國 立 交 通 大 學

應用數學系

碩 士 論 文

半徑為 3 以下的中點圖

Median graphs of radius at most three

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指導教授：翁志文　教授

中 華 民 國 一 ○ 一 年 七 月

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半徑為 3 以下的中點圖

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摘　　　要

中點圖已經被研究數十年，許多中點圖的理論及性質已經被發表，而其大部分的理論都和其具"凸"性質的某些子圖有關。我們試著用不同的角度去研究半徑不超過 3 的圖，過程中不用到"凸"性質並且試著找出這些圖是中點圖的充分且必要條件。

Median graphs with radius at most 3

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ABSTRACT

Median graphs have been studied for decades. Many important theorems and properties of median graphs have been found out and almost all of those theorems are relative to convex. We try to study median graphs in a different way. We consider median graphs with radius at most 3 and try to find out their necessary and sufficient conditions without using convex property.

誌謝

這篇文章的完成，首先要感謝的就是我的指導教授———翁志文教授。從大學時期就在代數這門課中，給了我不少啟發。也因為老師的鼓勵和支持，為我推薦了應用數學所的碩士班，我才有這個機會來完成這篇文章。也因為對於代數與圖論這兩個數學領域的熱愛，成為了翁老師的學生。在與老師討論研究的過程中，老師對於問題的敏銳的洞察力以及題目發展的延伸性，都是非常值得學習的地方。尤其往往我所寫出來複雜的論述，經過老師的修改之後，都能變成較於簡化易懂的形式，這是最令我佩服的一點。但由於我的懶散，以至於將論文拖到了第三年才完成，這點對於當初極力提拔我的老師，感到相當慚愧。但老師仍然盡心盡力地協助我完成這篇論文，甚至老師也答應了我在第三年下學期出國交換的計畫，幫我寫了推薦信，讓我在求學生涯獲得更豐富的經歷。實在相當地感謝翁老師，所有為我付出的一切。

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Median graphs of radius at most three

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中文摘要

中點圖已經被研究數十年，許多中點圖的理論及性質已經被發表，而其大部分的理論都和其具”凸”性質的某些子圖有關。我們試著用不同的角度去研究半徑不超過 3 的圖，過程中不用到”凸”性質並且試著找出這些圖是中點圖的充分且必要條件。

Abstract

Median graphs have been studied for decades. Many important theorems and properties of median graphs have been found out and almost all of those theorems are relative to convex. We try to study median graphs in a different way. We consider median graphs with radius at most 3 and try to find out their necessary and sufficient conditions without using convex property.

1 Introduction

A median of vertices u, v, and w is a vertex lies on the shortest paths between any two of them. A graph is called median graph if any triple of vertices has a unique median. Trees and n-cubes Q_n are well-known median graphs.

There are many important theorems and properties of median graphs today. Those theorems have been found out by some great mathematicians. In [1], Nebeský has proved a lot of basic properties and theorems of median graphs which let us have a basic understanding of median graphs. Mulder found out the structure of median graphs, which is median graphs could be obtained from a one-vertex graph by a so-called convex expansion procedure in [2]. Also, Mulder discovered the relations between n-cubes Q_n and median graphs in [3].

There are still many important theorems of median graphs. However, almost all of those theorems are relative to convex property. In order to avoid the complicated condition as convex, we try to study median graphs in a different way. We consider median graphs with radius 1, 2, and 3 and try to find out their necessary and sufficient conditions without using convex property.

In section 2, we introduce some definitions and notations as preliminary knowledge. They are needed in the rest of this paper. In section 3, the definition and some basic properties of median graphs are introduced. Also, we prove some properties which will help us to prove the necessary and sufficient conditions of median graphs with radius 2.

We start to prove those necessary and sufficient conditions of median graphs with radius 1 and 2 in section 4. In order to prove the part of median graphs with radius 2, we have used the method which mentioned by W.Imrich, S. Klavžar, H.M.Mulder in [5]. In section 5, we give two conditions and prove that they are sufficient conditions of median graphs with radius 3. We only prove a part of the necessary part but we believe that they are also the necessary conditions of median graphs with radius 3.

2 Preliminaries

At the beginning, we recall some definitions and notations needed in the rest of this paper. Given G a simple connected graph, $V(G)$ and $E(G)$ are vertex set and edge set of G, respectively.

