國 立 交 通 大 學

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碩士論文

完全圖的極大路徑填充

The maximal P_{k+1}-packings of K_n

研 究 生：徐瑩晏

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摘要

圖G的H−填充是一個蒐集一些G中邊兩兩互相不同的子圖的集合$\wp = \{H_1, H_2, \ldots, H_s\}$，其中每一個子圖$H_i$都和H同構。我們將那些沒有被$H_i$用到的邊集合所導出的子圖稱為此填充的殘留。若殘留的部分找不到一個子圖和H同構的話，則稱此填充為極大填充。每一個極大填充不一定擁有同樣多的基數。令$S(G; H)$為一蒐集圖G所有極大H−填充的基數的集合。如果圖G是一個有 n 個點的完全圖，則將$S(K_n; H)$簡化為$S(n; H)$。

在此篇論文中，我們將探討$S(n; P_{k+1})$，其中P_{k+1}是一條有$k + 1$個點的路徑。值得注意的是，一個K_n的極大P_{k+1}−填充若其基數為$S(n; P_{k+1})$中的最小值，則此填充為K_n的最小P_{k+1}−填充且其殘留恰會是一個限制子圖為P_{k+1}的極圖。反之，一個K_n的極大P_{k+1}−填充若其基數為$S(n; P_{k+1})$中的最大值，則此填充為K_n的最大P_{k+1}−填充。因此，我們有了以下結果：當$k = 3,4,5,6$時，我們確立了$S(n; P_{k+1})$。

中華民國一百年六月

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Abstract

An H-packing $\mathcal{P} = \{H_1, H_2, \ldots, H_s\}$ of a graph G is a set of edge-disjoint subgraphs of G in which each subgraph H_i is isomorphic to H. The leave L of $\mathcal{P}$ is the subgraph induced by the set of edges of G that does not occur in any H_i. $\mathcal{P}$ is a maximal H-packing if L contains no subgraph that is isomorphic to H. Let $S(G; H)$ denote the set of all possible cardinality of $\mathcal{P}$ such that $\mathcal{P}$ is a maximal H-packing of G. In case that G is the complete graph of order n, we use $S(n; H)$ to denote $S(K_n; H)$ for convenience.

In this thesis, we focus on the study of $S(n; P_{k+1})$ where P_{k+1} is a path with $k + 1$ vertices. Notice that the leave of the packing which attends $\min S(n; P_{k+1})$ is the extremal graph which forbids P_{k+1} and the packing which attends $\max S(n; P_{k+1})$ is a maximum packing of K_n with P_{k+1}'s. The main result obtained in this thesis is that we determine $S(n; P_{k+1})$ for $k = 3, 4, 5, 6$.

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Contents

List of Figures

1 Introduction and Preliminary

1.1 Motivation

The notation of "graph" was first mentioned by Euler at around 1736 in which he solved the well-known Königsberg seven bridges. But, it was until 50 years ago, "graph theory" found its importance in computer sciences. Since then, graph models were utilized in solving many discrete type problems, networking, scheduling, designs, . . . , etc. Especially, in recent decode, it was used in dealing several problems in computational molecular biology including DNA sequencing. Without a doubt, it is one of the most important branch of mathematics in 20^{th} and 21^{st} centuries.

Graph decomposition is one of the most popular topics studied in Graph Theory. Among many reasons in its applications "decomposing graphs into cliques" is most remarkable one since it is equivalent to obtaining combinatorial designs. Therefore, this topic attracts many researchers from many aspects of combinatorial theory, graph theorists, combinatorialists and also coding theorists.

1.2 Preliminaries

A graph $G = (V, E)$ consists of a vertex set $V(G)$, an edge set $E(G)$ and a relation that associates with each edge two vertices called its *endpoints*. If uv is a edge, we say u and v *adjacent*, and u (or v) *incident* to the edge. If there are more than one edge in the same pair of endpoints, these edges are called *multiple edges*. A *loop* is an edge which has the same endpoints. If a graph contains no multiple edges and loop, we call the graph a *simple graph*. All of the graphs considered in this thesis are simple graphs. For graph terminologies, we refer to [4].

The *order* of G, denoted by $|G|$, is the number of vertices of G. The *size* of G, denoted by $\|G\|$, is the number of edges of G. Consider $v \in V(G)$. The *degree* of v means the number of vertices adjacent to v. The *complement* of G is denoted by $\overline{G}$ where $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{uv \mid u, v \in V(G) \text{ and } uv \notin E(G)\}$. The *union* of the graphs G_1 and G_2 is denoted by $G_1 \cup G_2$ where

$$V(G_1 \cup G_2) = V(G_1) \cup V(G_2) \text{ and } E(G_1 \cup G_2) = E(G_1) \cup E(G_2),$$

if $E(G_1) \cap E(G_2) = \emptyset$. The union of t copies of the same graph G is denoted by G^t.

A *path* P_n is a simple graph where $V(P_n) = \{v_0, v_1, \ldots, v_{n-1}\}$ and $E(P_n) = \{v_i v_{i+1} \mid i = 0, 1, \ldots, n-2\}$. If there exists a path from u to v for all $u, v \in V(G)$, then G is *connected*. Let C be a connected subgraph of G. For all $u \in V(G) \backslash V(C)$, if we can't find $v \in V(C)$ such that there exist a path from u to v, then C is a *component* of G. If for all $u, v \in V(G)$, $uv \in E(G)$, then we say G is a *complete graph*. Let K_n denoted the complete graph of order n. A complete multigraph λK_n, is a complete graph K_n, in which every edge is taken λ times. The *complete m-partite graph* $K_{n_1,n_2,\ldots,n_m}$ is a simple graph has m partite sets V_i with order n_i, $1 \leq i \leq m$, respectively and two vertices are adjacent if and only if they are belonged to distinct partite sets. If $n_1 = n_2 = \ldots = n_m = n$ in $K_{n_1,n_2,\ldots,n_m}$, than the graph is denoted simply by $K_{m(n)}$. Let $X = \{x_0, x_1, \ldots, x_{n_1-1}\}$ and $Y = \{y_0, y_1, \ldots, y_{n_2-1}\}$ be the partite sets of K_{n_1,n_2}, and $X = \{x_0, x_1, \ldots, x_{n_1-1}\}$, $Y = \{y_0, y_1, \ldots, y_{n_2-1}\}$ and $Z = \{z_0, z_1, \ldots, z_{n_3-1}\}$ be the partite sets of K_{n_1,n_2,n_3} in later sections unless we give the other definition. The index of x, y or z will always be taken mod n_1, n_2 or n_3, respectively. The *bipartite difference* of an edge $x_i y_j$ in K_{n_1,n_2} as the value $j - i \pmod{n_2}$

If a graph H satisfies $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then we say H is a subgraph of G, denoted by $H \subseteq G$. Consider $E' \subseteq E$, an edge-induced subgraph H of G is defined by $H = (V', E')$ where $V' = \{v \in V \mid v \text{ is a endpoint of some } e \in E'\}$.

An *H-packing* of a graph G is a set $\mathcal{P} = \{H_1, H_2, \ldots, H_s\}$ such that H_i is isomorphic to H of G for $i = 1, 2, \ldots, s$ where H_i and H_j are edge-disjoint for all $i \neq j$. The *leave* L of a packing $\mathcal{P}$ is the subgraph induced by the set of edges of G that does not occur in any H_i. If L contains no edges, then G is said to be *H-decomposable*, denoted by $H \mid G$. A packing $\mathcal{P}$ is said to be *maximal* if the leave of $\mathcal{P}$ contains no subgraph that is isomorphic to H. The *size* of a packing $\mathcal{P}$, denoted by $|\mathcal{P}|$, is the cardinality of $\mathcal{P}$. $\mathcal{P}$ is a *maximum maximal packing* (or simply *maximum packing*) if $|\mathcal{P}| \geq |\mathcal{P}'|$ for all other maximal packing $\mathcal{P}'$. On the other hand, $\mathcal{P}$ is a *minimum maximal packing* (or simply *minimum packing*) if $|\mathcal{P}| \leq |\mathcal{P}'|$ for all other maximal packing $\mathcal{P}'$. The *spectrum* $S(G; H)$ denoted the set of all sizes such that there exists a maximal packing with this size. Clearly, $\max S(G;H)$ is the size of the maximum packing, and $\min S(G;H)$ is the size of the minimum packing. In case that G is the complete graph of order n, we use $S(n;H)$ to denote $S(K_n;H)$ for convenience.

