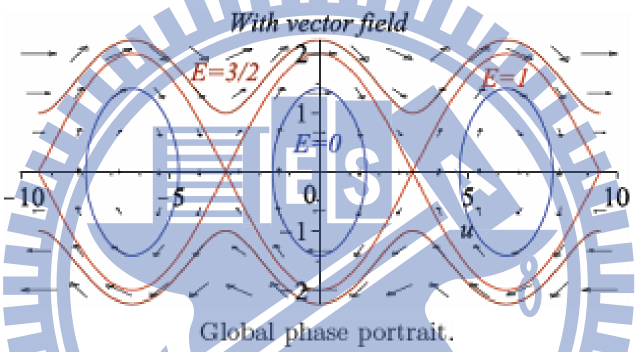



Figure 38. Global phase portrait.

We can make some conclusions from the Figure 38. :

1. There are three different kinds of phase portraits with different energy $E$. The outer curve corresponds to larger energy $E$. They are separated by the phase portrait with $E = 1$ and the phase curve is called the separatrix with periods $\infty$. The phase curves outer the separatrix are called the wave train and they has no period. The phase curves inside the separatrix are periodic and their period $T$ satisfies $2\pi < T < \infty$.

2. The direction of the phase curves which are upper the u-axis toward the right on the phase plane and it means the pendulum rotates counterclockwise. Similarly, the direction of the phase curves which are below the u-axis toward the left and it means the pendulum rotates clockwise.

3. The points $(n\pi, 0)$ are also the solutions for all $n \in \mathbb{Z}$. They are classified into two classes. The first is the points with $n$ is even. These points are stable and with energy $E = -1$. The other is the points with $n$ is odd and these points are unstable and with energy $E = 1$.

# 5 Appendix

We placed here the details of the previous examples , and placed here for more examples.

## 5.1 The details of the previous examples in section 2.1 .

**Example 64** *Evaluate the integrals of* $1/f(z)$ *over* $a_1$ *,* $a_2$ *and* $a_3$ *cycles where* $f(z) = \sqrt{(z+4)(z+2)(z-2)(z-4)(z-5)(z-7)(z-8)}$*. We analysis the integral in Mathematica and in theory to compare the result and using the result of angle to modify the computation to get value. Let* $z_1 = 8$ *,* $z_2 = 7$ *,* $z_3 = 5$ *,* $z_4 = 4$ *,* $z_5 = 2$ *,* $z_6 = -2$ *,* $z_7 = -4$*.*

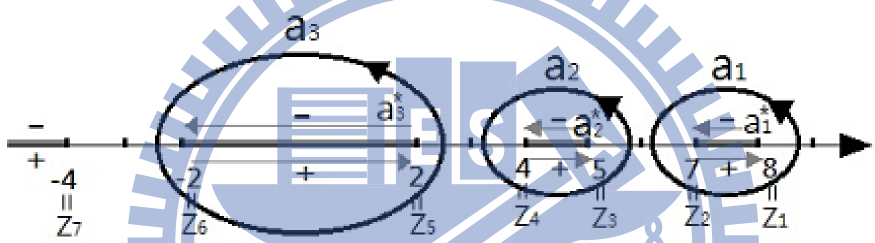



*Figure 39. a-cycles and their equivalent path* $a^*$*.*

***Solution 65*** *:*

$a_1$ *: Let* $a_1$ *is a cycle center at* $\frac{15}{2}$ *with radius* 1 *and enclosed the cut* $[7, 8]$*. So let* $z = \frac{15}{2} + e^{i\theta}$ *, we have*

$$\int_{a_1} \frac{1}{f(z)} dz = \int_{-\pi}^{\pi} \frac{ie^{i\theta}}{\prod_{k=1}^{7} \sqrt{\frac{15}{2} + e^{i\theta} - z_k}} d\theta = 0. + 0.0890282i$$

*By Cauchy Theorem. Since* $a_k$ *cycle is simple connected , we can use some equivalent paths , say* $a_k^*$ *, to easily compute the integrals for* $a_k$ *cycle.*

1. *If* $z \in a_1^*$ *on sheet-I in theory where* $a_1^* = 7 \xrightarrow{+} 8 \cup 7 \xleftarrow{-} 8$

   (a) $7 \xrightarrow{+} 8$ *: the path along* $x$*-axis from 7 to 8 on sheet-I with (+)-edge.* $z - 8 = -|z-8| = |z-8| e^{i(-\pi)}$ *then* $\frac{1}{\sqrt{z-8}} = |z-8|^{-\frac{1}{2}} e^{i(\frac{\pi}{2})} = i|z-8|^{-\frac{1}{2}}$

$z - z_k = |z - z_k|$ *then* $\frac{1}{\sqrt{z-z_k}} = |z - z_k|^{-\frac{1}{2}}$ , $k = 2, 3, 4, 5, 6, 7$

$$\int_{7 \xrightarrow{+} 8} \frac{1}{f(z)} dz = \int_7^8 i \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} \, dz$$

(b) $7 \xleftarrow{-} 8$ : *the path along x-axis from 8 to 7 on sheet-I with (−)-edge.*
$z - 8 = -|z - 8| = |z - 8| e^{i\pi}$ *then* $\frac{1}{\sqrt{z-8}} = |z - 8|^{-\frac{1}{2}} e^{i(-\frac{\pi}{2})} = (-i) |z - 8|^{-\frac{1}{2}}$
$z - z_k = |z - z_k|$ *then* $\frac{1}{\sqrt{z-z_k}} = |z - z_k|^{-\frac{1}{2}}$ , $k = 2, 3, 4, 5, 6, 7$

$$\int_{7 \xleftarrow{-} 8} \frac{1}{f(z)} dz = \int_8^7 (-i) \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} \, dz$$

*By (a) and (b) we obtain the value*

$$\begin{aligned} \int_{a_1^*} \frac{1}{f(z)} dz &= (2i) \int_7^8 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} \, dz \\ &\stackrel{Math.}{=} 0. + 0.0890282i \end{aligned}$$

2. *Analysis the integration over $a_1^*$ in Mathematica*

(a) $7 \xrightarrow{+} 8$ : *the path along x-axis from 7 to 8 on sheet-I with (+)-edge.*
$z - 8 = -|z - 8| = |z - 8| e^{i\pi}$ *then* $\frac{1}{\sqrt{z-8}} = |z - 8|^{-\frac{1}{2}} e^{i(-\frac{\pi}{2})} = (-i) |z - 8|^{-\frac{1}{2}}$
$z - z_k = |z - z_k|$ *then* $\frac{1}{\sqrt{z-z_k}} = |z - z_k|^{-\frac{1}{2}}$ , $k = 2, 3, 4, 5, 6, 7$

$$\int_{7 \xrightarrow{+} 8} \frac{1}{f(z)} dz = \int_7^8 (-i) \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} \, dz$$

*A difference of a minus sign with in sheet-I.*

(b) $7 \xleftarrow{-} 8$ : *the path along x-axis from 8 to 7 on sheet-I with (−)-edge.*
$z - 8 = -|z - 8| = |z - 8| e^{i\pi}$ *then* $\frac{1}{\sqrt{z-8}} = |z - 8|^{-\frac{1}{2}} e^{i(-\frac{\pi}{2})} = (-i) |z - 8|^{-\frac{1}{2}}$
$z - z_k = |z - z_k|$ *then* $\frac{1}{\sqrt{z-z_k}} = |z - z_k|^{-\frac{1}{2}}$ , $k = 2, 3, 4, 5, 6, 7$

$$\int_{7 \overset{-}{\leftarrow} 8} \frac{1}{f(z)} dz = \int_8^7 (-i) \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz$$

*as same as in sheet-I.*
*But in Mathematica*

$$\int_{a_1^*} \frac{1}{f(z)} dz = 0$$

3. *Using Lemma 6 to modify*

   (a) $7 \overset{+}{\rightarrow} 8$ *: the path along x-axis from 7 to 8 on sheet-I with* (+)*-edge.*
   $\arg(z - z_1) = -\pi$ *then* $\sqrt{z - z_1} \overset{Math.}{=} -\sqrt{z - z_1}$
   $\arg(z - z_k) = 0$ *then* $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}, k = 2, 3, 4, 5, 6, 7$
   *So* $f(z) \overset{Math.}{=} -f(z)$

   (b) $7 \overset{-}{\leftarrow} 8$ *: the path along x-axis from 8 to 7 on sheet-I with* (−)*-edge.*
   $\arg(z - z_1) = \pi$ *then* $\sqrt{z - z_1} \overset{Math.}{=} \sqrt{z - z_1}$
   $\arg(z - z_k) = 0$ *then* $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}, k = 2, 3, 4, 5, 6, 7$
   *So* $f(z) \overset{Math.}{=} f(z)$
   *We have*

$$\begin{aligned} \int_{a_1} \tfrac{1}{f(z)} dz &= \int_{a_1^*} \tfrac{1}{f(z)} dz \\ &\overset{Math.}{=} -2 \int_7^8 \tfrac{1}{f(z)} dz \\ &= 0. + 0.0890282i \end{aligned}$$

$a_2$ *: Let* $a_2$ *is a cycle center at* $\frac{9}{2}$ *with radius* 1 *and enclosed the cut* [4, 5]. *So let* $z = \frac{9}{2} + e^{i\theta}$ *, we have*

$$\begin{aligned} \int_{a_2} \frac{1}{f(z)} dz &= \int_{-\pi}^{\pi} \frac{ie^{i\theta}}{\prod_{k=1}^{7} \sqrt{\frac{9}{2} + e^{i\theta} - z_k}} d\theta \\ &= 0. + 0.1832730i \end{aligned}$$

*Same as* $a_1$ *, by Cauchy Theorem to compute equivalent path* $a_2^*$ *where* $a_2^* = 4 \overset{+}{\rightarrow} 5 \cup 4 \overset{-}{\leftarrow} 5$

1. *Analysis the integration over $a_2^*$ on sheet-I*

(a) $4 \xrightarrow{+} 5$ : *the path along $x$-axis from 4 to 5 on sheet-I with $(+)$-edge.*
$z - z_k = -|z - z_k| = |z - z_k|\, e^{i(-\pi)}$
*then* $\frac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}}\, e^{i\frac{\pi}{2}} = i\,|z - z_k|^{-\frac{1}{2}}, k = 1, 2, 3$
$z - z_k = |z - z_k|$ *then* $\frac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}}$ , $k = 4, 5, 6, 7$

$$\int_{4 \xrightarrow{+} 5} \frac{1}{f(z)} dz = \int_4^5 i^3 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}}\, dz$$

(b) $4 \xleftarrow{-} 5$ : *the path along $x$-axis from 5 to 4 on sheet-I with $(-)$-edge.*
$z - z_k = -|z - z_k| = |z - z_k|\, e^{i\pi}$
*then* $\frac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}}\, e^{i(-\frac{\pi}{2})} = (-i)\,|z - z_k|^{-\frac{1}{2}}, k = 1, 2, 3$
$z - z_k = |z - z_k|$ *then* $\frac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}}$ , $k = 4, 5, 6, 7$

$$\int_{4 \xleftarrow{-} 5} \frac{1}{f(z)} dz = \int_5^4 (-i)^3 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}}\, dz$$

*By (a) and (b) we obtain the value*

$$\int_{a_2^*} \frac{1}{f(z)} dz = (-2i) \int_4^5 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}}\, dz \overset{Math.}{=} 0. + 0.1832730i$$

2. *Analysis the integration over $a_2^*$ in Mathematica*

(a) $4 \xrightarrow{+} 5$ : *the path along $x$-axis from 4 to 5 on sheet-I with $(+)$-edge.*
$z - z_k = -|z - z_k| = |z - z_k|\, e^{i\pi}$
*then* $\frac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}}\, e^{i(-\frac{\pi}{2})} = (-i)\,|z - z_k|^{-\frac{1}{2}}, k = 1, 2, 3$
$z - z_k = |z - z_k|$ *then* $\frac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}}$ , $k = 2, 3, 4, 5, 6, 7$

$$\int_{4 \xrightarrow{+} 5} \frac{1}{f(z)} dz = \int_4^5 (-i)^3 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}}\, dz$$

(b) $4 \overset{-}{\leftarrow} 5$ : *the path along $x$-axis from 5 to 4 on sheet-I with* $(-)$*-edge.*

$z - z_k = -|z - z_k| = |z - z_k|\, e^{i\pi}$

*then* $\frac{1}{\sqrt{z-z_k}} = |z - z_k|^{-\frac{1}{2}}\, e^{i(-\frac{\pi}{2})} = (-i)\,|z - z_k|^{-\frac{1}{2}}\,, k = 1, 2, 3$

$z - z_k = |z - z_k|$ *then* $\frac{1}{\sqrt{z-z_k}} = |z - z_k|^{-\frac{1}{2}}\ ,\ k = 4, 5, 6, 7$

$$\int_{4 \overset{-}{\leftarrow} 5} \frac{1}{f(z)} dz = \int_5^4 (-i)^3 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}}\, dz$$

*By (a) and (b) we obtain different value in Mathematica*

$$\int_{a_2^*} \frac{1}{f(z)} dz = 0$$

3. *Using Lemma 6 to modify*

(a) $4 \overset{+}{\rightarrow} 5$ : *the path along $x$-axis from 4 to 5 on sheet-I with* $(+)$*-edge.*

$\arg(z - z_k) = -\pi$ *then* $\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}, k = 1, 2, 3$

$\arg(z - z_k) = 0$ *then* $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}, k = 4, 5, 6, 7$

*So* $f(z) \overset{Math.}{=} -f(z)$

(b) $4 \overset{-}{\leftarrow} 5$ : *the path along $x$-axis from 5 to 4 on sheet-I with (-) edge.*

$\arg(z - z_k) = \pi$ *then* $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}, k = 1, 2, 3$

$\arg(z - z_k) = 0$ *then* $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}, k = 4, 5, 6, 7$

*So* $f(z) \overset{Math.}{=} f(z)$

*We have*

$$\begin{aligned} \int_{a_2} \tfrac{1}{f(z)} dz &= \int_{a_2^*} \tfrac{1}{f(z)} dz \\ &\overset{Math.}{=} -2 \int_4^5 \tfrac{1}{f(z)} dz \\ &= 0. + 0.1832730i \end{aligned}$$

$a_3$ : *Let $a_3$ is a cycle center at 0 with radius $\frac{5}{2}$ and enclosed the cut* $[-2, 2]$. *So let* $z = \frac{5}{2} e^{i\theta}$ *, we have*

$$\begin{aligned} \int_{a_3} \frac{1}{f(z)} dz &= \int_{-\pi}^{\pi} \frac{(\frac{5}{2} i) e^{i\theta}}{\prod_{k=1}^{7} \sqrt{(\frac{5}{2}) e^{i\theta} - z_k}} d\theta \\ &= 0. + 0.1115720i \end{aligned}$$

*Same as* $a_1$ *, by Cauchy Theorem to compute equivalent path* $a_3^*$ *where* $a_3^* = -2 \xrightarrow{+} 2 \cup -2 \xleftarrow{-} 2$

1. *Analysis the integration over* $a_3^*$ *on sheet-I*

   (a) $-2 \xrightarrow{+} 2$ : *the path along x-axis from -2 to 2 on sheet-I with (+)-edge.*
   $z - z_k = -|z - z_k| = |z - z_k|\, e^{-i\pi}$
   *then* $\frac{1}{\sqrt{z-z_k}} = |z - z_k|^{-\frac{1}{2}}\, e^{i\frac{\pi}{2}} = i\,|z - z_k|^{-\frac{1}{2}}, k = 1, 2, 3, 4, 5$
   $z - z_k = |z - z_k|$ *then* $\frac{1}{\sqrt{z-z_k}} = |z - z_k|^{-\frac{1}{2}}$ , $k = 6, 7$

   $$\int_{-2\xrightarrow{+}2} \frac{1}{f(z)} dz = \int_{-2}^{2} i^5 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}}\, dz$$

   (b) $-2 \xleftarrow{-} 2$ : *the path along x-axis from 2 to -2 on sheet-I with (−)-edge.*
   $z - z_k = -|z - z_k| = |z - z_k|\, e^{i\pi}$
   *then* $\frac{1}{\sqrt{z-z_k}} = |z - z_k|^{-\frac{1}{2}}\, e^{i(-\frac{\pi}{2})} = (-i)\,|z - z_k|^{-\frac{1}{2}}, k = 1, 2, 3, 4, 5$
   $z - z_k = |z - z_k|$ *then* $\frac{1}{\sqrt{z-z_k}} = |z - z_k|^{-\frac{1}{2}}$ , $k = 6, 7$

   $$\int_{-2\xleftarrow{-}2} \frac{1}{f(z)} dz = \int_{2}^{-2} (-i)^5 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}}\, dz$$

   *By (a) and (b) we obtain the value*

   $$\int_{a_3^*} \frac{1}{f(z)} dz = (2i) \int_{-2}^{2} \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}}\, dz \overset{Math.}{=} 0. + 0.1115720i$$

2. *Analysis the integration over* $a_3^*$ *in Mathematica*

   (a) $-2 \xrightarrow{+} 2$ : *the path along x-axis from -2 to 2 on sheet-I with (+)-edge.*
   $z - z_k = -|z - z_k| = |z - z_k|\, e^{i\pi}$
   *then* $\frac{1}{\sqrt{z-z_k}} = |z - z_k|^{-\frac{1}{2}}\, e^{i(-\frac{\pi}{2})} = (-i)\,|z - z_k|^{-\frac{1}{2}}, k = 1, 2, 3, 4, 5$

$z - z_k = |z - z_k|$ *then* $\frac{1}{\sqrt{z-z_k}} = |z - z_k|^{-\frac{1}{2}}$ , $k = 6, 7$

$$\int_{-2 \overset{+}{\to} 2} \frac{1}{f(z)} dz = \int_{-2}^{2} (-i)^5 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz$$

(b) $-2 \overset{-}{\leftarrow} 2$ : *the path along x-axis from 2 to -2 on sheet-I with* $(-)$-*edge.*

$z - z_k = -|z - z_k| = |z - z_k| e^{i\pi}$

*then* $\frac{1}{\sqrt{z-z_k}} = |z - z_k|^{-\frac{1}{2}} e^{i(-\frac{\pi}{2})} = (-i) |z - z_k|^{-\frac{1}{2}}$ , $k = 1, 2, 3, 4, 5$

$z - z_k = |z - z_k|$ *then* $\frac{1}{\sqrt{z-z_k}} = |z - z_k|^{-\frac{1}{2}}$ , $k = 6, 7$

$$\int_{-2 \overset{-}{\leftarrow} 2} \frac{1}{f(z)} dz = \int_{2}^{-2} (-i)^5 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz$$

*By (a) , (b) we obtain different value in Mathematica*

$$\int_{a_3^*} \frac{1}{f(z)} dz = 0$$

3. *Using Lemma 6 to modify*

(a) $-2 \overset{+}{\to} 2$ : *the path along x-axis from -2 to 2 on sheet-I with* $(+)$-*edge.*

$\arg(z - z_k) = -\pi$ *then* $\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}, k = 1, 2, 3, 4, 5$

$\arg(z - z_k) = 0$ *then* $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}, k = 6, 7$

*So* $f(z) \overset{Math.}{=} -f(z)$

(b) $-2 \overset{-}{\leftarrow} 2$ : *the path along x-axis from 2 to -2 on sheet-I with* $(-)$-*edge.*

$\arg(z - z_k) = \pi$ *then* $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}, k = 1, 2, 3, 4, 5$

$\arg(z - z_k) = 0$ *then* $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}, k = 6, 7$

*So* $f(z) \overset{Math.}{=} f(z)$

*We have*

$$\begin{aligned} \int_{a_3} \tfrac{1}{f(z)} dz &= \int_{a_3^*} \tfrac{1}{f(z)} dz \\ &\overset{Math.}{=} -2 \int_{-2}^{2} \tfrac{1}{f(z)} dz \\ &= 0. + 0.1115720i \end{aligned}$$

**Example 66** *Evaluate the integrals of* $1/f(z)$ *over* $b_1$ *,* $b_2$ *and* $b_3$ *cycles where* $f(z)=\sqrt{(z+4)(z+2)(z-2)(z-4)(z-5)(z-7)(z-8)}$*. We analysis the integral in Mathematica and in theory to compare the result and using the result of angle to modify the computation to get value. Let* $z_1=8$ *,* $z_2=7$ *,* $z_3=5$ *,* $z_4=4$ *,* $z_5=2$ *,* $z_6=-2$ *,* $z_7=-4$*.*

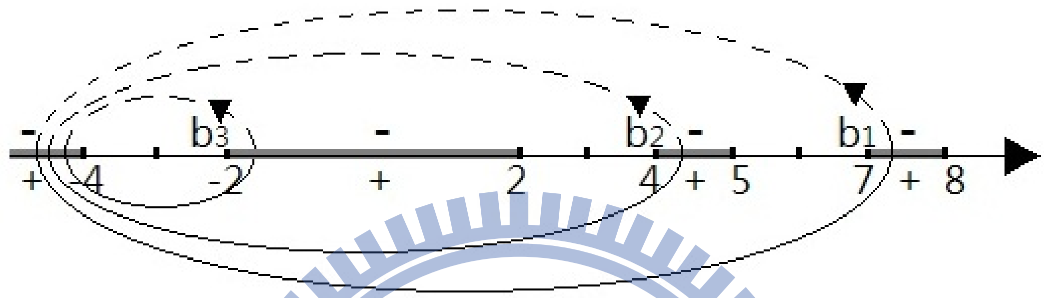



*Figure 40.* $b_1$ *,* $b_2$ *and* $b_3$ *cycles.*

***Solution 67*** *:*

$b_3$ *: Let* $b_3$ *is a cycle which center at* $-3$ *with radius 2. So we could write down the parameter, let* $z=-3+2e^{i\theta},\theta\in[-\pi,0)\cup[2\pi,3\pi)$ *. Notice that* $f(z)\mid_{(II)}=-f(z)\mid_{(I)}$*,so we have*

$$\begin{aligned}\int_{b_3}\frac{1}{f(z)}dz &= \int_{-\pi}^{0}\frac{2ie^{i\theta}}{\prod_{k=1}^{7}\sqrt{-3+2e^{i\theta}-z_k}}d\theta-\int_{0}^{\pi}\frac{2ie^{i\theta}}{\prod_{k=1}^{7}\sqrt{-3+2e^{i\theta}-z_k}}d\theta\\ &= 0.0372385\end{aligned}$$

*Since* $b_k$ *cycle is simple connected , we can use some equivalent paths , say* $b_k^*$ *, such that* $b_k\approx b_k^*$ *to easily compute the integrals for* $b_k$ *cycle. Here* $b_3\approx b_3^*$*.*

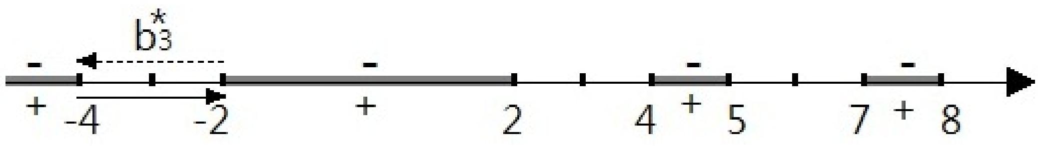



*Figure 41. The equivalent path* $b_3^*$*.*

1. *Analysis the integration over* $b_3^*$ *on sheet-I in theory*

(a) $-4 \rightarrow -2$ : *the path along x-axis from $-4$ to $-2$ on sheet-I.*

$z - z_k = |z - z_k|$ *then* $\frac{1}{\sqrt{z-z_k}} = |z - z_k|^{-\frac{1}{2}}, k = 7$

$z - z_k = -|z - z_k| = |z - z_k| e^{i(-\pi)}$ *then* $\frac{1}{\sqrt{z-z_k}} = |z - z_k|^{-\frac{1}{2}} e^{i(\frac{\pi}{2})} = i\,|z - z_k|^{-\frac{1}{2}}, k = 1, 2, 3, 4, 5, 6$

$$\begin{aligned}
\int_{-4\rightarrow -2} \frac{1}{f(z)} dz &= \int_{-4}^{-2} i^6 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz \\
&= -\int_{-4}^{-2} \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz
\end{aligned}$$

(b) $-4 \dashleftarrow -2$ : *the path along x-axis from $-2$ to $-4$ on sheet-II. We known in theory that $f(z)\,|_{(II)} = -f(z)\,|_{(I)}$ , so we consider* $-4 \leftarrow -2$ ,

$z - z_k = |z - z_k|$ *then* $\frac{1}{\sqrt{z-z_k}} = |z - z_k|^{-\frac{1}{2}}, k = 7$

$z - z_k = -|z - z_k| = |z - z_k| e^{i(-\pi)}$ *then* $\frac{1}{\sqrt{z-z_k}} = |z - z_k|^{-\frac{1}{2}} e^{i(\frac{\pi}{2})} = i\,|z - z_k|^{-\frac{1}{2}}, k = 1, 2, 3, 4, 5, 6$

$$\begin{aligned}
\int_{-4\dashleftarrow -2} \frac{1}{f(z)} dz &= -\int_{-4\leftarrow -2} \frac{1}{f(z)} dz \\
&= -\int_{-2}^{-4} i^6 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz \\
&= \int_{-2}^{-4} \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz \\
&= -\int_{-4}^{-2} \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz
\end{aligned}$$

*By (a) and (b) , we obtain*

$$\begin{aligned}
\int_{b_3} \frac{1}{f(z)} dz &= \int_{b_3^*} \frac{1}{f(z)} dz \\
&= \int_{-4\rightarrow -2} \frac{1}{f(z)} dz + \int_{-4\dashleftarrow -2} \frac{1}{f(z)} dz \\
&= -2\int_{-4}^{-2} \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz \\
&= 0.0372385
\end{aligned}$$

2. *Analysis the integration over $b_3^*$ in Mathematica*

(a) $-4 \to -2$ : *the path along x-axis from $-4$ to $-2$ on sheet-I.*
$z - z_k = |z - z_k|$ *then* $\frac{1}{\sqrt{z-z_k}} = |z - z_k|^{-\frac{1}{2}}, k = 7$
$z - z_k = -|z - z_k| = |z - z_k| e^{i\pi}$ *then* $\frac{1}{\sqrt{z-z_k}} = |z - z_k|^{-\frac{1}{2}} e^{i(-\frac{\pi}{2})} = (-i)|z - z_k|^{-\frac{1}{2}}, k = 1, 2, 3, 4, 5, 6$

$$\begin{aligned}\int_{-4\to-2} \frac{1}{f(z)} dz &= \int_{-4}^{-2} (-i)^6 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz \\ &= -\int_{-4}^{-2} \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz\end{aligned}$$

(b) $-4 \dashleftarrow -2$ : *the path along x-axis from $-2$ to $-4$ on sheet-II.*
$z - z_k = |z - z_k|$ *then* $\frac{1}{\sqrt{z-z_k}} = |z - z_k|^{-\frac{1}{2}}, k = 7$
$z - z_k = -|z - z_k| = |z - z_k| e^{i\pi}$ *then* $\frac{1}{\sqrt{z-z_k}} = |z - z_k|^{-\frac{1}{2}} e^{i(-\frac{\pi}{2})} = (-i)|z - z_k|^{-\frac{1}{2}}, k = 1, 2, 3, 4, 5, 6$

$$\begin{aligned}\int_{-4\dashleftarrow-2} \frac{1}{f(z)} dz &\overset{Math.}{=} \int_{-4\leftarrow-2} \frac{1}{f(z)} dz \\ &= \int_{-2}^{-4} (-i)^6 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz \\ &= \int_{-4}^{-2} \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz\end{aligned}$$

$\therefore$ *in Mathematica* $\int_{b_3^*} \frac{1}{f(z)} dz = 0$

3. *Using Lemma 6 to modify*

(a) $-4 \to -2$ : *the path along x-axis from $-4$ to $-2$ on sheet-I.*
$\arg(z - z_k) = -\pi$ *then* $\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}, k = 1, 2, 3, 4, 5, 6$
$\arg(z - z_k) = 0$ *then* $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}, k = 7$
$\therefore f(z) \overset{Math.}{=} f(z)$

(b) $-4 \dashleftarrow -2$ : *the path along x-axis from $-2$ to $-4$ on sheet-II. We known in theory that $f(z)\mid_{(II)} = -f(z)\mid_{(I)}$ , so we consider $-4 \leftarrow -2$*
$\arg(z - z_k) = -\pi$ *then* $\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}, k = 1, 2, 3, 4, 5, 6$

$\arg(z-z_k)=0$ *then* $\sqrt{z-z_k} \overset{Math.}{=} \sqrt{z-z_k}, k=7$

$\because f(z)\mid_{-4\leftarrow -2} \overset{Math.}{=} f(z)$

$\therefore f(z)\mid_{-4\leftarrow --- 2} = -f(z)\mid_{-4\leftarrow -2} \overset{Math.}{=} -f(z)$

$\therefore$

$$\begin{aligned}
\int_{b_3} \frac{1}{f(z)}dz &= \int_{b_3^*} \frac{1}{f(z)}dz \\
&= \int_{-4\rightarrow -2} \frac{1}{f(z)}dz + \int_{-4\leftarrow --- 2} \frac{1}{f(z)}dz \\
&\overset{Math.}{=} \int_{-4}^{-2} \prod_{k=1}^{7} |z-z_k|^{-\frac{1}{2}}\, dz - \int_{-2}^{-4} \prod_{k=1}^{7} |z-z_k|^{-\frac{1}{2}}\, dz \\
&= 2\int_{-4}^{-2} \prod_{k=1}^{7} |z-z_k|^{-\frac{1}{2}}\, dz \\
&= 0.0372385
\end{aligned}$$

$b_2$ : *Let* $b_2$ *is a cycle which center at* $0$ *with radius* $\frac{9}{2}$. *So we could write down the parameter, let* $z=\frac{9}{2}e^{i\theta}, \theta\in[-\pi,0)\cup[2\pi,3\pi)$ . *Notice that* $f(z)\mid_{(II)} = -f(z)\mid_{(I)}$,*so we have*

$$\begin{aligned}
\int_{b_2} \frac{1}{f(z)}dz &= \int_{-\pi}^{0} \frac{\frac{9}{2}ie^{i\theta}}{\prod_{k=1}^{7}\sqrt{\frac{9}{2}e^{i\theta}-z_k}}d\theta - \int_{0}^{\pi} \frac{\frac{9}{2}ie^{i\theta}}{\prod_{k=1}^{7}\sqrt{\frac{9}{2}e^{i\theta}-z_k}}d\theta \\
&= 0.2196815
\end{aligned}$$

*Using the same way before. Consider equivalent path* $b_2^* = b_3^* \cup -2 \overset{+}{\rightarrow} 2 \cup -2 \overset{-}{\leftarrow --} 2 \cup 2 \rightarrow 4 \cup 2 \leftarrow -- 4$

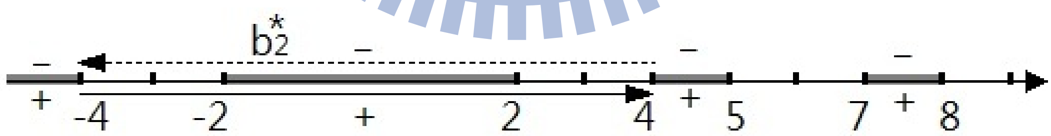



*Figure 42. The equivalent path* $b_2^*$.

1. *Analysis the integration over* $b_2^*$ *on sheet-I in theory*

   (a) $-2 \overset{+}{\rightarrow} 2$ : *the path along x-axis from* $-2$ *to* $2$ *on sheet-I.*
   $z-z_k=|z-z_k|$ *then* $\frac{1}{\sqrt{z-z_k}} = |z-z_k|^{-\frac{1}{2}}, k=6,7$
   $z-z_k = -|z-z_k| = |z-z_k|e^{i(-\pi)}$ *then* $\frac{1}{\sqrt{z-z_k}} = |z-z_k|^{-\frac{1}{2}} e^{i(\frac{\pi}{2})} = i\,|z-z_k|^{-\frac{1}{2}}, k=1,2,3,4,5,$

$$\int_{-2\overset{+}{\rightarrow}2}\frac{1}{f(z)}dz=\int_{-2}^{2}i^{5}\prod_{k=1}^{7}|z-z_k|^{-\frac{1}{2}}\,dz$$

(b) $-2 \overset{-}{\leftarrow\!\!-\,-} 2 \equiv -2 \overset{+}{\leftarrow} 2$ *i.e. the path on horizontal cut from* $2$ *to* $-2$ *on* $(-)$*-edge in sheet-II equals the path on horizontal cut from* $2$ *to* $-2$ *on* $(+)$*-edge in sheet-I. So we consider* $z \in -2 \overset{+}{\leftarrow} 2$.

