國立交通大學

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碩 士 論 文

柯倫布猜想在多項式的數值研究

Numerical study on Korenblum's conjecture for polynomials

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柯倫布猜想在多項式上的數值研究

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摘　要

本論文介紹柯倫布猜想（Korenblum conjecture）在多項式上的數值結果。柯倫布常數（Korenblum's constant ）在多項式上可以更容易地利用不同的解根和數值積分方法找出。最後，我們考慮柯倫布猜想在某些分式函數上，能找出目前最佳的柯倫布常數的上界。

關鍵詞：柯倫布猜想、柯倫布常數。

Numerical study on Korenblum's conjecture for polynomials

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Abstract

In this study we give a brief description of numerical results on Korenblum's conjecture for polynomials. The Korenblum's constant will be found out numerically by using different numerical integration methods and some methods for solving roots. It can be solved easily a little bit for Korenblum's conjecture under polynomials. Finally, we consider Korenblum's conjecture for some kinds of fractional functions and obtain a better upper bound of Korenblum's constant.

Keywords: Korenblum's conjecture, Korenblum's constant.

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終於，這篇論文總算完成，多虧這一路上多虧了有許多貴人們的適時相助，我才能順利地完成。最需要感謝我的指導老師張書銘 博士，在這一年半的時間裡，除了在研究上碰到瓶頸時適時的給我思考方向，一步一步的幫助我完成論文，就算我有所懈怠或者犯了錯誤，老師仍一如既往的給予督促與指正，並且對於做事的態度與人生經驗上的分享，更是讓我獲益良多。第二個要感謝的是程守慶 教授，感謝程老師讓我們作這個題目，在討論時細心的解釋問題並且也給予建議，對我的研究有著很大的幫助。

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黃柏綸

謹誌於交通大學

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1 Introduction

Definition 1.1. *Holomorphic [6]*

Given a complex-valued function f of a single complex variable, the derivative of f at point z_0 in its domain is defined by the limit

$$f^{'}(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

If f is complex differentiable at every point z_0 in U, we say that f is holomorphic on U.

Let $f(z)$ and $g(z)$ be holomorphic in the open unit disk $\mathbb{D}$ and let Z_f and Z_g be their zero sets. If for some c, $0 < c < 1$,

$$|f(z)| \le |g(z)| \quad (c < |z| < 1) \tag{1}$$

and

$$Z_f \supseteq Z_g, \tag{2}$$

then f/g is holomorphic, and the classical maximum principle implies that (1) holds in $\mathbb{D}$ and

$$\|f\| \le \|g\|, \tag{3}$$

where $\|\cdot\|$ is the Bergman norm:

$$\|f(z)\| = [\int_{\mathbb{D}} |f(z)|^2 dA(z)]^{1/2} < \infty,$$

where

$$dA(z) = \frac{1}{\pi} dxdy = \frac{1}{\pi} rdrd\theta \quad \text{and} \quad z = x + iy = re^{i\theta}$$

are the normalized Lebesgue area measure on $\mathbb{D}$.It is nature to ask whether (2) is necessarily for the implication from (1) to (3). Therefore, Korenblum made the following conjecture in 1991.

Conjecture 1.2. *(Korenblum) [9]*

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. The Bergman space $A^2(\mathbb{D})$ consists of holomorphic functions f and g in $\mathbb{D}$. Korenblun conjectured that there exists a numerical constant c, $0 < c < 1$, such that whenever $|f(z)| \le |g(z)|$ in the annulus $c < |z| < 1$, then

$\|f(z)\| \leq \|g(z)\|$. It would be interesting to know the sharp (i.e. the largest) value of c.

In 1999, Hayman [8] proved Korenblum's conjecture, and Hinkkanen [2] proved that Korenblum's conjecture is true for the Bergman space $A^p(\mathbb{D})(p \geq 1)$. But the sharp value of c even when $p = 2$(we call it Korenblum's constant) is still unknown. However, Hayman [8] gave a lower bound of c: $c \geq 0.04$. Hinkkanen [2] improved Hayman's result that $c \geq 0.15724\cdots$. Recently Schuster [7] has shown that $c \geq 0.21$ in terms of Möbius pseudodistance for the annulus. On the other hand, an upper bound on c can be found from Martin's example [4]: $c < 0.70450\cdots$. Wang [10] gave an upper bound on c: $c < 0.69472$.

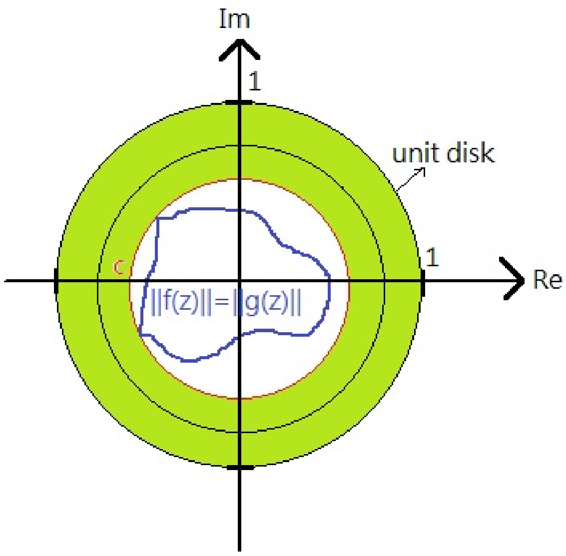

Figure 1: The green area is an annulus which satisfying $|f(z)| \leq |g(z)|$ and the maximum solution of $|f(z)| = |g(z)|$ is c.

Example 1.3. *[4]*

Define

$$f(z) = \frac{1}{\sqrt{2}} + \epsilon, \quad g(z) = z.$$

We have $|f(z)| \leq |g(z)|$ for $(\frac{1}{\sqrt{2}} + \epsilon) < |z| < 1$, and $\|f(z)\| = \frac{1}{2} + \epsilon(\sqrt{2} + \epsilon)$, $\|g(z)\| = \frac{1}{2}$. $\|f(z)\| \geq \|g(z)\|$ and Korenblum's conjecture fails, so c cannot larger than $\frac{1}{\sqrt{2}}$. Therefore, $\frac{1}{\sqrt{2}}$ is an upper bound of Korenblum's constant.