1.3 Known results

The problem of path decompositions of complete graphs was first mentioned in [7]. Earlier results on this topic are on the case when the paths have same size, and such that each vertex belongs to exactly l of these paths [9, 10, 11]. Tarsi proves that if n be odd or λ even, and $M = m_1, m_2, \ldots, m_s$ a sequence of natural numbers with $m_i \leq n-3$ and $\sum m_i = \lambda\frac{n(n-1)}{2}$, then there exists a P_M-decomposition [15]. He also proves the necessary and sufficient condition for the existence of a P_m-decomposition of a λK_n is $\lambda n(n-1) \equiv 0 \ (mod\ 2m)$ and $n \geq m+1$ [15]. Recently, Bryant [3] proves that Tarsi's result is also true for any positive integers n, λ and sequence $m_1, m_2, \ldots, m_s$.

There are numerous papers written on packing problem. The maximum number of K_k-packing of K_n had solved only in these cases $k=3$ [17] and $k=4$ [1]. The maximal C_k-packing of K_n had solved only when $k=3$, $k=4$ [18] and $k=5$ [14]. Roditty proved the conjecture saying that $\max S(n;T) = \lfloor \binom{n}{2}/h \rfloor$ (h is the number of edges of T) for all trees on at most 7 vertices [12, 13]. And then Caro and Yuster proved that conjecture for any trees if $n \geq n_0(T)$ [5]. In 1990, Fu, Huang and Shiue [8] find the spectrum $S(n;S_q)$ where S_q means the star with q edges. Chen, Fu and Huang studied the $(P_3 \cup P_2)$-packing of G different from $K_{1,1,3c+1}$ with $|G| \geq 5$, $\|G\| \geq 6$ and $\delta(G) \geq 2$ [6].

In this thesis, we study the problem of packing P_{k+1} into K_n, $n \geq k+1$. In Section 2, we present the maximum and minimum size of P_{k+1}-packing and then in Section 3, we obtain our main result of the P_{k+1}-packings of K_n.

2 Maximum and minimum P_{k+1}-packing of K_n

Review that if $\mathcal{P}$ is a P_{k+1}-packing of K_n, $\mathcal{P}$ is said to be maximal if there is no path $P_{k+1} \notin \mathcal{P}$ such that $\{P_{k+1}\} \cup \mathcal{P}$ is also a packing. In this section, we find the maximum and minimum number of the element in $S(n; P_{k+1})$.

Bryant [3] showed the following theorem.

Theorem 2.1. *Let n, λ and s be positive integers and let $m_1, m_2, \ldots, m_s$ be a sequence of positive integers. there exist s pairwise edge-disjoint paths of lengths $m_1, m_2, \ldots, m_s$ in λK_n if and only if $m_i \leq n-1$ for $i = 1, 2, \ldots, s$ and $m_1 + m_2 + \ldots + m_s \leq \lambda \frac{n(n-1)}{2}$.*

From theorem 1, we have the corollary.

Corollary 2.2. *There exists a maximum (maximal) P_{k+1}-packing of K_n. Moreover, the size of the packing is $\lfloor \binom{n}{2}/k \rfloor$, i.e., $\max S(n; P_{k+1}) = \lfloor \binom{n}{2}/k \rfloor$.*

Now, we consider the minimum packing. Obviously, a minimum packing has a maximum number of edges of leave. Note that, $\mathcal{P}$ is maximal if and only if L contains no P_{k+1}. Therefore, we consider the maximum number of edges of leave which contains no P_{k+1} for the minimum P_{k+1}-packing problem. For a given graph F, $ext(n; F)$ denote the maximum number of edges of a graph of order n not containing F as a subgraph.

Lemma 2.3. [2] *If $n = tk + r$, $0 \leq r < k$, then $ext(n; P_{k+1}) = \frac{tk(k-1)}{2} + \frac{r(r-1)}{2}$. Moreover, a graph G of order n has the edge number $ext(n; P_{k+1})$ if and only if G has $t+1$ connected components where one is K_r and the others are K_k, i.e. $G = K_k^t \cup K_r$.*

We have the corollary.

Corollary 2.4. *Consider $n = tk + r$, $0 \leq r < k$. If $\mathcal{P}$ is a minimum (maximal) P_{k+1}-packing of K_n, then $|\mathcal{P}| \geq \frac{tk(t-1)}{2} + tr$.*

Proof. If L is the leave of $\mathcal{P}$, then $\|L\| \leq \frac{tk(k-1)}{2} + \frac{r(r-1)}{2}$ since $\mathcal{P}$ is maximal. The number of edges of all P_{k+1} of $\mathcal{P}$ is at least $\frac{t(t-1)k^2}{2} + tkr$, and then $|\mathcal{P}| \geq \frac{tk(t-1)}{2} + tr$. □

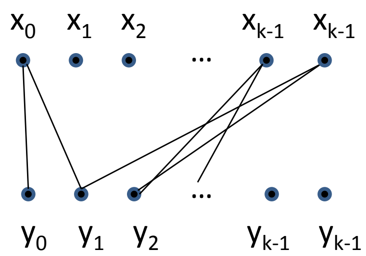

Figure 1: P_{k+1}-decomposition of $K_{k,k}$.

The following lemmas and Open problems are essential for finding the upper bound of the size of minimum packing. Following the Lemma 2.3, we consider whether exists a P_{k+1}-packing of K_n such that $L = K_k^t \cup K_r$. Hence we need to know whether $K_{k,r}, r \leq k$ has P_{k+1}-decomposition.

Lemma 2.5. *There exists a P_{k+1}-decomposition of $K_{k,k}$.*

Proof. For $0 \leq i \leq k-1$, let

$$p_i = \begin{cases} y_i x_i y_{i+1} x_{i-1} \cdots y_{i+(\frac{k+1}{2}-1)} x_{i-(\frac{k+1}{2}-1)}, & \text{if } k \text{ is odd; and} \\ y_i x_i y_{i+1} x_{i-1} \cdots y_{i+(\frac{k+2}{2}-2)} x_{i-(\frac{k+2}{2}-2)} y_{i+(\frac{k+2}{2}-1)}, & \text{if } k \text{ is even.} \end{cases}$$

(see Fig. 1). By the fact that all edges of p_i receive different bipartite labeling, $p_1, p_2, \ldots p_{k-1}$ are edge-disjoint paths of length k. Let $\mathcal{P} = \{p_i \mid 0 \leq i \leq k-1\}$, $\mathcal{P}$ is a P_{k+1}-decomposition of $K_{k,k}$. □

Lemma 2.6. *There exists no P_{k+1}-decomposition of $K_{k,r}$ if $1 \leq r < \lceil \frac{k}{2} \rceil$ or both k and r are odd.*

Proof. It is clear for $r < \lceil \frac{k}{2} \rceil$. Each P_{k+1} has one of its end vertices in X and the other one in Y when k is odd. Since all k vertices of X have odd degree r and there are only r paths in decomposing $K_{k,r}$ into P_{k+1}'s, we are done. □

Lemma 2.7. *There exists a P_{k+1}-decomposition of $K_{k,k,r}$ if $1 \leq r < \lceil \frac{k}{2} \rceil$.*

Proof. The P_{k+1}-decomposition of $K_{k,k,r}$ is obtained from the decomposition of $K_{k,k}$. Take a subgraph $K_{k,k}$ of $K_{k,k,r}$, $\mathcal{P} = \{p_i \mid 0 \leq i \leq k-1\}$ is a decomposition of $K_{k,k}$ where p_i is defined in Lemma 2.5. For each p_i, $0 \leq i \leq k-1$, delete the

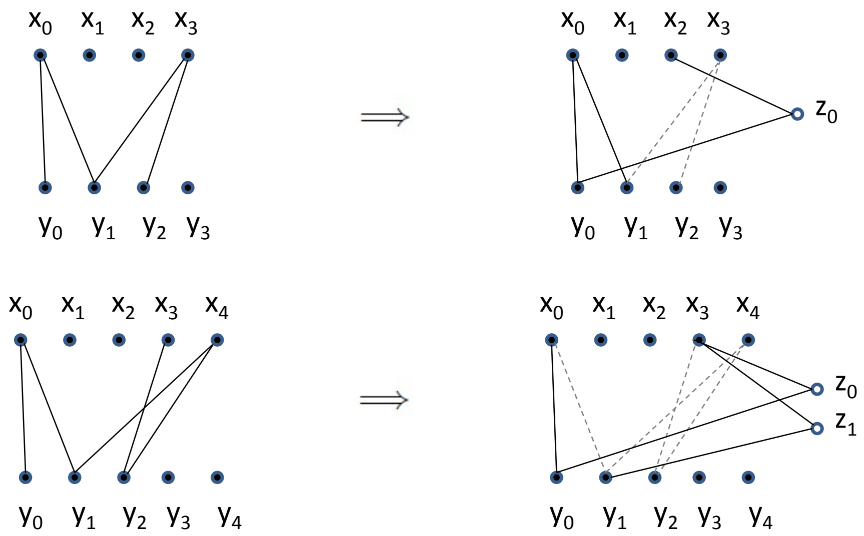

Figure 2: P_5-decomposition of $K_{4,4,1}$ and P_6-decomposition of $K_{5,5,2}$.