$z-z_k=|z-z_k|$ *then* $\frac{1}{\sqrt{z-z_k}}=|z-z_k|^{-\frac{1}{2}}, k=6,7$

$z-z_k=-|z-z_k|=|z-z_k|\,e^{i(-\pi)}$ *then* $\frac{1}{\sqrt{z-z_k}}=|z-z_k|^{-\frac{1}{2}}\,e^{i(\frac{\pi}{2})}=i\,|z-z_k|^{-\frac{1}{2}}, k=1,2,3,4,5$

$$\begin{aligned}\int_{-2\overset{-}{\leftarrow\,-\,-}2}\frac{1}{f(z)}dz &= \int_{-2\overset{+}{\leftarrow}2}\frac{1}{f(z)}dz\\ &= \int_{2}^{-2}i^{5}\prod_{k=1}^{7}|z-z_k|^{-\frac{1}{2}}\,dz\end{aligned}$$

(c) $2\rightarrow 4$ *: the path along x-axis from* $2$ *to* $4$ *on sheet-I.*

$z-z_k=|z-z_k|$ *then* $\frac{1}{\sqrt{z-z_k}}=|z-z_k|^{-\frac{1}{2}}, k=5,6,7$

$z-z_k=-|z-z_k|=|z-z_k|\,e^{i(-\pi)}$ *then* $\frac{1}{\sqrt{z-z_k}}=|z-z_k|^{-\frac{1}{2}}\,e^{i(\frac{\pi}{2})}=i\,|z-z_k|^{-\frac{1}{2}}, k=1,2,3,4,$

$$\int_{2\rightarrow 4}\frac{1}{f(z)}dz=\int_{2}^{4}i^{4}\prod_{k=1}^{7}|z-z_k|^{-\frac{1}{2}}\,dz$$

(d) $2 \leftarrow\!\!-\,-\ 4$ *: the path along x-axis from* $4$ *to* $2$ *on sheet-II. We known in theory that* $f(z)\,|_{(II)}=-f(z)\,|_{(I)}$ *, so we consider* $2\leftarrow 4$

$z-z_k=|z-z_k|$ *then* $\frac{1}{\sqrt{z-z_k}}=|z-z_k|^{-\frac{1}{2}}, k=5,6,7$

$z-z_k=-|z-z_k|=|z-z_k|\,e^{i(-\pi)}$ *then* $\frac{1}{\sqrt{z-z_k}}=|z-z_k|^{-\frac{1}{2}}\,e^{i(\frac{\pi}{2})}=i\,|z-z_k|^{-\frac{1}{2}}, k=1,2,3,4,$

$$\begin{aligned}\int_{2\leftarrow\,-\,-4}\frac{1}{f(z)}dz &= -\int_{2\leftarrow 4}\frac{1}{f(z)}dz\\ &= -\int_{4}^{2}i^{4}\prod_{k=1}^{7}|z-z_k|^{-\frac{1}{2}}\,dz\end{aligned}$$

*By (a) , (b) , (c) , (d) , we have*

$$\begin{aligned}\int_{b_2^*} \frac{1}{f(z)} dz &= \int_{b_3^*} \frac{1}{f(z)} dz + 2\int_2^4 i^4 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz \\ &= 0.2196815\end{aligned}$$

2. *Analysis the integration over $b_2^*$ in Mathematica*

   (a) $-2 \xrightarrow{+} 2$ :

   $z - z_k = |z - z_k|$ *then* $\frac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}}, k = 6, 7$

   $z - z_k = -|z - z_k| = |z - z_k| e^{i\pi}$ *then* $\frac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}} e^{i(-\frac{\pi}{2})} = (-i)|z - z_k|^{-\frac{1}{2}}, k = 1, 2, 3, 4, 5,$

   $$\int_{-2 \xrightarrow{+} 2} \frac{1}{f(z)} dz = \int_{-2}^{2} (-i)^5 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz$$

   (b) $-2 \overset{-}{\longleftarrow} 2$ :

   $z - z_k = |z - z_k|$ *then* $\frac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}}, k = 6, 7$

   $z - z_k = -|z - z_k| = |z - z_k| e^{i\pi}$ *then* $\frac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}} e^{i(-\frac{\pi}{2})} = (-i)|z - z_k|^{-\frac{1}{2}}, k = 1, 2, 3, 4, 5,$

   $$\int_{-2 \overset{-}{\longleftarrow} 2} \frac{1}{f(z)} dz = \int_{2}^{-2} (-i)^5 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz$$

   (c) $2 \rightarrow 4$ :

   $z - z_k = |z - z_k|$ *then* $\frac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}}, k = 5, 6, 7$

   $z - z_k = -|z - z_k| = |z - z_k| e^{i\pi}$ *then* $\frac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}} e^{i(-\frac{\pi}{2})} = (-i)|z - z_k|^{-\frac{1}{2}}, k = 1, 2, 3, 4,$

   $$\int_{2 \longleftarrow 4} \frac{1}{f(z)} dz = \int_{4}^{2} (-i)^4 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz$$

*But in Mathematica we obtain different value*

$$\int_{b_2^*} \frac{1}{f(z)} dz = 0$$

3. *Using Lemma 6 to modify*

   (a) $-2 \overset{+}{\rightarrow} 2$ :

   $\arg(z - z_k) = -\pi$ *then* $\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}, k = 1, 2, 3, 4, 5$

   $\arg(z - z_k) = 0$ *then* $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}, k = 6, 7$

   $\therefore f(z) \overset{Math.}{=} -f(z)$

   (b) $-2 \overset{-}{\dashleftarrow} 2 \equiv -2 \overset{+}{\leftarrow} 2$ *i.e. the path on horizontal cut from* $2$ *to* $-2$ *on* $(-)$*-edge in sheet-II equals the path on horizontal cut from* $2$ *to* $-2$ *on* $(+)$*-edge in sheet-I. So we consider* $z \in -2 \overset{+}{\leftarrow} 2$

   $\arg(z - z_k) = -\pi$ *then* $\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}, k = 1, 2, 3, 4, 5$

   $\arg(z - z_k) = 0$ *then* $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}, k = 6, 7$

   $\therefore f(z) \overset{Math.}{=} -f(z)$

   (c) $2 \rightarrow 4$ :

   $\arg(z - z_k) = -\pi$ *then* $\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}, k = 1, 2, 3, 4$

   $\arg(z - z_k) = 0$ *then* $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}, k = 5, 6, 7$

   $\therefore f(z) \overset{Math.}{=} f(z)$

   (d) $2 \dashleftarrow 4$ : *We known that* $f(z)\mid_{(II)} = -f(z)\mid_{(I)}$ *, so we consider* $2 \leftarrow 4$

   $\arg(z - z_k) = -\pi$ *then* $\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}, k = 1, 2, 3, 4$

   $\arg(z - z_k) = 0$ *then* $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}, k = 5, 6, 7$

   $\because f(z)\mid_{2 \leftarrow 4} \overset{Math.}{=} f(z)$

   $\therefore f(z)\mid_{2 \dashleftarrow 4} = -f(z)\mid_{2 \leftarrow 4} \overset{Math.}{=} -f(z)$

*By* 1. , 2. , 3. *and Cauchy Integral Theorem*

$$\begin{aligned} \int_{b_2} \tfrac{1}{f(z)} dz &= \int_{b_2^*} \tfrac{1}{f(z)} dz \\ &\overset{Math.}{=} \int_{b_3^*} \tfrac{1}{f(z)} dz + 2 \int_2^4 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz \\ &= 0.2196815 \end{aligned}$$

$b_1$ : *Let* $b_1$ *is a cycle which center at* $\frac{3}{2}$ *with radius* 6. *So we could write down the parameter, let* $z = \frac{3}{2} + 6e^{i\theta}, \theta \in [-\pi, 0) \cup [2\pi, 3\pi)$ . *Notice that* $f(z)\mid_{(II)} = -f(z)\mid_{(I)}$,*so we have*

$$\begin{aligned}\int_{b_1} \frac{1}{f(z)} dz &= \int_{-\pi}^{0} \frac{6ie^{i\theta}}{\prod_{k=1}^{7} \sqrt{(\frac{3}{2} + 6e^{i\theta}) - z_k}} d\theta - \int_{0}^{\pi} \frac{6ie^{i\theta}}{\prod_{k=1}^{7} \sqrt{(\frac{3}{2} + 6e^{i\theta}) - z_k}} d\theta \\ &= 0.4132335\end{aligned}$$

*Using the same way before. Consider equivalent path* $b_1^* = b_3^* \cup b_2^* \cup 4 \overset{+}{\longrightarrow} 5 \cup 4 \overset{-}{\dashleftarrow} 5 \cup 5 \longrightarrow 7 \cup 5 \dashleftarrow 7$

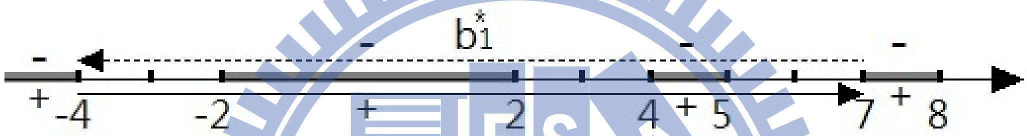



*Figure 43. The equivalent path* $b_1^*$.

1. *Analysis the integration over* $b_1^*$ *on sheet-I in theory*

   (a) $4 \overset{+}{\rightarrow} 5$ : *the path along* $x$*-axis from* 4 *to* 5 *on sheet-I.*
   $z - z_k = |z - z_k|$ *then* $\frac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}}, k = 4, 5, 6, 7$
   $z - z_k = -|z - z_k| = |z - z_k| e^{i(-\pi)}$ *then* $\frac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}} e^{i(\frac{\pi}{2})} = i|z - z_k|^{-\frac{1}{2}}, k = 1, 2, 3,$

$$\int_{4 \overset{+}{\rightarrow} 5} \frac{1}{f(z)} dz = \int_{4}^{5} i^3 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz$$

   (b) $4 \overset{-}{\dashleftarrow} 5 \equiv 4 \overset{+}{\leftarrow} 5$ *i.e. the path on horizontal cut from* 5 *to* 4 *on* $(-)$*-edge in sheet-II equals the path on horizontal cut from* 5 *to* 4 *on* $(+)$*-edge in sheet-I. So we consider* $z \in 4 \overset{+}{\leftarrow} 5$
   $z - z_k = |z - z_k|$ *then* $\frac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}}, k = 4, 5, 6, 7$
   $z - z_k = -|z - z_k| = |z - z_k| e^{i(-\pi)}$ *then* $\frac{1}{\sqrt{z - z_k}} = |z - z_k|^{-\frac{1}{2}} e^{i(\frac{\pi}{2})} = i|z - z_k|^{-\frac{1}{2}}, k = 1, 2, 3,$

$$
\begin{aligned}
\int_{4 \overset{-}{\leftarrow --} 5} \frac{1}{f(z)} dz &= \int_{4 \overset{+}{\leftarrow} 5} \frac{1}{f(z)} dz \\
&= \int_{5}^{4} i^{3} \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} \, dz
\end{aligned}
$$

(c) $5 \longrightarrow 7$ : *the path along $x$-axis from* $5$ *to* $7$ *on sheet-I.*

$z - z_k = |z - z_k|$ *then* $\frac{1}{\sqrt{z-z_k}} = |z - z_k|^{-\frac{1}{2}}, k = 3, 4, 5, 6, 7$

$z - z_k = -|z - z_k| = |z - z_k| \, e^{i(-\pi)}$ *then* $\frac{1}{\sqrt{z-z_k}} = |z - z_k|^{-\frac{1}{2}} \, e^{i(\frac{\pi}{2})} = i \, |z - z_k|^{-\frac{1}{2}}, k = 1, 2,$

$$
\int_{5 \rightarrow 7} \frac{1}{f(z)} dz = \int_{5}^{7} i^{2} \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} \, dz
$$

(d) $5 \leftarrow -- 7$ : *the path along $x$-axis from* $7$ *to* $5$ *on sheet-II. We known in theory that*

$f(z) \mid_{(II)} = -f(z) \mid_{(I)}$ *, so we consider* $5 \longleftarrow 7$

$z - z_k = |z - z_k|$ *then* $\frac{1}{\sqrt{z-z_k}} = |z - z_k|^{-\frac{1}{2}}, k = 3, 4, 5, 6, 7$

$z - z_k = -|z - z_k| = |z - z_k| \, e^{i(-\pi)}$ *then* $\frac{1}{\sqrt{z-z_k}} = |z - z_k|^{-\frac{1}{2}} \, e^{i(\frac{\pi}{2})} = i \, |z - z_k|^{-\frac{1}{2}}, k = 1, 2,$

$$
\begin{aligned}
\int_{5 \leftarrow -- 7} \frac{1}{f(z)} dz &= -\int_{5 \leftarrow 7} \frac{1}{f(z)} dz \\
&= -\int_{7}^{5} i^{2} \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} \, dz
\end{aligned}
$$

*By* (a) , (b) , (c) , (d) , *we have*

$$
\begin{aligned}
\int_{b_1^*} \frac{1}{f(z)} dz &= \int_{b_2^*} \frac{1}{f(z)} dz + 2 \int_{5}^{7} i^{2} \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} \, dz \\
&= 0.4132335
\end{aligned}
$$

2. *Analysis the integration over* $b_1^*$ *in Mathematica*

(a) $4 \xrightarrow{+} 5$ :

$z - z_k = |z - z_k|$ *then* $\frac{1}{\sqrt{z-z_k}} = |z - z_k|^{-\frac{1}{2}}, k = 4, 5, 6, 7$

$z - z_k = -|z - z_k| = |z - z_k| e^{i\pi}$ *then* $\frac{1}{\sqrt{z-z_k}} = |z - z_k|^{-\frac{1}{2}} e^{i(-\frac{\pi}{2})} = (-i)\, |z - z_k|^{-\frac{1}{2}}, k = 1, 2, 3,$

$$\int_{4 \xrightarrow{+} 5} \frac{1}{f(z)} dz = \int_4^5 (-i)^3 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}}\, dz$$

(b) $4 \overset{-}{\dashleftarrow} 5$ :

$z - z_k = |z - z_k|$ *then* $\frac{1}{\sqrt{z-z_k}} = |z - z_k|^{-\frac{1}{2}}, k = 4, 5, 6, 7$

$z - z_k = -|z - z_k| = |z - z_k| e^{i\pi}$ *then* $\frac{1}{\sqrt{z-z_k}} = |z - z_k|^{-\frac{1}{2}} e^{i(-\frac{\pi}{2})} = (-i)\, |z - z_k|^{-\frac{1}{2}}, k = 1, 2, 3,$

$$\int_{4 \overset{-}{\dashleftarrow} 5} \frac{1}{f(z)} dz = \int_5^4 (-i)^3 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}}\, dz$$

(c) $5 \rightarrow 7$ :

$z - z_k = |z - z_k|$ *then* $\frac{1}{\sqrt{z-z_k}} = |z - z_k|^{-\frac{1}{2}}, k = 3, 4, 5, 6, 7$

$z - z_k = -|z - z_k| = |z - z_k| e^{i\pi}$ *then* $\frac{1}{\sqrt{z-z_k}} = |z - z_k|^{-\frac{1}{2}} e^{i(-\frac{\pi}{2})} = (-i)\, |z - z_k|^{-\frac{1}{2}}, k = 1, 2,$

$$\int_{5 \rightarrow 7} \frac{1}{f(z)} dz = \int_5^7 (-i)^2 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}}\, dz$$

(d) $5 \dashleftarrow 7$ :

$z - z_k = |z - z_k|$ *then* $\frac{1}{\sqrt{z-z_k}} = |z - z_k|^{-\frac{1}{2}}, k = 3, 4, 5, 6, 7$

$z - z_k = -|z - z_k| = |z - z_k| e^{i\pi}$ *then* $\frac{1}{\sqrt{z-z_k}} = |z - z_k|^{-\frac{1}{2}} e^{i(-\frac{\pi}{2})} = (-i)\, |z - z_k|^{-\frac{1}{2}}, k = 1, 2,$

$$\int_{5 \dashleftarrow 7} \frac{1}{f(z)} dz = \int_7^5 (-i)^2 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}}\, dz$$

*But in Mathematica we obtain different value*

$$\int_{b_1^*} \frac{1}{f(z)} dz = 0$$

3. *Using Lemma 6 to modify*

   (a) $4 \overset{+}{\rightarrow} 5$ :

   $\arg(z - z_k) = -\pi$ *then* $\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}, k = 1, 2, 3,$

   $\arg(z - z_k) = 0$ *then* $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}, k = 4, 5, 6, 7$

   $\therefore f(z) \overset{Math.}{=} -f(z)$

   (b) $4 \overset{-}{\dashleftarrow} 5 \equiv 4 \overset{+}{\leftarrow} 5$ *i.e. the path on horizontal cut from* 5 *to* 4 *on (-)edge in sheet-II equals the path on horizontal cut from* 5 *to* 4 *on (+)edge in sheet-I. So we consider* $z \in 4 \overset{+}{\leftarrow} 5$

   $\arg(z - z_k) = -\pi$ *then* $\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}, k = 1, 2, 3,$

   $\arg(z - z_k) = 0$ *then* $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}, k = 4, 5, 6, 7$

   $\therefore f(z) \overset{Math.}{=} -f(z)$

   (c) $5 \rightarrow 7$ :

   $\arg(z - z_k) = -\pi$ *then* $\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}, k = 1, 2,$

   $\arg(z - z_k) = 0$ *then* $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}, k = 3, 4, 5, 6, 7$

   $\therefore f(z) \overset{Math.}{=} f(z)$

   (d) $5 \dashleftarrow 7$ : *We known that* $f(z)\mid_{(II)} = -f(z)\mid_{(I)}$ , *so we consider* $5 \leftarrow 7$

   $\arg(z - z_k) = -\pi$ *then* $\sqrt{z - z_k} \overset{Math.}{=} -\sqrt{z - z_k}, k = 1, 2,$

   $\arg(z - z_k) = 0$ *then* $\sqrt{z - z_k} \overset{Math.}{=} \sqrt{z - z_k}, k = 3, 4, 5, 6, 7$

   $\because f(z)\mid_{5 \leftarrow 7} \overset{Math.}{=} f(z)$

   $\therefore f(z)\mid_{5 \dashleftarrow 7} = -f(z)\mid_{5 \leftarrow 7} \overset{Math.}{=} -f(z)$

*By* 1. , 2. , 3. *and Cauchy Integral Theorem*

$$\begin{aligned}
\int_{b_1} \tfrac{1}{f(z)} dz &= \int_{b_1^*} \tfrac{1}{f(z)} dz \\
&\overset{Math.}{=} \int_{b_2^*} \tfrac{1}{f(z)} dz + 2 \int_5^7 \prod_{k=1}^{7} |z - z_k|^{-\frac{1}{2}} dz \\
&= 0.4132335
\end{aligned}$$

## 5.2 The details of the previous examples in section 2.2 .

**Example 68** *Evaluate the integrals of* $1/f(z)$ *over* $a, b$ *cycles for vertical cut where* $f(z) = \sqrt{(z-i)(z-2i)(z-3i)(z-5i)(z-6i)(z-8i)}$.

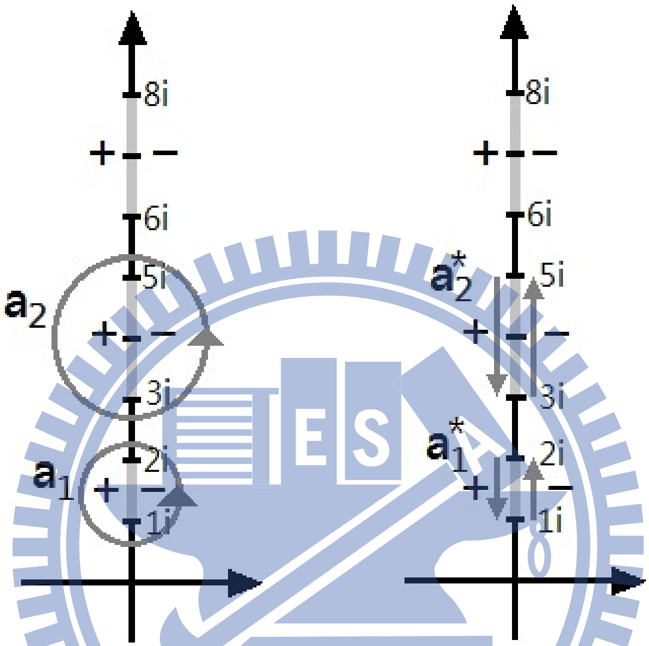



*Figure 44. path a and its equivalent path* $a^*$.

*Solution:*

1. *Compute* $\int_{a_1^*} \frac{1}{f(z)} dz$ *where* $a_1^*$ *is an equivalent path for* $a_1$ *and* $a_1^*$ *is the path along vertical cut from* $2i$ *to* $i$ *on* $(+)$*-edge of sheet-I (called* $a_{11}^*$*) and then back from* $i$ *to* $2i$ *on* $(-)$*-edge of sheet-I (called* $a_{12}^*$*).*

   (a) $a_{11}^*$ *: Let* $z = ri$ *where* $r : 2 \xrightarrow{+} 1$ *and* $dz = idr$

      i. *Analysis in theory :*
      *Since* $z - ki = |z-ki|\, e^{i\arg(z-ki)}$ *, so we consider* $\arg(z-ki)$.

$$\begin{aligned}\arg(z-i) &= -\frac{3}{2}\pi \Rightarrow \arg(\frac{1}{\sqrt{z-i}}) = \frac{3\pi}{4} \\ \arg(z-ki) &= -\frac{1}{2}\pi \Rightarrow \arg(\frac{1}{\sqrt{z-ki}}) = \frac{\pi}{4}, k = 2,3,5,6,8\end{aligned}$$

      *we have*

$$
\begin{aligned}
\frac{1}{f(z)} &= (\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}})(e^{\frac{3\pi}{4}})(e^{\frac{\pi}{4}})^5 \\
&= (\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}})(e^{2\pi}) \\
&= (\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}}) \\
&= R
\end{aligned}
$$

ii. *Analysis in Mathematica (no matter in which sheet) :*
*Since* $z-ki = |z-ki|\, e^{i\arg(z-ki)}$ *, so we consider* $\arg(z-ki)$*.*

$$
\begin{aligned}
\arg(z-i) &= \frac{1}{2}\pi \Rightarrow \arg(\frac{1}{\sqrt{z-i}}) = -\frac{\pi}{4} \\
\arg(z-ki) &= -\frac{1}{2}\pi \Rightarrow \arg(\frac{1}{\sqrt{z-i}}) = \frac{\pi}{4}, k = 2,3,5,6,8
\end{aligned}
$$

*we have*

$$
\begin{aligned}
\frac{1}{f(z)} &= (\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}})(e^{-\frac{\pi}{4}})(e^{\frac{\pi}{4}})^5 \\
&= (\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}})(e^{\pi}) \\
&= (\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}})(-1) \\
&= -R
\end{aligned}
$$

*Compare with (i.) and (ii.) we find that when we want to obtain true value, the value which we have from Mathematica should multiply (−1) , i.e.*

$$
sign(f(z)|_{(I)}) = (-1)sign(f(z)|_{Mathematica})
$$

iii. *Using the Lemma 15 to modify :*

$$
\begin{aligned}
\arg(z-i) &= -\frac{3}{2}\pi \Rightarrow \frac{1}{\sqrt{z-i}} \overset{Math.}{=} (-1)\frac{1}{\sqrt{z-i}} \\
\arg(z-ki) &= -\frac{1}{2}\pi \Rightarrow \frac{1}{\sqrt{z-i}} \overset{Math.}{=} \frac{1}{\sqrt{z-ki}}, k = 2,3,5,6,8
\end{aligned}
$$

*we have*

$$
\frac{1}{f(z)} \overset{Math.}{=} (-1)\frac{1}{f(z)}
$$

*The same result as above difference between in theory and in Mathematica , the difference is a minus sign.*

(b) $a_{12}^{*}$ : *Let* $z = ri$ *where* $r : 1 \overset{-}{\rightarrow} 2$ *and* $dz = idr$

i. *Analysis in theory :*
*Since* $z - ki = |z - ki|\, e^{i\arg(z-ki)}$ *, so we consider* $\arg(z - ki)$*.*

$$\begin{aligned} \arg(z-i) &= \frac{1}{2}\pi \Rightarrow \arg(\frac{1}{\sqrt{z-i}}) = -\frac{\pi}{4} \\ \arg(z-ki) &= -\frac{1}{2}\pi \Rightarrow \arg(\frac{1}{\sqrt{z-ki}}) = \frac{\pi}{4}, k = 2,3,5,6,8 \end{aligned}$$

*we have*

$$\begin{aligned} \frac{1}{f(z)} &= (\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}})(e^{-\frac{\pi}{4}})(e^{\frac{\pi}{4}})^5 \\ &= (\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}})(e^{\pi}) \\ &= (\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}})(-1) \\ &= -R \end{aligned}$$

ii. *Analysis in Mathematica (no matter in which sheet) :*
*Since* $z - ki = |z - ki|\, e^{i\arg(z-ki)}$ *, so we consider* $\arg(z - ki)$*.*

$$\begin{aligned} \arg(z-i) &= \frac{1}{2}\pi \Rightarrow \arg(\frac{1}{\sqrt{z-i}}) = -\frac{\pi}{4} \\ \arg(z-ki) &= -\frac{1}{2}\pi \Rightarrow \arg(\frac{1}{\sqrt{z-ki}}) = \frac{\pi}{4}, k = 2,3,5,6,8 \end{aligned}$$

*we have*

$$\begin{aligned} \frac{1}{f(z)} &= (\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}})(e^{-\frac{\pi}{4}})(e^{\frac{\pi}{4}})^5 \\ &= (\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}})(e^{\pi}) \\ &= (\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}})(-1) \\ &= -R \end{aligned}$$

*Compare with (i.) and (ii.) we find the value is same.*

iii. *Using the Lemma 15 to modify :*

$$\arg(z-i) = \frac{1}{2}\pi \Rightarrow \frac{1}{\sqrt{z-i}} \overset{Math.}{=} \frac{1}{\sqrt{z-i}}$$

$$\arg(z-ki) = -\frac{1}{2}\pi \Rightarrow \frac{1}{\sqrt{z-i}} \overset{Math.}{=} \frac{1}{\sqrt{z-ki}}, k=2,3,5,6,8$$

*we have*

$$\frac{1}{f(z)} \overset{Math.}{=} \frac{1}{f(z)}$$

*The same result as above.*

*By* $(a)$ *and* $(b)$ *above , we have*

$$\begin{aligned}
\int_{a_1} \frac{1}{f(z)}dz &= \int_{a_1^*} \frac{1}{f(z)}dz \\
&= \int_{a_{11}^*} \frac{1}{f(z)}dz + \int_{a_{12}^*} \frac{1}{f(z)}dz \\
&= -2\int_2^1 \Big(\prod_{k=1,k\neq 4,7}^{8} |ri-ki|^{-\frac{1}{2}}\Big) idr \\
&= 0. - 0.531987i
\end{aligned}$$

2. *Compute* $\int_{a_2^*} \frac{1}{f(z)}dz$ *where* $a_2^*$ *is an equivalent path for* $a_2$ *and* $a_2^*$ *is the path along vertical cut from* $5i$ *to* $3i$ *on* $(+)$*-edge of sheet-I (called* $a_{21}^*$*) and then back from* $3i$ *to* $5i$ *on* $(-)$*-edge of sheet-I (called* $a_{22}^*$*)*

   (a) $a_{21}^*$ *: Let* $z = ri$ *where* $r : 5 \xrightarrow{+} 3$ *and* $dz = idr$

      i. *Analysis in theory :*
      *Since* $z - ki = |z-ki|\, e^{i\arg(z-ki)}$ *, so we consider* $\arg(z-ki)$.

$$\arg(z-ki) = -\frac{3}{2}\pi \Rightarrow \arg\Big(\frac{1}{\sqrt{z-ki}}\Big) = \frac{3\pi}{4}, k=1,2,3$$

$$\arg(z-ki) = -\frac{1}{2}\pi \Rightarrow \arg\Big(\frac{1}{\sqrt{z-ki}}\Big) = \frac{\pi}{4}, k=5,6,8$$

*we have*

$$\begin{aligned}\frac{1}{f(z)} &= (\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}})(e^{\frac{3\pi}{4}})^3(e^{\frac{\pi}{4}})^3 \\ &= (\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}})(e^{3\pi}) \\ &= (\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}})(-1) \\ &= -R\end{aligned}$$

ii. *Analysis in Mathematica (no matter in which sheet) :*
*Since* $z-ki = |z-ki|\, e^{i\arg(z-ki)}$ *, so we consider* $\arg(z-ki)$*.*

$$\begin{aligned}\arg(z-ki) &= \frac{1}{2}\pi \Rightarrow \arg(\frac{1}{\sqrt{z-ki}}) = -\frac{\pi}{4}, k=1,2,3 \\ \arg(z-ki) &= -\frac{1}{2}\pi \Rightarrow \arg(\frac{1}{\sqrt{z-ki}}) = \frac{\pi}{4}, k=5,6,8\end{aligned}$$

*we have*

$$\begin{aligned}\frac{1}{f(z)} &= (\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}})(e^{-\frac{\pi}{4}})^3(e^{\frac{\pi}{4}})^3 \\ &= (\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}})(e^{0\pi}) \\ &= (\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}}) \\ &= R\end{aligned}$$

*Compare with (i.) and (ii.) we find that when we want to obtain true value, the value which we have from Mathematica should multiply (−1) , i.e.*

$$sign(f(z)|_{(I)}) = (-1)sign(f(z)|_{Mathematica})$$

iii. *Using the Lemma 15 to modify :*

$$\begin{aligned}\arg(z-ki) &= -\frac{3}{2}\pi \Rightarrow \frac{1}{\sqrt{z-ki}} \overset{Math.}{=} (-1)\frac{1}{\sqrt{z-ki}}, k=1,2,3 \\ \arg(z-ki) &= -\frac{1}{2}\pi \Rightarrow \frac{1}{\sqrt{z-ki}} \overset{Math.}{=} \frac{1}{\sqrt{z-ki}}, k=5,6,8\end{aligned}$$

*we have*

$$\frac{1}{f(z)} \overset{Math.}{=} (-1)\frac{1}{f(z)}$$

*The same result as above difference between in theory and in Mathematica , the difference is a minus sign.*

(b) $a^*_{22}$ *: Let* $z = ri$ *where* $r : 3 \xrightarrow{-} 5$ *and* $dz = idr$

i. *Analysis in theory :*
*Since* $z - ki = |z - ki|\, e^{i\arg(z-ki)}$ *, so we consider* $\arg(z - ki)$*.*

$$\begin{aligned}\arg(z-ki) &= \frac{1}{2}\pi \Rightarrow \arg(\frac{1}{\sqrt{z-ki}}) = -\frac{\pi}{4}, k = 1,2,3 \\ \arg(z-ki) &= -\frac{1}{2}\pi \Rightarrow \arg(\frac{1}{\sqrt{z-ki}}) = \frac{\pi}{4}, k = 5,6,8\end{aligned}$$

*we have*

$$\begin{aligned}\frac{1}{f(z)} &= (\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}})(e^{-\frac{\pi}{4}})^3(e^{\frac{\pi}{4}})^3 \\ &= (\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}})(e^{0\pi}) \\ &= (\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}}) \\ &= R\end{aligned}$$

ii. *Analysis in Mathematica (no matter in which sheet) :*
*Since* $z - ki = |z - ki|\, e^{i\arg(z-ki)}$ *, so we consider* $\arg(z - ki)$*.*

$$\begin{aligned}\arg(z-ki) &= \frac{1}{2}\pi \Rightarrow \arg(\frac{1}{\sqrt{z-ki}}) = -\frac{\pi}{4}, k = 1,2,3 \\ \arg(z-ki) &= -\frac{1}{2}\pi \Rightarrow \arg(\frac{1}{\sqrt{z-ki}}) = \frac{\pi}{4}, k = 5,6,8\end{aligned}$$

*we have*

$$\begin{aligned}\frac{1}{f(z)} &= (\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}})(e^{-\frac{\pi}{4}})^3(e^{\frac{\pi}{4}})^3 \\ &= (\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}})(e^{0\pi}) \\ &= (\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}}) \\ &= R\end{aligned}$$

*Compare with (i.) and (ii.) we find the value is same.*

iii. *Using the Lemma 15 to modify :*

$$\begin{aligned}\arg(z-ki) &= \frac{1}{2}\pi \Rightarrow \frac{1}{\sqrt{z-ki}} \overset{Math.}{=} \frac{1}{\sqrt{z-ki}}, k=1,2,3\\ \arg(z-ki) &= -\frac{1}{2}\pi \Rightarrow \frac{1}{\sqrt{z-ki}} \overset{Math.}{=} \frac{1}{\sqrt{z-ki}}, k=5,6,8\end{aligned}$$

*we have*

$$\frac{1}{f(z)} \overset{Math.}{=} \frac{1}{f(z)}$$

*The same result as above.*

*By* $(a)$ *and* $(b)$ *above , we have*

$$\begin{aligned}\int_{a_2}\frac{1}{f(z)}dz &= \int_{a_2^*}\frac{1}{f(z)}dz\\ &= \int_{a_{21}^*}\frac{1}{f(z)}dz + \int_{a_{22}^*}\frac{1}{f(z)}dz\\ &= -2\int_5^3\left(\prod_{k=1,k\neq 4,7}^{8}|ri-ki|^{-\frac{1}{2}}\right)idr\\ &= 0.+0.996888i\end{aligned}$$

3. *Compute* $\int_{b_2^*}\frac{1}{f(z)}dz$ *where* $b_2^*$ *is an equivalent path for* $b_2$ *and* $b_2^*$ *is the path along vertical line from* $6i$ *to* $5i$ *on sheet-I (called* $b_{21}^*$*) and then back from* $5i$ *to* $6i$ *on sheet-II (called* $b_{22}^*$*)*

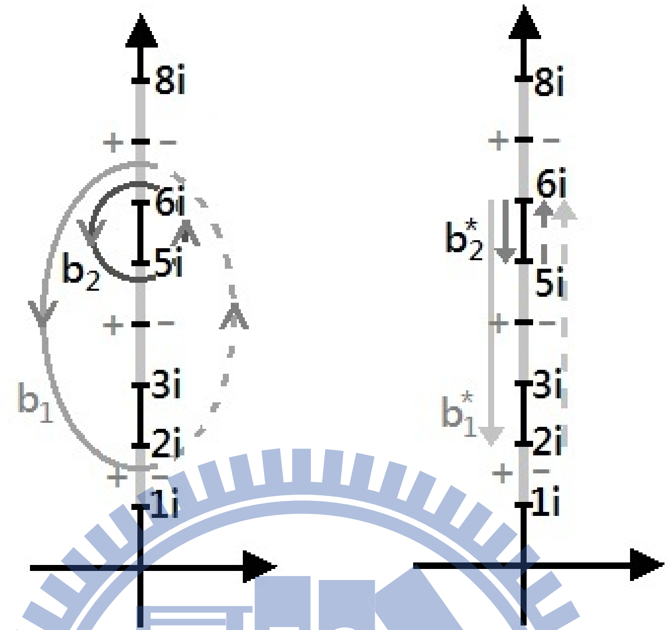



*Figure 45. b and equivalent path $b^*$.*

(a) $b_{21}^*$ : *Let* $z = ri$ *where* $r : 6 \to 5$ *and* $dz = idr$

i. *Analysis in theory :*

*Since* $z - ki = |z - ki|\, e^{i\arg(z-ki)}$ , *so we consider* $\arg(z - ki)$.