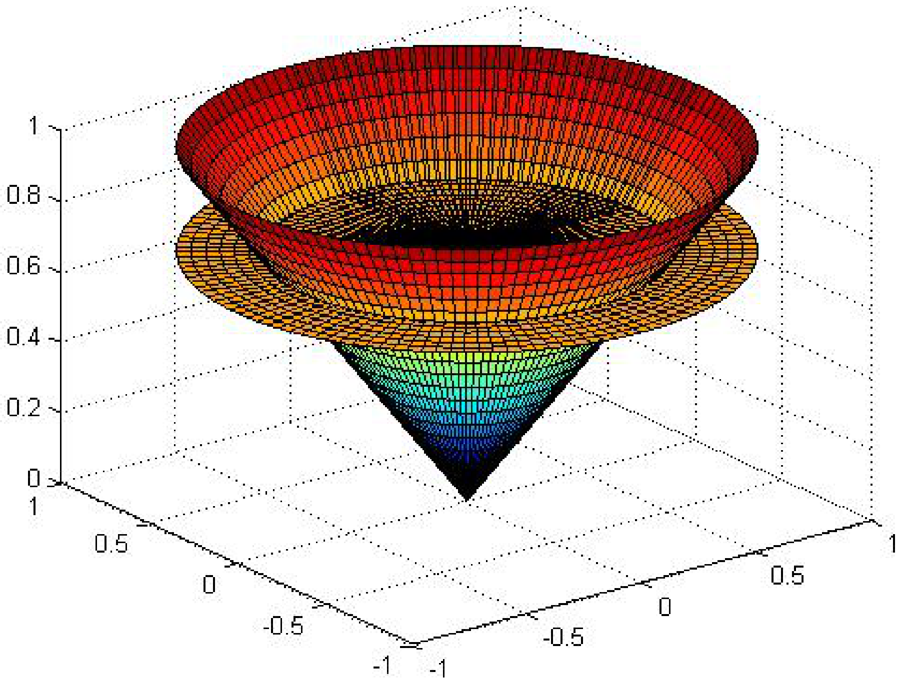

Figure 2: The functions $f(z) = \frac{1}{\sqrt{2}}$ and $g(z) = z$.

Theorem 1.4. *[5]*

Suppose $n \geq 0$ is an integer and f is an analytic function in $\mathbb{D}$. If $|f(z)| \leq |z^n|$ for $1/\sqrt{3} \leq |z| \leq 1$. then

$$\int_{\mathbb{D}} |f(z)|^2 dA(z) \leq \int_{\mathbb{D}} |z^n|^2 dA(z).$$

Proof. It is easy to see that this is a direct consequence of the following result.

Theorem 1.5. *[5]*

Suppose $n \geq 0$ is an integer and $F(x) = \sum_{k=0}^{\infty} a_k x^k$ with nonnegative coefficients. If $F(x) \leq x^n$ for $1/3 \leq x \leq 1$, then

$$\int_0^1 F(x)dx \leq \int_0^1 x^n dx = \frac{1}{n+1}.$$

Proof. The case $n = 0$ is trivial. We assume $n \geq 1$ in the rest of the proof. That $F(x) \leq x^n$ for $1/3 \leq x \leq 1$ implies that either $F(x) \leq x^n$ for all $x \in [0, 1]$ or there exists $c \in (0, 1/3]$ such that $F(c) = c^n$. In the first case the conclusion is obvious. So we assume that there exists $c \in (0, 1/3]$ such that $F(x) = c^n$ and $F(x) \leq x^n$ for $c \leq x \leq 1$. Thus we have

$$a_0 + a_1 + \cdots + a_n + a_{n+1} + \cdots = F(1),$$

$$a_0 + a_1 c + \cdots + a_n c^n + a_{n+1} c^{n+1} + \cdots = c^n$$

with $F(1) \leq 1$. We solve for a_n and a_{n+1} in terms of the other terms. Multiplying the first equation above by $-c^n$ and then adding the result to the second, we get

$$a_{n+1}(c^{n+1} - c^n) + \sum_{k \neq n, n+1} a_k(c^k - c^n) = c^n(1 - F(1)),$$

or

$$a_{n+1} = \sum_{k \neq n, n+1} a_k \frac{c^k - c^n}{c^n - c^{n+1}} - \frac{1 - F(1)}{1 - c}.$$

Similarly, we obtain

$$a_n = -\sum_{k \neq n, n+1} a_k \frac{c^k - c^{n+1}}{c^n - c^{n+1}} + \frac{1 - cF(1)}{1 - c}.$$

It follows that

$$\begin{aligned}
\int_0^1 F(x)dx &= \sum_{k=0}^{\infty} \frac{a_k}{k+1} \\
&= \sum_{k \neq n, n+1} \frac{a_k}{k+1} + \frac{a_n}{+1+} + \frac{a_{n+1}}{n+2} \\
&= \sum_{k \neq n, n+1} a_k \left[\frac{1}{k+1} - \frac{c^k - c^{n+1}}{(n+1)(c^n - c^{n+1})} + \frac{c^k * c^n}{(n+2)(c^n - c^{n+1})}\right] \\
&\quad + \frac{1}{n+1} + \frac{1 - F(1)}{1 - c}\left(\frac{c}{n+1} - \frac{1}{n+2}\right).
\end{aligned}$$

Since $c \leq 1/3$ and $n \geq 1$, we clearly have

$$\frac{c}{n+1} - \frac{1}{n+2} \leq 0.$$

But $1 - F(1) \geq 0$ and $1 - c > 0$, we obtain

$$\int_0^1 F(x)dx \leq \frac{1}{n+1} + \sum_{k \neq n, n+1} a_k \left[\frac{1}{k+1} - \frac{c^k - c^{n+1}}{(n+1)(c^n - c^{n+1})} + \frac{c^k - c^n}{(n+2)(c^n - c^{n+1})}\right].$$

Writing $c^k - c^n = c^k - c^{n+1} - (c^n - c^{n+1})$ in the above series, we easily get

$$\int_0^1 F(x) \leq \frac{1}{n+1} + \sum_{k \neq n, n+1} a_k \left[\frac{1}{k+1} - \frac{1}{n+2} - \frac{c^k - c^{n+1}}{(n+1)(n+2)(c^n - c^{n+1})}\right].$$

The desired result now follows from the following lemma.

Lemma 1.6. *[5]*

Let

$$D(k, n) = \frac{1}{k+1} - \frac{1}{n+2} - \frac{c^k - c^{n+1}}{(n+1)(n+2)(c^n - c^{n+1})}.$$

Then $D(k,n) \leq 0$ for all $n \geq 1$, $k \geq 0$, and $c \in (0, 1/3]$.

Proof. First note that $D(k,n) = 0$ if $k = n$ or $n+1$. If $k \geq n+2$, we have

$$\begin{aligned}
D(k,n) \leq & \frac{1}{n+3} - \frac{1}{n+2} + \frac{c^{n+1}}{(n+1)(n+2)(c^n - c^{n+1})} \\
= & \frac{c}{(n+1)(n+2)(1-c)} - \frac{1}{(n+2)(n+3)} \\
= & \frac{1}{(n+1)(n+2)(1-c)}(2c - \frac{n+1}{n+2}) \\
\leq & \frac{1}{(n+1)(n+2)(1-c)}(\frac{2}{3} - \frac{n+1}{n+2}) \\
\leq & 0
\end{aligned}$$