last $2r$ edges and combine with the path having length $2r$ in $K_{k,k,r}$ as following if k is odd :

$$\begin{cases} y_{i+(\frac{k+1}{2}-r)}z_{r-1}x_{i-(\frac{k+1}{2}-(r-1))}z_{r-2}\cdots y_{i+(\frac{k+1}{2}-2)}z_1x_{i-(\frac{k+1}{2}-1)}z_0y_i, & \text{if } r \text{ is even;}\\ x_{i-(\frac{k+1}{2}-r)}z_{r-1}y_{i+(\frac{k+1}{2}-(r-1))}z_{r-2}\cdots y_{i+(\frac{k+1}{2}-2)}z_1x_{i-(\frac{k+1}{2}-1)}z_0y_i, & \text{if } r \text{ is odd,}\end{cases}$$

or as following if k is even :

$$\begin{cases} y_{i+(\frac{k}{2}-r+1)}z_{r-1}x_{i-(\frac{k}{2}-r+1)}z_{r-2}\cdots y_{i+(\frac{k}{2}-1)}z_1x_{i-(\frac{k}{2}-1)}z_0y_i, & \text{if } r \text{ is even;}\\ x_{i-(\frac{k}{2}-r)}z_{r-1}y_{i+(\frac{k}{2}-(r-2))}\cup & \\ \quad y_{i+(\frac{k}{2}-(r-2))}z_{r-2}x_{i-(\frac{k}{2}-(r-2))}z_{r-3}\cdots y_{i+(\frac{k}{2}-1)}z_1x_{i-(\frac{k}{2}-1)}z_0y_i, & \text{if } r \text{ is odd.}\end{cases}$$

(There are examples for $k=4, r=1$ and $k=5, r=2$ in Fig. 2.) Let p_i' denote the new paths obtain from p_i for $0 \le i \le k-1$. Clearly, the edges both in p_i' and p_i be used only once since p_i is a decomposition. The edges adjacent to z_j, $0 \le j \le r-1$, are all used and only once. Next we consider the edges deleted above the edge set of those edges is

$$\{x_{i-(\frac{k+1}{2}-(j+1))}y_{i+(\frac{k+1}{2}-j)}, y_{i+(\frac{k+1}{2}-j)}x_{i-(\frac{k+1}{2}-j)} \mid 0 \le i \le k-1, 1 \le j \le r\}$$

if k is even or is

$$\{y_{i+(\frac{k}{2}-j)}x_{i-(\frac{k}{2}-j)}, x_{i-(\frac{k}{2}-j)}y_{i+(\frac{k}{2}-(j-1))} \mid 0 \le i \le k-1, 1 \le j \le r\}$$

if k is odd. Since $x_{i-(\frac{k+1}{2}-(j+1))} = x_{i+1-(\frac{k+1}{2}-j)}$ and $y_{i+(\frac{k}{2}-(j-1))} = y_{i+1+(\frac{k}{2}-j)}$ for $0 \le i \le k-1$, the edge set contains r edge-disjoint Hamilton cycles of $K_{k,k}$. For

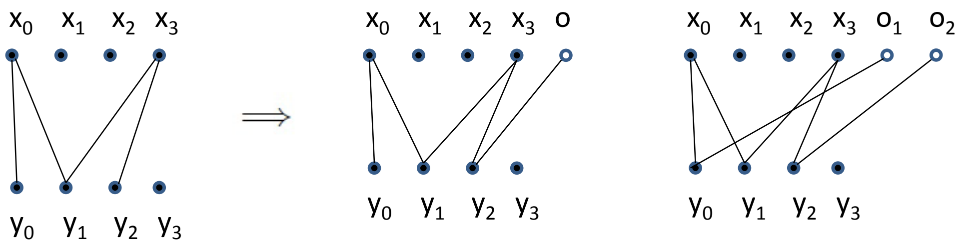

Figure 3: P_5-decomposition of $K_{5,4}$ and $K_{6,4}$.

each Hamilton cycle, we can decomposition it into two path of length k, and so we have $2r$'s P_{k+1}. Let $\mathcal{P}'$ be the set of all these paths and p_i' for $0 \leq i \leq k-1$, then $\mathcal{P}'$ is a P_{k+1}-decomposition of $K_{k,k,r}$. □

Open problem 1. Is $K_{k,k+r}$ P_{k+1}-decomposable if $1 \leq r < \lceil \frac{k}{2} \rceil$?

Open problem 2. Is $K_{k,k,r}$ P_{k+1}-decomposable if $\lceil \frac{k}{2} \rceil \leq r < k$ and both k and r are odd?

Open problem 3. Is $K_{k,r}$ P_{k+1}-decomposable if $\lceil \frac{k}{2} \rceil \leq r < k$ and at least one of k, r is even?

Remark. In Truszczynski's paper [16], he verified that $K_{k,r}$ can decomposed into P_{k+1}'s if $r \geq \lceil \frac{k}{2} \rceil$ and r is even. Hence the only unknown case in Open problem 3 is that k is even and r is odd. Note that the case k, r odd is not possible (Lemma 2.6).

When $r = k-1$ or $r = k-2$, the Open problem 3 can construct direct from the P_k-decomposition of $K_{k-1,k-1}$. (The figure 3 give the example for the construction of $K_{5,4}$ and $K_{6,4}$.) If $r = k-1$, let the vertex classes of $K_{k,r}$ (i.e. $K_{k,k-1}$) as follow:

$$X = \{x_0, x_1, \ldots, x_{k-2}, o\},\ Y = \{y_0, y_1, \ldots, y_{k-2}\} \text{ if } k \text{ is odd.}$$

$$X = \{x_0, x_1, \ldots, x_{k-2}\},\ Y = \{y_0, y_1, \ldots, y_{k-2}, o\} \text{ if } k \text{ is even.}$$

From Lemma 2.5, the P_k-decomposition of $K_{k-1,k-1}$ is $\{p_i \mid 0 \leq i \leq k-2\}$ where p_i defined as in Lemma 2.5. Let $p_i' = p_i \cup \{y_{i+(\frac{k+1}{2}-1)}o\}$ when k is odd and $p_i' = p_i \cup \{x_{i-(\frac{k}{2}-1)}o\}$ when k is even for $0 \leq i \leq k-2$. Then p_i' is a P_{k+1}-decomposition of $K_{k,k-1}$.

If $r = k-2$, both k and r are even and let the vertex classes of $K_{k,r}$ (i.e. $K_{k,k-2}$) be

$$X = \{x_0, x_1, \ldots, x_{k-3}, o_1, o_2\},\ Y = \{y_0, y_1, \ldots, y_{k-3}\}.$$

The P_{k-1}-decomposition of $K_{k-2,k-2}$ is $\{p_i \mid 0 \leq i \leq k-3\}$ where p_i defined as in Lemma 2.5, and then $\{\{o_1y_i\} \cup p_i \cup \{y_{i+(\frac{k}{2}-1)}o_2\} \mid 0 \leq i \leq k-3\}$ is a P_{k+1}-decomposition of $K_{k,k-2}$.

Now, we have a lemma about the size of minimum packing for some small n.

Corollary 2.8. *If $n = k + r$, $r < k$, and either $1 \leq r < \lceil \frac{k}{2} \rceil$ or both k and r are odd, there is no minimum (maximal) packing of K_n which has size r.*

Proof. From Lemma 2.6, we know that there is no P_{k+1}-decomposition of $K_{k,r}$. Therefore, we can't have a packing with the leave $K_k \cup K_r$ and then there is no packing with size $(\frac{(k+r)(k+r-1)}{2} - \frac{k(k-1)}{2} - \frac{r(r-1)}{2})/k = r$. □

3 The Spectrum of the maximal P_{k+1}-packing of K_n

In this section, we consider all the size of maximal P_{k+1}-packings of K_n where $k \le 6$. From Corollary 2.2, we have $\max S(n; P_{k+1}) = \lfloor \binom{n}{2}/k \rfloor$. Now, we consider the $\min S(n; P_{k+1})$.

We give a direct construction to prove the Open problem 1, 2 and 3 are right for $k \le 6$, and then have a minimum P_{k+1}-packing of K_n.

Lemma 3.1. *The Open problem 1, 2 and 3 are true for $k \le 6$. That is,*

1. *There is a P_{k+1}-decomposition of $K_{k,k+r}$ if $1 \le r < \lceil \frac{k}{2} \rceil$.*

2. *There is a P_{k+1}-decomposition of $K_{k,k,r}$ if $\lceil \frac{k}{2} \rceil \le r < k$ and both k and r are odd.*

3. *There is a P_{k+1}-decomposition of $K_{k,r}$ if $\lceil \frac{k}{2} \rceil \le r < k$ and at least one of k, r is even.*

Proof. Consider the cases $k = 3, 4, 5$ and 6, respectively.