$$\begin{aligned}\arg(z - ki) &= -\frac{3}{2}\pi \Rightarrow \arg\left(\frac{1}{\sqrt{z-ki}}\right) = \frac{3\pi}{4}, k = 1, 2, 3, 5 \\ \arg(z - ki) &= -\frac{1}{2}\pi \Rightarrow \arg\left(\frac{1}{\sqrt{z-ki}}\right) = \frac{\pi}{4}, k = 6, 8\end{aligned}$$

*we have*

$$\begin{aligned}\frac{1}{f(z)} &= \left(\prod_{k=1,k\neq 4,7}^{8} |z - ki|^{-\frac{1}{2}}\right)\left(e^{\frac{3\pi}{4}}\right)^4\left(e^{\frac{\pi}{4}}\right)^2 \\ &= \left(\prod_{k=1,k\neq 4,7}^{8} |z - ki|^{-\frac{1}{2}}\right)\left(e^{\frac{7}{2}\pi}\right) \\ &= \left(\prod_{k=1,k\neq 4,7}^{8} |z - ki|^{-\frac{1}{2}}\right)(-i) \\ &= (-i)R\end{aligned}$$

ii. *Analysis in Mathematica (no matter in which sheet) :*

*Since* $z - ki = |z - ki|\, e^{i\arg(z-ki)}$ , *so we consider* $\arg(z - ki)$.

$$\arg(z-ki) = \frac{1}{2}\pi \Rightarrow \arg(\frac{1}{\sqrt{z-ki}}) = -\frac{\pi}{4}, k=1,2,3,5$$
$$\arg(z-ki) = -\frac{1}{2}\pi \Rightarrow \arg(\frac{1}{\sqrt{z-ki}}) = \frac{\pi}{4}, k=6,8$$

*we have*

$$\begin{aligned}
\frac{1}{f(z)} &= (\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}})(e^{-\frac{\pi}{4}})^4(e^{\frac{\pi}{4}})^2 \\
&= (\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}})(e^{-\frac{1}{2}\pi}) \\
&= (\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}})(-i) \\
&= (-i)R
\end{aligned}$$

*Compare with (i.) and (ii.) we find the value is same.*

iii. *Using the Lemma 15 to modify :*

$$\arg(z-ki) = -\frac{3}{2}\pi \Rightarrow \frac{1}{\sqrt{z-ki}} \overset{Math.}{=} -\frac{1}{\sqrt{z-ki}}, k=1,2,3,5$$
$$\arg(z-ki) = -\frac{1}{2}\pi \Rightarrow \frac{1}{\sqrt{z-ki}} \overset{Math.}{=} \frac{1}{\sqrt{z-ki}}, k=6,8$$

*we have*

$$\frac{1}{f(z)} \overset{Math.}{=} \frac{1}{f(z)}$$

*The same result as above.*

(b) $b_{22}^{*}$ : *We known that* $f(z)\mid_{(II)} = -f(z)\mid_{(I)}$ , *so we can consider* $b_{22}^{**}$ *is the path along vertical line from* $5i$ *to* $6i$ *on sheet-I .*
*Let* $z = ri$ *where* $r: 5 \to 6$ *and* $dz = idr$

i. *Analysis in theory :*
*Since* $z-ki = |z-ki|\, e^{i\arg(z-ki)}$ , *so we consider* $\arg(z-ki)$.

$$\arg(z-ki) = -\frac{3}{2}\pi \Rightarrow \arg(\frac{1}{\sqrt{z-ki}}) = \frac{3\pi}{4}, k=1,2,3,5$$
$$\arg(z-ki) = -\frac{1}{2}\pi \Rightarrow \arg(\frac{1}{\sqrt{z-ki}}) = \frac{\pi}{4}, k=6,8$$

*we have*

$$
\begin{aligned}
\frac{1}{f(z)} &= \Big(\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}}\Big)(e^{\frac{3\pi}{4}})^4(e^{\frac{\pi}{4}})^2 \\
&= \Big(\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}}\Big)(e^{\frac{7}{2}\pi}) \\
&= \Big(\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}}\Big)(-i) \\
&= (-i)R
\end{aligned}
$$

*Hence*

$$
\begin{aligned}
\tfrac{1}{f(z)}\,|_{b_{22}^{*}} &= (-1)\tfrac{1}{f(z)}\,|_{b_{22}^{**}} \\
&= (-1)\Big(\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}}\Big)(-i) \\
&= (i)R
\end{aligned}
$$

ii. *Analysis in Mathematica (no matter in which sheet) :*
*Since* $z-ki = |z-ki|\, e^{i\arg(z-ki)}$ *, so we consider* $\arg(z-ki)$*.*

$$
\begin{aligned}
\arg(z-ki) &= \frac{1}{2}\pi \Rightarrow \arg\Big(\frac{1}{\sqrt{z-ki}}\Big) = -\frac{\pi}{4}, k=1,2,3,5 \\
\arg(z-ki) &= -\frac{1}{2}\pi \Rightarrow \arg\Big(\frac{1}{\sqrt{z-ki}}\Big) = \frac{\pi}{4}, k=6,8
\end{aligned}
$$

*we have*

$$
\begin{aligned}
\frac{1}{f(z)} &= \Big(\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}}\Big)(e^{-\frac{\pi}{4}})^4(e^{\frac{\pi}{4}})^2 \\
&= \Big(\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}}\Big)(e^{-\frac{1}{2}\pi}) \\
&= \Big(\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}}\Big)(-i) \\
&= (-i)R
\end{aligned}
$$

*Compare with (i.) and (ii.) we find that when we want to obtain true value, the value which we have from Mathematica should multiply* $(-1)$ *, i.e.*

$$
sign(f(z)|_{(I)}) = (-1)sign(f(z)|_{Mathematica})
$$

iii. *Using the Lemma 15 to modify :*

$$\arg(z-ki) = -\frac{3}{2}\pi \Rightarrow \frac{1}{\sqrt{z-ki}} \overset{Math.}{=} -\frac{1}{\sqrt{z-ki}}, k=1,2,3,5$$
$$\arg(z-ki) = -\frac{1}{2}\pi \Rightarrow \frac{1}{\sqrt{z-ki}} \overset{Math.}{=} \frac{1}{\sqrt{z-ki}}, k=6,8$$

*we have*

$$\frac{1}{f(z)}\Big|_{b_{22}^{**}} \overset{Math.}{=} \frac{1}{f(z)}$$

*and*

$$\frac{1}{f(z)}\Big|_{b_{22}^{*}} = -\frac{1}{f(z)}\Big|_{b_{22}^{**}} \overset{Math.}{=} -\frac{1}{f(z)}$$

*Hence*

$$\begin{aligned}
\int_{b_2}\frac{1}{f(z)}dz &= \int_{b_2^*}\frac{1}{f(z)}dz \\
&= \int_{b_{21}^*}\frac{1}{f(z)}dz + \int_{b_{22}^*}\frac{1}{f(z)}dz \\
&= \int_6^5 \frac{i}{f(ri)}dr + \int_5^6 -\frac{i}{f(ri)}dr \\
&= -2\int_5^6 \Big(\prod_{k=1,k\neq 4,7}^{8} |ri-ki|^{-\frac{1}{2}}\Big) i\,dr \\
&= -0.645057 + 0.i
\end{aligned}$$

4. *Compute* $\int_{b_1^*}\frac{1}{f(z)}dz$ *where* $b_1^*$ *is an equivalent path for* $b_1$ *and* $b_1^* = b_2^* \cup b_{11}^* \cup b_{12}^* \cup b_{13}^* \cup b_{14}^*$ *where* $b_{11}^*$ *is the path along vertical cut from* $5i$ *to* $3i$ *on* $(+)$*-edge of sheet-I ,* $b_{12}^*$ *is the path along vertical cut from* $3i$ *to* $5i$ *on* $(-)$*-edge of sheet-II ,* $b_{13}^*$ *is the path along vertical line from* $3i$ *to* $2i$ *on sheet-I ,* $b_{14}^*$ *is the path along vertical line from* $2i$ *to* $3i$ *on sheet-II .*

   (a) $b_{11}^* \equiv a_{21}^*$ *: Done .*

   (b) $b_{12}^* \equiv$ *the path along vertical cut from* $3i$ *to* $5i$ *on* $(+)$*-edge of sheet-I .*

   *Let* $z = ri$ *where* $r: 3 \to 5$ *and* $dz = idr$

      i. *Analysis in theory :*
      *Since* $z-ki = |z-ki|\, e^{i\arg(z-ki)}$ *, so we consider* $\arg(z-ki)$.

$$\begin{aligned}\arg(z-ki) &= -\frac{3}{2}\pi \Rightarrow \arg(\frac{1}{\sqrt{z-ki}}) = \frac{3\pi}{4}, k=1,2,3\\ \arg(z-ki) &= -\frac{1}{2}\pi \Rightarrow \arg(\frac{1}{\sqrt{z-ki}}) = \frac{\pi}{4}, k=5,6,8\end{aligned}$$

*we have*

$$\begin{aligned}\frac{1}{f(z)} &= (\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}})(e^{\frac{3\pi}{4}})^3(e^{\frac{\pi}{4}})^3\\ &= (\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}})(e^{3\pi})\\ &= (\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}})(-1)\\ &= (-1)R\end{aligned}$$

ii. *Analysis in Mathematica (no matter in which sheet) :*
*Since* $z-ki = |z-ki|\, e^{i\arg(z-ki)}$ *, so we consider* $\arg(z-ki)$*.*

$$\begin{aligned}\arg(z-ki) &= \frac{1}{2}\pi \Rightarrow \arg(\frac{1}{\sqrt{z-ki}}) = -\frac{\pi}{4}, k=1,2,3\\ \arg(z-ki) &= -\frac{1}{2}\pi \Rightarrow \arg(\frac{1}{\sqrt{z-ki}}) = \frac{\pi}{4}, k=5,6,8\end{aligned}$$

*we have*

$$\begin{aligned}\frac{1}{f(z)} &= (\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}})(e^{-\frac{\pi}{4}})^3(e^{\frac{\pi}{4}})^3\\ &= (\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}})(e^{0\pi})\\ &= (\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}})\\ &= R\end{aligned}$$

*Compare with (i.) and (ii.) we find that when we want to obtain true value, the value which we have from Mathematica should multiply* $(-1)$ *, i.e.*

$$sign(f(z)|_{(I)}) = (-1)sign(f(z)|_{Mathematica})$$

iii. *Using the Lemma 15 to modify :*

$$\arg(z-ki) = -\frac{3}{2}\pi \Rightarrow \frac{1}{\sqrt{z-ki}} \overset{Math.}{=} (-1)\frac{1}{\sqrt{z-ki}}, k=1,2,3$$
$$\arg(z-ki) = -\frac{1}{2}\pi \Rightarrow \frac{1}{\sqrt{z-ki}} \overset{Math.}{=} \frac{1}{\sqrt{z-ki}}, k=5,6,8$$

*we have*

$$\frac{1}{f(z)} \overset{Math.}{=} (-1)\frac{1}{f(z)}$$

*The same result as above difference between in theory and in Mathematica , the difference is a minus sign.*

(c) $b_{13}^*$ *: Let* $z = ri$ *where* $r : 3 \to 2$ *and* $dz = idr$

i. *Analysis in theory :*
*Since* $z - ki = |z-ki|\, e^{i\arg(z-ki)}$ *, so we consider* $\arg(z-ki)$.

$$\arg(z-ki) = -\frac{3}{2}\pi \Rightarrow \arg(\frac{1}{\sqrt{z-ki}}) = \frac{3\pi}{4}, k=1,2$$
$$\arg(z-ki) = -\frac{1}{2}\pi \Rightarrow \arg(\frac{1}{\sqrt{z-ki}}) = \frac{\pi}{4}, k=3,5,6,8$$

*we have*

$$\begin{aligned}\frac{1}{f(z)} &= (\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}})(e^{\frac{3\pi}{4}})^2(e^{\frac{\pi}{4}})^4 \\ &= (\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}})(e^{\frac{5}{2}\pi}) \\ &= (\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}})(i) \\ &= (i)R\end{aligned}$$

ii. *Analysis in Mathematica (no matter in which sheet) :*
*Since* $z - ki = |z-ki|\, e^{i\arg(z-ki)}$ *, so we consider* $\arg(z-ki)$.

$$\arg(z-ki) = \frac{1}{2}\pi \Rightarrow \arg(\frac{1}{\sqrt{z-ki}}) = -\frac{\pi}{4}, k=1,2$$
$$\arg(z-ki) = -\frac{1}{2}\pi \Rightarrow \arg(\frac{1}{\sqrt{z-ki}}) = \frac{\pi}{4}, k=3,5,6,8$$

*we have*

$$\begin{aligned}\frac{1}{f(z)} &= (\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}})(e^{-\frac{\pi}{4}})^2(e^{\frac{\pi}{4}})^4 \\ &= (\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}})(e^{\frac{1}{2}\pi}) \\ &= (\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}})(i) \\ &= (i)R\end{aligned}$$

*Compare with (i.) and (ii.) we find the value is same.*

iii. *Using the Lemma 15 to modify :*

$$\begin{aligned}\arg(z-ki) &= -\frac{3}{2}\pi \Rightarrow \frac{1}{\sqrt{z-ki}} \overset{Math.}{=} (-1)\frac{1}{\sqrt{z-ki}}, k=1,2 \\ \arg(z-ki) &= -\frac{1}{2}\pi \Rightarrow \frac{1}{\sqrt{z-ki}} \overset{Math.}{=} \frac{1}{\sqrt{z-ki}}, k=3,5,6,8\end{aligned}$$

*we have*

$$\frac{1}{f(z)} \overset{Math.}{=} \frac{1}{f(z)}$$

*The same result as above.*

(d) $b_{14}^{*}$ : *We known that* $f(z)\mid_{(II)}= -f(z)\mid_{(I)}$ *, so we can consider* $b_{14}^{**}$ *is the path along vertical line from* $2i$ *to* $3i$ *on sheet-I .*

*Let* $z = ri$ *where* $r : 2 \to 3$ *and* $dz = idr$

i. *Analysis in theory :*

*Since* $z - ki = |z-ki|\, e^{i\arg(z-ki)}$ *, so we consider* $\arg(z-ki)$.

$$\begin{aligned}\arg(z-ki) &= \frac{1}{2}\pi \Rightarrow \arg(\frac{1}{\sqrt{z-ki}}) = -\frac{\pi}{4}, k=1,2 \\ \arg(z-ki) &= -\frac{1}{2}\pi \Rightarrow \arg(\frac{1}{\sqrt{z-ki}}) = \frac{\pi}{4}, k=3,5,6,8\end{aligned}$$

*we have*

$$\begin{aligned}\frac{1}{f(z)} &= (\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}})(e^{-\frac{\pi}{4}})^2(e^{\frac{\pi}{4}})^4 \\ &= (\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}})(e^{\frac{1}{2}\pi}) \\ &= (\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}})(i) \\ &= (i)R\end{aligned}$$

*Hence*

$$\begin{aligned} \tfrac{1}{f(z)}\,|_{b_{14}^{*}} &= (-1)\tfrac{1}{f(z)}\,|_{b_{14}^{**}} \\ &= (-1)(\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}})(i) \\ &= (-i)R \end{aligned}$$

ii. *Analysis in Mathematica (no matter in which sheet) :*
*Since* $z-ki = |z-ki|\, e^{i\arg(z-ki)}$ *, so we consider* $\arg(z-ki)$*.*

$$\begin{aligned} \arg(z-ki) &= \frac{1}{2}\pi \Rightarrow \arg(\frac{1}{\sqrt{z-ki}}) = -\frac{\pi}{4}, k=1,2 \\ \arg(z-ki) &= -\frac{1}{2}\pi \Rightarrow \arg(\frac{1}{\sqrt{z-ki}}) = \frac{\pi}{4}, k=3,5,6,8 \end{aligned}$$

*we have*

$$\begin{aligned} \frac{1}{f(z)} &= (\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}})(e^{-\frac{\pi}{4}})^2(e^{\frac{\pi}{4}})^4 \\ &= (\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}})(e^{\frac{1}{2}\pi}) \\ &= (\prod_{k=1,k\neq 4,7}^{8} |z-ki|^{-\frac{1}{2}})(i) \\ &= (i)R \end{aligned}$$

*Compare with (i.) and (ii.) we find that when we want to obtain true value, the value which we have from Mathematica should multiply* $(-1)$ *, i.e.*

$$sign(f(z)|_{(I)}) = (-1)sign(f(z)|_{Mathematica})$$

iii. *Using the Lemma 15 to modify :*

$$\begin{aligned} \arg(z-ki) &= \frac{1}{2}\pi \Rightarrow \frac{1}{\sqrt{z-ki}} \overset{Math.}{=} \frac{1}{\sqrt{z-ki}}, k=1,2 \\ \arg(z-ki) &= -\frac{1}{2}\pi \Rightarrow \frac{1}{\sqrt{z-ki}} \overset{Math.}{=} \frac{1}{\sqrt{z-ki}}, k=3,5,6,8 \end{aligned}$$

*we have*

$$\frac{1}{f(z)}\,|_{b_{14}^{**}} \overset{Math.}{=} \frac{1}{f(z)}$$

*and*

$$\frac{1}{f(z)}\ |_{b_{14}^*}=(-1)\frac{1}{f(z)}\ |_{b_{14}^{**}}\overset{Math.}{=}(-1)\frac{1}{f(z)}$$

*Hence , by (a) , (b) , (c) , (d) and Cauchy Integral Theorem , we have*

$$\begin{aligned}\int_{b_1}\frac{1}{f(z)}dz &= \int_{b_1^*}\frac{1}{f(z)}dz\\ &= \left\{\int_{b_2^*}+\int_{b_{11}^*}+\int_{b_{12}^*}+\int_{b_{13}^*}+\int_{b_{14}^*}\right\}\frac{1}{f(z)}dz\\ &= (-2)\int_5^6\Big(\prod_{k=1,k\neq 4,7}^{8}|ri-ki|^{-\frac{1}{2}}\Big)idr\\ &\quad +(-2)\int_2^3\Big(\prod_{k=1,k\neq 4,7}^{8}|ri-ki|^{-\frac{1}{2}}\Big)idr\\ &= -1.40245+0.i\end{aligned}$$

**Example 69** *Compute the integrals of $1/f(z)$ over every cycles in the Figure below where*

$$f(z)=\sqrt{(z-z_1)(z-z_2)(z-z_3)(z-z_4)(z-z_5)(z-z_6)(z-z_7)(z-z_8)}$$

*for* $z_1=-2-i, z_2=-2+i, z_3=-1-i, z_4=-1+i, z_5=0+0i, z_6=0+i, z_7=1+i, z_8=1+2i.$

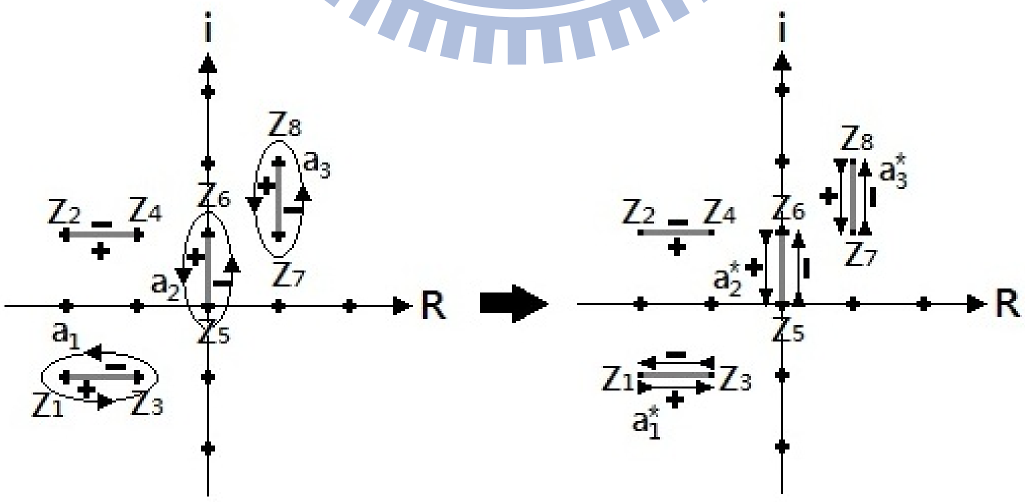



*Figure 46. cycles $a_1,a_2,a_3$ and equivalent pathes $a_1^*,a_2^*,a_3^*$.*

*Solution:*

1. $a_1$ *cycle : Let* $a_1^* = a_{11}^* \cup a_{12}^*$ *is the equivalent path for* $a_1$ *where* $a_{11}^*$ *is the path from* $Z_1 = -2-i$ *to* $Z_3 = -1-i$ *on* $(+)$*-edge of sheet-I ,* $a_{12}^*$ *is the path from* $Z_3 = -1-i$ *to* $Z_1 = -2-i$ *on* $(-)$*-edge of sheet-I.*

   (a) $a_{11}^* = -2-i \xrightarrow{+} -1-i.$
   *Let* $z = -2-i+r(1) = (r-2)-i$
   *where* $r : 0 \xrightarrow{+} 1$ *, and* $dz = dr$*.*
   *We have*

$$\begin{aligned}\int_{a_{11}^*} \frac{1}{f(z)}dz &= \int_{-2-i\xrightarrow{+}-1-i} \frac{1}{f(z)}dz \\ &\overset{Math.}{=} \int_{-2-i\rightarrow-1-i}(-1)\frac{1}{f(z)}dz \\ &= \int_0^1(-1)\frac{1}{f((r-2)-i)}dr\end{aligned}$$

   (b) $a_{12}^* = -1-i \xrightarrow{-} -2-i.$
   *Let* $z = -2-i+r(1) = (r-2)-i$
   *where* $r : 1 \xrightarrow{-} 0$ *, and* $dz = dr$*.*
   *We have*

$$\begin{aligned}\int_{a_{12}^*} \frac{1}{f(z)}dz &= \int_{-1-i\xrightarrow{-}-2-i} \frac{1}{f(z)}dz \\ &\overset{Math.}{=} \int_{-1-i\rightarrow-2-i}\frac{1}{f(z)}dz \\ &= \int_1^0\frac{1}{f((r-2)-i)}dr\end{aligned}$$

   *By (a) , (b) , we have*

$$\begin{aligned}\int_{a_1} \frac{1}{f(z)}dz &= \int_{a_1^*} \frac{1}{f(z)}dz \\ &= \int_{a_{11}^*} \frac{1}{f(z)}dz + \int_{a_{12}^*} \frac{1}{f(z)}dz \\ &\overset{Math.}{=} (-2)\int_0^1 \frac{1}{f((r-2)-i)}dr \\ &= 0.124401 - 0.0468335i\end{aligned}$$

2. $a_2$ *cycle : Let* $a_2^* = a_{21}^* \cup a_{22}^*$ *is the equivalent path for* $a_2$ *and where* $a_{21}^*$ *is the path from* $Z_6 = 0+i$ *to* $Z_5 = 0+0i$ *on* $(+)$*-edge of sheet-I ,* $a_{22}^*$ *is the path from* $Z_5 = 0+0i$ *to* $Z_6 = 0+i$ *on* $(-)$*-edge of sheet-I.*

(a) $a_{21}^* = 0 + i \xrightarrow{+} 0 + 0i$
Let $z = 0 + i + r(-i) = 0 + (1-r)i$
where $r : 0 \xrightarrow{+} 1$ , and $dz = (-1)idr$.
We have

$$\begin{aligned}
\int_{a_{21}^*} \frac{1}{f(z)} dz &= \int_{0+i \xrightarrow{+} 0+0i} \frac{1}{f(z)} dz \\
&\overset{Math.}{=} \int_{0+i \to 0+0i} (-1) \frac{1}{f(z)} dz \\
&= \int_0^1 (-1) \frac{1}{f(0+(1-r)i)} (-1) idr \\
&= (i) \int_0^1 \frac{1}{f(0+(1-r)i)} dr
\end{aligned}$$

(b) $a_{22}^* = 0 + 0i \xrightarrow{-} 0 + i$
Let $z = 0 + i + r(-i) = 0 + (1-r)i$
where $r : 1 \xrightarrow{-} 0$ , and $dz = (-1)idr$.
We have

$$\begin{aligned}
\int_{a_{22}^*} \frac{1}{f(z)} dz &= \int_{0+0i \xrightarrow{-} 0+i} \frac{1}{f(z)} dz \\
&\overset{Math.}{=} \int_{0+0i \to 0+i} \frac{1}{f(z)} dz \\
&= \int_1^0 \frac{1}{f(0+(1-r)i)} (-1) idr \\
&= (i) \int_0^1 \frac{1}{f(0+(1-r)i)} dr
\end{aligned}$$

By (a) , (b) , we have

$$\begin{aligned}
\int_{a_2} \frac{1}{f(z)} dz &= \int_{a_2^*} \frac{1}{f(z)} dz \\
&= \int_{a_{21}^*} \frac{1}{f(z)} dz + \int_{a_{22}^*} \frac{1}{f(z)} dz \\
&\overset{Math.}{=} (2i) \int_0^1 \frac{1}{f(0+(1-r)i)} dr \\
&= 1.22423 + 0.508246i
\end{aligned}$$

3. $a_3$ cycle : Let $a_3^* = a_{31}^* \cup a_{32}^*$ is the equivalent path for $a_3$ where $a_{31}^*$ is the path from $Z_8 = 1 + 2i$ to $Z_7 = 1 + i$ on (+)-edge of sheet-I , $a_{32}^*$ is the path from $Z_7 = 1 + i$ to $Z_8 = 1 + 2i$ on (−)-edge of sheet-I.

   (a) $a_{31}^* = 1 + 2i \xrightarrow{+} 1 + i$
   Let $z = 1 + 2i + r(-i) = 1 + (2-r)i$
   where $r : 0 \xrightarrow{+} 1$ , and $dz = (-1)idr$.

We have

$$\begin{aligned}
\int_{a_{31}^*} \frac{1}{f(z)}dz &= \int_{1+2i \overset{+}{\to} 1+i} \frac{1}{f(z)}dz \\
&\overset{Math.}{=} \int_{1+2i \to 1+i} (-1)\frac{1}{f(z)}dz \\
&= \int_0^1 (-1)\frac{1}{f(1+(2-r)i)}(-1)idr \\
&= (i)\int_0^1 \frac{1}{f(1+(2-r)i)}dr
\end{aligned}$$

(b) $a_{32}^* = 1+i \overset{-}{\to} 1+2i$

Let $z = 1+2i+r(-i) = 1+(2-r)i$

where $r : 1 \overset{-}{\to} 0$ , and $dz = (-1)idr$.

We have

$$\begin{aligned}
\int_{a_{32}^*} \frac{1}{f(z)}dz &= \int_{1+i \overset{-}{\to} 1+2i} \frac{1}{f(z)}dz \\
&\overset{Math.}{=} \int_{1+i \to 1+2i} \frac{1}{f(z)}dz \\
&= \int_1^0 \frac{1}{f(1+(2-r)i)}(-1)idr \\
&= (i)\int_0^1 \frac{1}{f(1+(2-r)i)}dr
\end{aligned}$$

By (a) , (b) , we have

$$\begin{aligned}
\int_{a_3} \frac{1}{f(z)}dz &= \int_{a_3^*} \frac{1}{f(z)}dz \\
&= \int_{a_{31}^*} \frac{1}{f(z)}dz + \int_{a_{32}^*} \frac{1}{f(z)}dz \\
&\overset{Math.}{=} (2i)\int_0^1 \frac{1}{f(1+(2-r)i)}dr \\
&=
\end{aligned}$$

**Example 70** *Compute the integrals of $1/f(z)$ over $b_2$ cycle in the Figure below where*

$$f(z) = \sqrt{(z-z_1)(z-z_2)(z-z_3)(z-z_4)(z-z_5)(z-z_6)(z-z_7)(z-z_8)}$$

*for* $z_1 = -2-i, z_2 = -2+i, z_3 = -1-i, z_4 = -1+i, z_5 = 0+0i, z_6 = 0+i, z_7 = 1+i, z_8 = 1+2i$.

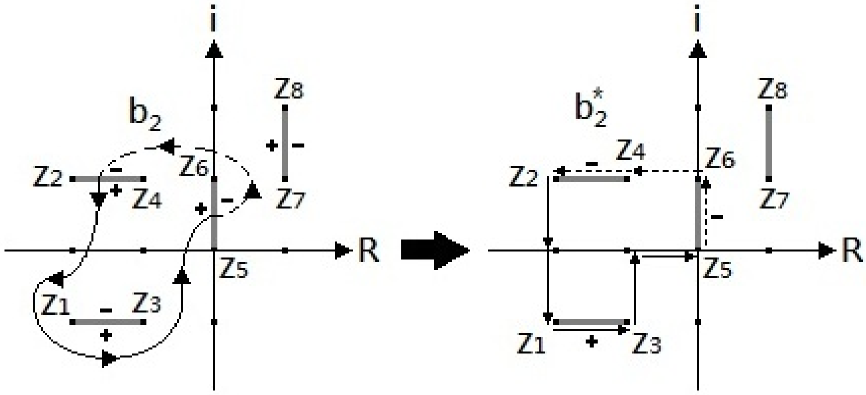



*Figure 47. Cycle $b_2$ and equivalent path $b_2^*$.*

*Solution:*

*Let* $b_2^* = b_{21}^* \cup b_{22}^* \cup b_{23}^* \cup b_{24}^* \cup b_{25}^* \cup b_{26}^* \cup b_{27}^* \cup b_{28}^*$ *is the equivalent path for* $b_2$ *where* $b_{21}^*$ *is the path from* $Z_2 = -2+i$ *to* $-2+0i$ *on sheet-I ,* $b_{22}^*$ *is the path from* $-2+0i$ *to* $Z_1 = -2-i$ *on sheet-I ,* $b_{23}^*$ *is the path from* $Z_1 = -2-i$ *to* $Z_3 = -1-i$ *on (+)-edge of sheet-I ,* $b_{24}^*$ *is the path from* $Z_3 = -1-i$ *to* $-1+0i$ *on sheet-I ,* $b_{25}^*$ *is the path from* $-1+0i$ *to* $Z_5 = 0+0i$ *on sheet-I ,* $b_{26}^*$ *is the path from* $Z_5 = 0+0i$ *to* $Z_6 = 0+i$ *on (−)-edge of sheet-II ,* $b_{27}^*$ *is the path from* $Z_6 = 0+i$ *to* $Z_4 = -1+i$ *on sheet-II ,* $b_{28}^*$ *is the path from* $Z_4 = -1+i$ *to* $Z_2 = -2+i$ *on (−)-edge of sheet-II.*

1. $b_{21}^* = -2+i \to -2+0i$

   *Let* $z = -2+i+r(-2i) = -2+(1-2r)i$

   *where* $r : 0 \to \frac{1}{2}$ *, and* $dz = (-2i)dr$.

   *We have*

$$\begin{aligned}
\int_{b_{21}^*} \frac{1}{f(z)}dz &= \int_{-2+i\to-2+0i} \frac{1}{f(z)}dz \\
&\overset{Math.}{=} \int_{-2+i\to-2+0i}(-1)\frac{1}{f(z)}dz \\
&= \int_0^{\frac{1}{2}}(-1)\frac{1}{f(-2+(1-2r)i)}(-2i)dr \\
&= (2i)\int_0^{\frac{1}{2}} \frac{1}{f(-2+(1-2r)i)}dr
\end{aligned}$$

2. $b_{22}^* = -2+0i \to -2-i$

   *Let* $z = -2+i+r(-2i) = -2+(1-2r)i$

   *where* $r : \frac{1}{2} \to 1$ *, and* $dz = (-2i)dr$.

   *We have*

$$
\begin{aligned}
\int_{b_{22}^{*}} \frac{1}{f(z)} dz &= \int_{-2+0i \to -2-i} \frac{1}{f(z)} dz \\
&\overset{Math.}{=} \int_{-2+0i \to -2-i} \frac{1}{f(z)} dz \\
&= \int_{\frac{1}{2}}^{1} \frac{1}{f(-2+(1-2r)i)} (-2i) dr \\
&= (-2i) \int_{\frac{1}{2}}^{1} \frac{1}{f(-2+(1-2r)i)} dr
\end{aligned}
$$

3. $b_{23}^{*} = -2 - i \xrightarrow{+} -1 - i$

   *Let* $z = -2 - i + r(1) = (r-2) - i$

   *where* $r : 0 \xrightarrow{+} 1$ *, and* $dz = dr$.

   *We have*

$$
\begin{aligned}
\int_{b_{23}^{*}} \frac{1}{f(z)} dz &= \int_{-2-i \xrightarrow{+} -1-i} \frac{1}{f(z)} dz \\
&\overset{Math.}{=} \int_{-2-i \to -1-i} (-1) \frac{1}{f(z)} dz \\
&= \int_{0}^{1} (-1) \frac{1}{f((r-2)-i)} dr \\
&= (-1) \int_{0}^{1} \frac{1}{f((r-2)-i)} dr
\end{aligned}
$$

4. $b_{24}^{*} = -1 - i \to -1 + 0i$

   *Let* $z = -1 - i + r(i) = -1 + (r-1)i$

   *where* $r : 0 \to 1$ *, and* $dz = idr$.