It remains to prove the result for $0 \leq k \leq n-1$. When $k = 0$ and $n = 1$, a little simplification shows that

$$D(0,1) = \frac{1}{2c}(c - \frac{1}{3}) \leq 0.$$

When $k = 0$ and $n \geq 2$, we have

$$\begin{aligned}
D(k,n) = & 1 - \frac{1}{n+2} - \frac{1 + c + \cdots + c^n}{(n+1)(n+2)c^n} \\
\leq & \frac{n+1}{n+2} - \frac{1}{(n+1)(n+2)c^n} \\
= & \frac{n+1}{n+2}c^n(c^n - \frac{1}{(n+1)^2}) \\
\leq & \frac{n+1}{n+2}c^n(\frac{1}{3^n} - \frac{1}{(n+1)^2}) \\
\leq & 0.
\end{aligned}$$

To prove the remaining case $1 \leq k \leq n-1$, we write

$$D(k,n) = \frac{n-k+1}{(k+1)(n+2)} - \frac{1 - c^{n-k+1}}{(n+1)(n+2)(1-c)c^{n-k}}.$$

Since $n - k + 1 \geq 2$, we have

$$1 - c^{n-k+1} \geq 1 - c^2 > 1 - c.$$

It follows that

$$\begin{aligned}
D(k,n) \leq & \frac{n-k+1}{(k+1)(n+2)} - \frac{1}{(n+1)(n+2)c^{n-k}} \\
= & \frac{n-k+1}{c^{n-k}(k+1)(n+2)}(c^{n-k} - \frac{k+1}{(n+1)(n-k+1)})
\end{aligned}$$

Let $m = n - k$, then $m \geq 1$ and

$$D(k,n) \leq \frac{m+1}{c^m(k+1)(m+k+2)}(c^m - \frac{k+1}{(m+1)(m+k+1)}).$$

It is clear that for $k \geq 1$

$$\frac{k+1}{(m+1)(m+k+1)} \geq \frac{2}{(m+1)(m+2)}.$$

Thus

$$\begin{aligned} D(k,n) \leq & \frac{m+1}{c^m(k+1)(m+k+2)(c^m - \frac{2}{(m+1)(m+2)})} \\ \leq & \frac{m+1}{c^m(k+1)(m+k+2)(\frac{1}{3^m} - \frac{2}{(m+1)(m+2)})} \\ \leq & 0, \end{aligned}$$

completing the proof of this lemma.

Lemma 1.7.

If $f(z) = \sum_{k=0}^{+\infty} a_k z^k \in A^2(\mathbb{D})$, then

$$\|f(z)\| = (\sum_{k=0}^{+\infty} \frac{|a_k|^2}{k+1})^{\frac{1}{2}}.$$

Proof.

$$\begin{aligned} ||f||^2 =& \int_D |f(z)|^2 dA(z) \\ =& \int_D |a_0 + a_1 z + \cdots + a_m z^m|^2 dA(z) \\ =& \int_0^1 \int_0^{2\pi} |a_0 + a_1(re^{i\theta}) + \cdots + a_m(re^{i\theta})^m|^2 r dr d\theta \\ =& \int_0^1 \int_0^{2\pi} (|a_0| + |a_1|(re^{i\theta}) + \cdots + |a_m|(re^{i\theta})^m)\overline{(|a_0| + |a_1|(re^{i\theta}) + \cdots + |a_m|(re^{i\theta})^m)} r dr d\theta \\ =& \int_0^1 \int_0^{2\pi} [|a_0| + |a_1|(cos\theta + isin\theta) + \cdots + |a_m|(cos(m\theta) + isin(m\theta))][|a_0| + \\ & |a_1|(cos\theta - isin\theta) + \cdots + |a_m|(cos(m\theta) - isin(m\theta))] r dr d\theta \\ =& \int_0^1 \int_0^{2\pi} (|a_0|^2 + |a_1|^2 + \cdots + |a_m|^2 + 2r(|a_0||a_1|cos\theta + \cdots + |a_{m-1}||a_m|cos((m \\ & -1)\theta)cos(m\theta))) r dr d\theta \\ =& |a_0|^2 + \frac{|a_1|^2}{2} + \cdots + \frac{|a_m|^2}{m+1}. \end{aligned}$$

Hence, $||f|| = (|a_0|^2 + \frac{|a_1|^2}{2} + \cdots + \frac{|a_m|^2}{m+1})^{\frac{1}{2}}$.

Since this conjecture is too difficult to prove, we try to find some counter examples to decrease the upper bound of c. To find the upper bound of c, we will find functions f and g which satisfies the following two conditions:

1. There exist an absolute constant c, $0 < c < 1$, such that $|f| \leq |g|$.

2. $||f|| > ||g||$.

Because the Bergman space is consist of all holomorphic functions and polynomial functions in z with complex-valued coefficients is the simplest holomorphic function, we try it at the beginning. From lemma 1.7, we can simplified the Bergman norm to be the sum of the absolute value of coefficients. It's interested that whenever we can find the example to prove or disprove c is larger than $1/\sqrt{3}$ from theorem 1.4. We want to start from

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \quad and \quad g(z) = z^n.$$

But it's too many coefficients, we decided from two terms of functions $f(z) = a_0 + a_1 z^m$ and $g(z) = z^n$ for complex-valued coefficients a_0, a_1 and $m, n \in \mathbb{N}$.

2 Theorem of numerical methods

2.1 Solving root method

Fixed Point theorem

Theorem 2.1. *Contraction Mapping Theorem [3]*

Let C be a closed subset of the real line. If F is a contractive mapping of C into C, then F has a unique fixed point. Moreover, this fixed point is the limit of every sequence obtained from the equation $x_{n+1} = F(x_n), n \geq 0$ with a starting point $x_0 \in C$.

Theorem 2.2. *[3]*

Let f be a function from $D \subset \mathbb{R}^n$ into $\mathbb{R}$ *and* $x_0 \in \mathcal{D}$ and . If all the partial derivatives of f exist and constants $\delta > 0$ and $K > 0$ exist so that whenever $||x - x_0|| < \delta$ and $x \in \mathcal{D}$, we have $|\frac{\partial f(x)}{\partial x_j}| \leq \mathcal{K}$,for each $j = 1, 2, \cdots, n$,Then f is continuous at x_0.

Theorem 2.3. *[3]*

Let $\mathcal{D} = (x_1, x_2, \cdots, x_n)^t | a_i \leq x_i \leq b_i \quad for \quad each \quad i = 1, 2, \cdots, n$ for some of constants $a_1, a_2, \cdots, a_n \quad and \quad b_1, b_2, \cdots, b_n$. Suppose G is a continuous function from $\mathcal{D} \subset \mathcal{R}^n$ into $\mathcal{R}^n$ with the property that $G(x) \in \mathcal{D} \quad whenever \quad x \in \mathcal{D}$. Then G has a fixed point in $\mathcal{D}$. Suppose ,in addition, that all the component functions of G have continuous partial derivatives and a constant $\mathcal{K} < 1$ exists with $|\frac{\partial g_i(x)}{\partial x_j}| \leq \frac{\mathcal{K}}{n} \quad whenever \quad x \in \mathcal{D}$, for each $j = 1, 2, \cdots, n$ and each component function g_i. Then the sequence $\{x^{(k)}\}_{k=0}^{\infty}$ defined by an arbitrarily selected a $x^{(0)}$ in $\mathcal{D}$ and generated by $x^{(k)} = G(x^{(k-1)})$, for each $\mathcal{K} \leq 1$ converges to the unique fixed point $p \in \mathcal{D}$ and $\|x^{(k)} - p\|_\infty \geq \frac{\mathcal{K}^k}{1-\mathcal{K}} \|x^{(1)-x(0)}\|_\infty$

Bisection (Interval Halving) Method [3]

If f is a continuous function on the interval [a,b] and if $f(a)g(b) < 0$, then f must have a zero in (a,b). Since $f(a)g(b) < 0$, the function f changes sign on the interval [a,b] and, therefore, it has at least one zero in the interval. This is a consequence of the **Intermediate-Value Theorem**.