Case 1. $k = 3$

The Open problem 2 can't occur in these case. When $k = 3$, there are only $K_{3,2}$ and $K_{3,4}$ be considered in Open problem 3 and Open problem 1, respectively. From the Remark of Open problem 3, we can construct a P_4-decomposition of $K_{3,2}$ direct. Decompose $K_{3,4}$ by two $K_{3,2}$'s, then we are done by the P_4-decomposition of $K_{3,2}$.

Case 2. $k = 4$

The Open problem 2 can't occur in the case. In Open problem 3, we only consider the graphs $K_{4,2}$ and $K_{4,3}$. We have a direct construction of P_5-decomposition of $K_{4,2}$ and $K_{4,3}$ from the Remark of Open problem 3. $K_{4,5}$ is the only case in Open problem 1. Since $K_{4,5}$ can decomposed to two graphs $K_{4,3}$ and $K_{4,2}$, we are done.

Case 3. $k = 5$

When $k = 5$, the graph in Open problem 2 is only $K_{5,5,3}$. Let

$$\begin{aligned}\mathcal{P} = &\{j_0z_0j_1z_2j_2z_1, j_2z_0j_3z_1j_4z_2 \mid j = x, y\}\\ &\cup \{y_1x_1z_1x_0z_2x_3, x_1y_2x_0y_3x_4z_0\}\\ &\cup \{y_ix_iy_{i+1}x_{i-1}y_{i+2}x_{i-2} \mid i = 0, 2, 3\}\\ &\cup \{x_2y_1z_1y_0z_2y_3, z_0y_4x_4y_0x_3y_1\},\end{aligned}$$

then $\mathcal{P}$ is a P_6-decomposition of $K_{5,5,3}$. Consider $K_{5,4}$, the only graph in Open problem 3, has the P_6-decomposition by the Remark of the Open problem 3. There are two kind of graphs, $K_{5,6}$ and $K_{5,7}$, in Open problem 1 when $k = 5$. Let $p_1 = x_1y_0x_3y_1x_0y_2$, $p_2 = x_1y_3x_3y_4x_2y_5$, $p_3 = x_3y_2x_1y_1x_4y_0$, $p_4 = x_3y_5x_1y_4x_4y_3$, $p_5 = x_4y_2x_2y_0x_0y_4$ and $p_6 = x_4y_5x_0y_3x_2y_1$, then $\{p_i \mid 1 \le i \le 6\}$ is a P_6-decomposition of $K_{5,6}$. Let $p'_i = y_ix_iy_{i+1}x_{i-1}y_{i+2}x_4$, $1 \le i \le 3$, $p'_4 = x_3y_4x_1y_5x_2y_6$, $p'_5 = x_1y_6x_3y_5x_0y_4$, $p'_6 = x_0y_1x_3y_2x_4y_5$ and $p'_7 = x_2y_4x_4y_6x_0y_0$. Then $\{p'_i \mid 1 \le i \le 7\}$ is a P_6-decomposition of $K_{5,7}$.

Case 4. $k = 6$

There is no graph in Open problem 2 and, we have three graphs $K_{6,3}$, $K_{6,4}$ and $K_{6,5}$ in Open problem 3 when $k = 6$. Clearly,

$$\{x_0y_0x_1y_2x_5y_1x_3, x_1y_1x_2y_0x_3y_2x_4, x_2y_2x_0y_1x_4y_0x_5\}$$

is a P_7-decomposition of $K_{6,3}$. $K_{6,4}$ and $K_{6,5}$ have the P_7-decomposition from the Remark of Open problem 3. Final, the graphs in Open problem 1 are $K_{6,7}$ and $K_{6,8}$. Similar to case 2, $K_{6,7}$ and $K_{6,8}$ have the P_7-decomposition by use P_7-decomposition of $K_{6,3}$ and $K_{6,4}$. □

Accordingly we have the theorem.

Theorem 3.2. *Consider $k \le 6$, $n = tk + r \ge k + 1$, $0 \le r < k$. There exist a minimum (maximal) P_{k+1}-packing $\mathcal{P}$ of K_n which with size*

$$|\mathcal{P}| = \begin{cases} r + 1, & \text{if } t = 1 \text{ and either } 1 \le r < \lceil \frac{k}{2} \rceil \text{ or } k \text{ and } r \text{ are odd;} \\ \frac{tk(t-1)}{2} + tr, & \text{otherwise.} \end{cases}$$

Proof. Let L denote the leave of $\mathcal{P}$. Note that, if we can find a minimum packing such that the leave with size $\frac{tk(k-1)}{2} + \frac{r(r-1)}{2}$, then $|\mathcal{P}| = (\frac{(tk+r)(tk+r-1)}{2} - \frac{tk(k-1)}{2} - \frac{r(r-1)}{2})/k = \frac{tk(t-1)}{2} + tr$.

First, consider $t = 1$ (i.e. $n = k + r$) and either $1 \leq r < \lceil \frac{k}{2} \rceil$ or both k and r are odd. $\frac{tk(t-1)}{2} + tr = r$ when $t = 1$. We have $|\mathcal{P}| \neq r$ from Corollary 2.8, and then $|\mathcal{P}| > r$ by Corollary 2.4. Consider $n = 4$ when $k = 3$, $n = 5$ when $k = 4$, $n = 6, 7, 8$ when $k = 5$ and $n = 7, 8$ when $k = 6$.

If we take one P_4 in K_4 (i.e. $r = 1$), the other edges is also form a P_4 clearly. So K_4 has a minimum P_4-packing of size 2 (i.e. $r + 1$). We know that the size of the maximal P_5-packing of K_5 more than $r = 1$, and let $\{a_i \mid 0 \leq i \leq 4\}$ be the vertex set of K_5. Since $\{a_0a_1a_2a_3a_4, a_0a_2a_4a_1a_3\}$ is a minimum P_5-packing of K_5 with leave P_3, we have a packing with size $r + 1 = 2$.

When $k = 5$, denote the vertex sets of K_6 (i.e. $r = 1$), K_7 (i.e. $r = 2$) and K_8 (i.e. $r = 3$) by $\{a_i \mid 0 \leq i \leq 5\}$, $\{b_i \mid 0 \leq i \leq 6\}$ and $\{c_i \mid 0 \leq i \leq 7\}$, respectively. $\{a_5a_0a_1a_3a_4a_2, a_2a_1a_5a_4a_0a_3\}$ is a minimum P_6-packing of K_6, $\{b_1b_5b_6b_4b_0b_3, b_3b_6b_2b_0b_1b_4, b_6b_1b_2b_3b_4b_5\}$ is a minimum P_6-packing of K_7, and $\{c_1c_4c_2c_5c_3c_6, c_2c_1c_3c_7c_4c_0, c_3c_0c_5c_1c_6c_7, c_4c_6c_0c_2c_7c_5\}$ is a minimum P_6-packing of K_8. The size of these packing is $r + 1$.

Consider K_7 and K_8 with the vertex sets $\{a_i \mid 0 \leq i \leq 6\}$ and $\{b_i \mid 0 \leq i \leq 7\}$, respectively. Since $\{a_2a_1a_3a_6a_4a_0a_5, a_3a_0a_2a_6a_5a_1a_4\}$ is a minimum P_7-packing of K_7 and $\{b_1b_0b_2b_7b_3b_4b_5, b_1b_6b_5b_7b_4b_0b_3, b_3b_2b_1b_7b_6b_0b_5\}$ are a minimum P_7-packing of K_8, we have packing with size $r + 1$. From above, we have $|\mathcal{P}| = r + 1$ when $n = k + r$ and either $1 \leq r < \lceil \frac{k}{2} \rceil$ or both k and r are odd.

Final, consider the other n. We have the proof by induction on n. If $n = k + r$ where $k > r \geq \lceil \frac{k}{2} \rceil$ and at least one of k, r is even, then there is a P_{k+1}-decomposition of $K_{k,r}$ from Lemma 3.1. In other words, we have a packing $\mathcal{P}$ with leave $L = K_k \cup K_k \cup K_r$ and $|\mathcal{P}| = \frac{2k(2-1)}{2} + 2r$. If $n = 2k$, we have a packing $\mathcal{P}$ with leave $L = K_k \cup K_k$ from Lemma 2.5 and then $|\mathcal{P}| = \frac{2k(2-1)}{2}$. If $n = 2k + r$, $1 \leq r < \lceil \frac{k}{2} \rceil$, we have a P_{k+1}-decomposition of $K_{k,k,r}$ from Lemma 2.7. That is, there is a minimum packing $\mathcal{P}$ with leave $L = K_k \cup K_k \cup K_r$ and then $|\mathcal{P}| = \frac{2k(2-1)}{2} + 2r$. If $n = 2k + r$ where $k > r \geq \lceil \frac{k}{2} \rceil$ and both k and r are odd, we have a P_{k+1}-decomposition of $K_{k,k,r}$ from Lemma 3.1. Accordingly, there is minimum packing $\mathcal{P}$ with leave $L = K_k \cup K_k \cup K_r$. In this case, $|\mathcal{P}| = \frac{2k(2-1)}{2} + 2r$.