   *We have*

$$
\begin{aligned}
\int_{b_{24}^{*}} \frac{1}{f(z)} dz &= \int_{-1-i \to -1+0i} \frac{1}{f(z)} dz \\
&\overset{Math.}{=} \int_{-1-i \to -1+0i} \frac{1}{f(z)} dz \\
&= \int_{0}^{1} \frac{1}{f(-1+(r-1)i)} idr \\
&= i \int_{0}^{1} \frac{1}{f(-1+(r-1)i)} dr
\end{aligned}
$$

5. $b_{25}^{*} = -1 + 0i \to 0 + 0i$

   *Let* $z = -1 + r(1) = r - 1$

   *where* $r : 0 \to 1$ *, and* $dz = dr$.

   *We have*

$$\begin{aligned}
\int_{b_{25}^*} \frac{1}{f(z)}dz &= \int_{-1+0i\rightarrow 0+0i} \frac{1}{f(z)}dz \\
&\overset{Math.}{=} \int_{-1+0i\rightarrow 0+0i} (-1)\frac{1}{f(z)}dz \\
&= \int_0^1 (-1)\frac{1}{f(r-1)}dr \\
&= (-1)\int_0^1 \frac{1}{f(r-1)}dr
\end{aligned}$$

6. $b_{26}^* = 0 + 0i \overset{-}{\dashrightarrow} 0 + i$

   Let $z = 0 + ri = ri$

   where $r : 0 \overset{-}{\dashrightarrow} 1$ , and $dz = idr$.

   We have

$$\begin{aligned}
\int_{b_{26}^*} \frac{1}{f(z)}dz &= \int_{0+0i\overset{-}{\dashrightarrow} 0+i} \frac{1}{f(z)}dz \\
&= \int_{0+0i\overset{+}{\rightarrow} 0+i} \frac{-1}{f(z)}dz \\
&\overset{Math.}{=} \int_{0+0i\overset{+}{\rightarrow} 0+i} (-1)\frac{1}{f(z)}dz \\
&= \int_0^1 (-1)\frac{1}{f(ri)}idr \\
&= (-i)\int_0^1 \frac{1}{f(ri)}dr
\end{aligned}$$

7. $b_{27}^* = 0 + i \dashrightarrow -1 + i$

   Let $z = i + r(-1) = -r + i$

   where $r : 0 \dashrightarrow 1$ , and $dz = (-1)dr$.

   We have

$$\begin{aligned}
\int_{b_{27}^*} \frac{1}{f(z)}dz &= \int_{0+i\dashrightarrow -1+i} \frac{1}{f(z)}dz \\
&= \int_{0+i\rightarrow -1+i} (-1)\frac{1}{f(z)}dz \\
&\overset{Math.}{=} \int_{0+i\rightarrow -1+i} \frac{1}{f(z)}dz \\
&= \int_0^1 \frac{1}{f(-r+i)}(-1)dr \\
&= (-1)\int_0^1 \frac{1}{f(-r+i)}dr
\end{aligned}$$

8. $b_{28}^* = -1 + i \overset{-}{\dashrightarrow} -2 + i$

   Let $z = -1 + i + r(-1) = (-1 - r) + i$

   where $r : 0 \overset{-}{\dashrightarrow} 1$ , and $dz = (-1)dr$.

*We have*

$$
\begin{aligned}
\int_{b_{28}^*} \frac{1}{f(z)}dz &= \int_{-1+i \overset{-}{\dashrightarrow} -2+i} \frac{1}{f(z)}dz \\
&= \int_{-1+i \overset{+}{\rightarrow} -2+i} \frac{1}{f(z)}dz \\
&\overset{Math.}{=} \int_{-1+i \rightarrow -2+i} \frac{1}{f(z)}dz \\
&= \int_0^1 \frac{1}{f((-1-r)+i)}(-1)dr \\
&= (-1)\int_0^1 \frac{1}{f((-1-r)+i)}dr
\end{aligned}
$$

*By* 1. , 2. , 3. , 4. , 5. , 6. , 7. , 8. , *we have*

$$
\begin{aligned}
\int_{b_2} \frac{1}{f(z)}dz &= \int_{b_2^*} \frac{1}{f(z)}dz \\
& \quad (2i)\int_0^{\frac{1}{2}} \frac{1}{f(-2+(1-2r)i)}dr + (-2i)\int_{\frac{1}{2}}^1 \frac{1}{f(-2+(1-2r)i)}dr \\
&\overset{Math.}{=} +(-1)\int_0^1 \frac{1}{f((r-2)-i)}dr + i\int_0^1 \frac{1}{f(-1+(r-1)i)}dr \\
& \quad +(-1)\int_0^1 \frac{1}{f(r-1)}dr + (-i)\int_0^1 \frac{1}{f(ri)}dr \\
& \quad +(-1)\int_0^1 \frac{1}{f(-r+i)}dr + (-1)\int_0^1 \frac{1}{f((-1-r)+i)}dr \\
&= 0.869165 - 0.577073i
\end{aligned}
$$

**Example 71** *Compute the integrals of* $1/f(z)$ *over* $b_3$ *cycle in the Figure below where*

$$f(z) = \sqrt{(z-z_1)(z-z_2)(z-z_3)(z-z_4)(z-z_5)(z-z_6)(z-z_7)(z-z_8)}$$

*for* $z_1 = -2-i, z_2 = -2+i, z_3 = -1-i, z_4 = -1+i, z_5 = 0+0i, z_6 = 0+i, z_7 = 1+i, z_8 = 1+2i.$

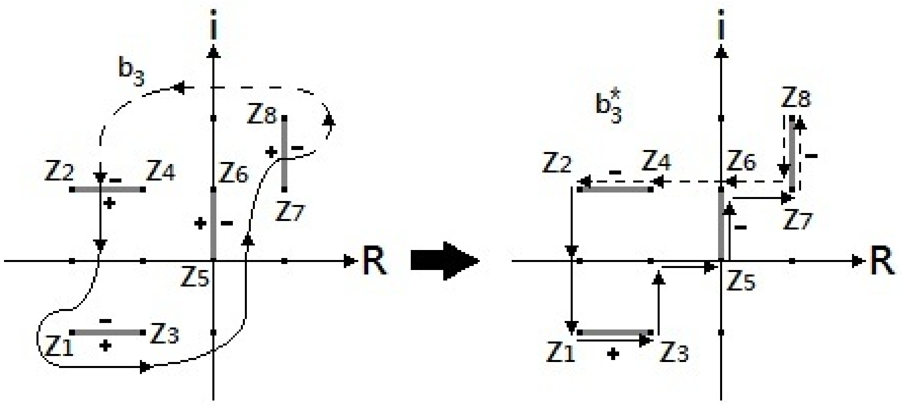



*Figure 48. Cycle* $b_3$ *and equivalent path* $b_3^*$.

*Solution:*

*Let* $b_3^* = b_{31}^* \cup b_{32}^* \cup b_{33}^* \cup b_{34}^* \cup b_{35}^* \cup b_{36}^* \cup b_{37}^* \cup b_{38}^* \cup b_{39}^* \cup b_{3a}^* \cup b_{3b}^* \cup b_{3c}^*$ *is the equivalent path for* $b_3$ *where* $b_{31}^*$ *is the path from* $Z_2 = -2+i$ *to* $-2+0i$ *on sheet-I ,* $b_{32}^*$ *is the path from* $-2+0i$ *to* $Z_1 = -2-i$ *on sheet-I ,* $b_{33}^*$ *is the path from* $Z_1 = -2-i$ *to* $Z_3 = -1-i$ *on* $(+)$*-edge of sheet-I ,* $b_{34}^*$ *is the path from* $Z_3 = -1-i$ *to* $-1+0i$ *on sheet-I ,* $b_{35}^*$ *is the path from* $-1+0i$ *to* $Z_5 = 0+0i$ *on sheet-I ,* $b_{36}^*$ *is the path from* $Z_5 = 0+0i$ *to* $Z_6 = 0+i$ *on* $(-)$*-edge of sheet-I ,* $b_{37}^*$ *is the path from* $Z_6 = 0+i$ *to* $Z_7 = 1+i$ *on sheet-I ,* $b_{38}^*$ *is the path from* $Z_7 = 1+i$ *to* $Z_8 = 1+2i$ *on* $(-)$*-edge of sheet-II ,* $b_{39}^*$ *is the path from* $Z_8 = 1+2i$ *to* $Z_7 = 1+i$ *on sheet-II ,* $b_{3a}^*$ *is the path from* $Z_7 = 1+i$ *to* $Z_6 = 0+i$ *on sheet-II ,* $b_{3b}^*$ *is the path from* $Z_6 = 0+i$ *to* $Z_4 = -1+i$ *on sheet-II ,* $b_{3c}^*$ *is the path from* $Z_4 = -1+i$ *to* $Z_2 = -2+i$ *on* $(-)$*-edge of sheet-II.*

1. $b_{31}^* = -2+i \rightarrow -2+0i$

   *Let* $z = -2+i+r(-2i) = -2+(1-2r)i$

   *where* $r : 0 \rightarrow \frac{1}{2}$ *, and* $dz = (-2i)dr$*.*

   *We have*

$$\begin{aligned}
\int_{b_{31}^*} \frac{1}{f(z)} dz &= \int_{-2+i\rightarrow -2+0i} \frac{1}{f(z)} dz \\
&\overset{Math.}{=} \int_{-2+i\rightarrow -2+0i} (-1) \frac{1}{f(z)} dz \\
&= \int_0^{\frac{1}{2}} (-1) \frac{1}{f(-2+(1-2r)i)} (-2i) dr \\
&= (2i) \int_0^{\frac{1}{2}} \frac{1}{f(-2+(1-2r)i)} dr
\end{aligned}$$

2. $b_{32}^* = -2+0i \rightarrow -2-i$

   *Let* $z = -2+i+r(-2i) = -2+(1-2r)i$

   *where* $r : \frac{1}{2} \rightarrow 1$ *, and* $dz = (-2i)dr$*.*

   *We have*

$$\begin{aligned}
\int_{b_{32}^*} \frac{1}{f(z)} dz &= \int_{-2+0i\rightarrow -2-i} \frac{1}{f(z)} dz \\
&\overset{Math.}{=} \int_{-2+0i\rightarrow -2-i} \frac{1}{f(z)} dz \\
&= \int_{\frac{1}{2}}^{1} \frac{1}{f(-2+(1-2r)i)} (-2i) dr \\
&= (-2i) \int_{\frac{1}{2}}^{1} \frac{1}{f(-2+(1-2r)i)} dr
\end{aligned}$$

3. $b^*_{33} = -2 - i \xrightarrow{+} -1 - i$

Let $z = -2 - i + r(1) = (r - 2) - i$

where $r : 0 \xrightarrow{+} 1$ , and $dz = dr$.

We have

$$\begin{aligned}
\int_{b^*_{33}} \frac{1}{f(z)} dz &= \int_{-2-i\xrightarrow{+}-1-i} \frac{1}{f(z)} dz \\
&\overset{Math.}{=} \int_{-2-i\rightarrow-1-i} (-1) \frac{1}{f(z)} dz \\
&= \int_0^1 (-1) \frac{1}{f((r-2)-i)} dr \\
&= (-1) \int_0^1 \frac{1}{f((r-2)-i)} dr
\end{aligned}$$

4. $b^*_{34} = -1 - i \rightarrow -1 + 0i$

Let $z = -1 - i + r(i) = -1 + (r - 1)i$

where $r : 0 \rightarrow 1$ , and $dz = idr$.

We have

$$\begin{aligned}
\int_{b^*_{34}} \frac{1}{f(z)} dz &= \int_{-1-i\rightarrow-1+0i} \frac{1}{f(z)} dz \\
&\overset{Math.}{=} \int_{-1-i\rightarrow-1+0i} \frac{1}{f(z)} dz \\
&= \int_0^1 \frac{1}{f(-1+(r-1)i)} idr \\
&= i \int_0^1 \frac{1}{f(-1+(r-1)i)} dr
\end{aligned}$$

5. $b^*_{35} = -1 + 0i \rightarrow 0 + 0i$

Let $z = -1 + r(1) = r - 1$

where $r : 0 \rightarrow 1$ , and $dz = dr$.

We have

$$\begin{aligned}
\int_{b^*_{35}} \frac{1}{f(z)} dz &= \int_{-1+0i\rightarrow0+0i} \frac{1}{f(z)} dz \\
&\overset{Math.}{=} \int_{-1+0i\rightarrow0+0i} (-1) \frac{1}{f(z)} dz \\
&= \int_0^1 (-1) \frac{1}{f(r-1)} dr \\
&= (-1) \int_0^1 \frac{1}{f(r-1)} dr
\end{aligned}$$

6. $b^*_{36} = 0 + 0i \overset{-}{\rightarrow} 0 + i$

Let $z = 0 + r(i) = ri$

where $r : 0 \overset{-}{\rightarrow} 1$ , and $dz = idr$.

We have

$$\begin{aligned}
\int_{b^*_{36}} \frac{1}{f(z)} dz &= \int_{0+0i \overset{-}{\rightarrow} 0+i} \frac{1}{f(z)} dz \\
&= \int_{0+0i \rightarrow 0+i} \frac{1}{f(z)} dz \\
&\overset{Math.}{=} \int_{0+0i \rightarrow 0+i} \frac{1}{f(z)} dz \\
&= \int_0^1 \frac{1}{f(ri)} idr \\
&= i \int_0^1 \frac{1}{f(ri)} dr
\end{aligned}$$

7. $b^*_{37} = 0 + i \rightarrow 1 + i$

Let $z = i + r(1) = r + i$

where $r : 0 \rightarrow 1$ , and $dz = dr$.

We have

$$\begin{aligned}
\int_{b^*_{37}} \frac{1}{f(z)} dz &= \int_{0+i \rightarrow 1+i} \frac{1}{f(z)} dz \\
&= \int_{0+i \rightarrow 1+i} \frac{1}{f(z)} dz \\
&\overset{Math.}{=} \int_{0+i \rightarrow 1+i} (-1) \frac{1}{f(z)} dz \\
&= \int_0^1 (-1) \frac{1}{f(r+i)} dr \\
&= (-1) \int_0^1 \frac{1}{f(r+i)} dr
\end{aligned}$$

8. $b^*_{38} = 1 + i \overset{-}{\dashrightarrow} 1 + 2i$

Let $z = 1 + i + r(i) = 1 + (r+1)i$

where $r : 0 \overset{-}{\dashrightarrow} 1$ , and $dz = (i)dr$.

We have

$$\begin{aligned}
\int_{b^*_{38}} \frac{1}{f(z)} dz &= \int_{1+i \overset{-}{\dashrightarrow} 1+2i} \frac{1}{f(z)} dz \\
&= \int_{1+i \overset{+}{\rightarrow} 1+2i} \frac{1}{f(z)} dz \\
&\overset{Math.}{=} \int_{1+i \rightarrow 1+2i} (-1) \frac{1}{f(z)} dz \\
&= \int_0^1 (-1) \frac{1}{f(1+(r+1)i)} (i) dr \\
&= (-i) \int_0^1 \frac{1}{f(1+(r+1)i)} dr
\end{aligned}$$

9. $b_{39}^{*} = 1 + 2i \overset{+}{\dashrightarrow} 1 + i$

   *Let* $z = 1 + i + r(i) = 1 + (r+1)i$

   *where* $r : 1 \overset{+}{\dashrightarrow} 0$ , *and* $dz = (i)dr$.

   *We have*

$$
\begin{aligned}
\int_{b_{39}^{*}} \frac{1}{f(z)} dz &= \int_{1+2i \overset{+}{\dashrightarrow} 1+i} \frac{1}{f(z)} dz \\
&= \int_{1+2i \overset{-}{\rightarrow} 1+i} \frac{1}{f(z)} dz \\
&\overset{Math.}{=} \int_{1+2i \rightarrow 1+i} \frac{1}{f(z)} dz \\
&= \int_{1}^{0} \frac{1}{f(1+(r+1)i)} (i) dr \\
&= (-i) \int_{0}^{1} \frac{1}{f(1+(r+1)i)} dr
\end{aligned}
$$

10. $b_{3a}^{*} = 1 + i \dashrightarrow 0 + i$

   *Let* $z = 1 + i + r(-1) = (1-r) + i$

   *where* $r : 0 \dashrightarrow 1$ , *and* $dz = (-1)dr$.

   *We have*

$$
\begin{aligned}
\int_{b_{3a}^{*}} \frac{1}{f(z)} dz &= \int_{1+i \dashrightarrow 0+i} \frac{1}{f(z)} dz \\
&= \int_{1+i \rightarrow 0+i} (-1) \frac{1}{f(z)} dz \\
&\overset{Math.}{=} \int_{1+i \rightarrow 0+i} \frac{1}{f(z)} dz \\
&= \int_{0}^{1} \frac{1}{f((1-r)+i)} (-1) dr \\
&= (-1) \int_{0}^{1} \frac{1}{f((1-r)+i)} dr
\end{aligned}
$$

11. $b_{3b}^{*} = 0 + i \dashrightarrow -1 + i$

   *Let* $z = 0 + i + r(-1) = -r + i$

   *where* $r : 0 \dashrightarrow 1$ , *and* $dz = (-1)dr$.

   *We have*

$$
\begin{aligned}
\int_{b_{3b}^{*}} \frac{1}{f(z)} dz &= \int_{0+i \dashrightarrow -1+i} \frac{1}{f(z)} dz \\
&= \int_{0+i \rightarrow -1+i} (-1) \frac{1}{f(z)} dz \\
&\overset{Math.}{=} \int_{0+i \rightarrow -1+i} \frac{1}{f(z)} dz \\
&= \int_{0}^{1} \frac{1}{f(-r+i)} (-1) dr \\
&= (-1) \int_{0}^{1} \frac{1}{f(-r+i)} dr
\end{aligned}
$$

*12.* $b_{3c}^{*} = -1 + i \overset{-}{\dashrightarrow} -2 + i$

*Let* $z = -1 + i + r(-1) = (-r-1) + i$

*where* $r : 0 \overset{-}{\dashrightarrow} 1$ *, and* $dz = (-1)dr$*.*

*We have*

$$\begin{aligned}
\int_{b_{3c}^{*}} \frac{1}{f(z)}dz &= \int_{-1+i \overset{-}{\dashrightarrow} -2+i} \frac{1}{f(z)}dz \\
&= \int_{-1+i \overset{+}{\rightarrow} -2+i} \frac{1}{f(z)}dz \\
&\overset{Math.}{=} \int_{-1+i \rightarrow -2+i} \frac{1}{f(z)}dz \\
&= \int_0^1 \frac{1}{f((-r-1)+i)}(-1)dr \\
&= (-1)\int_0^1 \frac{1}{f((-r-1)+i)}dr
\end{aligned}$$

*By* 1. , 2. , 3. , 4. , 5. , 6. , 7. , 8. , 9. , 10. , 11. , 12. , *we have*

$$\begin{aligned}
\int_{b_3} \frac{1}{f(z)}dz &= \int_{b_3^{*}} \frac{1}{f(z)}dz \\
&\overset{Math.}{=} (2i)\int_0^{\frac{1}{2}} \frac{1}{f(-2+(1-2r)i)}dr + (-2i)\int_{\frac{1}{2}}^1 \frac{1}{f(-2+(1-2r)i)}dr \\
&\quad +(-1)\int_0^1 \frac{1}{f((r-2)-i)}dr + i\int_0^1 \frac{1}{f(-1+(r-1)i)}dr \\
&\quad +(-1)\int_0^1 \frac{1}{f(r-1)}dr + i\int_0^1 \frac{1}{f(ri)}dr + (-1)\int_0^1 \frac{1}{f(r+i)}dr \\
&\quad +(-2i)\int_0^1 \frac{1}{f(1+(r+1)i)}dr + (-1)\int_0^1 \frac{1}{f((1-r)+i)}dr \\
&\quad +(-1)\int_0^1 \frac{1}{f(-r+i)}dr + (-1)\int_0^1 \frac{1}{f((-r-1)+i)}dr \\
&= -0.405194 - 0.115625i
\end{aligned}$$

## 5.3 The details of the previous computation in section 2.3 .

First , we will compute the integral of $1/f(z)$ over $a_1, a_2, a_3, a_4, a_5$ cycles in the Figure 45 below where

$$f(z) = \prod_{k=1}^{12} \sqrt{(z - z_k)}$$

and $Z_1 = -6.58948 + 5.23118i$ , $Z_2 = -6.58948 - 5.23118i$ , $Z_3 = -6.31381 + 1.46139i$ , $Z_4 = -6.31381 - 1.46139i$ , $Z_5 = -4.68652 + 0.0i$ , $Z_6 = -1.57080 + 0.0i$ , $Z_7 = 1.57080 + 0.0i$ , $Z_8 = 4.68652 + 0.0i$ , $Z_9 = 6.31381 + 1.46139i$ , $Z_{10} = 6.31381 - 1.46139i$ , $Z_{11} = 6.58948 + 5.23118i$ , $Z_{12} = 6.58948 - 5.23118i$.

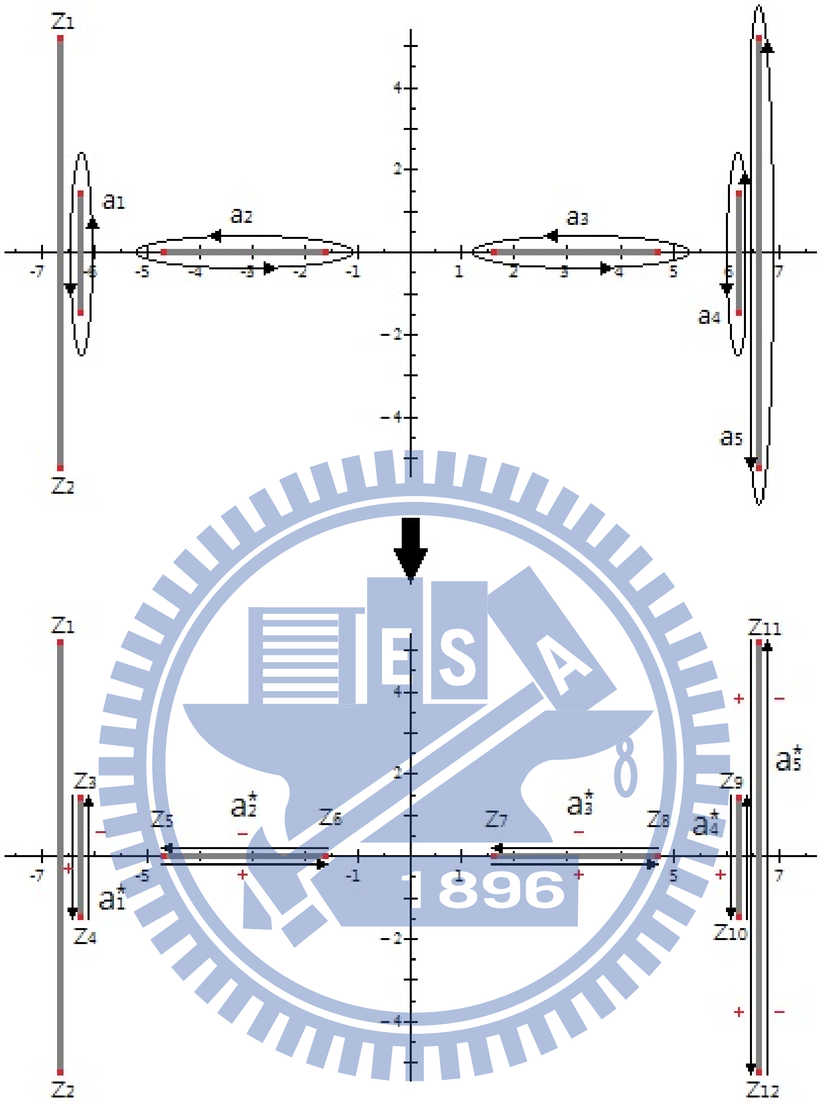



Figure 49. $a_1, a_2, a_3, a_4, a_5$ cycles and its equivalent path $a_1^*, a_2^*, a_3^*, a_4^*, a_5^*$.

Let $a_1^* = a_{11}^* \cup a_{12}^*$ is the equivalent path for $a_1$ where $a_{11}^*$ is the path from $Z_3 = -6.31381 + 1.46139i$ to $Z_4 = -6.31381 - 1.46139i$ on $(+)$-edge of sheet-I , $a_{12}^*$ is the path from $Z_4 = -6.31381 - 1.46139i$ to $Z_3 = -6.31381 + 1.46139i$ on $(-)$-edge of sheet-I.

1. $a_{11}^* = -6.31381 + 1.46139i \xrightarrow{+} -6.31381 - 1.46139i$

   Let

$$\begin{aligned} z &= -6.31381 + 1.46139i + r(-2.92278i) \\ &= -6.31381 + (1.46139 - 2.92278r)i \end{aligned}$$

where $r : 0 \xrightarrow{+} 1$ , and $dz = (-2.92278i)dr$.

We have

$$\begin{aligned}
\int_{a_{11}^*} \frac{1}{f(z)}dz &= \int_{-6.31381+1.46139i \xrightarrow{+} -6.31381-1.46139i} \frac{1}{f(z)}dz \\
&\overset{Math.}{=} \int_{-6.31381+1.46139i \rightarrow -6.31381-1.46139i} (-1)\frac{1}{f(z)}dz \\
&= \int_0^1 (-1)\frac{1}{f(-6.31381+(1.46139-2.92278r)i)}(-2.92278i)dr \\
&= (2.92278i)\int_0^1 \frac{1}{f(-6.31381+(1.46139-2.92278r)i)}dr
\end{aligned}$$

2. $a_{12}^* = -6.31381 - 1.46139i \xrightarrow{-} -6.31381 + 1.46139i$

Let

$$\begin{aligned}
z &= -6.31381 + 1.46139i + r(-2.92278i) \\
&= -6.31381 + (1.46139 - 2.92278r)i
\end{aligned}$$

where $r : 1 \xrightarrow{-} 0$ , and $dz = (-2.92278i)dr$.

We have

$$\begin{aligned}
\int_{a_{12}^*} \frac{1}{f(z)}dz &= \int_{-6.31381-1.46139i \xrightarrow{-} -6.31381+1.46139i} \frac{1}{f(z)}dz \\
&\overset{Math.}{=} \int_{-6.31381+1.46139i \rightarrow -6.31381-1.46139i} \frac{1}{f(z)}dz \\
&= \int_1^0 \frac{1}{f(-6.31381+(1.46139-2.92278r)i)}(-2.92278i)dr \\
&= (2.92278i)\int_0^1 \frac{1}{f(-6.31381+(1.46139-2.92278r)i)}dr
\end{aligned}$$

By 1. , 2. , we have

$$\begin{aligned}
\int_{a_1} \frac{1}{f(z)}dz &= \int_{a_1^*} \frac{1}{f(z)}dz \\
&\overset{Math.}{=} (2.92278i)\int_0^1 \frac{1}{f(-6.31381+(1.46139-2.92278r)i)}dr \\
&\quad +(2.92278i)\int_0^1 \frac{1}{f(-6.31381+(1.46139-2.92278r)i)}dr \\
&= (5.84556i)\int_0^1 \frac{1}{f(-6.31381+(1.46139-2.92278r)i)}dr \\
&= 9.52646 \times 10^{-18} + 0.000197837i
\end{aligned}$$

Let $a_2^* = a_{21}^* \cup a_{22}^*$ is the equivalent path for $a_2$ where $a_{21}^*$ is the path from $Z_5 = -4.68652 + 0.0i$ to $Z_6 = -1.57080 + 0.0i$ on $(+)$-edge of sheet-I , $a_{22}^*$ is the path from $Z_6 = -1.57080 + 0.0i$ to $Z_5 = -4.68652 + 0.0i$ on $(-)$-edge of sheet-I.

1. $a_{21}^* = -4.68652 + 0.0i \overset{+}{\rightarrow} -1.57080 + 0.0i$
   Let

$$\begin{aligned} z &= -4.68652 + r(3.11572) \\ &= -4.68652 + (3.11572)r \end{aligned}$$

   where $r : 0 \overset{+}{\rightarrow} 1$ , and $dz = (3.11572)dr$.
   We have

$$\begin{aligned} \int_{a_{21}^*} \frac{1}{f(z)} dz &= \int_{-4.68652+0.0i \overset{+}{\rightarrow} -1.57080+0.0i} \frac{1}{f(z)} dz \\ &\overset{Math.}{=} \int_{-4.68652+0.0i \rightarrow -1.57080+0.0i} (-1) \frac{1}{f(z)} dz \\ &= \int_0^1 (-1) \frac{1}{f(-4.68652+(3.11572)r)} (3.11572) dr \\ &= (-3.11572) \int_0^1 \frac{1}{f(-4.68652+(3.11572)r)} dr \end{aligned}$$

2. $a_{22}^* = -1.57080 + 0.0i \overset{-}{\rightarrow} -4.68652 + 0.0i$
   Let

$$\begin{aligned} z &= -4.68652 + r(3.11572) \\ &= -4.68652 + (3.11572)r \end{aligned}$$

   where $r : 1 \overset{-}{\rightarrow} 0$ , and $dz = (3.11572)dr$.
   We have

$$\begin{aligned} \int_{a_{22}^*} \frac{1}{f(z)} dz &= \int_{-1.57080+0.0i \overset{-}{\rightarrow} -4.68652+0.0i} \frac{1}{f(z)} dz \\ &\overset{Math.}{=} \int_{-1.57080+0.0i \rightarrow -4.68652+0.0i} \frac{1}{f(z)} dz \\ &= \int_1^0 \frac{1}{f(-4.68652+(3.11572)r)} (3.11572) dr \\ &= (-3.11572) \int_0^1 \frac{1}{f(-4.68652+(3.11572)r)} dr \end{aligned}$$

By 1. , 2. , we have

$$\begin{aligned} \int_{a_2} \frac{1}{f(z)} dz &= \int_{a_2^*} \frac{1}{f(z)} dz \\ &\overset{Math.}{=} (-3.11572) \int_0^1 \frac{1}{f(-4.68652+(3.11572)r)} dr \\ &\quad +(-3.11572) \int_0^1 \frac{1}{f(-4.68652+(3.11572)r)} dr \\ &= (-6.23144) \int_0^1 \frac{1}{f(-4.68652+(3.11572)r)} dr \\ &= 2.31913 \times 10^{-14} - 0.000472233i \end{aligned}$$

Let $a_3^* = a_{31}^* \cup a_{32}^*$ is the equivalent path for $a_3$ where $a_{31}^*$ is the path from $Z_7 = 1.57080 + 0.0i$ to $Z_8 = 4.68652 + 0.0i$ on $(+)$-edge of sheet-I , $a_{32}^*$ is the path from $Z_8 = 4.68652 + 0.0i$ to $Z_7 = 1.57080 + 0.0i$ on $(-)$-edge of sheet-I.

1. $a_{31}^* = 1.57080 + 0.0i \xrightarrow{+} 4.68652 + 0.0i$

   Let

$$\begin{aligned} z &= 1.57080 + r(3.11572) \\ &= 1.57080 + (3.11572)r \end{aligned}$$

   where $r : 0 \xrightarrow{+} 1$ , and $dz = (3.11572)dr$.

   We have

$$\begin{aligned} \int_{a_{31}^*} \frac{1}{f(z)} dz &= \int_{1.57080+0.0i \xrightarrow{+} 4.68652+0.0i} \frac{1}{f(z)} dz \\ &\overset{Math.}{=} \int_{1.57080+0.0i \to 4.68652+0.0i} (-1) \frac{1}{f(z)} dz \\ &= \int_0^1 (-1) \frac{1}{f(1.57080+(3.11572)r)} (3.11572) dr \\ &= (-3.11572) \int_0^1 \frac{1}{f(1.57080+(3.11572)r)} dr \end{aligned}$$

2. $a_{32}^* = 4.68652 + 0.0i \xrightarrow{-} 1.57080 + 0.0i$

   Let

$$\begin{aligned} z &= 1.57080 + r(3.11572) \\ &= 1.57080 + (3.11572)r \end{aligned}$$

   where $r : 1 \xrightarrow{-} 0$ , and $dz = (3.11572)dr$.

   We have

$$\begin{aligned} \int_{a_{32}^*} \frac{1}{f(z)} dz &= \int_{4.68652+0.0i \xrightarrow{-} 1.57080+0.0i} \frac{1}{f(z)} dz \\ &\overset{Math.}{=} \int_{4.68652+0.0i \to 1.57080+0.0i} \frac{1}{f(z)} dz \\ &= \int_1^0 \frac{1}{f(1.57080+(3.11572)r)} (3.11572) dr \\ &= (-3.11572) \int_0^1 \frac{1}{f(1.57080+(3.11572)r)} dr \end{aligned}$$

By 1. , 2. , we have

$$
\begin{aligned}
\int_{a_3} \frac{1}{f(z)}dz &= \int_{a_3^*} \frac{1}{f(z)}dz \\
&\overset{Math.}{=} (-3.11572)\int_0^1 \frac{1}{f(1.57080+(3.11572)r)}dr \\
&\quad +(-3.11572)\int_0^1 \frac{1}{f(1.57080+(3.11572)r)}dr \\
&= (-6.23144)\int_0^1 \frac{1}{f(1.57080+(3.11572)r)}dr \\
&= -2.23575\times 10^{-14} + 0.000472233i
\end{aligned}
$$

Let $a_4^* = a_{41}^* \cup a_{42}^*$ is the equivalent path for $a_4$ where $a_{41}^*$ is the path from $Z_9 = 6.31381 + 1.46139i$ to $Z_{10} = 6.31381 - 1.46139i$ on $(+)$-edge of sheet-I , $a_{42}^*$ is the path from $Z_{10} = 6.31381 - 1.46139i$ to $Z_9 = 6.31381 + 1.46139i$ on $(-)$-edge of sheet-I.