Theorem 2.4. *Bisection Method [3]*

If $[a_0, b_0], [a_1, b_1], \cdots, [a_m, b_m], \cdots$ denote the intervals in the bisection method, then the limits $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exists are equal, and represent a zero of f. If $r = \lim_{n\to\infty} c_n$ and $c_n = \frac{1}{2}(a_n + b_n)$, then $|r - c_n| \geq 2^{-(n+1)}(b_0 - a_0)$.

Newton's Method

Theorem 2.5. *Newton's Method [3]*

Let f^n be continuous and let r be a simple zero of f. Then there is a neighborhood of r and a constant c such that if Newton's method is started in a neighborhood, the successive points become steadily closer to r and satisfy

$$|x_{n+1} - r| \leq c(x_n - r)^2, \qquad (r \geq 0).$$

Theorem 2.6. *Newton's Method for a Convex Function [3]*

If f belongs to $\mathcal{C}^2(\mathbb{R})$, is increasing, is convex, and has a zero, then the zero is unique, and the Newton iteration will converge to it from any starting point.

Secant method

Theorem 2.7. *Secant method [3]*

The Newton iteration is defined by the equation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{4}$$

One of the drawbacks of Newton's method is that it involves the derivative of the function whose zero is sought. To overcome this disadvantage, a number of methods have been proposed. For example, Steffensen's iteration

$$x_{n+1} = x_n - \frac{[f(x_n)]^2}{f(x_n + f(x_n)) - f(x_n)},$$

gives one approach to this problem. Another is to replace $f'(x)$ in equation (1) by a difference quotient, such as

$$f'(x_n) \approx \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}. \tag{5}$$

The approximation given in equation (2) comes directly from the definition of f' as a limit; namely,

$$f'(x) = \lim_{u \to x} \frac{f(x) - f(u)}{x - u}.$$

When this replacement is made, the resulting algorithm is called the **secant method**. Its formula is

$$x_{n+1} = x_n - f(x_n)[\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}] \qquad (n \geq 1).$$

2.2 Integration method

Trapezoid Rule [3]

If the interval $[a, b]$ is uniformly partitioned like this

$$a = x_0 < x_1 < \cdots < x_n = b$$
$$a = x_0 < x_0 + h < x_0 + 2h < \cdots < x_0 + (n-1)h = b, \quad h = \frac{b-a}{n}.$$

then

$$\begin{aligned} \int_a^b f(x)dx = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(x)dx &\approx \frac{1}{2} \sum_{i=1}^{n} (x_i - x_{i-1})[f(x_i) + f(x_{i+1})] \\ &\approx \frac{h}{2}[f(a) + 2\sum_{i=1}^{n-1} f(a+ih) + f(b)], \end{aligned}$$

and the error term is

$$\begin{aligned}\epsilon =& \int_a^b f(x)dx - \frac{h}{2}[f(a) + 2\sum_{i=1}^{n-1} f(a+ih) + f(b)] \\ =& f(a)(x-a) + f(b)(b-x) - \frac{h}{2}[f(a) + 2\sum_{i=1}^{n-1} f(a+ih) + f(b)] \\ =& -\frac{h^2}{12}[f^{'}(b) - f^{'}(a)] \\ =& -\frac{h^2}{12}(b-a)f^{''}(\xi),\end{aligned}$$

where $\xi \in (a,b)$.

Composite Simpson's rule [3]

Let the interval $[a,b]$ divide into an even number of subintervals. Set $x_i = a + ih, h = \frac{b-a}{n}$ $(0 \le i \le n)$. Then

$$\begin{aligned}\int_a^b f(x)dx =& \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \cdots + \int_{x_{n-2}}^{n} f(x)dx \\ =& \sum_{i=1}^{n/2} \int_{x_{2i-2}}^{x_{2i}} f(x)dx \\ \approx& \sum_{i=1}^{n/2} \frac{h}{3}[f(x_{2i-2}) + 4f(x_{2i-1}) + f(x_{2i})],\end{aligned}$$

and the error term is

$$-\frac{1}{180}(b-a)h^4 f^{(4)}(\xi),$$

where $\xi \in (a,b)$.

Romberg Algorithm [3]

Letting $R(n,0)$ denote the trapezoid estimate with 2^n subintervals, we have

$$\begin{aligned}R(0,0) &= \frac{1}{2}(b-a)[f(a) + f(b)], \\ R(n,0) &= \frac{1}{2}R(n-1,0) + h_n \sum_{i=1}^{2^{n-1}} f(a + (2i-1)h_n).\end{aligned}$$

The estimate $R(0,0)$, $R(1,0)$, $R(2,0)$,$\cdots$, $R(M,0)$ are computed for a modest value of M, and there are no duplicate function evaluations. In the remainder of the Romberg algorithm, additional quantities $R(n,m)$ are to be computed. All of these can be interpreted as estimates of the integral I. Further evaluations of the integrand f are not necessary after the element

$R(M,0)$ has been computed. The subsequent columns of the $R-array$ for $n \geq 1$ and $m \geq 1$ are constructed from the formula

$$R(n,m) = R(n,m-1) + \frac{1}{4^m - 1}[R(n,m-1) - R(n-1,m-1)].$$

This calculation is very simple. It is used to provide a final array of the form

$$\begin{array}{lllll}
R(0,0) & & & & \\
R(1,0) & R(1,1) & & & \\
R(2,0) & R(2,1) & R(2,2) & & \\
R(3,0) & R(3,1) & R(3,2) & R(3,3) & \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
R(M,0) & R(M,1) & R(M,2) & \cdots & R(M,M)
\end{array}$$

3 Some numerical method and algorithms

3.1 Algorithm for solving polynomial equations

Direct Force Method

1. Start from radius r=1 and r is decreasing define $h(z) = |f(z)| - |g(z)|$.

2. If $h(z) < 0$,then repeat step1. until $h(z_n)h(z_{n-1}) > 0$

3. Then z_{n-1} is the c which we want.

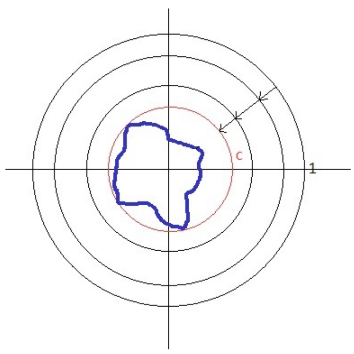

Figure 3: Using Direct Force method to find each radius of circles and the radius is decreasing until we find the first solution and this solution is c.