Suppose there exist a minimum packing of the complete graph with order smaller than n and the leave of the packing is $K_k \cup \ldots \cup K_k \cup K_r$. Let G be the complete graph with order n, there are four cases to be considered.

Case 1. $n = tk + r$ where $t \geq 3$ and $1 \leq r < \lceil \frac{k}{2} \rceil$.

Let $G' = G \backslash \{v_1, v_2, \ldots, v_k\}$, then G' is a complete graph with order $(t-1)k+r$. Since $(t-1)k + r < n$, by induction hypothesis, there exists a packing $\mathcal{P}'$ of G' with leave L' where $L' = K_k^{t-1} \cup K_r$ and $|\mathcal{P}'| = \frac{(t-1)k(t-2)}{2} + (t-1)r$. Obviously $\mathcal{P}'$ is also a packing of G. The leave of $\mathcal{P}'$ in G has edges $E(L') \cup \{uv_i | u \in V(G'), i = 1, 2, \ldots, k\} \cup \{v_i v_j | 1 \leq i < j \leq k\}$. Consider $K_{(t-1)k+r,k} = K_{k,k}^{t-2} \cup K_{k+r,k}$, $K_{(t-1)k+r,k}$ can decomposed to $(t-1)k + r$'s edge-disjoint paths of length k by Lemma 2.5 and Lemma 3.1. Let $\mathcal{P}''$ be the set of these paths and $\mathcal{P} = \mathcal{P}' \cup \mathcal{P}''$, then $\mathcal{P}$ is a P_{k+1}-packing of G with leave $L = L' \cup K_k$. Clearly, $\mathcal{P}$ is a minimal P_{k+1}-packing of K_n. Note that, $|\mathcal{P}| = \frac{(t-1)k(t-2)}{2} + (t-1)r + (t-1)k + r = \frac{tk(t-1)}{2} + tr$.

Case 2. $n = tk$ where $t \geq 3$.

The proof of this case is similar to case 1 (let $r = 0$) by use Lemma 2.5.

Case 3. $n = tk + r$ where $t \geq 3$, $k > r \geq \lceil \frac{k}{2} \rceil$ and both k and r are odd.

The case can be show as case 1. Note that, we consider $K_{(t-1)k+r,k} = K_{k,k}^{t-2} \cup K_{k-1,k} \cup K_{r+1,k}$ in this case. Since k and r are odd, $r \leq k-2$, there exist P_{k+1}-decompositions of $K_{k-1,k}$ and $K_{r+1,k}$ by Lemma 3.1. The graph $K_{k,k}$ has P_{k+1}-decomposition from Lemma 2.5.

Case 4. $n = tk + r$ where $t \geq 2$, $k > r \geq \lceil \frac{k}{2} \rceil$ and at least one of k, r is even.

The idea of case 4 is also like case 1. But in this case, we replaced $K_{(t-1)k+r,k}$ by $K_{k,k}^{t-1} \cup K_{r,k}$ and to complete it by Lemma 2.5 and Lemma 3.1.

Therefore, the proof concludes by mathematical induction. □

In our study of the spectrum $S(n; P_k)$, there are some special technique. The main technique used in the section need switching some edges of the paths in a given maximal P_{k+1}-packing of K_n with size s and the edges in the leave of the packing to produce a new P_{k+1}-packing. The goal is causing the new leave contains one only path of length k and add the path to the new P_k-packing, then we have a new maximal P_{k+1}-packing of K_n with size $s + 1$.

Let $A = \{n = tk + r \mid \text{either } r \geq \lceil \frac{k}{2} \rceil \text{and one of } k, r \text{ is even or } t > 1\}$. If $n \in A$, then there exist a minimum P_{k+1}-packing $\mathcal{P}$ of K_n with the construction as in Theorem 3.2. Note that the leave is $L = K_k^t \cup K_r$, that is, the graph induce by the packing from K_n is $K_{t(k),r}$. The way of the edge switching as the following step:

Step 1. Consider a subgraph H as $K_{k,r}$ or $K_{k,k,r}$, $0 \le r < k$, of $K_{k,\ldots,k,r}$ at a time.

Step 2. Take one or two paths from $\mathcal{P}$ which is also contain in H. Let $\mathcal{P}_1$ denote the set of these paths.

Step 3. Choose k edges from L, and rearranging the k edges and the paths took in step 2 to produce a new P_{k+1}-packing $\mathcal{P}_2$.

Step 4. Let $\mathcal{P}' = \mathcal{P} \setminus \mathcal{P}_1 \cup \mathcal{P}_2$. Since L contains no P_{k+1}, the leave of $\mathcal{P}'$ is also contains no P_{k+1}. Hence $\mathcal{P}'$ is a maximal P_{k+1}-packing of K_n with size $|\mathcal{P}|+1$.

Step 5. Repeat step 1 to 4, then we can have a maximal P_{k+1}-packing of K_n with desired size s until $s = \lfloor \binom{n}{2}/k \rfloor$.

Afterward we study the spectrum $S(n; P_k)$, $n = tk + r \ge k + 1$, $0 \le r < k$ for $k \le 6$. In our way, the subgraph $K_{k,r}$ or $K_{k,k,r}$ be considered only in first took and only took $K_{k,k}$ later in step 1. Note that, all subgraphs considered are edge disjoint. We only need verify the step 3 can always operate, then the following Lemmas are done.

Lemma 3.3. *Consider $n = 3t + r \ge 4$, $0 \le r < 3$.*

1. $S(n; P_4) = \{s \mid r + 1 \le s \le \lfloor \binom{n}{2}/3 \rfloor\} = \{2\}$ *if* $n = 4$.
2. $S(n; P_4) = \{s \mid \frac{3t(t-1)}{2} + tr \le s \le \lfloor \binom{n}{2}/3 \rfloor\}$ *if* $n \ge 5$.

Proof. First, consider $n = 4$. Since $r = 1$ and $\lfloor \binom{n}{2}/3 \rfloor = 2 = r + 1$ is the size of minimum P_4-packing of K_4, we are done.

Second, consider $n \ge 5$. We have a minimum P_4-packing $\mathcal{P}$ of K_n with leave L as defined in the Theorem 3.2. Study the three cases.

Case 1. $n \equiv 0 \pmod 3$.

Clearly, $K_{3,3}$ is a subgraph of K_n. Recall that, $y_0x_0y_1x_2$ and $y_2x_2y_0x_1$ are paths of $\mathcal{P}$. Let $\mathcal{P}' = \mathcal{P} \setminus \{y_0x_0y_1x_2\} \cup \{x_1x_0y_1x_2, y_0x_0x_2x_1\}$, then $\mathcal{P}'$ is a maximal P_4-packing of K_n with leave L'. Note that $y_2x_2y_0x_1$ is still in $\mathcal{P}'$ and there are three edges y_0y_1, y_1y_2 and y_2y_0 which in $E(\overline{K_{3,3}})$ not be used. Let $\mathcal{P}'' = \mathcal{P}' \setminus \{y_2x_2y_0x_1\} \cup \{x_2y_2y_0x_1, x_2y_0y_1y_2\}$, then $\mathcal{P}''$ is a maximal P_4-packing of K_n with leave L''. If $n = 6$, there is no edge in L'' and then $|\mathcal{P}''| = |\mathcal{P}'| + 1 = |\mathcal{P}| + 2 = 5 = \lfloor \binom{n}{2}/3 \rfloor$.

Reuse the subgraph $K_{3,3}$ as well as the way of the edge switching when $n \geq 9$, then we have the maximal P_4-packing with desired size until the size of leave less than k.

Case 2. $n \equiv 2 \pmod 3$.

Consider $K_{3,2} \subseteq K_n$ with vertex classes $X = \{x_0, x_1, o\}$ and $Y = \{y_0, y_1\}$. Clerly, $y_0x_0y_1o$ is a path of $\mathcal{P}$. Let $\mathcal{P}' = \mathcal{P} \setminus \{y_0x_0y_1o\} \cup \{x_1x_0y_1o, y_0x_0ox_1\}$, then $\mathcal{P}'$ is a maximal P_4-packing of K_n with leave L'. If $n = 5$, the leave L' only contain edge $\{y_0y_1\}$ and then $|\mathcal{P}'| = |\mathcal{P}| + 1 = 3 = \lfloor \binom{n}{2}/3 \rfloor$. If $n \geq 8$, we can use the subgraph $K_{3,2}$ and $K_{3,3}$ (see case 1) to produce a maximal P_4-packing with desired size until the size of leave less than k.

Case 3. $n \equiv 1 \pmod 3$.