Let $u, v \in V(G)$. If $uv \in E(G)$, we say u is *incident* to v, denoted by $u \sim v$. By a *path* from u to v of length t, we mean a sequence of vertices $u_0 = u$, u_1, $\ldots$, $u_t = v$ such that u_i are distinct with possible $u_0 = u_t$ and $u_i u_{i+1} \in E(G)$ for $0 \le i \le t-1$, where u and v are called the *start* vertex and the *end* vertex of the path respectively. The *distance function* $d(u, v)$ means the length of a shortest length among paths from u to v. A *cycle* with length n, denoted by C_n, is a path with same start vertex and end vertex and has length n, $n \ge 3$. We call a C_n odd cycle if $n = 2k+1$, and even cycle if $n = 2k+2$ for $k \in \mathbb{N}$. The *interval* between u and v is the set

$$I(u, v) = \{w \in V \mid d(u, w) + d(w, v) = d(u, v)\},$$

i.e. those vertices on the shortest paths from u to v. The set of neighbors of u is denoted as $N(u)$ and defined as $N(u) = \{x \in V(G) \mid d(u, x) = 1\}$. The number $d(u) = |N(u)|$ is called the *degree* of u and we call those vertices with degree 1 in G the *leaves*.

Give two graphs G and H. We say H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Given $X \subseteq V(G)$, we call $< X >$ an *induced subgraph* of G if $u, v \in X$ and $uv \in E(< X >)$ if and only if $uv \in E(G)$. A subgraph H is an *isometric subgraph* if $d_G(u, v) = d_H(u, v)$, for all $u, v \in V(H)$, where $d_G(u, v)$ is the distance if u and v in G and $d_H(u, v)$ is the distance of u and v in H. A subgraph H is *convex* if for all $u, v \in V(H)$, $I(u, v) \subseteq V(H)$.

Give $x \in V(G)$. The *eccentricity* of x is denoted by $e(x)$ and defined as $e(x) = max\{d(x, v) \mid v \in V(G)\}$. The *radius* of G is denoted by $r(G)$ and defined as $r(G) = min\{e(x) \mid x \in V(G)\}$. A vertex $c \in V(G)$ is a *central vertex* of G if $e(c) = r(G)$. By *periphery* of G, is a set consists of all vertices in G which has distance $r(G)$ from some central vertex $c \in V(G)$.

Given two graphs G and H. The *Cartesian product* of G and H, denoted by $G \Box H$, is the graph with vertex set $V(G) \times V(H)$ and $(a,x)(b,y) \in E(G \Box H)$ whenever $ab \in E(G)$ and $x = y$, or $a = b$ and $xy \in E(H)$.

A *tree* is a simple connected graph which has no cycle. A *star* is a graph with a unique center c, and $E(G) = \{vc \mid v \in V(G) \setminus \{c\}\}$. Obviously, a star is also a tree. A cube of size n, denoted by Q_n, is defined inductively as $Q_n = Q_{n-1} \Box Q_{n-1}$, $n \geq 2$, where Q_0 is a vertex, Q_1 is an edge.

For an edge $e = uv$ in a graph G, the *subdivision* of e is obtained by replacing the edge e by a new vertex adjacent to both u and v. For convenience, we denote the new vertex by e and the new edges by ue and ev.

A graph G is a *bipartite* graph if there are two set A and B such that $A \neq \emptyset$, $B \neq \emptyset$, $V(G) = A \cup B$ and $A \cap B = \emptyset$. Also, uv is not an edge if $u, v \in A$ or $u, v \in B$. It is well-known that G is a bipartite graph if and only if G contains no odd-cycle.

3 Median graphs

Let G be a simple connected graph, and $V(G)$, $E(G)$ are vertex set and edge set of G, respectively. For $u, v, w \in V(G)$ we use the abbreviation

$$I(u,v,w) = I(u,v) \cap I(u,w) \cap I(v,w),$$

and for $m \in V(G)$, we call m a *median* of u, v and w if $m \in I(u,v,w)$, i.e. m lies on the paths between each two of these three vertices. A connected graph G is a *median graph* if there is exactly a median for all $u, v, w \in V(G)$, i.e. $|I(u,v,w)| = 1$. Trees and n-cubes Q_n are well-known median graphs.

Lemma 3.1. *Suppose $u, v, w \in V(G)$. Then $v \in I(u,w)$ if and only if $I(u,v,w) = \{v\}$.*

Proof. ($\Rightarrow$) If $a \in I(u,v,w)$ then

$$(d(u,a) + d(a,v)) + (d(v,a) + d(a,w)) = d(u,v) + d(v,w) = d(u,w),$$

so $2d(a,v) = 0$. This implies $a = v$ to have the lemma.