Neighborhood Searching Method(We call it NS method here.)

1. Given a initial point x_0 and computing $h(x_0) = f(x_0) - g(x_0)$.
2. Search the neighborhood of x_0 and given perturbs for several directions and the only one point such that h is the smallest, and this is our new point x_1.
3. Repeat Step 2. until h is small enough.

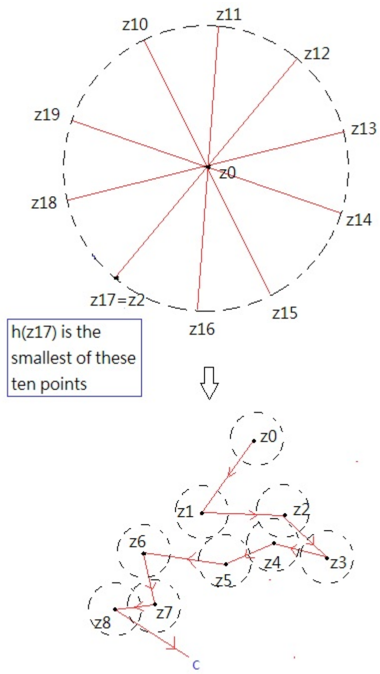

Figure 4: Neighborhood Searching method is searching the neighborhood of previous point and the minimum of the solution is the next point.

Bisection (Interval Halving) Method

1. Define the upper number "up" and lower number "down" and the middle point "$mid = \frac{up+down}{2}$".
2. Compute $h_{up} = f(up) - g(up)$ and $h_{down} = f(down) - g(down)$ and $h_{mid} = f(mid) - g(mid)$.
3. (a) If $h_{up}h_{mid} < 0$, implies the solution exists between up and mid by intermediate value theorem.

   (b) If $h_{down}h_{mid} < 0$, implies the solution exists between down and mid.

(c) If $h_{down}h_{up} > 0$, implies there is no solution between up and down.

4. choose new up down and mid to repeat Step 2. until h is small enough.

Secant Method

1. Choose a initial point x_0, d=0.1 and max times k.

2. Calculate $x_{n+1} = x_n - f(x_n)[\frac{d}{f(x_n)-f(x_n-d)}]$.

3. Repeat step 2 until k times.

Continuation method

1. Choose a first angle $\theta_0 = 0$ and use bisection method to compute the solution z_0^* such that $||f(z)|| = ||g(z)||$.

2. For $i = 1, 2, \cdots, n$ do step3-4.

3. Choose $\theta_1 = \theta$ be a initial point and compute (dy,dx) be the direct vector to find the next point z_1 on angle θ_1.

4. Let z_1 be center point and choose a neighborhood with radius r and use bisection method to find the solution z_1^*.

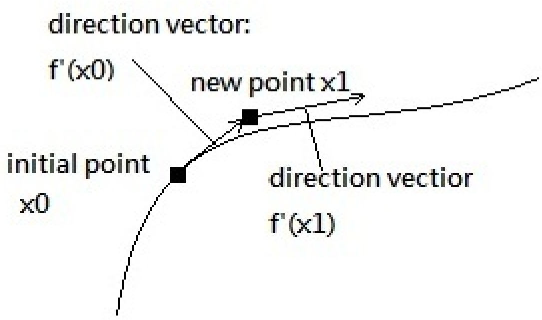

Figure 5: Continuation method is using the derivative of previous two points as the direction for searching solution.

3.2 Algorithms for integration

Trapezoid rule

1. uniformly partitioned the interval [a,b] into n subintervals.

2. calculate

$$\frac{h}{2}[f(a) + 2\sum_{i=1}^{n-1} f(a+ih) + f(b)], \text{ where } h = \frac{b-a}{2}$$

Composite Simpson's rule

1. uniformly partitioned the interval [a,b] into n subintervals and n is even number.

2. calculate

$$\frac{h}{3}\sum_{i=1}^{n/2}[f(x_{2i-2}) + 4f(x_{2i-1}) + f(x_{2i})]$$

Romberg Algorithm

1. Let $h_0 = b - a$ and $R(0,0) = \frac{1}{2}h[f(a) + f(b)]$.

2. $h_n = h_{n-1}/2$ and $R(n,0) = \frac{1}{2}R(n-1,0) + h_n \sum_{i=1}^{2^{n-1}} f(a + (2i-1)h)$.

3. $R(n,m) = R(n,m-1) + [R(n,m-1) - R(n-1,m-1)]/(4^m - 1)$.

4. repeat step2. and step3. and the $R(n,m)$ is our answer.

4 Numerical results

Table 1: Compared the running time with different solving root methods.

running step	bisection	Direct force	NS	continuation
10^{-3}	0.080619	0.007448	0.081113	0.006089
10^{-4}	0.084511	0.014932	x	0.009197
10^{-5}	0.095309	0.066767	x	0.004317
10^{-6}	0.100648	1.302836	x	0.007237
10^{-7}	0.106015	13.000338	x	0.008396

From table 1, continuation cost less time when the error is 10^{-7} and it's a better choice for solving roots. The Neighborhood Searching method only can find the root with error 10^{-3} and cost too many time, so this method is not our decision. The Direct Force method might be more precise but cost most time of those four methods. Because the Newton's method and secant method may be some error at the numerical differentiable, we give up using these methods.

Table 2: Compared with different integration methods.

number of partition	Simpson method	Trapezoid method	Romberg
4	3.7013×10^{-5}	6.8851×10^{-1}	5.7932×10^{-4}
8	2.3262×10^{-6}	3.4202×10^{-1}	8.5947×10^{-7}
16	1.4559×10^{-7}	1.7045×10^{-1}	3.3548×10^{-10}
32	9.1027×10^{-9}	8.5086×10^{-2}	3.2862×10^{-14}
64	5.6896×10^{-10}	4.2508×10^{-2}	4.4408×10^{-15}

This table shows that the Romberg integration has minimum error with the same partition, so the better choice are composite Simpson rule and Romberg integration. If the integration are costing too much time, we will abandon some accuracy and use composite Simpson rule.

Table 3: Compared with different degrees of $f(z) = a_0 + a_1 z^m$ and $g(z) = z^n$.

m n	1	2	3	4	5	6
1	0.710865	0.843565	0.892874	0.918579	0.934325	0.944985
2	0.710378	0.843656	0.893056	0.918792	0.934538	0.945168
3	0.709799	0.843565	0.893148	0.918884	0.934660	0.945289
4	0.709221	0.843412	0.893148	0.918944	0.934721	0.945350
5	0.708733	0.843230	0.893087	0.918944	0.934751	0.945411
10	0.707485	0.842224	0.892569	0.918731	0.934691	0.945442
15	0.707180	0.841554	0.892021	0.918366	0.934447	0.945320
20	0.707119	0.841219	0.891625	0.918000	0.934203	0.945107
25	0.707089	0.841037	0.891351	0.917726	0.933960	0.944893
30	0.707089	0.840945	0.891168	0.917513	0.933746	0.944711

From table 3, the degree of f is large and the degree of g is 1 will has the better c.