1. $a_{41}^* = 6.31381 + 1.46139i \xrightarrow{+} 6.31381 - 1.46139i$

   Let

$$
\begin{aligned}
z &= 6.31381 + 1.46139i + r(-2.92278i) \\
&= 6.31381 + (1.46139 - 2.92278r)i
\end{aligned}
$$

   where $r : 0 \xrightarrow{+} 1$ , and $dz = (-2.92278i)dr$.

   We have

$$
\begin{aligned}
\int_{a_{41}^*} \frac{1}{f(z)}dz &= \int_{6.31381+1.46139i \xrightarrow{+} 6.31381-1.46139i} \frac{1}{f(z)}dz \\
&\overset{Math.}{=} \int_{6.31381+1.46139i \to 6.31381-1.46139i} \frac{1}{f(z)}dz \\
&= \int_0^1 \frac{1}{f(6.31381+(1.46139-2.92278r)i)}(-2.92278i)dr \\
&= (-2.92278i)\int_0^1 \frac{1}{f(6.31381+(1.46139-2.92278r)i)}dr
\end{aligned}
$$

2. $a_{42}^* = 6.31381 - 1.46139i \xrightarrow{-} 6.31381 + 1.46139i$

   Let

$$
\begin{aligned}
z &= 6.31381 + 1.46139i + r(-2.92278i) \\
&= 6.31381 + (1.46139 - 2.92278r)i
\end{aligned}
$$

   where $r : 1 \xrightarrow{-} 0$ , and $dz = (-2.92278i)dr$.

   We have

$$
\begin{aligned}
\int_{a_{42}^*} \frac{1}{f(z)} dz &= \int_{6.31381-1.46139i \xrightarrow{-} 6.31381+1.46139i} \frac{1}{f(z)} dz \\
&\overset{Math.}{=} \int_{6.31381-1.46139i \rightarrow 6.31381+1.46139i} (-1) \frac{1}{f(z)} dz \\
&= \int_1^0 (-1) \frac{1}{f(6.31381+(1.46139-2.92278r)i)} (-2.92278i) dr \\
&= (-2.92278i) \int_0^1 \frac{1}{f(6.31381+(1.46139-2.92278r)i)} dr
\end{aligned}
$$

By 1. , 2. , we have

$$
\begin{aligned}
\int_{a_4} \frac{1}{f(z)} dz &= \int_{a_4^*} \frac{1}{f(z)} dz \\
&\overset{Math.}{=} (-2.92278i) \int_0^1 \frac{1}{f(6.31381+(1.46139-2.92278r)i)} dr \\
&\quad + (-2.92278i) \int_0^1 \frac{1}{f(6.31381+(1.46139-2.92278r)i)} dr \\
&= (-5.84556i) \int_0^1 \frac{1}{f(6.31381+(1.46139-2.92278r)i)} dr \\
&= 9.52151 \times 10^{-18} - 0.000197837i
\end{aligned}
$$

Let $a_5^* = a_{51}^* \cup a_{52}^*$ is the equivalent path for $a_5$ where $a_{51}^*$ is the path from $Z_{11} = 6.58948 + 5.23118i$ to $Z_{12} = 6.58948 - 5.23118i$ on $(+)$-edge of sheet-I , $a_{52}^*$ is the path from $Z_{12} = 6.58948 - 5.23118i$ to $Z_{11} = 6.58948 + 5.23118i$ on $(-)$-edge of sheet-I.

1. $a_{51}^* = 6.58948 + 5.23118i \xrightarrow{+} 6.58948 - 5.23118i$

   Let

$$
\begin{aligned}
z &= 6.58948 + 5.23118i + r(-10.46236i) \\
&= 6.58948 + (5.23118 - 10.46236r)i
\end{aligned}
$$

   where $r : 0 \xrightarrow{+} 1$ , and $dz = (-10.46236i)dr$.

   We have

$$
\begin{aligned}
\int_{a_{51}^*} \frac{1}{f(z)} dz &= \int_{6.58948+5.23118i \xrightarrow{+} 6.58948-5.23118i} \frac{1}{f(z)} dz \\
&\overset{Math.}{=} \int_{6.58948+5.23118i \rightarrow 6.58948-5.23118i} (-1) \frac{1}{f(z)} dz \\
&= \int_0^1 (-1) \frac{1}{f(6.58948+(5.23118-10.46236r)i)} (-10.46236i) dr \\
&= (10.46236i) \int_0^1 \frac{1}{f(6.58948+(5.23118-10.46236r)i)} dr
\end{aligned}
$$

2. $a^*_{52} = 6.58948 - 5.23118i \xrightarrow{-} 6.58948 + 5.23118i$
Let

$$\begin{aligned} z &= 6.58948 + 5.23118i + r(-10.46236i) \\ &= 6.58948 + (5.23118 - 10.46236r)i \end{aligned}$$

where $r : 1 \xrightarrow{-} 0$ , and $dz = (-10.46236i)dr$.
We have

$$\begin{aligned} \int_{a^*_{52}} \frac{1}{f(z)}dz &= \int_{6.58948-5.23118i \xrightarrow{-} 6.58948+5.23118i} \frac{1}{f(z)}dz \\ &\overset{Math.}{=} \int_{6.58948-5.23118i \to 6.58948+5.23118i} \frac{1}{f(z)}dz \\ &= \int_1^0 \frac{1}{f(6.58948+(5.23118-10.46236r)i)}(-10.46236i)dr \\ &= (10.46236i) \int_0^1 \frac{1}{f(6.58948+(5.23118-10.46236r)i)}dr \end{aligned}$$

By 1. , 2. , we have

$$\begin{aligned} \int_{a_5} \frac{1}{f(z)}dz &= \int_{a^*_5} \frac{1}{f(z)}dz \\ &\overset{Math.}{=} (10.46236i) \int_0^1 \frac{1}{f(6.58948+(5.23118-10.46236r)i)}dr \\ &\quad +(10.46236i) \int_0^1 \frac{1}{f(6.58948+(5.23118-10.46236r)i)}dr \\ &= (20.92472i) \int_0^1 \frac{1}{f(6.58948+(5.23118-10.46236r)i)}dr \\ &= -1.54107 \times 10^{-17} + 0.000262034i \end{aligned}$$

Second , we will compute the integral of $1/f(z)$ over $b_1, b_2, b_3, b_4, b_5$ cycles in the Figure 46 below where

$$f(z) = \prod_{k=1}^{12} \sqrt{(z - z_k)}$$

and $Z_1 = -6.58948 + 5.23118i$ , $Z_2 = -6.58948 - 5.23118i$ , $Z_3 = -6.31381 + 1.46139i$ , $Z_4 = -6.31381 - 1.46139i$ , $Z_5 = -4.68652 + 0.0i$ , $Z_6 = -1.57080 + 0.0i$ , $Z_7 = 1.57080 + 0.0i$ , $Z_8 = 4.68652 + 0.0i$ , $Z_9 = 6.31381 + 1.46139i$ , $Z_{10} = 6.31381 - 1.46139i$ , $Z_{11} = 6.58948 + 5.23118i$ , $Z_{12} = 6.58948 - 5.23118i$.

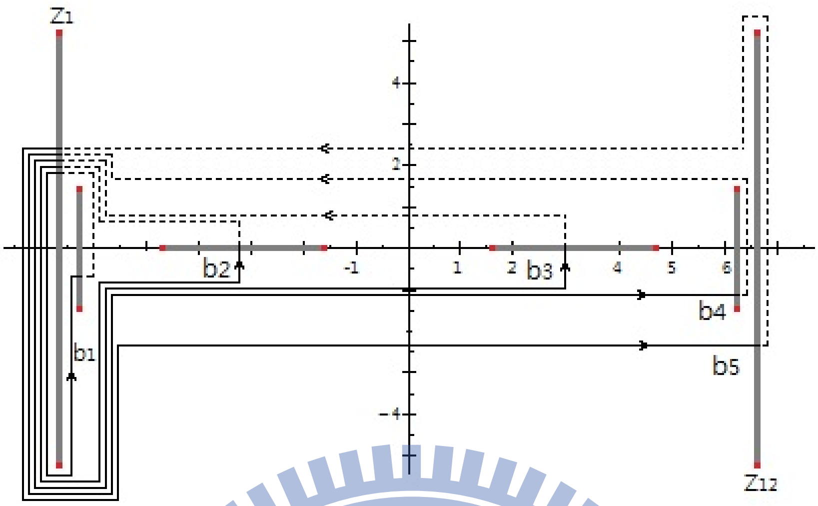



Figure 50. $b_1, b_2, b_3, b_4, b_5$ cycles.

Let $b_1^* = b_{11}^* \cup b_{12}^* \cup b_{13}^* \cup b_{14}^* \cup b_{15}^* \cup b_{16}^* \cup b_{17}^* \cup b_{18}^*$ is the equivalent path for $b_1$ where $b_{11}^*$ is the path from $Z_1 = -6.58948 + 5.23118i$ to $-6.58948 + 1.46139i$ on (+)-edge of sheet-I , $b_{12}^*$ is the path from $-6.58948+1.46139i$ to $-6.58948-1.46139i$ on (+)-edge of sheet-I , $b_{13}^*$ is the path from $-6.58948 - 1.46139i$ to $Z_2 = -6.58948 - 5.23118i$ on (+)-edge of sheet-I , $b_{14}^*$ is the path from $Z_2 = -6.58948 - 5.23118i$ to $-6.58948 - 1.46139i$ on (−)-edge of sheet-I , $b_{15}^*$ is the path from $-6.58948 - 1.46139i$ to $Z_4 = -6.31381 - 1.46139i$ on sheet-I , $b_{16}^*$ is the path from $Z_4 = -6.31381 - 1.46139i$ to $Z_3 = -6.31381 + 1.46139i$ on (−)-edge of sheet-II , $b_{17}^*$ is the path from $Z_3 = -6.31381 + 1.46139i$ to $-6.58948 + 1.46139i$ on sheet-II , $b_{18}^*$ is the path from $-6.58948 + 1.46139i$ to $Z_1 = -6.58948 + 5.23118i$ on (−)-edge of sheet-II.

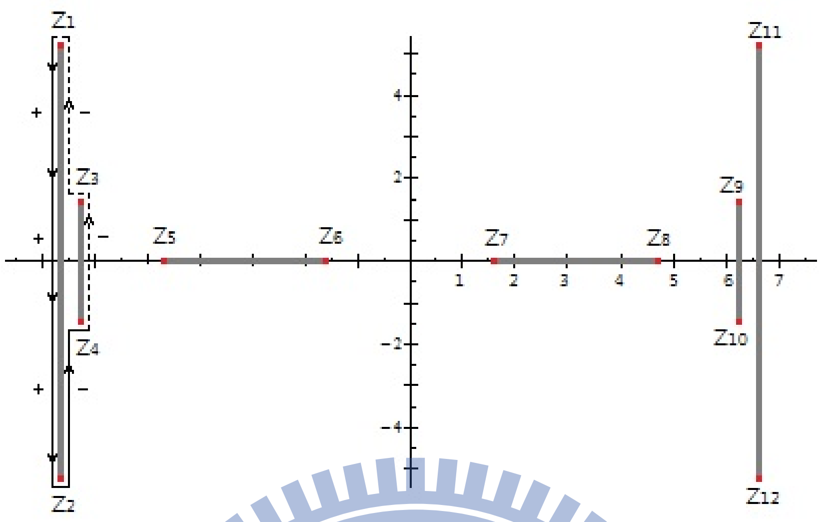



Figure 51. $b_1^*$ path.

1. $b_{11}^* = -6.58948 + 5.23118i \xrightarrow{+} -6.58948 + 1.46139i$

   Let

$$\begin{aligned} z &= -6.58948 + 5.23118i + r(-3.76979i) \\ &= -6.58948 + (5.23118 - 3.76979r)i \end{aligned}$$

   where $r : 0 \xrightarrow{+} 1$ , and $dz = (-3.76979i)dr$.

   We have

$$\begin{aligned} \int_{b_{11}^*} \frac{1}{f(z)} dz &= \int_{-6.58948+5.23118i \xrightarrow{+} -6.58948+1.46139i} \frac{1}{f(z)} dz \\ &\overset{Math.}{=} \int_{-6.58948+5.23118i \rightarrow -6.58948+1.46139i} \frac{1}{f(z)} dz \\ &= \int_0^1 \frac{1}{f(-6.58948+(5.23118-3.76979r)i)} (-3.76979i) dr \\ &= (-3.76979i) \int_0^1 \frac{1}{f(-6.58948+(5.23118-3.76979r)i)} dr \end{aligned}$$

2. $b_{12}^* = -6.58948 + 1.46139i \xrightarrow{+} -6.58948 - 1.46139i$

   Let

$$\begin{aligned} z &= -6.58948 + 1.46139i + r(-2.92278i) \\ &= -6.58948 + (1.46139 - 2.92278r)i \end{aligned}$$

   where $r : 0 \xrightarrow{+} 1$ , and $dz = (-2.92278i)dr$.

We have

$$\begin{aligned}
\int_{b_{12}^*} \frac{1}{f(z)} dz &= \int_{-6.58948+1.46139i \xrightarrow{+} -6.58948-1.46139i} \frac{1}{f(z)} dz \\
&\overset{Math.}{=} \int_{-6.58948+1.46139i \rightarrow -6.58948-1.46139i} \frac{1}{f(z)} dz \\
&= \int_0^1 \frac{1}{f(-6.58948+(1.46139-2.92278r)i)} (-2.92278i) dr \\
&= (-2.92278i) \int_0^1 \frac{1}{f(-6.58948+(1.46139-2.92278r)i)} dr
\end{aligned}$$

3. $b_{13}^* = -6.58948 - 1.46139i \xrightarrow{+} -6.58948 - 5.23118i$

   Let

$$\begin{aligned}
z &= -6.58948 - 1.46139i + r(-3.76979i) \\
&= -6.58948 + (-1.46139 - 3.76979r)i
\end{aligned}$$

   where $r : 0 \xrightarrow{+} 1$ , and $dz = (-3.76979i)dr$.

   We have

$$\begin{aligned}
\int_{b_{13}^*} \frac{1}{f(z)} dz &= \int_{-6.58948-1.46139i \xrightarrow{+} -6.58948-5.23118i} \frac{1}{f(z)} dz \\
&\overset{Math.}{=} \int_{-6.58948-1.46139i \rightarrow -6.58948-5.23118i} \frac{1}{f(z)} dz \\
&= \int_0^1 \frac{1}{f(-6.58948+(-1.46139-3.76979r)i)} (-3.76979i) dr \\
&= (-3.76979i) \int_0^1 \frac{1}{f(-6.58948+(-1.46139-3.76979r)i)} dr
\end{aligned}$$

4. $b_{14}^* = -6.58948 - 5.23118i \xrightarrow{-} -6.58948 - 1.46139i$

   Let

$$\begin{aligned}
z &= -6.58948 - 1.46139i + r(-3.76979i) \\
&= -6.58948 + (-1.46139 - 3.76979r)i
\end{aligned}$$

   where $r : 1 \xrightarrow{-} 0$ , and $dz = (-3.76979i)dr$.

   We have

$$\begin{aligned}
\int_{b_{14}^*} \frac{1}{f(z)} dz &= \int_{-6.58948-5.23118i \xrightarrow{-} -6.58948-1.46139i} \frac{1}{f(z)} dz \\
&\overset{Math.}{=} \int_{-6.58948-5.23118i \rightarrow -6.58948-1.46139i} (-1) \frac{1}{f(z)} dz \\
&= \int_1^0 (-1) \frac{1}{f(-6.58948+(-1.46139-3.76979r)i)} (-3.76979i) dr \\
&= (-3.76979i) \int_0^1 \frac{1}{f(-6.58948+(-1.46139-3.76979r)i)} dr
\end{aligned}$$

5. $b^*_{15} = -6.58948 - 1.46139i \to -6.31381 - 1.46139i$

Let

$$\begin{aligned} z &= -6.58948 - 1.46139i + r(0.27567) \\ &= (-6.58948 + 0.27567r) + (-1.46139)i \end{aligned}$$

where $r : 0 \to 1$ , and $dz = (0.27567)dr$.

We have

$$\begin{aligned} \int_{b^*_{15}} \frac{1}{f(z)} dz &= \int_{-6.58948-1.46139i \to -6.31381-1.46139i} \frac{1}{f(z)} dz \\ &\overset{Math.}{=} \int_{-6.58948-1.46139i \to -6.31381-1.46139i} (-1) \frac{1}{f(z)} dz \\ &= \int_0^1 (-1) \frac{1}{f((-6.58948+0.27567r)+(-1.46139)i)} (0.27567) dr \\ &= (-0.27567) \int_0^1 \frac{1}{f((-6.58948+0.27567r)+(-1.46139)i)} dr \end{aligned}$$

6. $b^*_{16} = -6.31381 - 1.46139i \overset{-}{\dashrightarrow} -6.31381 + 1.46139i$

Let

$$\begin{aligned} z &= -6.31381 - 1.46139i + r(2.92278i) \\ &= -6.31381 + (-1.46139 + 2.92278r)i \end{aligned}$$

where $r : 0 \overset{-}{\dashrightarrow} 1$ , and $dz = (2.92278i)dr$.

We have

$$\begin{aligned} \int_{b^*_{16}} \frac{1}{f(z)} dz &= \int_{-6.31381-1.46139i \overset{-}{\dashrightarrow} -6.31381+1.46139i} \frac{1}{f(z)} dz \\ &= \int_{-6.31381-1.46139i \overset{+}{\to} -6.31381+1.46139i} \frac{1}{f(z)} dz \\ &\overset{Math.}{=} \int_{-6.31381-1.46139i \to -6.31381+1.46139i} (-1) \frac{1}{f(z)} dz \\ &= \int_0^1 (-1) \frac{1}{f(-6.31381+(-1.46139+2.92278r)i)} (2.92278i) dr \\ &= (-2.92278i) \int_0^1 \frac{1}{f(-6.31381+(-1.46139+2.92278r)i)} dr \end{aligned}$$

7. $b^*_{17} = -6.31381 + 1.46139i \dashrightarrow -6.58948 + 1.46139i$

Let

$$\begin{aligned} z &= -6.31381 + 1.46139i + r(-0.27567i) \\ &= -6.31381 + (1.46139 - 0.27567r)i \end{aligned}$$

where $r : 0 \dashrightarrow 1$ , and $dz = (-0.27567i)dr$.

We have

$$
\begin{aligned}
\int_{b^*_{17}} \frac{1}{f(z)}dz &= \int_{-6.31381+1.46139i \dashrightarrow -6.58948+1.46139i} \frac{1}{f(z)}dz \\
&= \int_{-6.31381+1.46139i \to -6.58948+1.46139i} (-1)\frac{1}{f(z)}dz \\
&\overset{Math.}{=} \int_{-6.31381+1.46139i \to -6.58948+1.46139i} \frac{1}{f(z)}dz \\
&= \int_0^1 \frac{1}{f(-6.31381+(1.46139-0.27567r)i)}(-0.27567i)dr \\
&= (-0.27567i)\int_0^1 \frac{1}{f(-6.31381+(1.46139-0.27567r)i)}dr
\end{aligned}
$$

8. $b^*_{18} = -6.58948 + 1.46139i \overset{-}{\dashrightarrow} -6.58948 + 5.23118i$

Let

$$
\begin{aligned}
z &= -6.58948 + 1.46139i + r(3.76979i) \\
&= -6.58948 + (1.46139 + 3.76979r)i
\end{aligned}
$$

where $r : 0 \overset{-}{\dashrightarrow} 1$ , and $dz = (3.76979i)dr$.

We have

$$
\begin{aligned}
\int_{b^*_{18}} \frac{1}{f(z)}dz &= \int_{-6.58948+1.46139i \overset{-}{\dashrightarrow} -6.58948+5.23118i} \frac{1}{f(z)}dz \\
&= \int_{-6.58948+1.46139i \overset{+}{\to} -6.58948+5.23118i} \frac{1}{f(z)}dz \\
&\overset{Math.}{=} \int_{-6.58948+1.46139i \to -6.58948+5.23118i} \frac{1}{f(z)}dz \\
&= \int_0^1 \frac{1}{f(-6.58948+(1.46139+3.76979r)i)}(3.76979i)dr \\
&= (3.76979i)\int_0^1 \frac{1}{f(-6.58948+(1.46139+3.76979r)i)}dr
\end{aligned}
$$

By 1. , 2. , 3. , 4. , 5. , 6. , 7. , 8. , we have

$$
\begin{aligned}
\int_{b_1} \frac{1}{f(z)}dz &= \int_{b^*_1} \frac{1}{f(z)}dz \\
&= \int_{b^*_{11}} \frac{1}{f(z)}dz + \int_{b^*_{12}} \frac{1}{f(z)}dz + \int_{b^*_{13}} \frac{1}{f(z)}dz \\
&\quad + \int_{b^*_{14}} \frac{1}{f(z)}dz + \int_{b^*_{15}} \frac{1}{f(z)}dz + \int_{b^*_{16}} \frac{1}{f(z)}dz \\
&\quad + \int_{b^*_{17}} \frac{1}{f(z)}dz + \int_{b^*_{18}} \frac{1}{f(z)}dz \\
&= (-0.0000106043 - 0.0000764721i)
\end{aligned}
$$

Let $b_2^* = b_{21}^* \cup b_{22}^* \cup b_{23}^* \cup b_{24}^* \cup b_{25}^* \cup b_{26}^* \cup b_{27}^* \cup b_{28}^* \cup b_{29}^* \cup b_{2a}^* \cup b_{2b}^* \cup b_{2c}^* \cup b_{2d}^*$ is the equivalent path for $b_2$ where $b_{21}^*$ is the path from $Z_1 = -6.58948 + 5.23118i$ to $-6.58948 + 1.46139i$ on $(+)$-edge of sheet-I , $b_{22}^*$ is the path from $-6.58948+1.46139i$ to $-6.58948-1.46139i$ on $(+)$-edge of sheet-I , $b_{23}^*$ is the path from $-6.58948 - 1.46139i$ to $Z_2 = -6.58948 - 5.23118i$ on $(+)$-edge of sheet-I , $b_{24}^*$ is the path from $Z_2 = -6.58948-5.23118i$ to $-6.58948-1.46139i$ on $(-)$-edge of sheet-I , $b_{25}^*$ is the path from $-6.58948 - 1.46139i$ to $Z_4 = -6.31381-1.46139i$ on sheet-I , $b_{26}^*$ is the path from $Z_4 = -6.31381-1.46139i$ to $-6.31381 + 0i$ on $(-)$-edge of sheet-I , $b_{27}^*$ is the path from $-6.31381 + 0i$ to $Z_5 = -4.68652 + 0i$ on sheet-I , $b_{28}^*$ is the path from $Z_5 = -4.68652 + 0i$ to $Z_6 = -1.57080 + 0i$ on $(+)$-edge of sheet-I , $b_{29}^*$ is the path from $Z_6 = -1.57080+0i$ to $Z_5 = -4.68652+0i$ on $(-)$-edge of sheet-II , $b_{2a}^*$ is the path from $Z_5 = -4.68652 + 0i$ to $-6.31381 + 0i$ on sheet-I , $b_{2b}^*$ is the path from $-6.31381 + 0i$ to $Z_3 = -6.31381 + 1.46139i$ on $(-)$-edge of sheet-II , $b_{2c}^*$ is the path from $Z_3 = -6.31381 + 1.46139i$ to $-6.58948 + 1.46139i$ on sheet-II , $b_{2d}^*$ is the path from $-6.58948 + 1.46139i$ to $Z_1 = -6.58948 + 5.23118i$ on $(-)$-edge of sheet-II.

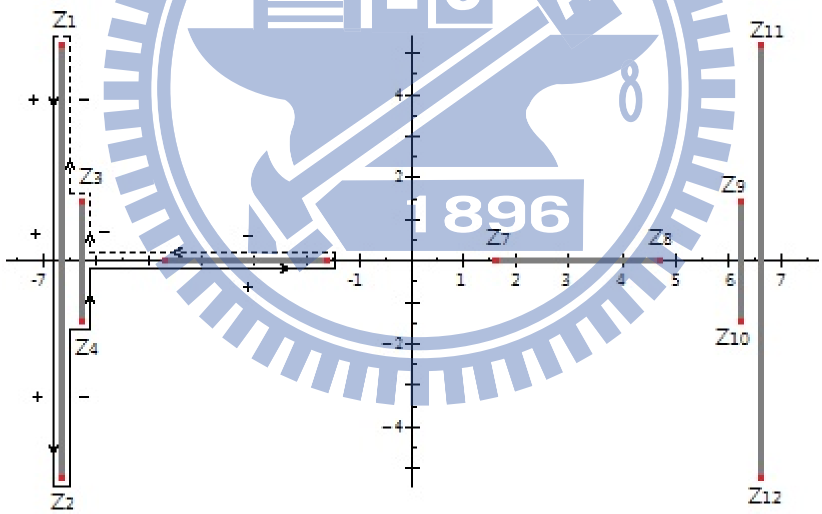



Figure 52. $b_2^*$ path.

1. $b_{21}^* = -6.58948 + 5.23118i \xrightarrow{+} -6.58948 + 1.46139i$

   Let

$$\begin{aligned} z &= -6.58948 + 5.23118i + r(-3.76979i) \\ &= -6.58948 + (5.23118 - 3.76979r)i \end{aligned}$$

   where $r : 0 \xrightarrow{+} 1$ , and $dz = (-3.76979i)dr$.

We have

$$\begin{aligned}
\int_{b_{21}^*} \frac{1}{f(z)} dz &= \int_{-6.58948+5.23118i \xrightarrow{+} -6.58948+1.46139i} \frac{1}{f(z)} dz \\
&\overset{Math.}{=} \int_{-6.58948+5.23118i \rightarrow -6.58948+1.46139i} \frac{1}{f(z)} dz \\
&= \int_0^1 \frac{1}{f(-6.58948+(5.23118-3.76979r)i)} (-3.76979i) dr \\
&= (-3.76979i) \int_0^1 \frac{1}{f(-6.58948+(5.23118-3.76979r)i)} dr
\end{aligned}$$

2. $b_{22}^* = -6.58948 + 1.46139i \xrightarrow{+} -6.58948 - 1.46139i$

   Let

$$\begin{aligned}
z &= -6.58948 + 1.46139i + r(-2.92278i) \\
&= -6.58948 + (1.46139 - 2.92278r)i
\end{aligned}$$

   where $r : 0 \xrightarrow{+} 1$ , and $dz = (-2.92278i)dr$.

   We have

$$\begin{aligned}
\int_{b_{22}^*} \frac{1}{f(z)} dz &= \int_{-6.58948+1.46139i \xrightarrow{+} -6.58948-1.46139i} \frac{1}{f(z)} dz \\
&\overset{Math.}{=} \int_{-6.58948+1.46139i \rightarrow -6.58948-1.46139i} \frac{1}{f(z)} dz \\
&= \int_0^1 \frac{1}{f(-6.58948+(1.46139-2.92278r)i)} (-2.92278i) dr \\
&= (-2.92278i) \int_0^1 \frac{1}{f(-6.58948+(1.46139-2.92278r)i)} dr
\end{aligned}$$

3. $b_{23}^* = -6.58948 - 1.46139i \xrightarrow{+} -6.58948 - 5.23118i$

   Let

$$\begin{aligned}
z &= -6.58948 - 1.46139i + r(-3.76979i) \\
&= -6.58948 + (-1.46139 - 3.76979r)i
\end{aligned}$$

   where $r : 0 \xrightarrow{+} 1$ , and $dz = (-3.76979i)dr$.

   We have

$$\begin{aligned}
\int_{b_{23}^*} \frac{1}{f(z)} dz &= \int_{-6.58948-1.46139i \xrightarrow{+} -6.58948-5.23118i} \frac{1}{f(z)} dz \\
&\overset{Math.}{=} \int_{-6.58948-1.46139i \rightarrow -6.58948-5.23118i} \frac{1}{f(z)} dz \\
&= \int_0^1 \frac{1}{f(-6.58948+(-1.46139-3.76979r)i)} (-3.76979i) dr \\
&= (-3.76979i) \int_0^1 \frac{1}{f(-6.58948+(-1.46139-3.76979r)i)} dr
\end{aligned}$$

4. $b^*_{24} = -6.58948 - 5.23118i \stackrel{-}{\rightarrow} -6.58948 - 1.46139i$

Let

$$\begin{aligned} z &= -6.58948 - 1.46139i + r(-3.76979i) \\ &= -6.58948 + (-1.46139 - 3.76979r)i \end{aligned}$$

where $r : 1 \stackrel{-}{\rightarrow} 0$ , and $dz = (-3.76979i)dr$.

We have

$$\begin{aligned} \int_{b^*_{24}} \frac{1}{f(z)}dz &= \int_{-6.58948-5.23118i \stackrel{-}{\rightarrow} -6.58948-1.46139i} \frac{1}{f(z)}dz \\ &\stackrel{Math.}{=} \int_{-6.58948-5.23118i \rightarrow -6.58948-1.46139i} (-1)\frac{1}{f(z)}dz \\ &= \int_1^0 (-1)\frac{1}{f(-6.58948+(-1.46139-3.76979r)i)}(-3.76979i)dr \\ &= (-3.76979i)\int_0^1 \frac{1}{f(-6.58948+(-1.46139-3.76979r)i)}dr \end{aligned}$$

5. $b^*_{25} = -6.58948 - 1.46139i \rightarrow -6.31381 - 1.46139i$

Let

$$\begin{aligned} z &= -6.58948 - 1.46139i + r(0.27567) \\ &= (-6.58948 + 0.27567r) + (-1.46139)i \end{aligned}$$

where $r : 0 \rightarrow 1$ , and $dz = (0.27567)dr$.

We have

$$\begin{aligned} \int_{b^*_{25}} \frac{1}{f(z)}dz &= \int_{-6.58948-1.46139i \rightarrow -6.31381-1.46139i} \frac{1}{f(z)}dz \\ &\stackrel{Math.}{=} \int_{-6.58948-1.46139i \rightarrow -6.31381-1.46139i} (-1)\frac{1}{f(z)}dz \\ &= \int_0^1 (-1)\frac{1}{f((-6.58948+0.27567r)+(-1.46139)i)}(0.27567)dr \\ &= (-0.27567)\int_0^1 \frac{1}{f((-6.58948+0.27567r)+(-1.46139)i)}dr \end{aligned}$$

6. $b^*_{26} = -6.31381 - 1.46139i \stackrel{-}{\rightarrow} -6.31381 + 0i$

Let

$$\begin{aligned} z &= -6.31381 - 1.46139i + r(1.46139i) \\ &= -6.31381 + (-1.46139 + 1.46139r)i \end{aligned}$$

where $r : 0 \xrightarrow{-} 1$ , and $dz = (1.46139i)dr$.

We have

$$\begin{aligned}
\int_{b_{26}^*} \frac{1}{f(z)} dz &= \int_{-6.31381-1.46139i \xrightarrow{-} -6.31381+0i} \frac{1}{f(z)} dz \\
&= \int_{-6.31381-1.46139i \xrightarrow{-} -6.31381+0i} \frac{1}{f(z)} dz \\
&\overset{Math.}{=} \int_{-6.31381-1.46139i \rightarrow -6.31381+0i} \frac{1}{f(z)} dz \\
&= \int_0^1 \frac{1}{f(-6.31381+(-1.46139+1.46139r)i)} (1.46139i) dr \\
&= (1.46139i) \int_0^1 \frac{1}{f(-6.31381+(-1.46139+1.46139r)i)} dr
\end{aligned}$$

7. $b_{27}^* = -6.31381 + 0i \rightarrow -4.68652 + 0i$

   Let

$$\begin{aligned}
z &= -6.31381 + 0i + r(1.62729) \\
&= -6.31381 + 1.62729r
\end{aligned}$$

   where $r : 0 \rightarrow 1$ , and $dz = (1.62729)dr$.

   We have

$$\begin{aligned}
\int_{b_{27}^*} \frac{1}{f(z)} dz &= \int_{-6.31381+0i \rightarrow -4.68652+0i} \frac{1}{f(z)} dz \\
&= \int_{-6.31381+0i \rightarrow -4.68652+0i} \frac{1}{f(z)} dz \\
&\overset{Math.}{=} \int_{-6.31381+0i \rightarrow -4.68652+0i} \frac{1}{f(z)} dz \\
&= \int_0^1 \frac{1}{f(-6.31381+1.62729r)} (1.62729) dr \\
&= (1.62729) \int_0^1 \frac{1}{f(-6.31381+1.62729r)} dr
\end{aligned}$$

8. $b_{28}^* = -4.68652 + 0i \xrightarrow{+} -1.57080 + 0i$

   Let

$$\begin{aligned}
z &= -4.68652 + 0i + r(3.11572) \\
&= -4.68652 + 3.11572r
\end{aligned}$$

   where $r : 0 \xrightarrow{+} 1$ , and $dz = (3.11572)dr$.

   We have

$$
\begin{aligned}
\int_{b_{28}^*} \frac{1}{f(z)}dz &= \int_{-4.68652+0i \xrightarrow{+} -1.57080+0i} \frac{1}{f(z)}dz \\
&= \int_{-4.68652+0i \xrightarrow{+} -1.57080+0i} \frac{1}{f(z)}dz \\
&\overset{Math.}{=} \int_{-4.68652+0i \rightarrow -1.57080+0i} (-1)\frac{1}{f(z)}dz \\
&= \int_0^1 (-1)\frac{1}{f(-4.68652+3.11572r)}(3.11572)dr \\
&= (-3.11572)\int_0^1 \frac{1}{f(-4.68652+3.11572r)}dr
\end{aligned}
$$

9. $b_{29}^* = -1.57080 + 0i \overset{-}{\dashrightarrow} -4.68652 + 0i$

   Let

$$
\begin{aligned}
z &= -1.57080 + 0i + r(-3.11572) \\
&= -1.57080 + (-3.11572r)
\end{aligned}
$$

   where $r : 0 \overset{-}{\dashrightarrow} 1$ , and $dz = (-3.11572)dr$.