Since $n \geq 5$ (i.e. $n \geq 7$ in these case), we have $K_{3,3,1}$ is a subgraph of K_n. We have $x_2z_0y_0x_0$ and $x_1z_0y_2x_2$ are paths of $\mathcal{P}$. Let $\mathcal{P}' = \mathcal{P} \setminus \{x_2z_0y_0x_0\} \cup \{x_2x_0y_0z_0, x_0x_1x_2z_0\}$, then $\mathcal{P}'$ is a maximal P_4-packing of K_n with leave L'. Note that $x_1z_0y_2x_2$ is still in $\mathcal{P}'$ and there are three edges y_0y_1, y_1y_2 and y_2y_0 which in $E(\overline{K_{3,3,1}})$ not be used. We will let the three edges be used. Consider $\mathcal{P}'' = \mathcal{P}' \setminus \{x_1z_0y_2x_2\} \cup \{x_1z_0y_2y_0, y_0y_1y_2x_2\}$, then $\mathcal{P}''$ is a maximal P_4-packing of K_n with leave L''. If $n = 7$, we are done. If $n \geq 10$, we can produce a maximal P_4-packing with desired size by the subgraph $K_{3,3,1}$ and $K_{3,3}$ (see case 1) until the size of leave less than k. □

Lemma 3.4. *Consider $n = 4t + r \geq 5$, $0 \leq r < 4$.*

1. $S(n; P_5) = \{s \mid r + 1 \leq s \leq \lfloor \binom{n}{2}/4 \rfloor\} = \{2\}$ *if* $n = 5$.

2. $S(n; P_5) = \{s \mid \frac{4t(t-1)}{2} + tr \leq s \leq \lfloor \binom{n}{2}/4 \rfloor\}$ *if* $n \geq 6$.

Proof. When $n = 5$, we have a minimum P_5-packing of K_5 with size $r + 1 = \lfloor \binom{n}{2}/4 \rfloor\} = \{2\}$ from Theorem 3.2.

If $n \geq 5$, there is a minimum P_5-packing $\mathcal{P}$ of K_n with leave L as defined in the Theorem 3.2. Consider the four cases.

Case 1. $n \equiv 0 \pmod 4$.

Take the subgraph $K_{4,4}$ of K_n. Clearly, $\{y_ix_iy_{i+1}x_{i-1}y_{i+2} \mid 0 \leq i \leq 3\} \subseteq \mathcal{P}$ is a packing of $K_{4,4}$. Let $\mathcal{P}' = \mathcal{P} \setminus \{y_0x_0y_1x_3y_2\} \cup \{x_1x_0y_1x_3y_2, y_0x_0x_3x_2x_1\}$, $\mathcal{P}'' = \mathcal{P}' \setminus \{y_1x_1y_2x_0y_3, y_2x_2y_3x_1y_0\} \cup \{x_3x_1y_2x_0y_3, x_2y_2y_0x_1y_3, x_0x_2y_3y_1x_1\}$ and $\mathcal{P}''' = \mathcal{P}'' \setminus$

$\{y_3x_3y_0x_2y_1\} \cup \{x_3y_3y_0x_2y_1, x_3y_0y_1y_2y_3\}$, then $\mathcal{P}'$, $\mathcal{P}''$ and $\mathcal{P}'''$ are maximal P_5-packing of K_n. If $n = 8$, there is no edge in L'' and then we have the packings with size from $\frac{4t(t-1)}{2} + tr = 4$ to $\lfloor \binom{n}{2}/4 \rfloor = 7$. If $n \geq 12$, we can use the subgraph $K_{4,4}$ to produce a maximal P_5-packing with desired size until the size of leave less than k.

Case 2. $n \equiv 3 \pmod 4$.

Consider the subgraph $K_{4,3}$ of K_n with the vertex classes $X = \{x_0, x_1, x_2\}$ and $Y = \{y_0, y_1, y_2, o\}$. $y_0x_0y_1x_2o, y_1x_1y_2x_0o$ are paths of $K_{4,3}$. Let $\mathcal{P}' = \mathcal{P} \setminus \{y_0x_0y_1x_2o\} \cup \{oy_0x_0y_1x_2, y_0y_1y_2ox_2\}$, then $\mathcal{P}'$ is a maximal P_5-packing of K_n. Let $\mathcal{P}'' = \mathcal{P}' \setminus \{y_1x_1y_2x_0o\} \cup \{x_0oy_1x_1y_2, y_0y_2x_0x_1x_2\}$, then $\mathcal{P}''$ is a maximal P_5-packing of K_n with leave L''. Note that there are only one edge which in $E(\overline{K_{4,3}})$ not be used. If $n = 7$, we are done. If $n \geq 11$, we can produce a maximal P_5-packing with desired size by the subgraph $K_{4,3}$ and $K_{4,4}$ (see case 1) until the size of leave less than k.

Case 3. $n \equiv 2 \pmod 4$.

$K_{4,2}$ is the subgraph of K_n with vertex classes $X = \{x_0, x_1, o_1, o_2\}$ and $Y = \{y_0, y_1\}$. Recall that $\mathcal{P}$ contains a path $o_1y_0x_0y_1o_2$. Let $\mathcal{P}' = \mathcal{P} \setminus \{o_1y_0x_0y_1o_2\} \cup \{o_2o_1y_0x_0y_1, y_1o_2x_0x_1o_1\}$, then $\mathcal{P}'$ is a maximal P_5-packing of K_n with leave L'. If $n = 6$, we are done since $E(L') = \{x_0o_1, x_1o_2, y_0y_1\}$. Reuse the edge switching of subgraph $K_{4,2}$ and $K_{4,4}$ (see case 1) when $n \geq 10$, then we have the maximal P_5-packing with desired size until the size of leave less than k.

Case 4. $n \equiv 1 \pmod 4$.

In the case, $K_{4,4,1}$ is a subgraph of K_n. Recall that $\{x_{i-1}z_0y_ix_iy_{i+1} \mid 0 \leq i \leq 3\} \subseteq \mathcal{P}$.Let

$$\mathcal{P}' = \mathcal{P} \setminus \{x_3z_0y_0x_0y_1\} \cup \{x_3z_0y_0x_0x_1, x_1x_2x_3x_0y_1\},$$
$$\mathcal{P}'' = \mathcal{P}' \setminus \{x_2z_0y_3x_3y_0\} \cup \{x_2z_0y_3y_0x_3, y_0y_1y_2y_3x_3\} \text{ and,}$$
$$\mathcal{P}''' = \mathcal{P}'' \setminus \{x_0z_0y_1x_1y_2, x_1z_0y_2x_2y_3\} \cup \{z_0y_1x_1y_2y_0, z_0y_2x_2y_3y_1, x_2x_0z_0x_1x_3\},$$

then $\mathcal{P}'$,$\mathcal{P}''$ and $\mathcal{P}'''$ all are the maximal P_5-packing of K_n. Clearly, there is no edge in the leave of $\mathcal{P}'''$. If $n = 9$, we are done. If $n \geq 13$, we can produce a maximal P_5-packing with desired size by the subgraph $K_{4,4,1}$ and $K_{4,4}$ (see case 1) until the size of leave less than k. □

Lemma 3.5. *Consider* $n = 5t + r \geq 6$, $0 \leq r < 5$.

1. $S(n; P_6) = \{s \mid r + 1 \leq s \leq \lfloor \binom{n}{2}/5 \rfloor\}$ *if* $n = 6, 7$ *or* 8.

2. $S(n; P_6) = \{s \mid \frac{5t(t-1)}{2} + tr \leq s \leq \lfloor \binom{n}{2}/5 \rfloor\}$ *if* $n \geq 9$.

Proof. First, consider $n = 6, 7$ or 8. In fact

$$\{r + 1 \leq s \leq \lfloor \binom{n}{2}/5 \rfloor\} = \begin{cases} \{2, 3\}, & \text{if } n = 6 \\ \{3, 4\}, & \text{if } n = 7 \\ \{4, 5\}, & \text{if } n = 8. \end{cases}$$

There are a minimum P_6-packing of K_6 with size 2, a minimum P_6-packing of K_7 with size 3 and a minimum P_6-packing of K_8 with size 4 from Theorem 3.2. Besides, we have a maximum P_6-packing of K_6 with size 3, a maximum P_6-packing of K_7 with size 4 and a maximum P_6-packing of K_8 with size 5 from Corollary 2.2.

Final, consider $n \geq 9$. We have a minimum P_6-packing $\mathcal{P}$ of K_n with leave L as defined in Theorem 3.2. There are five cases of n.

Case 1. $n \equiv 0 \pmod 5$.