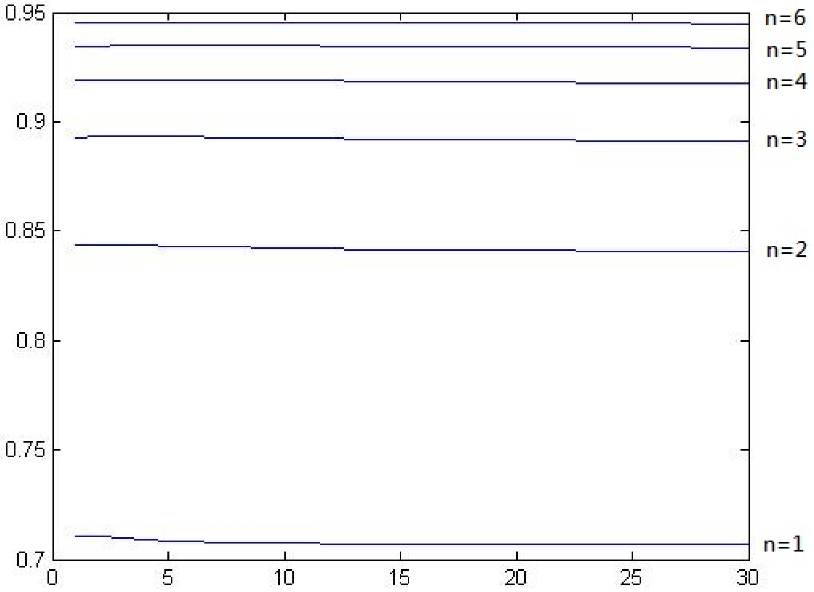

Figure 6: The change of c with different m and n.

5 Some results

5.1 Maximum modulus principle

Theorem 5.1. *Maximum modulus principle [6]*

Let $U \subseteq \mathbb{C}$ be a domain, and let f be an analytic function on U. Then if there is a point $z_0 \in U$ at which $|f|$ has a local maximum, then f is constant. Furthermore, let $U \subseteq \mathbb{C}$ be a bounded domain, and let f be a continuous function on the closed set $\overline{U}$ that is analytic on U. Then the maximum value of $|f|$ on $\overline{U}$ (which always exists) occurs on the boundary ∂U. In other words,

$$\underset{\overline{U}}{max}|f| = \underset{\partial U}{max}|f|.$$

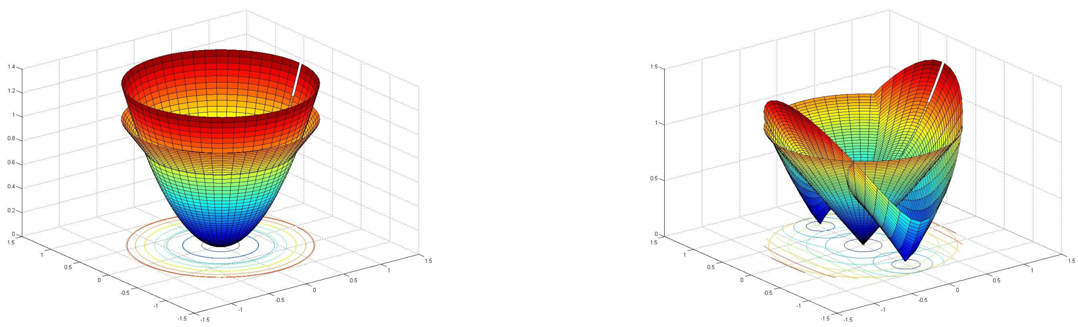

Figure 7: The left figure shows that when $|a_0|$ is closed to $\frac{1}{\sqrt{2}}$, we will have the annulus which are we wanted; the right figure shows that when $|a_0|$ is not large enough, we cannot obtain the annulus.

Considering the real-valued coefficients of polynomial function and applying the maximum modulus principle, the problem can be easier to solve. Define a new function $\gamma(z) = f(z)/g(z)$ which satisfies $|f(1)| = |g(1)|$ and $|f(c)| = |g(c)|$ for $c \in (0, 1)$ and $h(z)$ is analytic in $(c, 1)$. Applying maximum modulus principle, it's easily seen that $|h(z)| \leq 1$ (i.e. $|f(z)| \leq |g(z)|$) for $c < |z| < 1$. Adding the condition $\|f(z)\| = \|g(z)\|$, we can find the coefficients of functions to have a better value of c.

Example 5.2. *[9]*

Let

$$f(z) = a + z^n \quad g(z) = z(1 + az^n),$$

where $a = 3\sqrt{6}/11$, $n = 10$. Then $\|f\| = \|g\|$ and $|f| \leq |g|$ in $c < |z| < 1$, where $c = 0.679501...$ is the real root in $(0, 1)$ of the equation

$$\frac{3\sqrt{6}}{11} + z^{10} = z + \frac{3\sqrt{6}}{11} z^{11}.$$

The upper bound of c is $0.679501\cdots$.

Solving the equation $\|f(z)\| = \|g(z)\|$, then we have $|a|^2 + \frac{1}{11} = \frac{1}{2} + \frac{|a|^2}{12}$. the solution of a is $3\sqrt{6}/11$. Define $\gamma(z) = \underset{|z|=r}{max} \frac{|f(z)|}{|g(z)|}$ which satisfy $\gamma(c) = \gamma(1)$ for $0 < c < 1$, and the maximum modulus principle implies $|f(z)| \leq |g(z)|$ for $c < |z| < 1$. The following equations is using the same method to find its a and corresponding to c.

Example 5.3. *[9]*

Let

$$f(z)=\frac{a+z^n}{(1-az^n)^b}\quad g(z)=\frac{z(1+az^n)}{(1-az^n)^b},$$

where $0<a<1$, $b\geq 0$, $n\in\mathbb{N}$. Then $|f(z)\leq|g(z)|$ in $c<|z|<1$,where c is the real root in $(0,1)$ of the equation

$$a+z^n=z(1+az^n).$$

Moreover,when $a=0.666707$, $b=0.4768$ and $n=10$. we have $c=0.67794\cdots$. and $\|f\|>\|g\|$. The upper bound of c is $0.67794\cdots$..

Example 5.4. *[1]*

Let

$$f(z)=\frac{a+z^n}{2-az^n},\quad g(z)=\frac{z(1+az^n)}{2-az^n}$$

where $a=0.6666714$ and $n=10$. Then $\|f(z)\|>\|g(z)\|$ and $|f(z)|\leq|g(z)|$ in $c<|z|<1$, where $c=0.6779049274\cdots$ is the real root of the equation $f(z)=g(z)$. The upper bound of c is $0.6779049274\cdots$.