   We have

$$
\begin{aligned}
\int_{b_{29}^*} \frac{1}{f(z)}dz &= \int_{-1.57080+0i \overset{-}{\dashrightarrow} -4.68652+0i} \frac{1}{f(z)}dz \\
&= \int_{-1.57080+0i \xrightarrow{+} -4.68652+0i} \frac{1}{f(z)}dz \\
&\overset{Math.}{=} \int_{-1.57080+0i \rightarrow -4.68652+0i} (-1)\frac{1}{f(z)}dz \\
&= \int_0^1 (-1)\frac{1}{f(-1.57080+(-3.11572r))}(-3.11572)dr \\
&= (3.11572)\int_0^1 \frac{1}{f(-1.57080+(-3.11572r))}dr
\end{aligned}
$$

10. $b_{2a}^* = -4.68652 + 0i \dashrightarrow -6.31381 + 0i$

   Let

$$
\begin{aligned}
z &= -4.68652 + 0i + r(-1.62729) \\
&= -4.68652 + (-1.62729r)
\end{aligned}
$$

   where $r : 0 \dashrightarrow 1$ , and $dz = (-1.62729)dr$.

   We have

$$
\begin{aligned}
\int_{b_{2a}^*} \frac{1}{f(z)}dz &= \int_{-4.68652+0i \dashrightarrow -6.31381+0i} \frac{1}{f(z)}dz \\
&= \int_{-4.68652+0i \rightarrow -6.31381+0i} (-1)\frac{1}{f(z)}dz \\
&\overset{Math.}{=} \int_{-4.68652+0i \rightarrow -6.31381+0i} (-1)\frac{1}{f(z)}dz \\
&= \int_0^1 (-1)\frac{1}{f(-4.68652+(-1.62729r))}(-1.62729)dr \\
&= (1.62729)\int_0^1 \frac{1}{f(-4.68652+(-1.62729r))}dr
\end{aligned}
$$

11. $b_{2b}^* = -6.31381 + 0i \overset{-}{\dashrightarrow} -6.31381 + 1.46139i$

Let

$$\begin{aligned} z &= -6.31381 + 0i + r(1.46139i) \\ &= -6.31381 + (1.46139r)i \end{aligned}$$

where $r : 0 \overset{-}{\dashrightarrow} 1$ , and $dz = (1.46139i)dr$.

We have

$$\begin{aligned} \int_{b_{2b}^*} \frac{1}{f(z)} dz &= \int_{-6.31381+0i \overset{-}{\dashrightarrow} -6.31381+1.46139i} \frac{1}{f(z)} dz \\ &= \int_{-6.31381+0i \overset{+}{\to} -6.31381+1.46139i} \frac{1}{f(z)} dz \\ &\overset{Math.}{=} \int_{-6.31381+0i \to -6.31381+1.46139i} (-1) \frac{1}{f(z)} dz \\ &= \int_0^1 (-1) \frac{1}{f(-6.31381+(1.46139r)i)} (1.46139i) dr \\ &= (-1.46139i) \int_0^1 \frac{1}{f(-6.31381+(1.46139r)i)} dr \end{aligned}$$

12. $b_{2c}^* = -6.31381 + 1.46139i \dashrightarrow -6.58948 + 1.46139i$

Let

$$\begin{aligned} z &= -6.31381 + 1.46139i + r(-0.27567) \\ &= (-6.31381 - 0.27567r) + 1.46139i \end{aligned}$$

where $r : 0 \dashrightarrow 1$ , and $dz = (-0.27567)dr$.

We have

$$\begin{aligned} \int_{b_{2c}^*} \frac{1}{f(z)} dz &= \int_{-6.31381+1.46139i \dashrightarrow -6.58948+1.46139i} \frac{1}{f(z)} dz \\ &= \int_{-6.31381+1.46139i \to -6.58948+1.46139i} (-1) \frac{1}{f(z)} dz \\ &\overset{Math.}{=} \int_{-6.31381+1.46139i \to -6.58948+1.46139i} \frac{1}{f(z)} dz \\ &= \int_0^1 \frac{1}{f((-6.31381-0.27567r)+1.46139i)} (-0.27567) dr \\ &= (-0.27567) \int_0^1 \frac{1}{f((-6.31381-0.27567r)+1.46139i)} dr \end{aligned}$$

13. $b_{2d}^* = -6.58948 + 1.46139i \overset{-}{\dashrightarrow} -6.58948 + 5.23118i$

Let

$$\begin{aligned} z &= -6.58948 + 1.46139i + r(3.76979i) \\ &= -6.58948 + (1.46139 + 3.76979r)i \end{aligned}$$

where $r : 0 \overset{-}{\dashrightarrow} 1$ , and $dz = (3.76979i)dr$.

We have

$$
\begin{aligned}
\int_{b_{2d}^*} \frac{1}{f(z)}dz &= \int_{-6.58948+1.46139i \overset{-}{\dashrightarrow} -6.58948+5.23118i} \frac{1}{f(z)}dz \\
&= \int_{-6.58948+1.46139i \overset{+}{\rightarrow} -6.58948+5.23118i} \frac{1}{f(z)}dz \\
&\overset{Math.}{=} \int_{-6.58948+1.46139i \rightarrow -6.58948+5.23118i} \frac{1}{f(z)}dz \\
&= \int_0^1 \frac{1}{f(-6.58948+(1.46139+3.76979r)i)}(3.76979i)dr \\
&= (3.76979i) \int_0^1 \frac{1}{f(-6.58948+(1.46139+3.76979r)i)}dr
\end{aligned}
$$

By 1. , 2. , 3. , 4. , 5. , 6. , 7. , 8. , 9. , 10. , 11. , 12. , 13. , we have

$$
\begin{aligned}
\int_{b_2} \frac{1}{f(z)}dz &= \int_{b_2^*} \frac{1}{f(z)}dz \\
&= \int_{b_{21}^*} \frac{1}{f(z)}dz + \int_{b_{22}^*} \frac{1}{f(z)}dz + \int_{b_{23}^*} \frac{1}{f(z)}dz \\
&\quad + \int_{b_{24}^*} \frac{1}{f(z)}dz + \int_{b_{25}^*} \frac{1}{f(z)}dz + \int_{b_{26}^*} \frac{1}{f(z)}dz \\
&\quad + \int_{b_{27}^*} \frac{1}{f(z)}dz + \int_{b_{28}^*} \frac{1}{f(z)}dz + \int_{b_{29}^*} \frac{1}{f(z)}dz \\
&\quad + \int_{b_{2a}^*} \frac{1}{f(z)}dz + \int_{b_{2b}^*} \frac{1}{f(z)}dz + \int_{b_{2c}^*} \frac{1}{f(z)}dz \\
&\quad + \int_{b_{2d}^*} \frac{1}{f(z)}dz \\
&= (0.00025277 + 0.0000169501i)
\end{aligned}
$$

Let $b_3^* = b_{31}^* \cup b_{32}^* \cup b_{33}^* \cup b_{34}^* \cup b_{35}^* \cup b_{36}^* \cup b_{37}^* \cup b_{38}^* \cup b_{39}^* \cup b_{3a}^* \cup b_{3b}^* \cup b_{3c}^* \cup b_{3d}^* \cup b_{3e}^* \cup b_{3f}^* \cup b_{3g}^* \cup b_{3h}^*$ is the equivalent path for $b_3$ where $b_{31}^*$ is the path from $Z_1 = -6.58948 + 5.23118i$ to $-6.58948 + 1.46139i$ on $(+)$-edge of sheet-I , $b_{32}^*$ is the path from $-6.58948 + 1.46139i$ to $-6.58948 - 1.46139i$ on $(+)$-edge of sheet-I , $b_{33}^*$ is the path from $-6.58948 - 1.46139i$ to $Z_2 = -6.58948 - 5.23118i$ on $(+)$-edge of sheet-I , $b_{34}^*$ is the path from $Z_2 = -6.58948 - 5.23118i$ to $-6.58948 - 1.46139i$ on $(-)$-edge of sheet-I , $b_{35}^*$ is the path from $-6.58948 - 1.46139i$ to $Z_4 = -6.31381 - 1.46139i$ on sheet-I , $b_{36}^*$ is the path from $Z_4 = -6.31381 - 1.46139i$ to $-6.31381 + 0i$ on $(-)$-edge of sheet-I , $b_{37}^*$ is the path from $-6.31381 + 0i$ to $Z_5 = -4.68652 + 0i$ on sheet-I , $b_{38}^*$ is the path from $Z_5 = -4.68652 + 0i$ to $Z_6 = -1.57080 + 0i$ on $(+)$-edge of sheet-I , $b_{39}^*$

is the path from $Z_6 = -1.57080 + 0i$ to $Z_7 = 1.57080 + 0i$ on sheet-I , $b_{3a}^*$ is the path from $Z_7 = 1.57080 + 0i$ to $Z_8 = 4.68652 + 0i$ on (+)-edge of sheet-I , $b_{3b}^*$ is the path from $Z_8 = 4.68652 + 0i$ to $Z_7 = 1.57080 + 0i$ on $(-)$-edge of sheet-II , $b_{3c}^*$ is the path from $Z_7 = 1.57080 + 0i$ to $Z_6 = -1.57080 + 0i$ on sheet-II , $b_{3d}^*$ is the path from $Z_6 = -1.57080 + 0i$ to $Z_5 = -4.68652 + 0i$ on $(-)$-edge of sheet-II , $b_{3e}^*$ is the path from $Z_5 = -4.68652+0i$ to $-6.31381+0i$ on sheet-II , $b_{3f}^*$ is the path from $-6.31381 + 0i$ to $Z_3 = -6.31381 + 1.46139i$ on $(-)$-edge of sheet-II , $b_{3g}^*$ is the path from $Z_3 = -6.31381 + 1.46139i$ to $-6.58948 + 1.46139i$ on sheet-II , $b_{3h}^*$ is the path from $-6.58948 + 1.46139i$ to $Z_1 = -6.58948 + 5.23118i$ on $(-)$-edge of sheet-II.

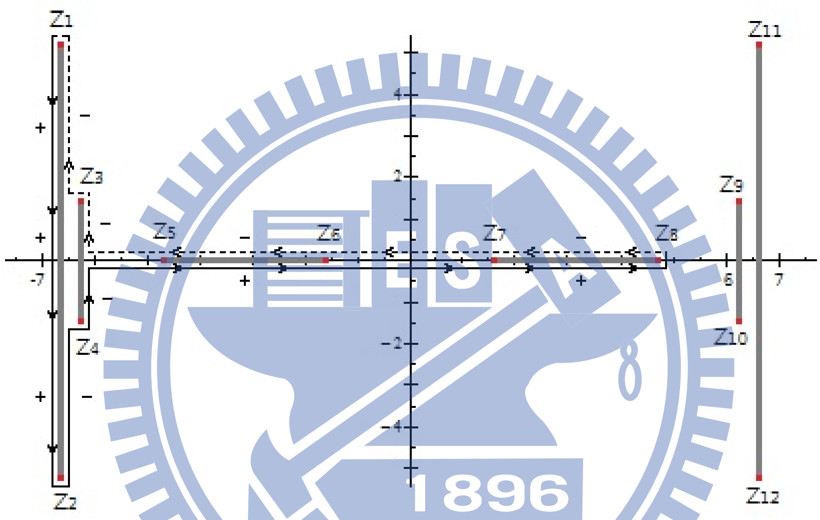



Figure 53. $b_3^*$ path.

1. $b_{31}^* = -6.58948 + 5.23118i \xrightarrow{+} -6.58948 + 1.46139i$

   Let

$$\begin{aligned} z &= -6.58948 + 5.23118i + r(-3.76979i) \\ &= -6.58948 + (5.23118 - 3.76979r)i \end{aligned}$$

   where $r : 0 \xrightarrow{+} 1$ , and $dz = (-3.76979i)dr$.

   We have

$$\begin{aligned} \int_{b_{31}^*} \frac{1}{f(z)} dz &= \int_{-6.58948+5.23118i \xrightarrow{+} -6.58948+1.46139i} \frac{1}{f(z)} dz \\ &\overset{Math.}{=} \int_{-6.58948+5.23118i \to -6.58948+1.46139i} \frac{1}{f(z)} dz \\ &= \int_0^1 \frac{1}{f(-6.58948+(5.23118-3.76979r)i)} (-3.76979i) dr \\ &= (-3.76979i) \int_0^1 \frac{1}{f(-6.58948+(5.23118-3.76979r)i)} dr \end{aligned}$$

2. $b^*_{32} = -6.58948 + 1.46139i \xrightarrow{+} -6.58948 - 1.46139i$

Let

$$\begin{aligned} z &= -6.58948 + 1.46139i + r(-2.92278i) \\ &= -6.58948 + (1.46139 - 2.92278r)i \end{aligned}$$

where $r : 0 \xrightarrow{+} 1$ , and $dz = (-2.92278i)dr$.

We have

$$\begin{aligned} \int_{b^*_{32}} \frac{1}{f(z)} dz &= \int_{-6.58948+1.46139i \xrightarrow{+} -6.58948-1.46139i} \frac{1}{f(z)} dz \\ &\overset{Math.}{=} \int_{-6.58948+1.46139i \rightarrow -6.58948-1.46139i} \frac{1}{f(z)} dz \\ &= \int_0^1 \frac{1}{f(-6.58948+(1.46139-2.92278r)i)} (-2.92278i) dr \\ &= (-2.92278i) \int_0^1 \frac{1}{f(-6.58948+(1.46139-2.92278r)i)} dr \end{aligned}$$

3. $b^*_{33} = -6.58948 - 1.46139i \xrightarrow{+} -6.58948 - 5.23118i$

Let

$$\begin{aligned} z &= -6.58948 - 1.46139i + r(-3.76979i) \\ &= -6.58948 + (-1.46139 - 3.76979r)i \end{aligned}$$

where $r : 0 \xrightarrow{+} 1$ , and $dz = (-3.76979i)dr$.

We have

$$\begin{aligned} \int_{b^*_{33}} \frac{1}{f(z)} dz &= \int_{-6.58948-1.46139i \xrightarrow{+} -6.58948-5.23118i} \frac{1}{f(z)} dz \\ &\overset{Math.}{=} \int_{-6.58948-1.46139i \rightarrow -6.58948-5.23118i} \frac{1}{f(z)} dz \\ &= \int_0^1 \frac{1}{f(-6.58948+(-1.46139-3.76979r)i)} (-3.76979i) dr \\ &= (-3.76979i) \int_0^1 \frac{1}{f(-6.58948+(-1.46139-3.76979r)i)} dr \end{aligned}$$

4. $b^*_{34} = -6.58948 - 5.23118i \xrightarrow{-} -6.58948 - 1.46139i$

Let

$$\begin{aligned} z &= -6.58948 - 1.46139i + r(-3.76979i) \\ &= -6.58948 + (-1.46139 - 3.76979r)i \end{aligned}$$

where $r : 1 \overset{-}{\rightarrow} 0$ , and $dz = (-3.76979i)dr$.

We have

$$\begin{aligned}
\int_{b_{34}^*} \frac{1}{f(z)} dz &= \int_{-6.58948-5.23118i \overset{-}{\rightarrow} -6.58948-1.46139i} \frac{1}{f(z)} dz \\
&\overset{Math.}{=} \int_{-6.58948-5.23118i \rightarrow -6.58948-1.46139i} (-1) \frac{1}{f(z)} dz \\
&= \int_1^0 (-1) \frac{1}{f(-6.58948+(-1.46139-3.76979r)i)} (-3.76979i) dr \\
&= (-3.76979i) \int_0^1 \frac{1}{f(-6.58948+(-1.46139-3.76979r)i)} dr
\end{aligned}$$

5. $b_{35}^* = -6.58948 - 1.46139i \rightarrow -6.31381 - 1.46139i$

   Let

$$\begin{aligned}
z &= -6.58948 - 1.46139i + r(0.27567) \\
&= (-6.58948 + 0.27567r) + (-1.46139)i
\end{aligned}$$

   where $r : 0 \rightarrow 1$ , and $dz = (0.27567)dr$.

   We have

$$\begin{aligned}
\int_{b_{35}^*} \frac{1}{f(z)} dz &= \int_{-6.58948-1.46139i \rightarrow -6.31381-1.46139i} \frac{1}{f(z)} dz \\
&\overset{Math.}{=} \int_{-6.58948-1.46139i \rightarrow -6.31381-1.46139i} (-1) \frac{1}{f(z)} dz \\
&= \int_0^1 (-1) \frac{1}{f((-6.58948+0.27567r)+(-1.46139)i)} (0.27567) dr \\
&= (-0.27567) \int_0^1 \frac{1}{f((-6.58948+0.27567r)+(-1.46139)i)} dr
\end{aligned}$$

6. $b_{36}^* = -6.31381 - 1.46139i \overset{-}{\rightarrow} -6.31381 + 0i$

   Let

$$\begin{aligned}
z &= -6.31381 - 1.46139i + r(1.46139i) \\
&= -6.31381 + (-1.46139 + 1.46139r)i
\end{aligned}$$

   where $r : 0 \overset{-}{\rightarrow} 1$ , and $dz = (1.46139i)dr$.

   We have

$$\begin{aligned}
\int_{b_{36}^*} \frac{1}{f(z)} dz &= \int_{-6.31381-1.46139i \overset{-}{\rightarrow} -6.31381+0i} \frac{1}{f(z)} dz \\
&= \int_{-6.31381-1.46139i \overset{-}{\rightarrow} -6.31381+0i} \frac{1}{f(z)} dz \\
&\overset{Math.}{=} \int_{-6.31381-1.46139i \rightarrow -6.31381+0i} \frac{1}{f(z)} dz \\
&= \int_0^1 \frac{1}{f(-6.31381+(-1.46139+1.46139r)i)} (1.46139i) dr \\
&= (1.46139i) \int_0^1 \frac{1}{f(-6.31381+(-1.46139+1.46139r)i)} dr
\end{aligned}$$

7. $b_{37}^{*} = -6.31381 + 0i \rightarrow -4.68652 + 0i$

Let

$$\begin{aligned} z &= -6.31381 + 0i + r(1.62729) \\ &= -6.31381 + 1.62729r \end{aligned}$$

where $r : 0 \rightarrow 1$ , and $dz = (1.62729)dr$.

We have

$$\begin{aligned} \int_{b_{37}^{*}} \frac{1}{f(z)} dz &= \int_{-6.31381+0i \rightarrow -4.68652+0i} \frac{1}{f(z)} dz \\ &= \int_{-6.31381+0i \rightarrow -4.68652+0i} \frac{1}{f(z)} dz \\ &\overset{Math.}{=} \int_{-6.31381+0i \rightarrow -4.68652+0i} \frac{1}{f(z)} dz \\ &= \int_0^1 \frac{1}{f(-6.31381+1.62729r)} (1.62729) dr \\ &= (1.62729) \int_0^1 \frac{1}{f(-6.31381+1.62729r)} dr \end{aligned}$$

8. $b_{38}^{*} = -4.68652 + 0i \xrightarrow{+} -1.57080 + 0i$

Let

$$\begin{aligned} z &= -4.68652 + 0i + r(3.11572) \\ &= -4.68652 + 3.11572r \end{aligned}$$

where $r : 0 \xrightarrow{+} 1$ , and $dz = (3.11572)dr$.

We have

$$\begin{aligned} \int_{b_{38}^{*}} \frac{1}{f(z)} dz &= \int_{-4.68652+0i \xrightarrow{+} -1.57080+0i} \frac{1}{f(z)} dz \\ &= \int_{-4.68652+0i \xrightarrow{+} -1.57080+0i} \frac{1}{f(z)} dz \\ &\overset{Math.}{=} \int_{-4.68652+0i \rightarrow -1.57080+0i} (-1) \frac{1}{f(z)} dz \\ &= \int_0^1 (-1) \frac{1}{f(-4.68652+3.11572r)} (3.11572) dr \\ &= (-3.11572) \int_0^1 \frac{1}{f(-4.68652+3.11572r)} dr \end{aligned}$$

9. $b_{39}^{*} = -1.57080 + 0i \rightarrow 1.57080 + 0i$

Let

$$\begin{aligned} z &= -1.57080 + 0i + r(3.1416) \\ &= -1.57080 + 3.1416r \end{aligned}$$

where $r : 0 \rightarrow 1$ , and $dz = (3.1416)dr$.

We have

$$\begin{aligned}
\int_{b_{39}^*} \frac{1}{f(z)} dz &= \int_{-1.57080+0i \rightarrow 1.57080+0i} \frac{1}{f(z)} dz \\
&= \int_{-1.57080+0i \rightarrow 1.57080+0i} \frac{1}{f(z)} dz \\
&\overset{Math.}{=} \int_{-1.57080+0i \rightarrow 1.57080+0i} \frac{1}{f(z)} dz \\
&= \int_0^1 \frac{1}{f(-1.57080+3.1416r)} (3.1416) dr \\
&= (3.1416) \int_0^1 \frac{1}{f(-1.57080+3.1416r)} dr
\end{aligned}$$

10. $b_{3a}^* = 1.57080 + 0i \xrightarrow{+} 4.68652 + 0i$

Let

$$\begin{aligned}
z &= 1.57080 + 0i + r(3.11572) \\
&= 1.57080 + 3.11572r
\end{aligned}$$

where $r : 0 \xrightarrow{+} 1$ , and $dz = (3.11572)dr$.

We have

$$\begin{aligned}
\int_{b_{3a}^*} \frac{1}{f(z)} dz &= \int_{1.57080+0i \xrightarrow{+} 4.68652+0i} \frac{1}{f(z)} dz \\
&= \int_{1.57080+0i \xrightarrow{+} 4.68652+0i} \frac{1}{f(z)} dz \\
&\overset{Math.}{=} \int_{1.57080+0i \rightarrow 4.68652+0i} (-1) \frac{1}{f(z)} dz \\
&= \int_0^1 (-1) \frac{1}{f(1.57080+3.11572r)} (3.11572) dr \\
&= (-3.11572) \int_0^1 \frac{1}{f(1.57080+3.11572r)} dr
\end{aligned}$$

11. $b_{3b}^* = 4.68652 + 0i \overset{-}{\dashrightarrow} 1.57080 + 0i$

Let

$$\begin{aligned}
z &= 4.68652 + 0i + r(-3.11572) \\
&= 4.68652 + (-3.11572r)
\end{aligned}$$

where $r : 0 \overset{-}{\dashrightarrow} 1$ , and $dz = (-3.11572)dr$.

We have

$$\begin{aligned}
\int_{b_{3b}^{*}} \frac{1}{f(z)} dz &= \int_{4.68652+0i \overset{-}{\dashrightarrow} 1.57080+0i} \frac{1}{f(z)} dz \\
&= \int_{4.68652+0i \overset{+}{\rightarrow} 1.57080+0i} \frac{1}{f(z)} dz \\
&\overset{Math.}{=} \int_{4.68652+0i \rightarrow 1.57080+0i} (-1) \frac{1}{f(z)} dz \\
&= \int_0^1 (-1) \frac{1}{f(4.68652+(-3.11572r))} (-3.11572) dr \\
&= (3.11572) \int_0^1 \frac{1}{f(4.68652+(-3.11572r))} dr
\end{aligned}$$

12. $b_{3c}^{*} = 1.57080 + 0i \dashrightarrow -1.57080 + 0i$

Let

$$\begin{aligned}
z &= 1.57080 + 0i + r(-3.1416) \\
&= 1.57080 + (-3.1416r)
\end{aligned}$$

where $r : 0 \dashrightarrow 1$ , and $dz = (-3.1416)dr$.

We have

$$\begin{aligned}
\int_{b_{3c}^{*}} \frac{1}{f(z)} dz &= \int_{1.57080+0i \dashrightarrow -1.57080+0i} \frac{1}{f(z)} dz \\
&= \int_{1.57080+0i \rightarrow -1.57080+0i} (-1) \frac{1}{f(z)} dz \\
&\overset{Math.}{=} \int_{1.57080+0i \rightarrow -1.57080+0i} (-1) \frac{1}{f(z)} dz \\
&= \int_0^1 (-1) \frac{1}{f(1.57080+(-3.1416r))} (-3.1416) dr \\
&= (3.1416) \int_0^1 \frac{1}{f(1.57080+(-3.1416r))} dr
\end{aligned}$$

13. $b_{3d}^{*} = -1.57080 + 0i \overset{-}{\dashrightarrow} -4.68652 + 0i$

Let

$$\begin{aligned}
z &= -1.57080 + 0i + r(-3.11572) \\
&= -1.57080 + (-3.11572r)
\end{aligned}$$

where $r : 0 \overset{-}{\dashrightarrow} 1$ , and $dz = (-3.11572)dr$.

We have

$$\begin{aligned}
\int_{b_{3d}^{*}} \frac{1}{f(z)} dz &= \int_{-1.57080+0i \overset{-}{\dashrightarrow} -4.68652+0i} \frac{1}{f(z)} dz \\
&= \int_{-1.57080+0i \overset{+}{\rightarrow} -4.68652+0i} \frac{1}{f(z)} dz \\
&\overset{Math.}{=} \int_{-1.57080+0i \rightarrow -4.68652+0i} (-1) \frac{1}{f(z)} dz \\
&= \int_0^1 (-1) \frac{1}{f(-1.57080+(-3.11572r))} (-3.11572) dr \\
&= (3.11572) \int_0^1 \frac{1}{f(-1.57080+(-3.11572r))} dr
\end{aligned}$$

14. $b_{3e}^{*} = -4.68652 + 0i \dashrightarrow -6.31381 + 0i$

Let

$$\begin{aligned} z &= -4.68652 + 0i + r(-1.62729) \\ &= -4.68652 + (-1.62729r) \end{aligned}$$

where $r : 0 \dashrightarrow 1$ , and $dz = (-1.62729)dr$.

We have

$$\begin{aligned} \int_{b_{3e}^{*}} \frac{1}{f(z)} dz &= \int_{-4.68652+0i \dashrightarrow -6.31381+0i} \frac{1}{f(z)} dz \\ &= \int_{-4.68652+0i \to -6.31381+0i} (-1) \frac{1}{f(z)} dz \\ &\overset{Math.}{=} \int_{-4.68652+0i \to -6.31381+0i} (-1) \frac{1}{f(z)} dz \\ &= \int_0^1 (-1) \frac{1}{f(-4.68652+(-1.62729r))} (-1.62729) dr \\ &= (1.62729) \int_0^1 \frac{1}{f(-4.68652+(-1.62729r))} dr \end{aligned}$$

15. $b_{3f}^{*} = -6.31381 + 0i \overset{-}{\dashrightarrow} -6.31381 + 1.46139i$

Let

$$\begin{aligned} z &= -6.31381 + 0i + r(1.46139i) \\ &= -6.31381 + (1.46139r)i \end{aligned}$$

where $r : 0 \overset{-}{\dashrightarrow} 1$ , and $dz = (1.46139i)dr$.

We have

$$\begin{aligned} \int_{b_{3f}^{*}} \frac{1}{f(z)} dz &= \int_{-6.31381+0i \overset{-}{\dashrightarrow} -6.31381+1.46139i} \frac{1}{f(z)} dz \\ &= \int_{-6.31381+0i \overset{+}{\to} -6.31381+1.46139i} \frac{1}{f(z)} dz \\ &\overset{Math.}{=} \int_{-6.31381+0i \to -6.31381+1.46139i} (-1) \frac{1}{f(z)} dz \\ &= \int_0^1 (-1) \frac{1}{f(-6.31381+(1.46139r)i)} (1.46139i) dr \\ &= (-1.46139i) \int_0^1 \frac{1}{f(-6.31381+(1.46139r)i)} dr \end{aligned}$$

16. $b_{3g}^{*} = -6.31381 + 1.46139i \dashrightarrow -6.58948 + 1.46139i$

Let

$$\begin{aligned} z &= -6.31381 + 1.46139i + r(-0.27567) \\ &= (-6.31381 - 0.27567r) + 1.46139i \end{aligned}$$

where $r : 0 \dashrightarrow 1$ , and $dz = (-0.27567)dr$.

We have

$$\begin{aligned}
\int_{b_{3g}^*} \frac{1}{f(z)} dz &= \int_{-6.31381+1.46139i \dashrightarrow -6.58948+1.46139i} \frac{1}{f(z)} dz \\
&= \int_{-6.31381+1.46139i \rightarrow -6.58948+1.46139i} (-1) \frac{1}{f(z)} dz \\
&\overset{Math.}{=} \int_{-6.31381+1.46139i \rightarrow -6.58948+1.46139i} \frac{1}{f(z)} dz \\
&= \int_0^1 \frac{1}{f((-6.31381-0.27567r)+1.46139i)} (-0.27567) dr \\
&= (-0.27567) \int_0^1 \frac{1}{f((-6.31381-0.27567r)+1.46139i)} dr
\end{aligned}$$

17. $b_{3h}^* = -6.58948 + 1.46139i \overset{-}{\dashrightarrow} -6.58948 + 5.23118i$

Let

$$\begin{aligned}
z &= -6.58948 + 1.46139i + r(3.76979i) \\
&= -6.58948 + (1.46139 + 3.76979r)i
\end{aligned}$$

where $r : 0 \overset{-}{\dashrightarrow} 1$ , and $dz = (3.76979i)dr$.

We have

$$\begin{aligned}
\int_{b_{3h}^*} \frac{1}{f(z)} dz &= \int_{-6.58948+1.46139i \overset{-}{\dashrightarrow} -6.58948+5.23118i} \frac{1}{f(z)} dz \\
&= \int_{-6.58948+1.46139i \overset{+}{\rightarrow} -6.58948+5.23118i} \frac{1}{f(z)} dz \\
&\overset{Math.}{=} \int_{-6.58948+1.46139i \rightarrow -6.58948+5.23118i} \frac{1}{f(z)} dz \\
&= \int_0^1 \frac{1}{f(-6.58948+(1.46139+3.76979r)i)} (3.76979i) dr \\
&= (3.76979i) \int_0^1 \frac{1}{f(-6.58948+(1.46139+3.76979r)i)} dr
\end{aligned}$$

By 1. , 2. , 3. , 4. , 5. , 6. , 7. , 8. , 9. , 10. , 11. , 12. , 13. , 14. , 15. , 16. , 17. , we have

$$
\begin{aligned}
\int_{b_3} \frac{1}{f(z)} dz &= \int_{b_3^*} \frac{1}{f(z)} dz \\
&= \int_{b_{31}^*} \frac{1}{f(z)} dz + \int_{b_{32}^*} \frac{1}{f(z)} dz + \int_{b_{33}^*} \frac{1}{f(z)} dz \\
&\quad + \int_{b_{34}^*} \frac{1}{f(z)} dz + \int_{b_{35}^*} \frac{1}{f(z)} dz + \int_{b_{36}^*} \frac{1}{f(z)} dz \\
&\quad + \int_{b_{37}^*} \frac{1}{f(z)} dz + \int_{b_{38}^*} \frac{1}{f(z)} dz + \int_{b_{39}^*} \frac{1}{f(z)} dz \\
&\quad + \int_{b_{3a}^*} \frac{1}{f(z)} dz + \int_{b_{3b}^*} \frac{1}{f(z)} dz + \int_{b_{3c}^*} \frac{1}{f(z)} dz \\
&\quad + \int_{b_{3d}^*} \frac{1}{f(z)} dz + \int_{b_{3e}^*} \frac{1}{f(z)} dz + \int_{b_{3f}^*} \frac{1}{f(z)} dz \\
&\quad + \int_{b_{3g}^*} \frac{1}{f(z)} dz + \int_{b_{3h}^*} \frac{1}{f(z)} dz \\
&= (-0.000226449 + 0.0000169501 i)
\end{aligned}
$$

Let $b_4^* = b_{41}^* \cup b_{42}^* \cup b_{43}^* \cup b_{44}^* \cup b_{45}^* \cup b_{46}^* \cup b_{47}^* \cup b_{48}^* \cup b_{49}^* \cup b_{4a}^* \cup b_{4b}^* \cup b_{4c}^* \cup b_{4d}^* \cup b_{4e}^* \cup b_{4f}^* \cup b_{4g}^* \cup b_{4h}^* \cup b_{4i}^* \cup b_{4j}^* \cup b_{4k}^* \cup b_{4l}^* \cup b_{4m}^*$ is the equivalent path for $b_4$ where $b_{41}^*$ is the path from $Z_1 = -6.58948 + 5.23118i$ to $-6.58948 + 1.46139i$ on $(+)$-edge of sheet-I , $b_{42}^*$ is the path from $-6.58948 + 1.46139i$ to $-6.58948 - 1.46139i$ on $(+)$-edge of sheet-I , $b_{43}^*$ is the path from $-6.58948 - 1.46139i$ to $Z_2 = -6.58948 - 5.23118i$ on $(+)$-edge of sheet-I , $b_{44}^*$ is the path from $Z_2 = -6.58948 - 5.23118i$ to $-6.58948 - 1.46139i$ on $(-)$-edge of sheet-I , $b_{45}^*$ is the path from $-6.58948 - 1.46139i$ to $Z_4 = -6.31381 - 1.46139i$ on sheet-I , $b_{46}^*$ is the path from $Z_4 = -6.31381 - 1.46139i$ to $-6.31381 + 0i$ on $(-)$-edge of sheet-I , $b_{47}^*$ is the path from $-6.31381 + 0i$ to $Z_5 = -4.68652 + 0i$ on sheet-I , $b_{48}^*$ is the path from $Z_5 = -4.68652 + 0i$ to $Z_6 = -1.57080 + 0i$ on $(+)$-edge of sheet-I , $b_{49}^*$ is the path from $Z_6 = -1.57080 + 0i$ to $Z_7 = 1.57080 + 0i$ on sheet-I , $b_{4a}^*$ is the path from $Z_7 = 1.57080 + 0i$ to $Z_8 = 4.68652 + 0i$ on $(+)$-edge of sheet-I , $b_{4b}^*$ is the path from $Z_8 = 4.68652 + 0i$ to $6.31381 + 0i$ on sheet-I , $b_{4c}^*$ is the path from $6.31381 + 0i$ to $Z_{10} = 6.31381 - 1.46139i$ on $(+)$-edge of sheet-I , $b_{4d}^*$ is the path from $Z_{10} = 6.31381 - 1.46139i$ to $Z_9 = 6.31381 + 1.46139i$ on $(-)$-edge of sheet-II , $b_{4e}^*$ is the path from $Z_9 = 6.31381 + 1.46139i$ to $6.31381 + 0i$ on $(+)$-edge of sheet-II , $b_{4f}^*$ is the path from $6.31381 + 0i$ to $Z_8 = 4.68652 + 0i$ on sheet-II , $b_{4g}^*$ is the path from $Z_8 = 4.68652 + 0i$ to $Z_7 = 1.57080 + 0i$ on $(-)$-edge of sheet-II , $b_{4h}^*$ is the path from $Z_7 = 1.57080 + 0i$ to $Z_6 = -1.57080 + 0i$ on sheet-II , $b_{4i}^*$ is the path from $Z_6 = -1.57080 + 0i$ to $Z_5 = -4.68652 + 0i$ on

(−)-edge of sheet-II , $b^*_{4j}$ is the path from $Z_5 = -4.68652 + 0i$ to $-6.31381 + 0i$ on sheet-II , $b^*_{4k}$ is the path from $-6.31381 + 0i$ to $Z_3 = -6.31381 + 1.46139i$ on (−)-edge of sheet-II , $b^*_{4l}$ is the path from $Z_3 = -6.31381 + 1.46139i$ to $-6.58948 + 1.46139i$ on sheet-II , $b^*_{4m}$ is the path from $-6.58948 + 1.46139i$ to $Z_1 = -6.58948 + 5.23118i$ on (−)-edge of sheet-II.