($\Leftarrow$) Clearly $\{v\} = I(u,v,w) \subseteq I(u,w)$. □

The following lemma is also proved in paper [1].

Lemma 3.2. *Give a simple connected graph G, $u, v, w \in V(G)$ and $vw \in E(G)$. If $\{u, v, w\}$ has a median m then either $m = v$ or $m = w$, not both.*

Proof. Since $I(u,v,w) \subseteq I(v,w) = \{v, w\}$, the lemma follows from Lemma 3.1. □

Proposition 3.3. *A median graph is bipartite.*

Proof. We suppose that G is a median graph but not bipartite, i.e. G contains an odd cycle, C_{2k+1} for some $k \in \mathbb{N}$. If it is not isometric, it must contain a smaller isometric odd cycle $C_{2\ell+1}$ for $\ell \in \mathbb{N}$ with $l < k$. Hence we just suppose it is isometric. Then we pick vertices $v, w \in V(C_{2k+1})$ where vw is an edge. Because C_{2k+1} is an odd cycle, there is a vertex $u \in V(C_{2k+1})$ such that $d(u, v) = d(u, w) = k$. By Lemma 3.2, the median of $\{u, v, w\}$ must be v or w, without loss of generality, say v. By definition of median, we have $d(w, v) + d(v, u) = d(w, u)$ which is a contradiction to $d(w, u) = d(v, u)$. Therefore, we proved that if G is a median graph then G is also a bipartite graph. □

From Proposition 3.3, if a median graph contains a cycle C_4 or a complete multiple graph $K_{2,3}$, then these two subgraphs are indeed the induced subgraphs.

Lemma 3.4. *A median graph does not contain* $K_{2,3}$.

Proof. Suppose the graph does contain $K_{2,3}$ which has bipartition $\{u, v, w\} \cup \{s, t\}$. Then $I(u, v, w) = \{s, t\}$, a contradiction to the median graph definition. □

Since a median graph is well-known a bipartite graph, the graphs mentioned in this paper are supposed to be connected bipartite graphs.

Definition 3.5. G_1,G_2 are two graphs, $x \in V(G_1)$, $y \in V(G_2)$, we defined the operation *coalescence* $\dot{+}$ as $\dot{+}(G_1, G_2, x, y)$ which is a function combine G_1 and G_2 to a new graph $G_1 \dot{+}_{xy} G_2$ by deleting y and adds edges between x and $N(y)$.

From the above definition $V(G_1 \dot{+}_{xy} G_2) = V(G_1) \cup V(G_2) \setminus \{y\}$. Another way to view the new edges set of $G_1 \dot{+}_{xy} G_2$ is as $E(G_1 \dot{+}_{xy} G_2) = E(G_1) \cup E(G_2)$, where we replace those edges xz by yz when $z \in N(y)$.

Lemma 3.6. *$G_1 \dot{+}_{xy} G_2$ is a median graph if G_1, G_2 are median graphs.*

Proof. To complete this lemma, we need to show that every three vertices u, v, w in $V(G_1 \dot{+}_{xy} G_2)$ has a unique median. Obviously, it holds when $u, v, w \in V(G_1)$ or $u, v, w \in (V(G_2) \setminus \{y\}) \cup \{x\}$ since $< V(G_1) >, < (V(G_2) \setminus \{y\}) \cup \{x\} >$ are convex subgraphs in $G_1 \dot{+}_{xy} G_2$.

Therefore, we may assume, without loss of generality, $u, v \in V(G_1)$, $w \in (V(G_2) \setminus \{y\}) \cup \{x\}$. Since $< V(G_1) >, < (V(G_2) \setminus \{y\}) \cup \{x\} >$ are convex, any path from w to u or v must pass x. By this fact, we get $I(w, u) = I(w, x) \cup I(x, u)$ and $I(w, v) = I(w, x) \cup I(x, v)$. Since $< V(G_1) >$ is convex, we have $I(u, v) \cap I(x, w) = \{x\}$ or $\emptyset$. Therefore,

$$\begin{aligned} I(u, v, w) &= I(u, v) \cap I(u, w) \cap I(v, w) \\ &= I(u, v) \cap (I(u, x) \cup I(x, w)) \cap (I(v, x) \cup I(x, w)) \\ &= (I(u, v) \cap I(u, x) \cap I(v, x)) \cup (I(u, v) \cap I(x, w)) \\ &= I(u, v, x), \end{aligned}$$

where the last equality is by Lemma 3.1. Since $u, v, x \in V(G_1)$ and $< V(G_1) >$ is a median graph, we have

$$|I(u, v, w)| = |I(u, v, x)| = 1.$$

$\square$

Corollary 3.7. *$G \dotplus T$ is a median graph, where G is a median graph and T is a tree.*