From above examples, we may consider that whether the Korenblum's constant c will decrease by dividing the same analytic function for $f(z)=a+z^n$ and $g(z)=z(1+az^n)$. Because c is decreasing when the coefficient a is decreasing , we use some kinds of theorems and methods to find out the minimum of a and corresponding b.

5.2 Some methods for evaluating the minimum of a

Divide $(b-az^{10})$

Starting from dividing $b-az^{10}$ at the same time, and we have following two functions

$$f(z)=\frac{a+z^{10}}{b-az^{10}}\quad \textit{and}\quad g(z)=\frac{z+az^{11}}{b-az^{10}}.$$

Since the fractional functions are hard to evaluate its L^2 norm and the polynomials can be calculated by hand by lemma 1.7, Taylor's expansion will be a good choice for us.
Define $h(z)=\frac{1}{b}+\frac{a}{b^2}z^{10}+\frac{a^2}{b^3}z^{20}+\frac{a^3}{b^4}z^{30}+\cdots$ be the infinitely terms Taylor's expansion for

$\dfrac{1}{b-az^{10}}$, and we have the new functions f(z) and g(z) by multiplying h(z) at the same time.

$$\begin{aligned}
f(z) =&(a+z^{10})h(z)\\
=&(a+z^{10})(\frac{1}{b}+\frac{a}{b^2}z^{10}+\frac{a^2}{b^3}z^{20}+\frac{a^3}{b^4}z^{30}+\cdots)\\
=&\frac{a}{b}+(\frac{a^2}{b^2}+\frac{1}{b})z^{10}+(\frac{a^3}{b^3}+\frac{a}{b^2})z^{20}+(\frac{a^4}{b^4}+\frac{a^2}{b^3})z^{30}+(\frac{a^3}{b^4})z^{40}+\cdots,
\end{aligned}$$

$$\begin{aligned}
g(z) =&(z+az^{11})h(z)\\
=&(z+az^{11})(\frac{1}{b}+\frac{a}{b^2}z^{10}+\frac{a^2}{b^3}z^{20}+\frac{a^3}{b^4}z^{30}+\cdots)\\
=&(\frac{1}{b})z+(\frac{a}{b^2}+\frac{a}{b})z^{11}+(\frac{a^2}{b^3}+\frac{a^2}{b^2})z^{21}+(\frac{a^3}{b^4}+\frac{a^3}{b^3})z^{31}+(\frac{a^4}{b^4})z^{41}+\cdots.
\end{aligned}$$

Therefore, we have

$$\|f(z)\| =(\frac{a}{b})^2+\frac{(\frac{a^2}{b^2}+\frac{1}{b})^2}{11}+\frac{(\frac{a^3}{b^3}+\frac{a}{b^2})^2}{21}+\frac{(\frac{a^4}{b^4}+\frac{a^2}{b^3})^2}{31}+\cdots,$$

$$\|g(z)\| =\frac{(\frac{1}{b})^2}{2}+\frac{(\frac{a}{b^2}+\frac{a}{b})^2}{12}+\frac{(\frac{a^2}{b^3}+\frac{a^2}{b^2})^2}{22}+\frac{(\frac{a^3}{b^4}+\frac{a^3}{b^3})^2}{32}+\cdots.$$

Next, define $F(a,b)=a$ and $G(a,b)=\|f(z)\|-\|g(z)\|$ and Lagrange's function $L(a,b,\lambda)=F(a,b)+G(a,b)$. Our destination is finding the minimum value of a on the condition $G(a,b)=0$. Applying Lagrange multiplier and we the following three equations.

$$\begin{aligned}
&\frac{\partial L(a,b,\lambda)}{\partial a}=1+\lambda\frac{\partial G(a,b)}{\partial a}=0,\\
&\frac{\partial L(a,b,\lambda)}{\partial b}=\lambda\frac{\partial G(a,b)}{\partial b}=0,\\
&G(a,b)=0.
\end{aligned}$$

Solving above three equations and we will obtain b such that a is minimum.

Finally, Let $\gamma(r)=max_{|z|=r}|\frac{f(z)}{g(z)}|=\frac{a+r^{10}}{r(1+ar^{10})}$. Then, $\gamma(c)=\gamma(1)=1$. Since $f(z)/g(z)$ is analytic in $c\leq|z|\leq 1$, the maximum modulus principle implies that $\gamma(z)\leq 1$ and then $|f(z)|\leq|g(z)|$ in $c<|z|<1$. We obtain that c is the real root of the equation $a+z^{10}=z+az^{11}$.

Compared with different expansion order

There are some results for different order of Taylor's expansion.

1. Choose $h = \frac{1}{b} + \frac{a}{b^2}z^{10}$
   When $a = 0.6666984038$ and $b = 2.047$, the Korenblum's constant c is $0.67793633153\cdots$.

2. Choose $h = \frac{1}{b} + \frac{a}{b^2}z^{10} + \frac{a^2}{b^3}z^{20}$
   When $a = 0.6666714726$ and $b = 2.028$, the Korenblum's constant c is $0.67790501184\cdots$.

3. Choose $h = \frac{1}{b} + \frac{a}{b^2}z^{10} + \frac{a^2}{b^3}z^{20} + \frac{a^3}{b^4}z^{30}$
   When $a = 0.6666704664$ and $b = 2.027$, the Korenblum's constant c is $0.67790384171\cdots$.

4. Choose $h = \frac{1}{b} + \frac{a}{b^2}z^{10} + \frac{a^2}{b^3}z^{20} + \frac{a^3}{b^4}z^{30} + \frac{a^4}{b^5}z^{40}$
   When $a = 0.6666704169$ and $b = 2.027$, the Korenblum's constant c is $0.67790378415\cdots$.

5. Choose $h = \frac{1}{b} + \frac{a}{b^2}z^{10} + \frac{a^2}{b^3}z^{20} + \frac{a^3}{b^4}z^{30} + \frac{a^4}{b^5}z^{40} + \frac{a^5}{b^6}z^{50}$
   When $a = 0.6666704141$ and $b = 2.027$, the Korenblum's constant c is $0.67790378089\cdots$.

6. Choose $h = \frac{1}{b} + \frac{a}{b^2}z^{10} + \frac{a^2}{b^3}z^{20} + \frac{a^3}{b^4}z^{30} + \frac{a^4}{b^5}z^{40} + \frac{a^5}{b^6}z^{50} + \frac{a^6}{b^7}z^{60}$
   When $a = 0.6666704139$ and $b = 2.027$, the Korenblum's constant c is $0.67790378066\cdots$.

Since the results of expansion order of $h(z)$ larger than 50 is small enough, we have the following conclusion. Let $f(z) = \frac{a + z^{10}}{b - az^{10}}$ and $g(z) = \frac{z + az^{11}}{b - az^{10}}$. where $a = 0.6666704139$ and $b = 2.027$. Then $||f(z)|| > ||g(z)||$ and $|f(z)| \leq |g(z)|$ in $c < |z| < 1$,where $c = 0.67790378066\cdots$.