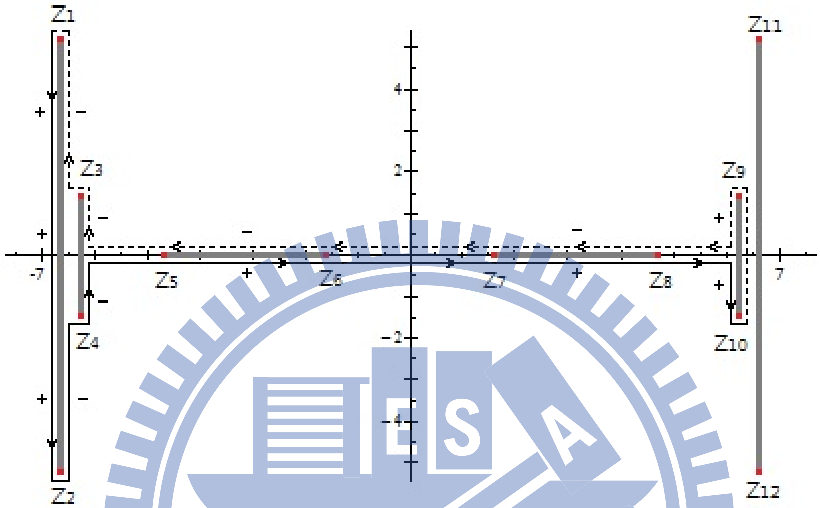



Figure 54. $b^*_4$ path.

1. $b^*_{41} = -6.58948 + 5.23118i \xrightarrow{+} -6.58948 + 1.46139i$

   Let

$$\begin{aligned} z &= -6.58948 + 5.23118i + r(-3.76979i) \\ &= -6.58948 + (5.23118 - 3.76979r)i \end{aligned}$$

   where $r : 0 \xrightarrow{+} 1$ , and $dz = (-3.76979i)dr$.

   We have

$$\begin{aligned} \int_{b^*_{41}} \frac{1}{f(z)} dz &= \int_{-6.58948+5.23118i \xrightarrow{+} -6.58948+1.46139i} \frac{1}{f(z)} dz \\ &\overset{Math.}{=} \int_{-6.58948+5.23118i \rightarrow -6.58948+1.46139i} \frac{1}{f(z)} dz \\ &= \int_0^1 \frac{1}{f(-6.58948+(5.23118-3.76979r)i)} (-3.76979i) dr \\ &= (-3.76979i) \int_0^1 \frac{1}{f(-6.58948+(5.23118-3.76979r)i)} dr \end{aligned}$$

2. $b^*_{42} = -6.58948 + 1.46139i \xrightarrow{+} -6.58948 - 1.46139i$

Let

$$\begin{aligned} z &= -6.58948 + 1.46139i + r(-2.92278i) \\ &= -6.58948 + (1.46139 - 2.92278r)i \end{aligned}$$

where $r : 0 \xrightarrow{+} 1$ , and $dz = (-2.92278i)dr$.

We have

$$\begin{aligned} \int_{b_{42}^*} \frac{1}{f(z)}dz &= \int_{-6.58948+1.46139i \xrightarrow{+} -6.58948-1.46139i} \frac{1}{f(z)}dz \\ &\overset{Math.}{=} \int_{-6.58948+1.46139i \rightarrow -6.58948-1.46139i} \frac{1}{f(z)}dz \\ &= \int_0^1 \frac{1}{f(-6.58948+(1.46139-2.92278r)i)}(-2.92278i)dr \\ &= (-2.92278i) \int_0^1 \frac{1}{f(-6.58948+(1.46139-2.92278r)i)}dr \end{aligned}$$

3. $b_{43}^* = -6.58948 - 1.46139i \xrightarrow{+} -6.58948 - 5.23118i$

   Let

$$\begin{aligned} z &= -6.58948 - 1.46139i + r(-3.76979i) \\ &= -6.58948 + (-1.46139 - 3.76979r)i \end{aligned}$$

   where $r : 0 \xrightarrow{+} 1$ , and $dz = (-3.76979i)dr$.

   We have

$$\begin{aligned} \int_{b_{43}^*} \frac{1}{f(z)}dz &= \int_{-6.58948-1.46139i \xrightarrow{+} -6.58948-5.23118i} \frac{1}{f(z)}dz \\ &\overset{Math.}{=} \int_{-6.58948-1.46139i \rightarrow -6.58948-5.23118i} \frac{1}{f(z)}dz \\ &= \int_0^1 \frac{1}{f(-6.58948+(-1.46139-3.76979r)i)}(-3.76979i)dr \\ &= (-3.76979i) \int_0^1 \frac{1}{f(-6.58948+(-1.46139-3.76979r)i)}dr \end{aligned}$$

4. $b_{44}^* = -6.58948 - 5.23118i \xrightarrow{-} -6.58948 - 1.46139i$

   Let

$$\begin{aligned} z &= -6.58948 - 1.46139i + r(-3.76979i) \\ &= -6.58948 + (-1.46139 - 3.76979r)i \end{aligned}$$

where $r : 1 \overset{-}{\rightarrow} 0$ , and $dz = (-3.76979i)dr$.

We have

$$\begin{aligned}
\int_{b_{44}^{*}} \frac{1}{f(z)}dz &= \int_{-6.58948-5.23118i \overset{-}{\rightarrow} -6.58948-1.46139i} \frac{1}{f(z)}dz \\
&\overset{Math.}{=} \int_{-6.58948-5.23118i \rightarrow -6.58948-1.46139i} (-1)\frac{1}{f(z)}dz \\
&= \int_{1}^{0} (-1)\frac{1}{f(-6.58948+(-1.46139-3.76979r)i)}(-3.76979i)dr \\
&= (-3.76979i)\int_{0}^{1} \frac{1}{f(-6.58948+(-1.46139-3.76979r)i)}dr
\end{aligned}$$

5. $b_{45}^{*} = -6.58948 - 1.46139i \rightarrow -6.31381 - 1.46139i$

Let

$$\begin{aligned}
z &= -6.58948 - 1.46139i + r(0.27567) \\
&= (-6.58948 + 0.27567r) + (-1.46139)i
\end{aligned}$$

where $r : 0 \rightarrow 1$ , and $dz = (0.27567)dr$.

We have

$$\begin{aligned}
\int_{b_{45}^{*}} \frac{1}{f(z)}dz &= \int_{-6.58948-1.46139i \rightarrow -6.31381-1.46139i} \frac{1}{f(z)}dz \\
&\overset{Math.}{=} \int_{-6.58948-1.46139i \rightarrow -6.31381-1.46139i} (-1)\frac{1}{f(z)}dz \\
&= \int_{0}^{1} (-1)\frac{1}{f((-6.58948+0.27567r)+(-1.46139)i)}(0.27567)dr \\
&= (-0.27567)\int_{0}^{1} \frac{1}{f((-6.58948+0.27567r)+(-1.46139)i)}dr
\end{aligned}$$

6. $b_{46}^{*} = -6.31381 - 1.46139i \overset{-}{\rightarrow} -6.31381 + 0i$

Let

$$\begin{aligned}
z &= -6.31381 - 1.46139i + r(1.46139i) \\
&= -6.31381 + (-1.46139 + 1.46139r)i
\end{aligned}$$

where $r : 0 \overset{-}{\rightarrow} 1$ , and $dz = (1.46139i)dr$.

We have

$$\begin{aligned}
\int_{b_{46}^{*}} \frac{1}{f(z)}dz &= \int_{-6.31381-1.46139i \overset{-}{\rightarrow} -6.31381+0i} \frac{1}{f(z)}dz \\
&= \int_{-6.31381-1.46139i \overset{-}{\rightarrow} -6.31381+0i} \frac{1}{f(z)}dz \\
&\overset{Math.}{=} \int_{-6.31381-1.46139i \rightarrow -6.31381+0i} \frac{1}{f(z)}dz \\
&= \int_{0}^{1} \frac{1}{f(-6.31381+(-1.46139+1.46139r)i)}(1.46139i)dr \\
&= (1.46139i)\int_{0}^{1} \frac{1}{f(-6.31381+(-1.46139+1.46139r)i)}dr
\end{aligned}$$

7. $b^*_{47} = -6.31381 + 0i \rightarrow -4.68652 + 0i$

Let

$$\begin{aligned} z &= -6.31381 + 0i + r(1.62729) \\ &= -6.31381 + 1.62729r \end{aligned}$$

where $r : 0 \rightarrow 1$ , and $dz = (1.62729)dr$.

We have

$$\begin{aligned} \int_{b^*_{47}} \frac{1}{f(z)}dz &= \int_{-6.31381+0i\rightarrow-4.68652+0i} \frac{1}{f(z)}dz \\ &= \int_{-6.31381+0i\rightarrow-4.68652+0i} \frac{1}{f(z)}dz \\ &\overset{Math.}{=} \int_{-6.31381+0i\rightarrow-4.68652+0i} \frac{1}{f(z)}dz \\ &= \int_0^1 \frac{1}{f(-6.31381+1.62729r)}(1.62729)dr \\ &= (1.62729)\int_0^1 \frac{1}{f(-6.31381+1.62729r)}dr \end{aligned}$$

8. $b^*_{48} = -4.68652 + 0i \xrightarrow{+} -1.57080 + 0i$

Let

$$\begin{aligned} z &= -4.68652 + 0i + r(3.11572) \\ &= -4.68652 + 3.11572r \end{aligned}$$

where $r : 0 \xrightarrow{+} 1$ , and $dz = (3.11572)dr$.

We have

$$\begin{aligned} \int_{b^*_{48}} \frac{1}{f(z)}dz &= \int_{-4.68652+0i\xrightarrow{+}-1.57080+0i} \frac{1}{f(z)}dz \\ &= \int_{-4.68652+0i\xrightarrow{+}-1.57080+0i} \frac{1}{f(z)}dz \\ &\overset{Math.}{=} \int_{-4.68652+0i\rightarrow-1.57080+0i} (-1)\frac{1}{f(z)}dz \\ &= \int_0^1 (-1)\frac{1}{f(-4.68652+3.11572r)}(3.11572)dr \\ &= (-3.11572)\int_0^1 \frac{1}{f(-4.68652+3.11572r)}dr \end{aligned}$$

9. $b^*_{49} = -1.57080 + 0i \rightarrow 1.57080 + 0i$

Let

$$\begin{aligned} z &= -1.57080 + 0i + r(3.1416) \\ &= -1.57080 + 3.1416r \end{aligned}$$

where $r : 0 \to 1$ , and $dz = (3.1416)dr$.

We have

$$\begin{aligned}
\int_{b_{49}^*} \frac{1}{f(z)}dz &= \int_{-1.57080+0i \to 1.57080+0i} \frac{1}{f(z)}dz \\
&= \int_{-1.57080+0i \to 1.57080+0i} \frac{1}{f(z)}dz \\
&\overset{Math.}{=} \int_{-1.57080+0i \to 1.57080+0i} \frac{1}{f(z)}dz \\
&= \int_0^1 \frac{1}{f(-1.57080+3.1416r)}(3.1416)dr \\
&= (3.1416)\int_0^1 \frac{1}{f(-1.57080+3.1416r)}dr
\end{aligned}$$

10. $b_{4a}^* = 1.57080 + 0i \xrightarrow{+} 4.68652 + 0i$

    Let

$$\begin{aligned}
z &= 1.57080 + 0i + r(3.11572) \\
&= 1.57080 + 3.11572r
\end{aligned}$$

where $r : 0 \xrightarrow{+} 1$ , and $dz = (3.11572)dr$.

We have

$$\begin{aligned}
\int_{b_{4a}^*} \frac{1}{f(z)}dz &= \int_{1.57080+0i \xrightarrow{+} 4.68652+0i} \frac{1}{f(z)}dz \\
&= \int_{1.57080+0i \xrightarrow{+} 4.68652+0i} \frac{1}{f(z)}dz \\
&\overset{Math.}{=} \int_{1.57080+0i \to 4.68652+0i} (-1)\frac{1}{f(z)}dz \\
&= \int_0^1 (-1)\frac{1}{f(1.57080+3.11572r)}(3.11572)dr \\
&= (-3.11572)\int_0^1 \frac{1}{f(1.57080+3.11572r)}dr
\end{aligned}$$

11. $b_{4b}^* = 4.68652 + 0i \to 6.31381 + 0i$

    Let

$$\begin{aligned}
z &= 4.68652 + 0i + r(1.62729) \\
&= 4.68652 + 1.62729r
\end{aligned}$$

where $r : 0 \to 1$ , and $dz = (1.62729)dr$.

We have

$$
\begin{aligned}
\int_{b_{4b}^*} \frac{1}{f(z)} dz &= \int_{4.68652+0i \to 6.31381+0i} \frac{1}{f(z)} dz \\
&= \int_{4.68652+0i \to 6.31381+0i} \frac{1}{f(z)} dz \\
&\overset{Math.}{=} \int_{4.68652+0i \to 6.31381+0i} \frac{1}{f(z)} dz \\
&= \int_0^1 \frac{1}{f(4.68652+1.62729r)} (1.62729) dr \\
&= (1.62729) \int_0^1 \frac{1}{f(4.68652+1.62729r)} dr
\end{aligned}
$$

12. $b_{4c}^* = 6.31381 + 0i \xrightarrow{+} 6.31381 - 1.46139i$

    Let

$$
\begin{aligned}
z &= 6.31381 + 0i + r(-1.46139i) \\
&= 6.31381 + (-1.46139r)i
\end{aligned}
$$

where $r : 0 \xrightarrow{+} 1$ , and $dz = (-1.46139i)dr$.

We have

$$
\begin{aligned}
\int_{b_{4c}^*} \frac{1}{f(z)} dz &= \int_{6.31381+0i \xrightarrow{+} 6.31381-1.46139i} \frac{1}{f(z)} dz \\
&= \int_{6.31381+0i \xrightarrow{+} 6.31381-1.46139i} \frac{1}{f(z)} dz \\
&\overset{Math.}{=} \int_{6.31381+0i \to 6.31381-1.46139i} \frac{1}{f(z)} dz \\
&= \int_0^1 \frac{1}{f(6.31381+(-1.46139r)i)} (-1.46139i) dr \\
&= (-1.46139i) \int_0^1 \frac{1}{f(6.31381+(-1.46139r)i)} dr
\end{aligned}
$$

13. $b_{4d}^* = 6.31381 - 1.46139i \overset{-}{\dashrightarrow} 6.31381 + 1.46139i$

    Let

$$
\begin{aligned}
z &= 6.31381 - 1.46139i + r(2.92278i) \\
&= 6.31381 + (-1.46139 + 2.92278r)i
\end{aligned}
$$

where $r : 0 \overset{-}{\dashrightarrow} 1$ , and $dz = (2.92278i)dr$.

We have

$$
\begin{aligned}
\int_{b_{4d}^*} \frac{1}{f(z)}dz &= \int_{6.31381-1.46139i \overset{-}{\dashrightarrow} 6.31381+1.46139i} \frac{1}{f(z)}dz \\
&= \int_{6.31381-1.46139i \overset{+}{\rightarrow} 6.31381+1.46139i} \frac{1}{f(z)}dz \\
&\overset{Math.}{=} \int_{6.31381-1.46139i \rightarrow 6.31381+1.46139i} \frac{1}{f(z)}dz \\
&= \int_0^1 \frac{1}{f(6.31381+(-1.46139+2.92278r)i)}(2.92278i)dr \\
&= (2.92278i)\int_0^1 \frac{1}{f(6.31381+(-1.46139+2.92278r)i)}dr
\end{aligned}
$$

14. $b_{4e}^* = 6.31381 + 1.46139i \overset{+}{\dashrightarrow} 6.31381 + 0i$

Let

$$
\begin{aligned}
z &= 6.31381 + 1.46139i + r(-1.46139i) \\
&= 6.31381 + (1.46139 - 1.46139r)i
\end{aligned}
$$

where $r : 0 \overset{+}{\dashrightarrow} 1$ , and $dz = (-1.46139i)dr$.

We have

$$
\begin{aligned}
\int_{b_{4e}^*} \frac{1}{f(z)}dz &= \int_{6.31381+1.46139i \overset{+}{\dashrightarrow} 6.31381+0i} \frac{1}{f(z)}dz \\
&= \int_{6.31381+1.46139i \overset{-}{\rightarrow} 6.31381+0i} \frac{1}{f(z)}dz \\
&\overset{Math.}{=} \int_{6.31381+1.46139i \rightarrow 6.31381+0i} (-1)\frac{1}{f(z)}dz \\
&= \int_0^1 (-1)\frac{1}{f(6.31381+(1.46139-1.46139r)i)}(-1.46139i)dr \\
&= (1.46139i)\int_0^1 \frac{1}{f(6.31381+(1.46139-1.46139r)i)}dr
\end{aligned}
$$

15. $b_{4f}^* = 6.31381 + 0i \dashrightarrow 4.68652 + 0i$

Let

$$
\begin{aligned}
z &= 6.31381 + 0i + r(-1.62729) \\
&= 6.31381 - 1.62729r
\end{aligned}
$$

where $r : 0 \dashrightarrow 1$ , and $dz = (-1.62729)dr$.

We have

$$
\begin{aligned}
\int_{b_{4f}^*} \frac{1}{f(z)}dz &= \int_{6.31381+0i \dashrightarrow 4.68652+0i} \frac{1}{f(z)}dz \\
&= \int_{6.31381+0i \rightarrow 4.68652+0i} (-1)\frac{1}{f(z)}dz \\
&\overset{Math.}{=} \int_{6.31381+0i \rightarrow 4.68652+0i} (-1)\frac{1}{f(z)}dz \\
&= \int_0^1 (-1)\frac{1}{f(6.31381-1.62729r)}(-1.62729)dr \\
&= (1.62729)\int_0^1 \frac{1}{f(6.31381-1.62729r)}dr
\end{aligned}
$$

16. $b^*_{4g} = 4.68652 + 0i \overset{-}{\dashrightarrow} 1.57080 + 0i$

Let

$$\begin{aligned} z &= 4.68652 + 0i + r(-3.11572) \\ &= 4.68652 + (-3.11572r) \end{aligned}$$

where $r : 0 \overset{-}{\dashrightarrow} 1$ , and $dz = (-3.11572)dr$.

We have

$$\begin{aligned} \int_{b^*_{4g}} \frac{1}{f(z)} dz &= \int_{4.68652+0i \overset{-}{\dashrightarrow} 1.57080+0i} \frac{1}{f(z)} dz \\ &= \int_{4.68652+0i \overset{+}{\rightarrow} 1.57080+0i} \frac{1}{f(z)} dz \\ &\overset{Math.}{=} \int_{4.68652+0i \rightarrow 1.57080+0i} (-1) \frac{1}{f(z)} dz \\ &= \int_0^1 (-1) \frac{1}{f(4.68652+(-3.11572r))} (-3.11572) dr \\ &= (3.11572) \int_0^1 \frac{1}{f(4.68652+(-3.11572r))} dr \end{aligned}$$

17. $b^*_{4h} = 1.57080 + 0i \dashrightarrow -1.57080 + 0i$

Let

$$\begin{aligned} z &= 1.57080 + 0i + r(-3.1416) \\ &= 1.57080 + (-3.1416r) \end{aligned}$$

where $r : 0 \dashrightarrow 1$ , and $dz = (-3.1416)dr$.

We have

$$\begin{aligned} \int_{b^*_{4h}} \frac{1}{f(z)} dz &= \int_{1.57080+0i \dashrightarrow -1.57080+0i} \frac{1}{f(z)} dz \\ &= \int_{1.57080+0i \rightarrow -1.57080+0i} (-1) \frac{1}{f(z)} dz \\ &\overset{Math.}{=} \int_{1.57080+0i \rightarrow -1.57080+0i} (-1) \frac{1}{f(z)} dz \\ &= \int_0^1 (-1) \frac{1}{f(1.57080+(-3.1416r))} (-3.1416) dr \\ &= (3.1416) \int_0^1 \frac{1}{f(1.57080+(-3.1416r))} dr \end{aligned}$$

18. $b^*_{4i} = -1.57080 + 0i \overset{-}{\dashrightarrow} -4.68652 + 0i$

Let

$$\begin{aligned} z &= -1.57080 + 0i + r(-3.11572) \\ &= -1.57080 + (-3.11572r) \end{aligned}$$

where $r : 0 \overset{-}{\dashrightarrow} 1$ , and $dz = (-3.11572)dr$.

We have

$$\begin{aligned}
\int_{b^*_{4i}} \frac{1}{f(z)}dz &= \int_{-1.57080+0i \overset{-}{\dashrightarrow} -4.68652+0i} \frac{1}{f(z)}dz \\
&= \int_{-1.57080+0i \overset{+}{\to} -4.68652+0i} \frac{1}{f(z)}dz \\
&\overset{Math.}{=} \int_{-1.57080+0i \to -4.68652+0i} (-1)\frac{1}{f(z)}dz \\
&= \int_0^1 (-1)\frac{1}{f(-1.57080+(-3.11572r))}(-3.11572)dr \\
&= (3.11572)\int_0^1 \frac{1}{f(-1.57080+(-3.11572r))}dr
\end{aligned}$$

19. $b^*_{4j} = -4.68652 + 0i \dashrightarrow -6.31381 + 0i$

    Let

$$\begin{aligned}
z &= -4.68652 + 0i + r(-1.62729) \\
&= -4.68652 + (-1.62729r)
\end{aligned}$$

where $r : 0 \dashrightarrow 1$ , and $dz = (-1.62729)dr$.

We have

$$\begin{aligned}
\int_{b^*_{4j}} \frac{1}{f(z)}dz &= \int_{-4.68652+0i \dashrightarrow -6.31381+0i} \frac{1}{f(z)}dz \\
&= \int_{-4.68652+0i \to -6.31381+0i} (-1)\frac{1}{f(z)}dz \\
&\overset{Math.}{=} \int_{-4.68652+0i \to -6.31381+0i} (-1)\frac{1}{f(z)}dz \\
&= \int_0^1 (-1)\frac{1}{f(-4.68652+(-1.62729r))}(-1.62729)dr \\
&= (1.62729)\int_0^1 \frac{1}{f(-4.68652+(-1.62729r))}dr
\end{aligned}$$

20. $b^*_{4k} = -6.31381 + 0i \overset{-}{\dashrightarrow} -6.31381 + 1.46139i$

    Let

$$\begin{aligned}
z &= -6.31381 + 0i + r(1.46139i) \\
&= -6.31381 + (1.46139r)i
\end{aligned}$$

where $r : 0 \overset{-}{\dashrightarrow} 1$ , and $dz = (1.46139i)dr$.

We have

$$
\begin{aligned}
\int_{b^*_{4k}} \frac{1}{f(z)}dz &= \int_{-6.31381+0i \overset{-}{\dashrightarrow} -6.31381+1.46139i} \frac{1}{f(z)}dz \\
&= \int_{-6.31381+0i \overset{\pm}{\rightarrow} -6.31381+1.46139i} \frac{1}{f(z)}dz \\
&\overset{Math.}{=} \int_{-6.31381+0i \rightarrow -6.31381+1.46139i} (-1)\frac{1}{f(z)}dz \\
&= \int_0^1 (-1)\frac{1}{f(-6.31381+(1.46139r)i)}(1.46139i)dr \\
&= (-1.46139i)\int_0^1 \frac{1}{f(-6.31381+(1.46139r)i)}dr
\end{aligned}
$$

21. $b^*_{4l} = -6.31381 + 1.46139i \dashrightarrow -6.58948 + 1.46139i$

Let

$$
\begin{aligned}
z &= -6.31381 + 1.46139i + r(-0.27567) \\
&= (-6.31381 - 0.27567r) + 1.46139i
\end{aligned}
$$

where $r : 0 \dashrightarrow 1$ , and $dz = (-0.27567)dr$.

We have

$$
\begin{aligned}
\int_{b^*_{4l}} \frac{1}{f(z)}dz &= \int_{-6.31381+1.46139i \dashrightarrow -6.58948+1.46139i} \frac{1}{f(z)}dz \\
&= \int_{-6.31381+1.46139i \rightarrow -6.58948+1.46139i} (-1)\frac{1}{f(z)}dz \\
&\overset{Math.}{=} \int_{-6.31381+1.46139i \rightarrow -6.58948+1.46139i} \frac{1}{f(z)}dz \\
&= \int_0^1 \frac{1}{f((-6.31381-0.27567r)+1.46139i)}(-0.27567)dr \\
&= (-0.27567)\int_0^1 \frac{1}{f((-6.31381-0.27567r)+1.46139i)}dr
\end{aligned}
$$

22. $b^*_{4m} = -6.58948 + 1.46139i \overset{-}{\dashrightarrow} -6.58948 + 5.23118i$

Let

$$
\begin{aligned}
z &= -6.58948 + 1.46139i + r(3.76979i) \\
&= -6.58948 + (1.46139 + 3.76979r)i
\end{aligned}
$$

where $r : 0 \overset{-}{\dashrightarrow} 1$ , and $dz = (3.76979i)dr$.

We have

$$
\begin{aligned}
\int_{b^*_{4m}} \frac{1}{f(z)}dz &= \int_{-6.58948+1.46139i \overset{-}{\dashrightarrow} -6.58948+5.23118i} \frac{1}{f(z)}dz \\
&= \int_{-6.58948+1.46139i \overset{\pm}{\rightarrow} -6.58948+5.23118i} \frac{1}{f(z)}dz \\
&\overset{Math.}{=} \int_{-6.58948+1.46139i \rightarrow -6.58948+5.23118i} \frac{1}{f(z)}dz \\
&= \int_0^1 \frac{1}{f(-6.58948+(1.46139+3.76979r)i)}(3.76979i)dr \\
&= (3.76979i)\int_0^1 \frac{1}{f(-6.58948+(1.46139+3.76979r)i)}dr
\end{aligned}
$$

By 1. , 2. , 3. , 4. , 5. , 6. , 7. , 8. , 9. , 10. , 11. , 12. , 13. , 14. , 15. , 16. , 17. , 18. , 19. , 20. , 21. , 22. , we have

$$\begin{aligned}
\int_{b_4} \frac{1}{f(z)}dz &= \int_{b_4^*} \frac{1}{f(z)}dz \\
&= \int_{b_{41}^*} \frac{1}{f(z)}dz + \int_{b_{42}^*} \frac{1}{f(z)}dz + \int_{b_{43}^*} \frac{1}{f(z)}dz \\
&\quad + \int_{b_{44}^*} \frac{1}{f(z)}dz + \int_{b_{45}^*} \frac{1}{f(z)}dz + \int_{b_{46}^*} \frac{1}{f(z)}dz \\
&\quad + \int_{b_{47}^*} \frac{1}{f(z)}dz + \int_{b_{48}^*} \frac{1}{f(z)}dz + \int_{b_{49}^*} \frac{1}{f(z)}dz \\
&\quad + \int_{b_{4a}^*} \frac{1}{f(z)}dz + \int_{b_{4b}^*} \frac{1}{f(z)}dz + \int_{b_{4c}^*} \frac{1}{f(z)}dz \\
&\quad + \int_{b_{4d}^*} \frac{1}{f(z)}dz + \int_{b_{4e}^*} \frac{1}{f(z)}dz + \int_{b_{4f}^*} \frac{1}{f(z)}dz \\
&\quad + \int_{b_{4g}^*} \frac{1}{f(z)}dz + \int_{b_{4h}^*} \frac{1}{f(z)}dz + \int_{b_{4i}^*} \frac{1}{f(z)}dz \\
&\quad + \int_{b_{4j}^*} \frac{1}{f(z)}dz + \int_{b_{4k}^*} \frac{1}{f(z)}dz + \int_{b_{4l}^*} \frac{1}{f(z)}dz \\
&\quad + \int_{b_{4m}^*} \frac{1}{f(z)}dz \\
&= (0.0000523241 + 0.000115868i)
\end{aligned}$$

Let $b_5^* = b_{51}^* \cup b_{52}^* \cup b_{53}^* \cup b_{54}^* \cup b_{55}^* \cup b_{56}^* \cup b_{57}^* \cup b_{58}^* \cup b_{59}^* \cup b_{5a}^* \cup b_{5b}^* \cup b_{5c}^* \cup b_{5d}^* \cup b_{5e}^* \cup b_{5f}^* \cup b_{5g}^* \cup b_{5h}^* \cup b_{5i}^* \cup b_{5j}^* \cup b_{5k}^* \cup b_{5l}^* \cup b_{5m}^* \cup b_{5n}^* \cup b_{5o}^* \cup b_{5p}^* \cup b_{5q}^*$ is the equivalent path for $b_5$ where $b_{51}^*$ is the path from $Z_1 = -6.58948+5.23118i$ to $-6.58948+1.46139i$ on (+)-edge of sheet-I , $b_{52}^*$ is the path from $-6.58948+1.46139i$ to $-6.58948-1.46139i$ on (+)-edge of sheet-I , $b_{53}^*$ is the path from $-6.58948 - 1.46139i$ to $Z_2 = -6.58948 - 5.23118i$ on (+)-edge of sheet-I , $b_{54}^*$ is the path from $Z_2 = -6.58948-5.23118i$ to $-6.58948-1.46139i$ on (−)-edge of sheet-I , $b_{55}^*$ is the path from $-6.58948-1.46139i$ to $Z_4 = -6.31381-1.46139i$ on sheet-I , $b_{56}^*$ is the path from $Z_4 = -6.31381-1.46139i$ to $-6.31381+0i$ on (−)-edge of sheet-I , $b_{57}^*$ is the path from $-6.31381+0i$ to $Z_5 = -4.68652+0i$ on sheet-I , $b_{58}^*$ is the path from $Z_5 = -4.68652+0i$ to $Z_6 = -1.57080+0i$ on (+)-edge of sheet-I , $b_{59}^*$ is the path from $Z_6 = -1.57080+0i$ to $Z_7 = 1.57080+0i$ on sheet-I , $b_{5a}^*$ is the path from $Z_7 = 1.57080 + 0i$ to $Z_8 = 4.68652 + 0i$ on (+)-edge of sheet-I , $b_{5b}^*$ is the path from $Z_8 = 4.68652 + 0i$ to $6.31381 + 0i$ on sheet-I ,

$b_{5c}^*$ is the path from $6.31381+0i$ to $Z_{10}=6.31381-1.46139i$ on $(+)$-edge of sheet-I , $b_{5d}^*$ is the path from $Z_{10}=6.31381-1.46139i$ to $6.58948-1.46139i$ on sheet-I , $b_{5e}^*$ is the path from $6.58948-1.46139i$ to $Z_{12}=6.58948-5.23118i$ on $(+)$-edge of sheet-I , $b_{5f}^*$ is the path from $Z_{12}=6.58948-5.23118i$ to $Z_{11}=6.58948+5.23118i$ on $(-)$-edge of sheet-II , $b_{5g}^*$ is the path from $Z_{11}=6.58948+5.23118i$ to $6.58948+1.46139i$ on $(+)$-edge of sheet-II , $b_{5h}^*$ is the path from $6.58948+1.46139i$ to $Z_9=6.31381+1.46139i$ on sheet-II , $b_{5i}^*$ is the path from $Z_9=6.31381+1.46139i$ to $6.31381+0i$ on $(+)$-edge of sheet-II , $b_{5j}^*$ is the path from $6.31381+0i$ to $Z_8=4.68652+0i$ on sheet-II , $b_{5k}^*$ is the path from $Z_8=4.68652+0i$ to $Z_7=1.57080+0i$ on $(-)$-edge of sheet-II , $b_{5l}^*$ is the path from $Z_7=1.57080+0i$ to $Z_6=-1.57080+0i$ on sheet-II , $b_{5m}^*$ is the path from $Z_6=-1.57080+0i$ to $Z_5=-4.68652+0i$ on $(-)$-edge of sheet-II , $b_{5n}^*$ is the path from $Z_5=-4.68652+0i$ to $-6.31381+0i$ on sheet-II , $b_{5o}^*$ is the path from $-6.31381+0i$ to $Z_3=-6.31381+1.46139i$ on $(-)$-edge of sheet-II , $b_{5p}^*$ is the path from $Z_3=-6.31381+1.46139i$ to $-6.58948+1.46139i$ on sheet-II , $b_{5q}^*$ is the path from $-6.58948+1.46139i$ to $Z_1=-6.58948+5.23118i$ on $(-)$-edge of sheet-II.