Proof. Since tree is also a median graph, it is immediately proved by Lemma 3.6. $\square$

Lemma 3.8. *Give two graphs G, H and two vertices $x \in V(G)$ and $y \in V(H)$. If H is not a median graph, then $G \dotplus_{xy} H$ is not a median graph.*

Proof. Suppose H is not a median graph because $|I(u, v, w)| \neq 1$, where $u, v, w \in V(H)$. Since $< (V(H) \setminus \{y\}) \cup \{x\} >$ is a convex subgraph in $G \dotplus_{xy} H$, i.e. $I(a, b) \subset (V(H) \setminus \{y\}) \cup \{x\}$, for all $a, b \in (V(H) \setminus \{y\}) \cup \{x\}$. The convex property keeps the result $|I(u, v, w)| \neq 1$ in graph $G \dotplus_{xy} H$. Therefore, we have proved that $G \dotplus_{xy} H$ is not a median graph. $\square$

Corollary 3.9. *$G_1 \dotplus_{xy} G_2$ is a median graph if and only if both G_1, G_2 are median graphs.*

Proof. By Lemma 3.6 and Lemma 3.8. $\square$

Lemma 3.10. *Let G be a median graph. Let $G' = G \setminus \{v \in V(G) \mid d(v) = 1\}$, i.e. G' is the graph obtained from G by deleting all the leaves in G. Then G' is also a median graph.*

Proof. We can see that G is the result of repeating the operation $\dotplus$ between G' and those edges incident to leaves in G. Therefore, by the fact of edges are median graphs, this lemma are proved by Corollary 3.9. $\square$

4 Median graphs with radius 2

Throughout the remaining of the thesis fix a simple connected graph $G = (V(G), E(G))$ with at least three vertices and a center $c \in V(G)$. Note that the degree $d(c)$ of c is at least 2. We shall define some notions needed for the rest of this paper. Let

$$L_i = \{x \mid x \in V(G), d(x, c) = i\}$$

and $\ell(p) = i$ if $p \in L_i$. For $p \in L_i$ set

$$\begin{aligned} p^+ &= \{u \mid p \in I(u, c)\}, \\ p^- &= \{u \mid u \in I(p, c)\}. \end{aligned}$$

Proposition 4.1. *If G is a bipartite graph with radius 1. Then G is a median graph.*

Proof. Since G is bipartite. Bipartite graph with radius 1 is a star which is a median graph. □

Before mentioning median graphs with radius 2, we have to see some concepts from [5]. Let $G = (V, E)$ be a graph with $|V| = n$, $|E| = m$. The graph $\hat{G}$ is obtained from G by subdividing all edges of G and adding a new vertex c joined to all the original vertices of G. So we have $\hat{V} = V \cup E \cup \{c\}$ and

$$\hat{E} = \{cv \mid v \in V\} \cup \{ue \mid e \in E, u \in V \text{and } u \text{ is incident with } e \text{ in } G\}.$$

Furthermore, the paper proves the following result:

Lemma 4.2. *A graph G is triangle-free if and only if its associated graph $\hat{G}$ is a median graph.* □

Theorem 4.3. *Let G be a bipartite graph with radius 2. Then G is a median graph if and only if the following (i)-(ii) hold.*

(i) G does not contain the induced subgraph $K_{2,3}$.

(ii) G does not contain the induced subgraph $C_6 \subseteq L_1 \cup L_2$.

Proof. The necessity (i) follows from Lemma 3.4. For (ii), if G does contain induced subgraph $C_6 \subseteq L_1 \cup L_2$ then the three vertices in $C_6 \cap L_2$ has a median m in L_1, and then m together with any two of the three vertices in $C_6 \cap L_1$ has c as a median and another median in L_2, a contradiction.