Consider the subgraph $K_{5,5}$. We have $\{y_i x_i y_{i+1} x_{i-1} y_{i+2} x_{i-2} \mid 0 \leq i \leq 4\} \subseteq \mathcal{P}$ from Lemma 2.5 and Theorem 3.1. Switch the edges as following:

- Replace $\{y_0 x_0 y_1 x_4 y_2 x_3\}$ by $\{x_1 x_0 y_1 x_4 y_2 x_3, y_0 x_0 x_4 x_3 x_2 x_1\}$.
- Replace $\{y_1 x_1 y_2 x_0 y_3 x_4\}$ by $\{x_3 x_1 y_2 x_0 y_3 x_4, y_1 x_1 x_4 x_2 x_0 x_3\}$.
- Replace $\{y_3 x_3 y_4 x_2 y_0 x_1\}$ by $\{y_3 x_3 y_4 x_2 y_0 y_2, y_2 y_4 y_1 y_3 y_0 x_1\}$.
- Replace $\{y_4 x_4 y_0 x_3 y_1 x_2\}$ by $\{x_4 y_4 y_0 x_3 y_1 x_2, x_4 y_0 y_1 y_2 y_3 y_4\}$.

We can produce some P_6-packings of K_n and the new packings are all maximal. If $n \geq 10$, the step 3 can operate until the leave contains no edge by use subgraph $K_{5,5}$.

Case 2. $n \equiv 4 \pmod 5$.

Take the subgraph $K_{5,4}$ of K_n and then $\{y_i x_i y_{i+1} x_{i-1} y_{i+2} o \mid 0 \leq i \leq 3\} \subseteq \mathcal{P}$ from the remark of Open problem 3 and Theorem 3.1. Switch the edges as following:

- Replace $\{y_0 x_0 y_1 x_3 y_2 o\}$ by $\{x_1 x_0 y_1 x_3 y_2 o, y_0 x_0 o x_3 x_2 x_1\}$.

- Replace $\{y_1x_1y_2x_0y_3o\}$ by $\{x_3x_1y_2x_0y_3o, y_1x_1ox_2x_0x_3\}$.

- Replace $\{y_2x_2y_3x_1y_0o, y_3x_3y_0x_2y_1o\}$ by $\{y_1x_2y_3x_1y_0o, y_2y_1y_3x_3y_0x_2, oy_1y_0y_3y_2 x_2\}$.

We produce some P_6-packings of K_n and the new packings are all maximal. If $n = 9$, it's finished. If $n \geq 14$, the step 3 can operate until the leave contains only one edge by use subgraph $K_{5,4}$ and $K_{5,5}$ (see case 1).

Case 3. $n \equiv 3 \pmod 5$.

$K_{5,5,3}$ is a subgraph of K_n in the case. We have $y_0x_0y_1x_4y_2x_3$, $x_1y_2x_0y_3x_4z_0$, $y_3x_3y_4x_2y_0x_1$ and $z_0y_4x_4y_0x_3y_1$ are the paths in $\mathcal{P}$. Switch the edges as following:

- Replace $\{y_0x_0y_1x_4y_2x_3\}$ by $\{x_1x_0y_1x_4y_2x_3, y_0x_0x_4x_3x_2x_1\}$.

- Replace $\{x_1y_2x_0y_3x_4z_0\}$ by $\{x_1x_3x_0y_3x_4x_2, x_2x_0y_2x_1x_4z_0\}$.

- Replace $\{y_3x_3y_4x_2y_0x_1\}$ by $\{y_3x_3y_4x_2y_0y_2, y_2y_4y_1y_3y_0x_1\}$.

- Replace $\{z_0y_4x_4y_0x_3y_1\}$ by $\{y_3y_4x_4y_0x_3y_1, z_0y_4y_0y_1y_2y_3\}$.

We produce some P_6-packings of K_n and the new packings are all maximal. Hence, it's trivial for $n = 13$. Consider the subgraph $K_{5,5,3}$ and $K_{5,5}$ (see case 1) in step 1 if $n \geq 18$, then the step 3 can operate until the leave contains only three edges.

Case 4. $n \equiv 2 \pmod 5$.

If $n = 12$, we only consider the subgraph $K_{5,5,2}$ in step 1. We have a packing $\{y_{i+1}z_1x_{i-2}z_0y_ix_i \mid 0 \leq i \leq 4\}$ of $K_{5,5,2}$ from Lemma 2.7. Switch the edges as following:

- Replace $\{y_1z_1x_3z_0y_0x_0\}$ by $\{x_1x_2x_3z_0y_0x_0, y_1z_1x_3x_4x_0x_1\}$.

- Replace $\{y_2z_1x_4z_0y_1x_1\}$ by $\{x_1x_2x_4z_0y_1x_1, y_2z_1x_4x_1x_3x_0\}$.

- Replace $\{y_4z_1x_1z_0y_3x_3\}$ by $\{y_4z_1x_1x_0y_3x_1, y_1y_4y_2y_0y_3x_3\}$.

- Replace $\{y_0z_1x_2z_0y_4x_4\}$ by $\{y_0z_1x_2z_0y_4y_3, x_4y_4y_0y_1y_2y_3\}$.

We produce some P_6-packings of K_n and the new packings are all maximal. Since the edge number of the leave of the last new packing is one, we are done. If $n \geq 17$, consider the subgraph $K_{5,5,2}$ and $K_{5,5}$ (see case 1). Switch the edges in the subgraph by the same way, then we have packing with desired size.

Case 5. $n \equiv 1 \pmod 5$.

Similarly to the other cases, $K_{5,5,1}$ is a subgraph of K_n. From Lemma 2.7 and Theorem 3.1, $\{x_{i-2}z_0y_ix_iy_{i+1}x_{i-1} \mid 0 \le i \le 4 \mid 0 \le i \le 3\} \subseteq \mathcal{P}$. Switch the edges as following:

- Replace $\{x_3z_0y_0x_0y_1x_4\}$ by $\{z_0y_0x_0y_1x_4x_3, x_4x_0x_1x_2x_3z_0\}$.
- Replace $\{x_4z_0y_1x_1y_2x_0\}$ by $\{y_1x_1y_2x_0x_2x_4, x_0x_3x_1x_4z_0y_1\}$.
- Replace $\{x_1z_0y_3x_3y_4x_2\}$ by $\{x_1z_0y_3x_3y_4y_1, y_1y_3y_0y_2y_4x_2\}$.
- Replace $\{x_2z_0y_4x_4y_0x_3\}$ by $\{x_2z_0y_4x_4y_0y_1, y_1y_2y_3y_4y_0x_3\}$.

We produce some P_6-packings of K_n and the new packings are all maximal. If $n = 11$, we are done since the leave of the last new packing contains no edge. Consider the subgraph $K_{5,5,1}$ and $K_{5,5}$ (see case 1) if $n \ge 16$, then we have packing with desired size by edge switching. □

Lemma 3.6. *Consider $n = 6t + r \ge 7$, $0 \le r < 6$.*

1. *$S(n; P_7) = \{s \mid r+1 \le s \le \lfloor \binom{n}{2}/6 \rfloor\}$ if $n = 7, 8$.*
2. *$S(n; P_7) = \{s \mid \frac{6t(t-1)}{2} + tr \le s \le \lfloor \binom{n}{2}/6 \rfloor\}$ if $n \ge 9$.*

Proof. First, consider $n = 7, 8$. Note that,

$$\{r+1 \le s \le \lfloor \binom{n}{2}/6 \rfloor\} = \begin{cases} \{2,3\}, & \text{if } n = 7 \\ \{3,4\}, & \text{if } n = 8. \end{cases}$$

Since the minimum number of the set is the size of minimum P_7-packing of K_n and the maximum number of the set is the size of maximum P_7-packing of K_n, we are done from Theorem 3.2 and Corollary 2.2.

We have a minimum P_7-packing $\mathcal{P}$ of K_n with leave L as defined in Theorem 3.2 if $n \ge 9$. Consider the edge switching of K_n by the following six cases of n.

Case 1. $n \equiv 0 \pmod 6$.

Consider $K_{6,6} \subseteq K_n$, $\{y_ix_iy_{i+1}x_{i-1}y_{i+2}x_{i-2}y_{i+3} \mid 0 \le i \le 4\} \subseteq \mathcal{P}$ from Lemma 2.5 and Theorem 3.1. Switch the edges as following:

- Replace $\{y_0x_0y_1x_5y_2x_4y_3\}$ by $\{x_1x_0y_1x_5y_2x_4y_3, y_0x_0x_5x_4x_3x_2x_1\}$.
- Replace $\{y_1x_1y_2x_0y_3x_5y_4\}$ by $\{x_4x_1y_2x_0y_3x_5y_4, y_1x_1x_3x_0x_4x_2x_5\}$.
- Replace $\{y_2x_2y_3x_1y_4x_0y_5, y_3x_3y_4x_2y_5x_1y_0\}$ by $\{x_3y_3x_1y_4x_0y_5x_2, y_5x_1x_5x_3y_4x_2x_0, x_1y_0y_2x_2y_3y_5x_1\}$.
- Replace $\{y_4x_4y_5x_3y_0x_2y_1\}$ by $\{y_1y_4x_4y_5x_3y_0x_2, x_2y_1y_3y_0y_4y_2y_5\}$.
- Replace $\{y_5x_5y_0x_4y_1x_3y_2\}$ by $\{x_5y_5y_0x_4y_1x_3y_2, x_5y_0y_1y_2y_3y_4y_5\}$.