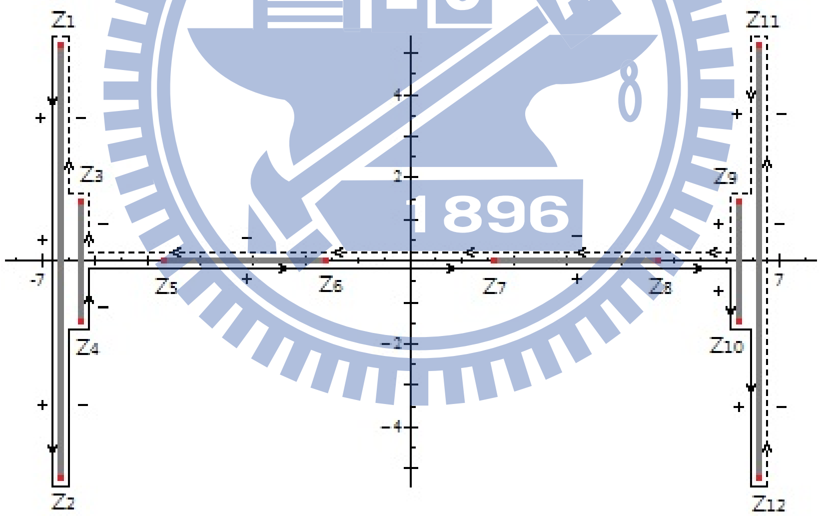



Figure 55. $b_5^*$ path.

1. $b_{51}^* = -6.58948+5.23118i \xrightarrow{+} -6.58948+1.46139i$

   Let

$$\begin{aligned} z &= -6.58948+5.23118i+r(-3.76979i) \\ &= -6.58948+(5.23118-3.76979r)i \end{aligned}$$

   where $r: 0 \xrightarrow{+} 1$ , and $dz=(-3.76979i)dr$.

We have

$$\begin{aligned}
\int_{b_{51}^*} \frac{1}{f(z)} dz &= \int_{-6.58948+5.23118i \xrightarrow{+} -6.58948+1.46139i} \frac{1}{f(z)} dz \\
&\overset{Math.}{=} \int_{-6.58948+5.23118i \rightarrow -6.58948+1.46139i} \frac{1}{f(z)} dz \\
&= \int_0^1 \frac{1}{f(-6.58948+(5.23118-3.76979r)i)} (-3.76979i) dr \\
&= (-3.76979i) \int_0^1 \frac{1}{f(-6.58948+(5.23118-3.76979r)i)} dr
\end{aligned}$$

2. $b_{52}^* = -6.58948 + 1.46139i \xrightarrow{+} -6.58948 - 1.46139i$

   Let

$$\begin{aligned}
z &= -6.58948 + 1.46139i + r(-2.92278i) \\
&= -6.58948 + (1.46139 - 2.92278r)i
\end{aligned}$$

   where $r : 0 \xrightarrow{+} 1$ , and $dz = (-2.92278i)dr$.

   We have

$$\begin{aligned}
\int_{b_{52}^*} \frac{1}{f(z)} dz &= \int_{-6.58948+1.46139i \xrightarrow{+} -6.58948-1.46139i} \frac{1}{f(z)} dz \\
&\overset{Math.}{=} \int_{-6.58948+1.46139i \rightarrow -6.58948-1.46139i} \frac{1}{f(z)} dz \\
&= \int_0^1 \frac{1}{f(-6.58948+(1.46139-2.92278r)i)} (-2.92278i) dr \\
&= (-2.92278i) \int_0^1 \frac{1}{f(-6.58948+(1.46139-2.92278r)i)} dr
\end{aligned}$$

3. $b_{53}^* = -6.58948 - 1.46139i \xrightarrow{+} -6.58948 - 5.23118i$

   Let

$$\begin{aligned}
z &= -6.58948 - 1.46139i + r(-3.76979i) \\
&= -6.58948 + (-1.46139 - 3.76979r)i
\end{aligned}$$

   where $r : 0 \xrightarrow{+} 1$ , and $dz = (-3.76979i)dr$.

   We have

$$\begin{aligned}
\int_{b_{53}^*} \frac{1}{f(z)} dz &= \int_{-6.58948-1.46139i \xrightarrow{+} -6.58948-5.23118i} \frac{1}{f(z)} dz \\
&\overset{Math.}{=} \int_{-6.58948-1.46139i \rightarrow -6.58948-5.23118i} \frac{1}{f(z)} dz \\
&= \int_0^1 \frac{1}{f(-6.58948+(-1.46139-3.76979r)i)} (-3.76979i) dr \\
&= (-3.76979i) \int_0^1 \frac{1}{f(-6.58948+(-1.46139-3.76979r)i)} dr
\end{aligned}$$

4. $b^*_{54} = -6.58948 - 5.23118i \overset{-}{\rightarrow} -6.58948 - 1.46139i$

   Let

$$\begin{aligned} z &= -6.58948 - 1.46139i + r(-3.76979i) \\ &= -6.58948 + (-1.46139 - 3.76979r)i \end{aligned}$$

   where $r : 1 \overset{-}{\rightarrow} 0$ , and $dz = (-3.76979i)dr$.

   We have

$$\begin{aligned} \int_{b^*_{54}} \frac{1}{f(z)} dz &= \int_{-6.58948-5.23118i \overset{-}{\rightarrow} -6.58948-1.46139i} \frac{1}{f(z)} dz \\ &\overset{Math.}{=} \int_{-6.58948-5.23118i \rightarrow -6.58948-1.46139i} (-1) \frac{1}{f(z)} dz \\ &= \int_1^0 (-1) \frac{1}{f(-6.58948+(-1.46139-3.76979r)i)} (-3.76979i) dr \\ &= (-3.76979i) \int_0^1 \frac{1}{f(-6.58948+(-1.46139-3.76979r)i)} dr \end{aligned}$$

5. $b^*_{55} = -6.58948 - 1.46139i \rightarrow -6.31381 - 1.46139i$

   Let

$$\begin{aligned} z &= -6.58948 - 1.46139i + r(0.27567) \\ &= (-6.58948 + 0.27567r) + (-1.46139)i \end{aligned}$$

   where $r : 0 \rightarrow 1$ , and $dz = (0.27567)dr$.

   We have

$$\begin{aligned} \int_{b^*_{55}} \frac{1}{f(z)} dz &= \int_{-6.58948-1.46139i \rightarrow -6.31381-1.46139i} \frac{1}{f(z)} dz \\ &\overset{Math.}{=} \int_{-6.58948-1.46139i \rightarrow -6.31381-1.46139i} (-1) \frac{1}{f(z)} dz \\ &= \int_0^1 (-1) \frac{1}{f((-6.58948+0.27567r)+(-1.46139)i)} (0.27567) dr \\ &= (-0.27567) \int_0^1 \frac{1}{f((-6.58948+0.27567r)+(-1.46139)i)} dr \end{aligned}$$

6. $b^*_{56} = -6.31381 - 1.46139i \overset{-}{\rightarrow} -6.31381 + 0i$

   Let

$$\begin{aligned} z &= -6.31381 - 1.46139i + r(1.46139i) \\ &= -6.31381 + (-1.46139 + 1.46139r)i \end{aligned}$$

where $r : 0 \xrightarrow{-} 1$ , and $dz = (1.46139i)dr$.

We have

$$
\begin{aligned}
\int_{b_{56}^*} \frac{1}{f(z)}dz &= \int_{-6.31381-1.46139i \xrightarrow{-} -6.31381+0i} \frac{1}{f(z)}dz \\
&= \int_{-6.31381-1.46139i \xrightarrow{-} -6.31381+0i} \frac{1}{f(z)}dz \\
&\overset{Math.}{=} \int_{-6.31381-1.46139i \rightarrow -6.31381+0i} \frac{1}{f(z)}dz \\
&= \int_0^1 \frac{1}{f(-6.31381+(-1.46139+1.46139r)i)}(1.46139i)dr \\
&= (1.46139i) \int_0^1 \frac{1}{f(-6.31381+(-1.46139+1.46139r)i)}dr
\end{aligned}
$$

7. $b_{57}^* = -6.31381 + 0i \rightarrow -4.68652 + 0i$

   Let

$$
\begin{aligned}
z &= -6.31381 + 0i + r(1.62729) \\
&= -6.31381 + 1.62729r
\end{aligned}
$$

where $r : 0 \rightarrow 1$ , and $dz = (1.62729)dr$.

We have

$$
\begin{aligned}
\int_{b_{57}^*} \frac{1}{f(z)}dz &= \int_{-6.31381+0i \rightarrow -4.68652+0i} \frac{1}{f(z)}dz \\
&= \int_{-6.31381+0i \rightarrow -4.68652+0i} \frac{1}{f(z)}dz \\
&\overset{Math.}{=} \int_{-6.31381+0i \rightarrow -4.68652+0i} \frac{1}{f(z)}dz \\
&= \int_0^1 \frac{1}{f(-6.31381+1.62729r)}(1.62729)dr \\
&= (1.62729) \int_0^1 \frac{1}{f(-6.31381+1.62729r)}dr
\end{aligned}
$$

8. $b_{58}^* = -4.68652 + 0i \xrightarrow{+} -1.57080 + 0i$

   Let

$$
\begin{aligned}
z &= -4.68652 + 0i + r(3.11572) \\
&= -4.68652 + 3.11572r
\end{aligned}
$$

where $r : 0 \xrightarrow{+} 1$ , and $dz = (3.11572)dr$.

We have

$$\begin{aligned}
\int_{b_{58}^*} \frac{1}{f(z)} dz &= \int_{-4.68652+0i \xrightarrow{+} -1.57080+0i} \frac{1}{f(z)} dz \\
&= \int_{-4.68652+0i \xrightarrow{+} -1.57080+0i} \frac{1}{f(z)} dz \\
&\overset{Math.}{=} \int_{-4.68652+0i \rightarrow -1.57080+0i} (-1) \frac{1}{f(z)} dz \\
&= \int_0^1 (-1) \frac{1}{f(-4.68652+3.11572r)} (3.11572) dr \\
&= (-3.11572) \int_0^1 \frac{1}{f(-4.68652+3.11572r)} dr
\end{aligned}$$

9. $b_{59}^* = -1.57080 + 0i \rightarrow 1.57080 + 0i$

   Let

$$\begin{aligned}
z &= -1.57080 + 0i + r(3.1416) \\
&= -1.57080 + 3.1416r
\end{aligned}$$

   where $r : 0 \rightarrow 1$ , and $dz = (3.1416)dr$.

   We have

$$\begin{aligned}
\int_{b_{59}^*} \frac{1}{f(z)} dz &= \int_{-1.57080+0i \rightarrow 1.57080+0i} \frac{1}{f(z)} dz \\
&= \int_{-1.57080+0i \rightarrow 1.57080+0i} \frac{1}{f(z)} dz \\
&\overset{Math.}{=} \int_{-1.57080+0i \rightarrow 1.57080+0i} \frac{1}{f(z)} dz \\
&= \int_0^1 \frac{1}{f(-1.57080+3.1416r)} (3.1416) dr \\
&= (3.1416) \int_0^1 \frac{1}{f(-1.57080+3.1416r)} dr
\end{aligned}$$

10. $b_{5a}^* = 1.57080 + 0i \xrightarrow{+} 4.68652 + 0i$

    Let

$$\begin{aligned}
z &= 1.57080 + 0i + r(3.11572) \\
&= 1.57080 + 3.11572r
\end{aligned}$$

    where $r : 0 \xrightarrow{+} 1$ , and $dz = (3.11572)dr$.

    We have

$$\begin{aligned}
\int_{b_{5a}^*} \frac{1}{f(z)} dz &= \int_{1.57080+0i \xrightarrow{+} 4.68652+0i} \frac{1}{f(z)} dz \\
&= \int_{1.57080+0i \xrightarrow{+} 4.68652+0i} \frac{1}{f(z)} dz \\
&\overset{Math.}{=} \int_{1.57080+0i \rightarrow 4.68652+0i} (-1) \frac{1}{f(z)} dz \\
&= \int_0^1 (-1) \frac{1}{f(1.57080+3.11572r)} (3.11572) dr \\
&= (-3.11572) \int_0^1 \frac{1}{f(1.57080+3.11572r)} dr
\end{aligned}$$

11. $b^*_{5b} = 4.68652 + 0i \rightarrow 6.31381 + 0i$

Let

$$\begin{aligned} z &= 4.68652 + 0i + r(1.62729) \\ &= 4.68652 + 1.62729r \end{aligned}$$

where $r : 0 \rightarrow 1$ , and $dz = (1.62729)dr$.

We have

$$\begin{aligned} \int_{b^*_{5b}} \frac{1}{f(z)} dz &= \int_{4.68652+0i \rightarrow 6.31381+0i} \frac{1}{f(z)} dz \\ &= \int_{4.68652+0i \rightarrow 6.31381+0i} \frac{1}{f(z)} dz \\ &\overset{Math.}{=} \int_{4.68652+0i \rightarrow 6.31381+0i} \frac{1}{f(z)} dz \\ &= \int_0^1 \frac{1}{f(4.68652+1.62729r)} (1.62729) dr \\ &= (1.62729) \int_0^1 \frac{1}{f(4.68652+1.62729r)} dr \end{aligned}$$

12. $b^*_{5c} = 6.31381 + 0i \xrightarrow{+} 6.31381 - 1.46139i$

Let

$$\begin{aligned} z &= 6.31381 + 0i + r(-1.46139i) \\ &= 6.31381 + (-1.46139r)i \end{aligned}$$

where $r : 0 \xrightarrow{+} 1$ , and $dz = (-1.46139i)dr$.

We have

$$\begin{aligned} \int_{b^*_{5c}} \frac{1}{f(z)} dz &= \int_{6.31381+0i \xrightarrow{+} 6.31381-1.46139i} \frac{1}{f(z)} dz \\ &= \int_{6.31381+0i \xrightarrow{+} 6.31381-1.46139i} \frac{1}{f(z)} dz \\ &\overset{Math.}{=} \int_{6.31381+0i \rightarrow 6.31381-1.46139i} \frac{1}{f(z)} dz \\ &= \int_0^1 \frac{1}{f(6.31381+(-1.46139r)i)} (-1.46139i) dr \\ &= (-1.46139i) \int_0^1 \frac{1}{f(6.31381+(-1.46139r)i)} dr \end{aligned}$$

13. $b^*_{5d} = 6.31381 - 1.46139i \rightarrow 6.58948 - 1.46139i$

Let

$$\begin{aligned} z &= 6.31381 - 1.46139i + r(0.27567) \\ &= (6.31381 + 0.27567r) - 1.46139i \end{aligned}$$

where $r : 0 \rightarrow 1$ , and $dz = (0.27567)dr$.

We have

$$\begin{aligned}
\int_{b^*_{5d}} \frac{1}{f(z)}dz &= \int_{6.31381-1.46139i \rightarrow 6.58948-1.46139i} \frac{1}{f(z)}dz \\
&= \int_{6.31381-1.46139i \rightarrow 6.58948-1.46139i} \frac{1}{f(z)}dz \\
&\overset{Math.}{=} \int_{6.31381-1.46139i \rightarrow 6.58948-1.46139i} (-1)\frac{1}{f(z)}dz \\
&= \int_0^1 (-1)\frac{1}{f((6.31381+0.27567r)-1.46139i)}(0.27567)dr \\
&= (0.27567)\int_0^1 \frac{1}{f((6.31381+0.27567r)-1.46139i)}dr
\end{aligned}$$

14. $b^*_{5e} = 6.58948 - 1.46139i \overset{+}{\rightarrow} 6.58948 - 5.23118i$

Let

$$\begin{aligned}
z &= 6.58948 - 1.46139i + r(-3.76979i) \\
&= 6.58948 + (-1.46139 - 3.76979r)i
\end{aligned}$$

where $r : 0 \overset{+}{\rightarrow} 1$ , and $dz = (-3.76979i)dr$.

We have

$$\begin{aligned}
\int_{b^*_{5e}} \frac{1}{f(z)}dz &= \int_{6.58948-1.46139i \overset{+}{\rightarrow} 6.58948-5.23118i} \frac{1}{f(z)}dz \\
&= \int_{6.58948-1.46139i \overset{+}{\rightarrow} 6.58948-5.23118i} \frac{1}{f(z)}dz \\
&\overset{Math.}{=} \int_{6.58948-1.46139i \rightarrow 6.58948-5.23118i} (-1)\frac{1}{f(z)}dz \\
&= \int_0^1 (-1)\frac{1}{f(6.58948+(-1.46139-3.76979r)i)}(-3.76979i)dr \\
&= (3.76979i)\int_0^1 \frac{1}{f(6.58948+(-1.46139-3.76979r)i)}dr
\end{aligned}$$

15. $b^*_{5f} = 6.58948 - 5.23118i \overset{-}{\dashrightarrow} 6.58948 + 5.23118i$

Let

$$\begin{aligned}
z &= 6.58948 - 5.23118i + r(10.46236i) \\
&= 6.58948 + (-5.23118 + 10.46236r)i
\end{aligned}$$

where $r : 0 \overset{-}{\dashrightarrow} 1$ , and $dz = (10.46236i)dr$.

We have

$$
\begin{aligned}
\int_{b^*_{5f}} \frac{1}{f(z)}dz &= \int_{6.58948-5.23118i \overset{-}{\dashrightarrow} 6.58948+5.23118i} \frac{1}{f(z)}dz \\
&= \int_{6.58948-5.23118i \overset{+}{\to} 6.58948+5.23118i} \frac{1}{f(z)}dz \\
&\overset{Math.}{=} \int_{6.58948-5.23118i \to 6.58948+5.23118i} (-1) f(z)dz \\
&= \int_0^1 (-1)\frac{1}{f(6.58948+(-5.23118+10.46236r)i)}(10.46236i)dr \\
&= (-10.46236i)\int_0^1 \frac{1}{f(6.58948+(-5.23118+10.46236r)i)}dr
\end{aligned}
$$

16. $b^*_{5g} = 6.58948 + 5.23118i \overset{+}{\dashrightarrow} 6.58948 + 1.46139i$

    Let

$$
\begin{aligned}
z &= 6.58948 + 5.23118i + r(-3.76979i) \\
&= 6.58948 + (5.23118 - 3.76979r)i
\end{aligned}
$$

where $r : 0 \overset{+}{\dashrightarrow} 1$ , and $dz = (-3.76979i)dr$.

We have

$$
\begin{aligned}
\int_{b^*_{5g}} \frac{1}{f(z)}dz &= \int_{6.58948+5.23118i \overset{+}{\dashrightarrow} 6.58948+1.46139i} \frac{1}{f(z)}dz \\
&= \int_{6.58948+5.23118i \overset{-}{\to} 6.58948+1.46139i} \frac{1}{f(z)}dz \\
&\overset{Math.}{=} \int_{6.58948+5.23118i \to 6.58948+1.46139i} \frac{1}{f(z)}dz \\
&= \int_0^1 \frac{1}{f(6.58948+(5.23118-3.76979r)i)}(-3.76979i)dr \\
&= (-3.76979i)\int_0^1 \frac{1}{f(6.58948+(5.23118-3.76979r)i)}dr
\end{aligned}
$$

17. $b^*_{5h} = 6.58948 + 1.46139i \dashrightarrow 6.31381 + 1.46139i$

    Let

$$
\begin{aligned}
z &= 6.58948 + 1.46139i + r(-0.27567) \\
&= (6.58948 - 0.27567r) + 1.46139i
\end{aligned}
$$

where $r : 0 \dashrightarrow 1$ , and $dz = (-0.27567)dr$.

We have

$$
\begin{aligned}
\int_{b^*_{5h}} \frac{1}{f(z)}dz &= \int_{6.58948+1.46139i \dashrightarrow 6.31381+1.46139i} \frac{1}{f(z)}dz \\
&= \int_{6.58948+1.46139i \to 6.31381+1.46139i} (-1)\frac{1}{f(z)}dz \\
&\overset{Math.}{=} \int_{6.58948+1.46139i \to 6.31381+1.46139i} \frac{1}{f(z)}dz \\
&= \int_0^1 \frac{1}{f((6.58948-0.27567r)+1.46139i)}(-0.27567)dr \\
&= (-0.27567)\int_0^1 \frac{1}{f((6.58948-0.27567r)+1.46139i)}dr
\end{aligned}
$$

18. $b^*_{5i} = 6.31381 + 1.46139i \overset{+}{\dashrightarrow} 6.31381 + 0i$

Let

$$\begin{aligned} z &= 6.31381 + 1.46139i + r(-1.46139i) \\ &= 6.31381 + (1.46139 - 1.46139r)i \end{aligned}$$

where $r : 0 \overset{+}{\dashrightarrow} 1$ , and $dz = (-1.46139i)dr$.

We have

$$\begin{aligned} \int_{b^*_{5i}} \frac{1}{f(z)}dz &= \int_{6.31381+1.46139i \overset{+}{\dashrightarrow} 6.31381+0i} \frac{1}{f(z)}dz \\ &= \int_{6.31381+1.46139i \overset{-}{\rightarrow} 6.31381+0i} \frac{1}{f(z)}dz \\ &\overset{Math.}{=} \int_{6.31381+1.46139i \rightarrow 6.31381+0i} (-1)\frac{1}{f(z)}dz \\ &= \int_0^1 (-1)\frac{1}{f(6.31381+(1.46139-1.46139r)i)}(-1.46139i)dr \\ &= (1.46139i)\int_0^1 \frac{1}{f(6.31381+(1.46139-1.46139r)i)}dr \end{aligned}$$

19. $b^*_{5j} = 6.31381 + 0i \dashrightarrow 4.68652 + 0i$

Let

$$\begin{aligned} z &= 6.31381 + 0i + r(-1.62729) \\ &= 6.31381 - 1.62729r \end{aligned}$$

where $r : 0 \dashrightarrow 1$ , and $dz = (-1.62729)dr$.

We have

$$\begin{aligned} \int_{b^*_{5j}} \frac{1}{f(z)}dz &= \int_{6.31381+0i \dashrightarrow 4.68652+0i} \frac{1}{f(z)}dz \\ &= \int_{6.31381+0i \rightarrow 4.68652+0i} (-1)\frac{1}{f(z)}dz \\ &\overset{Math.}{=} \int_{6.31381+0i \rightarrow 4.68652+0i} (-1)\frac{1}{f(z)}dz \\ &= \int_0^1 (-1)\frac{1}{f(6.31381-1.62729r)}(-1.62729)dr \\ &= (1.62729)\int_0^1 \frac{1}{f(6.31381-1.62729r)}dr \end{aligned}$$

20. $b^*_{5k} = 4.68652 + 0i \overset{-}{\dashrightarrow} 1.57080 + 0i$

Let

$$\begin{aligned} z &= 4.68652 + 0i + r(-3.11572) \\ &= 4.68652 + (-3.11572r) \end{aligned}$$

where $r : 0 \overset{-}{\dashrightarrow} 1$ , and $dz = (-3.11572)dr$.

We have

$$\begin{aligned}
\int_{b^*_{5k}} \frac{1}{f(z)}dz &= \int_{4.68652+0i \overset{-}{\dashrightarrow} 1.57080+0i} \frac{1}{f(z)}dz \\
&= \int_{4.68652+0i \overset{+}{\rightarrow} 1.57080+0i} \frac{1}{f(z)}dz \\
&\overset{Math.}{=} \int_{4.68652+0i \rightarrow 1.57080+0i} (-1)\frac{1}{f(z)}dz \\
&= \int_0^1 (-1)\frac{1}{f(4.68652+(-3.11572r))}(-3.11572)dr \\
&= (3.11572)\int_0^1 \frac{1}{f(4.68652+(-3.11572r))}dr
\end{aligned}$$

21. $b^*_{5l} = 1.57080 + 0i \dashrightarrow -1.57080 + 0i$

Let

$$\begin{aligned}
z &= 1.57080 + 0i + r(-3.1416) \\
&= 1.57080 + (-3.1416r)
\end{aligned}$$

where $r : 0 \dashrightarrow 1$ , and $dz = (-3.1416)dr$.

We have

$$\begin{aligned}
\int_{b^*_{5l}} \frac{1}{f(z)}dz &= \int_{1.57080+0i \dashrightarrow -1.57080+0i} \frac{1}{f(z)}dz \\
&= \int_{1.57080+0i \rightarrow -1.57080+0i} (-1)\frac{1}{f(z)}dz \\
&\overset{Math.}{=} \int_{1.57080+0i \rightarrow -1.57080+0i} (-1)\frac{1}{f(z)}dz \\
&= \int_0^1 (-1)\frac{1}{f(1.57080+(-3.1416r))}(-3.1416)dr \\
&= (3.1416)\int_0^1 \frac{1}{f(1.57080+(-3.1416r))}dr
\end{aligned}$$

22. $b^*_{5m} = -1.57080 + 0i \overset{-}{\dashrightarrow} -4.68652 + 0i$

Let

$$\begin{aligned}
z &= -1.57080 + 0i + r(-3.11572) \\
&= -1.57080 + (-3.11572r)
\end{aligned}$$

where $r : 0 \overset{-}{\dashrightarrow} 1$ , and $dz = (-3.11572)dr$.

We have

$$
\begin{aligned}
\int_{b_{5m}^*} \frac{1}{f(z)} dz &= \int_{-1.57080+0i \overset{-}{\dashrightarrow} -4.68652+0i} \frac{1}{f(z)} dz \\
&= \int_{-1.57080+0i \overset{+}{\rightarrow} -4.68652+0i} \frac{1}{f(z)} dz \\
&\overset{Math.}{=} \int_{-1.57080+0i \rightarrow -4.68652+0i} (-1) \frac{1}{f(z)} f(z) dz \\
&= \int_0^1 (-1) \frac{1}{f(-1.57080+(-3.11572r))} (-3.11572) dr \\
&= (3.11572) \int_0^1 \frac{1}{f(-1.57080+(-3.11572r))} dr
\end{aligned}
$$

23. $b_{5n}^* = -4.68652 + 0i \dashrightarrow -6.31381 + 0i$

Let

$$
\begin{aligned}
z &= -4.68652 + 0i + r(-1.62729) \\
&= -4.68652 + (-1.62729r)
\end{aligned}
$$

where $r : 0 \dashrightarrow 1$ , and $dz = (-1.62729)dr$.

We have

$$
\begin{aligned}
\int_{b_{5n}^*} \frac{1}{f(z)} dz &= \int_{-4.68652+0i \dashrightarrow -6.31381+0i} \frac{1}{f(z)} dz \\
&= \int_{-4.68652+0i \rightarrow -6.31381+0i} (-1) \frac{1}{f(z)} dz \\
&\overset{Math.}{=} \int_{-4.68652+0i \rightarrow -6.31381+0i} (-1) \frac{1}{f(z)} dz \\
&= \int_0^1 (-1) \frac{1}{f(-4.68652+(-1.62729r))} (-1.62729) dr \\
&= (1.62729) \int_0^1 \frac{1}{f(-4.68652+(-1.62729r))} dr
\end{aligned}
$$

24. $b_{5o}^* = -6.31381 + 0i \overset{-}{\dashrightarrow} -6.31381 + 1.46139i$

Let

$$
\begin{aligned}
z &= -6.31381 + 0i + r(1.46139i) \\
&= -6.31381 + (1.46139r)i
\end{aligned}
$$

where $r : 0 \overset{-}{\dashrightarrow} 1$ , and $dz = (1.46139i)dr$.

We have

$$
\begin{aligned}
\int_{b_{5o}^*} \frac{1}{f(z)} dz &= \int_{-6.31381+0i \overset{-}{\dashrightarrow} -6.31381+1.46139i} \frac{1}{f(z)} dz \\
&= \int_{-6.31381+0i \overset{+}{\rightarrow} -6.31381+1.46139i} \frac{1}{f(z)} dz \\
&\overset{Math.}{=} \int_{-6.31381+0i \rightarrow -6.31381+1.46139i} (-1) \frac{1}{f(z)} dz \\
&= \int_0^1 (-1) \frac{1}{f(-6.31381+(1.46139r)i)} (1.46139i) dr \\
&= (-1.46139i) \int_0^1 \frac{1}{f(-6.31381+(1.46139r)i)} dr
\end{aligned}
$$

25. $b_{5p}^{*} = -6.31381 + 1.46139i \dashrightarrow -6.58948 + 1.46139i$

Let

$$\begin{aligned} z &= -6.31381 + 1.46139i + r(-0.27567) \\ &= (-6.31381 - 0.27567r) + 1.46139i \end{aligned}$$

where $r : 0 \dashrightarrow 1$ , and $dz = (-0.27567)dr$.

We have

$$\begin{aligned} \int_{b_{5p}^{*}} \frac{1}{f(z)}dz &= \int_{-6.31381+1.46139i \dashrightarrow -6.58948+1.46139i} \frac{1}{f(z)}dz \\ &= \int_{-6.31381+1.46139i \rightarrow -6.58948+1.46139i} (-1)\frac{1}{f(z)}dz \\ &\overset{Math.}{=} \int_{-6.31381+1.46139i \rightarrow -6.58948+1.46139i} \frac{1}{f(z)}dz \\ &= \int_0^1 \frac{1}{f((-6.31381-0.27567r)+1.46139i)}(-0.27567)dr \\ &= (-0.27567)\int_0^1 \frac{1}{f((-6.31381-0.27567r)+1.46139i)}dr \end{aligned}$$

26. $b_{5q}^{*} = -6.58948 + 1.46139i \overset{-}{\dashrightarrow} -6.58948 + 5.23118i$

Let

$$\begin{aligned} z &= -6.58948 + 1.46139i + r(3.76979i) \\ &= -6.58948 + (1.46139 + 3.76979r)i \end{aligned}$$

where $r : 0 \overset{-}{\dashrightarrow} 1$ , and $dz = (3.76979i)dr$.

We have

$$\begin{aligned} \int_{b_{5q}^{*}} \frac{1}{f(z)}dz &= \int_{-6.58948+1.46139i \overset{-}{\dashrightarrow} -6.58948+5.23118i} \frac{1}{f(z)}dz \\ &= \int_{-6.58948+1.46139i \overset{+}{\rightarrow} -6.58948+5.23118i} \frac{1}{f(z)}dz \\ &\overset{Math.}{=} \int_{-6.58948+1.46139i \rightarrow -6.58948+5.23118i} \frac{1}{f(z)}dz \\ &= \int_0^1 \frac{1}{f(-6.58948+(1.46139+3.76979r)i)}(3.76979i)dr \\ &= (3.76979i)\int_0^1 \frac{1}{f(-6.58948+(1.46139+3.76979r)i)}dr \end{aligned}$$

By 1. , 2. , 3. , 4. , 5. , 6. , 7. , 8. , 9. , 10. , 11. , 12. , 13. , 14. , 15. , 16. , 17. , 18. , 19. , 20. , 21. , 22. , 23. , 24. , 25. , 26. , we have

$$
\begin{aligned}
\int_{b_5} \frac{1}{f(z)}dz &= \int_{b_5^*} \frac{1}{f(z)}dz \\
&= \int_{b_{51}^*} \frac{1}{f(z)}dz + \int_{b_{52}^*} \frac{1}{f(z)}dz + \int_{b_{53}^*} \frac{1}{f(z)}dz \\
&\quad + \int_{b_{54}^*} \frac{1}{f(z)}dz + \int_{b_{55}^*} \frac{1}{f(z)}dz + \int_{b_{56}^*} \frac{1}{f(z)}dz \\
&\quad + \int_{b_{57}^*} \frac{1}{f(z)}dz + \int_{b_{58}^*} \frac{1}{f(z)}dz + \int_{b_{59}^*} \frac{1}{f(z)}dz \\
&\quad + \int_{b_{5a}^*} \frac{1}{f(z)}dz + \int_{b_{5b}^*} \frac{1}{f(z)}dz + \int_{b_{5c}^*} \frac{1}{f(z)}dz \\
&\quad + \int_{b_{5d}^*} \frac{1}{f(z)}dz + \int_{b_{5e}^*} \frac{1}{f(z)}dz + \int_{b_{5f}^*} \frac{1}{f(z)}dz \\
&\quad + \int_{b_{5g}^*} \frac{1}{f(z)}dz + \int_{b_{5h}^*} \frac{1}{f(z)}dz + \int_{b_{5i}^*} \frac{1}{f(z)}dz \\
&\quad + \int_{b_{5j}^*} \frac{1}{f(z)}dz + \int_{b_{5k}^*} \frac{1}{f(z)}dz + \int_{b_{5l}^*} \frac{1}{f(z)}dz \\
&\quad + \int_{b_{5m}^*} \frac{1}{f(z)}dz + \int_{b_{5n}^*} \frac{1}{f(z)}dz + \int_{b_{5o}^*} \frac{1}{f(z)}dz \\
&\quad + \int_{b_{5p}^*} \frac{1}{f(z)}dz + \int_{b_{5q}^*} \frac{1}{f(z)}dz \\
&= (0.0000523241 + 0.000115868i)
\end{aligned}
$$

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