To prove sufficiency, first note that the no $K_{2,3}$ assumption and radius 2 of G assumption imply that each vertex in L_2 has degree at most 2, and there is no induced subgraph C_4 in $L_1 \cup L_2$. We delete those leaves in G which are $\{v \mid v \in V(G), d(v) = 1\}$ and this will not impact the median property by Lemma 3.10. Thus, we can assume $d(y) \geq 2$ for all $v \in V(G)$. Now we try to make G to a new graph G' by doing below steps. We delete the vertex c which is the center of G. We let $V(G') = \{u \mid u \in L_1\}$ and u, v are incident if they have a common neighbor in L_2. Since there is no C_4 in $L_1 \cup L_2$, there are no multiple edges in G'. Also, since there is no C_6 in $L_1 \cup L_2$, there is no triangle in G'. Thus, by Lemma 4.2, $G = \hat{G}'$ is a median graph. Now we have proved this whole theorem. □

5 Median graph with radius 3

To study those median graphs of higher radius, we need to introduce some more definitions and notations. In [4], it mentioned the following definitions. For $uv \in E(G)$, we call uv an *up-edge* of u if $d(u, c) < d(v, c)$, that is, $\ell(u) < \ell(v)$. Otherwise, we call uv a *down-edge* of u. Notice that G is a bipartite graph so that there is no edge uv such that $d(u, c) = d(v, c)$. Therefore, each edge

uv is either a up-edge or a down-edge to u. Let *down-degree* $\underline{d}(u)$ (resp. *up-degree* $\overline{d}(u)$) denote the number of down-edges (resp. up-edges) of u, that is, the number of those neighbors of u in $L_{l(u)-1}$. By [4], we have the proposition below

Proposition 5.1. *Let G be a median graph and let $v \in L_i$ with $\underline{d}(v) = k$. Then $i \geq k$ and v and its down-edges are contained in a cube of dimension k which meets the levels $L_i, L_{i-1}, ..., L_{i-k}$.* □

This proposition give us some clues to develop median graphs with radius 3.

Lemma 5.2. *Let G be a bipartite graph of radius 3. Suppose the following (a), (b) hold.*

- *(a) (forbidden condition) G does not contain the induced subgraph $K_{3,2}$.*
- *(b) (enforcing condition) Every induced subgraph C_6 is contained in an induced cube of dimension 3.*

Then the following (i)-(v) hold.

- *(i) If $x \in L_3$, then $d(x) = \underline{d}(x) = k \leq 3$. Also, x and its down-edges are contained in a cube of dimension k which contains an element in L_{3-k}.*
- *(ii) $|p^+ \cap q^+ \cap L_{i+1}| \leq 1$, $p, q \in L_i$, $i = 1, 2$;*
- *(iii) $|p^- \cap q^+ \cap L_i| \leq 2$, $p \in L_{i+1}$, $q \in L_{i-1}$, $i = 1, 2$;*
- *(iv) If there is a induced subgraph C_6 in $L_1 \cup L_2$, then there is a vertex a such that $\{a\} \cup \{c\} \cup C_6$ is a cube, where $a \in L_3$ and there is no C_6 in $L_2 \cup L_3$ and*
- *(v) If there is a induced subgraph C_6 $u - x - v - c - w - z - u$, where $u \in L_3$, $x, z \in L_2$, $w, v \in L_1$. Then there are two vertices a and b such that $\{a\} \cup \{b\} \cup C_6$ is a cube, where $a \in L_2$, $b \in L_1$.*

Proof. (i) This is clear if $d(x) = 1$. Suppose $d(x) \geq 2$. Pick distinct $a_1, a_2 \in N(x)$. If there exists $e \in a_1^- \cap a_2^- \cap L_1$, then the subgraph induced on $\{x, a_1, e, a_2\}$ is C_4. This finishes the proof when $d(x) = 2$. Suppose $a_1^- \cap a_2^- \cap L_1 = \emptyset$. Then we find $b_1, b_2 \in L_1$ such that the subgraph induced on $\{x, a_1, b_1, c, b_2, a_2\}$ is C_6. By the enforcing condition we find $b_3 \in L_1$ and $a_3 \in L_2$ such that the subgraph induced on $\{x, a_1, b_1, c, b_2, a_2, s, a_3, b_3\}$ is a cube of dimension 3. This finishes the proof when $d(x) = 3$. Suppose that there is a vertex in $a_4 \in N(x) - \{a_1, a_2, a_3\}$. Note that a_4 is not adjacent to b_i for $1 \leq i \leq 3$, otherwise there is a $K_{3,2}$. Choose $b_4 \in P_1$ such that the subgraph induced on $\{x, a_2, b_3, c, b_4, a_4\}$ is C_6. Use enforcing condition again we find $b_5 \in L_1$ and $a_5 \in L_2$ such that the subgraph induced on $\{x, a_2, b_3, c, b_4, a_4, b_5, a_5\}$ is a cube of dimension 3. Note that $a_5 \neq a_i$ for $1 \leq i \leq 4$ and $x \sim a_5$ and $a_5 b_3 \in E(G)$. Then the induced subgraph on

$\{x, b_3, a_1, a_2, a_5\}$ is a $K_{3,2}$, a contradiction to the forbidden condition. Hence $d(x) \leq 3$.