Divide $(b - az^{10})^n$

Dividing $(b - az^{10})$ can decrease the value of Korenblum's constant c, so we consider the case that dividing $(b - az^{10})^n$. Applying Taylor's expansion and Lagrange multiplier with previous methods, we can have

n	a	b	korenblum's constant
1	$0.66667\underline{04139}$	2.027	$0.677\underline{90378066}\cdots$
2	$0.66666\underline{66892}$	4	$0.677\underline{89944915}\cdots$
3	$0.66666\underline{70927}$	5.987	$0.677\underline{89991839}\cdots$
4	$0.66666\underline{76043}$	7.978	$0.677\underline{90051333}\cdots$

Table 4: Compared with different coefficients a, b and n, and the corresponding of c.

Theorem 5.5. *[11]*

Suppose that $m \geq 4$ is an integer, $a \geq 0$ and $b = \sqrt{\frac{2}{(m-1)(m-2)}}$. Let

$$f(z) = \frac{a + z^m}{(1 - bz^m)^2}, \quad g(z) = \frac{z(1 + az^m)}{(1 - bz^m)^2}.$$

Then $\|f\| > \|g\|$ if and only if $a > \sqrt{\frac{m-2}{2m-2}}$.

Proof. Note that

$$\frac{1}{(1-z)^2} = \sum_{k=0}^{\infty}(k+1)z^k, z \in \mathbb{D}. \tag{6}$$

We have

$$\begin{aligned} f(z) =& a + \sum_{k=1}^{\infty}(kb^{k-1} + a(k+1)b^k)z^{mk}, \\ g(z) =& z + \sum_{k=1}^{\infty}((k+1)b^k + akb^{k-1})z^{mk+1}. \end{aligned}$$

It follows from Lemma 1.7 and equation (6) that, when $b = \sqrt{\frac{2}{(m-1)(m-2)}}$ and $a = \sqrt{\frac{m-2}{2m-2}}$,

$$\begin{aligned} &\|f\|^2 - \|g\|^2 \\ &= a^2 - \frac{1}{2} + \sum_{k=1}^{\infty}[\frac{(kb^{k-1} + a(k+1)b^k)^2}{mk+1} - \frac{((k+1)b^k + akb^{k-1})^2}{mk+2}] \\ &= a^2 - \frac{1}{2} + \sum_{k=1}^{\infty} b^{2k-2}[\frac{mk+1}{(m-1)^2} - \frac{mk+2}{2(m-1)(m-2)}] \\ &= a^2 - \frac{1}{2} + \frac{1}{2(m-1)^2(m-2)}[m(m-3)\sum_{k=1}^{\infty} kb^{2k-2} - 2\sum_{k=1}^{\infty} b^{2k-2}] \\ &= a^2 - \frac{1}{2} + \frac{1}{2(m-1)^2(m-2)}[\frac{m(m-3)}{(1-b^2)^2}] - \frac{2}{1-b^2}] \\ &= 0 \end{aligned}$$

Since $\|f\|^2 - \|g\|^2$ is an increasing function of a on $[0, \infty)$. Hence when $a > \sqrt{\frac{m-2}{2m-2}}$ we have $\|f\| > \|g\|$, and when $0 \leq a < \sqrt{\frac{m-2}{2m-2}}$ we have $\|f\| < \|g\|$.

Theorem 5.6. *[11]*

Suppose that $m \geq 4$ is an integer. Let

$$f(z) = \frac{a + z^m}{(1 - bz^m)^2}, \quad g(z) = \frac{z(1 + az^m)}{(1 - bz^m)^2}$$

where $a = \sqrt{\frac{m-2}{2m-2}}$ and $b = \sqrt{\frac{2}{(m-1)(m-2)}}$. Then $\|f\| = \|g\|$ and $|f(z)| \leq |g(z)|$ in $c < |z| < 1$, where c is the real root in (0,1) of the equation

$$a + z^m = z(1 + az^m).$$

In particular, when m=10 we have $c = 0.67789942295 \cdots$.

This c is very close to the results of table 4. Wang is using analysis method to analyze when does the L^2 norm will be equal, and giving an explicit relation between a, b and m. Our method is using Taylor series to expand the function, and the more terms we expand the more precise c we get. The value of a is tends to 2/3 and the result will be match with Wang's example.

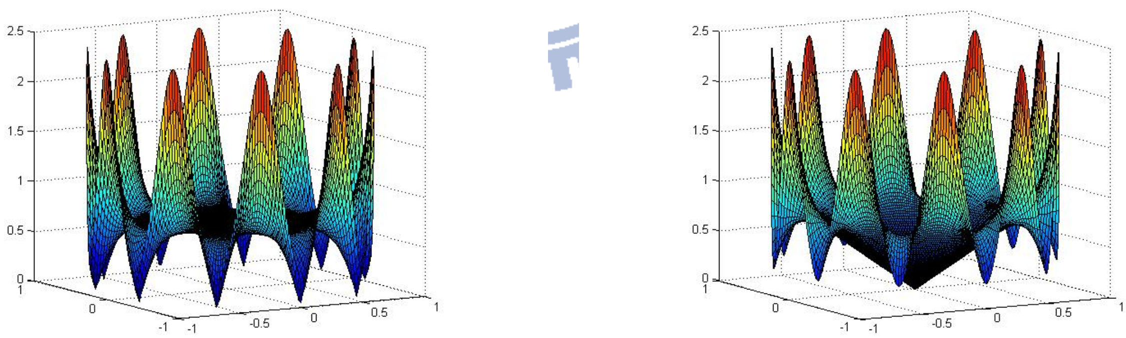

Figure 8: The left figure is $f(z)$ and the right is $g(z)$ in Theorem 5.6

6 Conclusion and future work

At first, we just try all the possible coefficients a_k of the type of functions we choose which satisfies $|f(z)| \leq |g(z)|$ and $\|f(z)\| > \|g(z)\|$. Defining $z = re^{i\theta}$, and c is the maximum solutions of $|f(z)| = |g(z)|$ for every angle θ. Suppose $f(z)$ and $g(z)$ are analytic functions on (0,1) and satisfying following three conditions.

1. $f(1) = g(1)$.

2. $f(c) = g(c)$ for $c \in (0, 1)$.

3. $f(z)/g(z)$ is analytic in $(c, 1)$.

Applying maximum modulus principle, and functions which satisfying previous two conditions can be easily proved that $|f(z)| \leq |g(z)|$ on $c \leq |z| \leq 1$, and Taylor's expansion and Lagrange multiplier may be useful for the fractional functions to find the minimum of the coefficients. We still want to known whether there exists functions $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = z^n$ such that$|f(z)| \leq |g(z)|$ for $1/\sqrt{3} < |z| < 1$, then $\|f(z)\| \leq \|g(z)\|$. We are considering that the coefficients of polynomial function f may has some rules. The geometric series or some series can be easily calculate its sum is our ideal for the sum of coefficients.

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