All the new packings created after the edge switching are maximal since L contains no P_7 and then we have the packing with the desired size when $n \geq 12$.
Case 2. $n \equiv 5 \pmod 6$.

We have a subgraph $K_{6,5}$ with vertex classes $X = \{x_0, x_1, x_2, x_3, x_4\}$ and $Y = \{y_0, y_1, y_2, y_3, y_4, o\}$ in the case. $\{y_ix_iy_{i+1}x_{i-1}y_{i+2}x_{i-2}o \mid 0 \leq i \leq 4\} \subseteq \mathcal{P}$. Switch the edges as following:

- Replace $\{y_0x_0y_1x_4y_2x_3o\}$ by $\{oy_0x_0y_1x_4y_2x_3, x_3oy_4y_3y_2y_1y_0\}$.
- Replace $\{y_1x_1y_2x_0y_3x_4o\}$ by $\{y_4y_1x_1y_2x_0y_3x_4, x_4oy_2y_4y_0y_3y_1\}$.
- Replace $\{y_3x_3y_4x_2y_0x_1o\}$ by $\{x_1oy_3x_3y_4x_2y_0, y_2y_0x_1x_4x_2x_3x_0\}$.
- Replace $\{y_4x_4y_0x_3y_1x_2o\}$ by $\{x_0x_4y_0x_3y_1x_2x_1, y_4x_4x_3x_1x_0x_2o\}$.

If $n = 11$, we can have the maximal P_7-packing with the desired size. Consider the subgraph $K_{6,5}$ and $K_{6,6}$ (see case 1) if $n \geq 17$, then we are done.
Case 3. $n \equiv 4 \pmod 6$.

$K_{6,4}$ is a subgraph of K_n. Recall that the vertex classes of $K_{6,4}$ are $X = \{x_0, x_1, x_2, x_3, o_1, o_2\}$ and $Y = \{y_0, y_1, y_2, y_3\}$, and $\{o_1y_ix_iy_{i+1}x_{i-1}y_{i+2}o_2 \mid 0 \leq i \leq 3\} \subseteq \mathcal{P}$. Switch the edges as following:

- Replace $\{o_1y_0x_0y_1x_3y_2o_2\}$ by $\{o_2o_1y_0x_0y_1x_3y_2, y_2o_2x_0x_1x_2x_3o_1\}$.
- Replace $\{o_1y_1x_1y_2x_0y_3o_2\}$ by $\{x_2o_1y_1x_1y_2x_0y_3, o_1x_1x_3x_0x_2o_2y_3\}$.
- Replace $\{o_1y_2x_2y_3x_1y_0o_2\}$ by $\{x_0o_1y_2x_2y_3y_0y_1, y_1y_2y_3x_1y_0o_2x_3\}$.

If $n = 10$, we are done. If $n \geq 16$, we can produce a maximal P_7-packing with desired size by consider the subgraph $K_{6,4}$ and $K_{6,6}$ (see case 1).

Case 4. $n \equiv 3 \pmod{6}$.

Consider the subgraph $K_{6,3}$ if $n = 9$. According the construction of $\mathcal{P}$, we have a subset $\{x_0y_0x_1y_2x_5y_1x_3, x_1y_1x_2y_0x_3y_2x_4, x_2y_2x_0y_1x_4y_0x_5\}$ of $\mathcal{P}$ and it is is a packing of $K_{6,3}$. Switch the edges as following:

- Replace $\{x_0y_0x_1y_2x_5y_1x_3\}$ by $\{x_0x_1y_2x_5y_1x_3x_2, x_3x_4x_5x_0y_0x_1x_2\}$.
- Replace $\{x_1y_1x_2y_0x_3y_2x_4\}$ by $\{x_0x_4x_1y_1x_2y_0y_2, y_0x_3y_2x_4x_2x_5x_1\}$.
- Replace $\{x_2y_2x_0y_1x_4y_0x_5\}$ by $\{y_2x_0y_1x_4y_0x_5x_3, x_1x_3x_0x_2y_2y_1y_0\}$.

After the edge switching, we have the maximal P_7-packing with the desired size. If $n \geq 15$, the step 3 can operate by use the edge switching of subgraph $K_{6,3}$ and $K_{6,6}$ (see case 1).

Case 5. $n \equiv 2 \pmod{6}$.

In step 1, we take the subgraph $K_{6,6,2}$ and $K_{6,6}$ (see case 1) if $n \geq 14$. We have $\{y_{i+2}z_1x_{i-2}z_0y_ix_iy_{i+1} \mid 0 \leq i \leq 5\} \cup \{y_2x_4y_3x_5y_4x_0y_5, y_4x_2y_5x_3y_0x_4y_2\} \subseteq \mathcal{P}$ is a packing of $K_{6,6,2}$ from Lemma 2.7 and Theorem 3.1. In step 3, switch the edges as following:

- Replace $\{y_2z_1x_4z_0y_0x_0y_1\}$ by $\{y_2z_1x_4z_0y_0x_0x_1, x_1x_2x_3x_4x_5x_0y_1\}$.
- Replace $\{y_3z_1x_5z_0y_1x_1y_2\}$ by $\{x_0x_3x_5z_0y_1x_1y_2, y_3z_1x_5x_2x_4x_1x_3\}$.
- Replace $\{y_2x_4y_3x_5y_4x_0y_5, y_4x_2y_5x_3y_0x_4y_1\}$ by $\{x_2y_4x_0x_4y_3x_5x_1, x_0x_2y_5x_3y_0y_2x_4, x_0y_5y_1x_4y_0y_4x_5\}$.
- Replace $\{y_0z_1x_2z_0y_4x_4y_5\}$ by $\{z_1x_2z_0y_4x_4y_5y_3, y_5y_2y_4y_1y_3y_0z_1\}$.
- Replace $\{y_1z_1x_3z_0y_5x_5y_0\}$ by $\{y_0y_1z_1x_3z_0y_5x_5, x_5y_0y_5y_4y_3y_2y_1\}$.

The P_7-packings still maximal after the edge switching, we can produce the packing with the desired size.

Case 6. $n \equiv 1 \pmod{6}$.

We will consider $K_{6,6,1}$ in the last case. Recall that, $K_{6,6,1}$ has a packing $\{x_{i-2}z_0y_ix_iy_{i+1}x_{i-1}y_{i+2} \mid 0 \leq i \leq 5\} \subseteq \mathcal{P}$. Switch the edges as following:

- Replace $\{x_4z_0y_0x_0y_1x_5y_2\}$ by $\{x_3x_4z_0y_0x_0x_5y_2, x_3x_2x_1x_0y_1x_5x_4\}$.
- Replace $\{x_5z_0y_1x_1y_2x_0y_3\}$ by $\{x_3x_5z_0y_1x_1y_2x_0, y_3x_0x_3x_1x_4x_2x_5\}$.
- Replace $\{x_0z_0y_2x_2y_3x_1y_4, x_1z_0y_3x_3y_4x_2y_5\}$ by $\{x_0x_2y_3x_1y_4y_0y_2, x_5x_1z_0y_3x_3y_4x_2, y_1y_5x_2y_2z_0x_0x_4\}$.
- Replace $\{x_2z_0y_4x_4y_5x_3y_0\}$ by $\{x_2z_0y_4x_3y_5y_3y_0,\ y_3y_1y_4y_2y_5x_3y_0\}$.
- Replace $\{x_3z_0y_5x_5y_0x_4y_1\}$ by $\{x_3z_0y_5x_5y_0y_1y_2,\ y_2y_3y_4y_5y_0x_4y_1\}$.

We have some maximal P_7-packings of K_n with different size. It is take only one subgraph $K_{6,6,1}$ if $n = 13$, and take subgraph $K_{6,6,1}$ and $K_{6,6}$ (see case 1) both in step 1. We are done. □

From Lemma 3.3 to Lemma 3.6, we have the theorem.

Theorem 3.7. *Suppose $n = tk + r \geq k + 1$, $0 \leq r < k$. If $k \leq 6$, then*

1. $S(n; P_{k+1}) = \{s \mid r + 1 \leq s \leq \lfloor \binom{n}{2}/k \rfloor\}$, *if $t = 1$ and either $1 \leq r < \lceil \frac{k}{2} \rceil$ or both k and r are odd.*
2. $S(n; P_{k+1}) = \{s \mid \frac{tk(t-1)}{2} + tr \leq s \leq \lfloor \binom{n}{2}/k \rfloor\}$, *otherwise.*

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