(ii) Assume that there exist two distinct $s, t \in p^+ \cap q^+ \cap P_{c(i+1)}$ for $i = 1$ or 2. In the case $i = 1$ we find $K_{2,3}$ on the set $\{c, p, q, s, t\}$. For the case $i = 2$ if there exists a vertex $u \in P_{c1}$ adjacent to p and q, we still find $K_{2,3}$ on the set $\{u, p, q, s, t\}$. Suppose that there exists no vertex in P_{c1} adjacent to p and q. Then we find $d, e \in L_1$ such that the subgraph induced on $\{s, p, d, c, e, q, s\}$ is C_6. By the enforcing condition we find $b \in L_1$ and $a \in L_2$ such that the subgraph induced on $\{s, p, d, c, e, q, s, a, b\}$ is a cube of dimension 3. Then the subgraph induced on $\{s, t, b, p, q\}$ is $K_{2,3}$, a contradiction.

(iii) This is clear from the forbidden condition assumption.

(iv)-(v) It is clear from the enforcing condition.

□

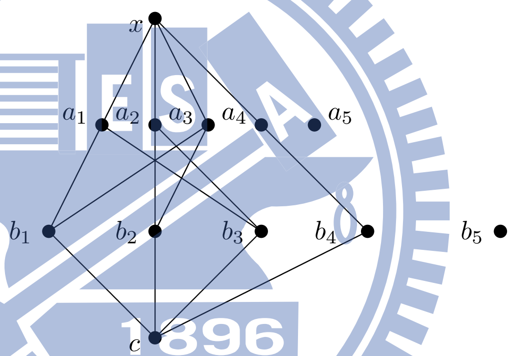

Figure 1. A diagram to illustrate the above proof.

The following is our conjecture.

Conjecture 5.3. Let G be a bipartite graph with radius 3. Then G is a median graph if and only if the following conditions (a), (b) hold.

(a) (forbidden condition) G does not contain the induced subgraph $K_{3,2}$.

(b) (enforcing condition) Every induced subgraph C_6 is contained in an induced cube of dimension 3.

In fact the necessary condition of the above Conjecture holds for any median graphs.

Theorem 5.4. *Let G be a median graph. Then the forbidden condition and the enforcing condition hold in G.*

Proof. We have proved in Lemma 3.4 for the forbidden of K_{23} in a median graph. $a_1, a_2, a_3, a_4, a_5, a_6, a_1$ be an induced C_6 in G. Let $m_1 \in I(a_1, a_3, a_5)$ and $m_2 \in I(a_2, a_4, a_6)$. Clearly $m_1 \notin \{a_1, a_3, a_5\}$ by Lemma 3.1, and $m_1 \notin \{a_2, a_4, a_6\}$ since the the subgraph C_6 is induced. Similarly, $m_2 \notin \{a_1, a_2, a_3, a_4, a_5, a_6\}$. Then the subgraph induced on $\{a_1, a_2, a_3, a_4, a_5, a_6, m_1, m_2\}$ is an induced cube Q_3. □

To prove the sufficient condition of our Conjecture holds, we need more tools.

Definition 5.5. A path $u = u_0, u_1, \ldots, u_t = v$ is called a *down-path* from u to v if there exist an integer $0 \leq k \leq t-1$ such that $\ell(u_0) > \ell(u_1) > \cdots > \ell(u_k)$ and $\ell(u_k) < \ell(u_{k+1}) < \cdots < \ell(u_t)$, and it is denoted by $\hat{P}_{u,v}$.

Lemma 5.6. *Let G be a bipartite graph such that for any two vertices u, v of length two there exists a down-path for u to v. Then for any vertices $u, v \in V(G)$ there exists a down-path $\hat{P}_{u,v}$.*

Proof. We prove by induction on $d(u, v)$. The case $d(u, v) \leq 1$ is clear since a path of length at most 1 is a down-path. The case $d(u, v) = 2$ follows from our assumption. Suppose $d(u, v) = t > 2$. Pick a vertex $u'_{t-1} \in I(u, v) \cap N(v)$. By induction there exists a down path $u = u'_0, u'_1, \ldots, u'_{t-1}$ from u to u'_{t-1}. Note that $u'_1 \in u^-$ and $d(u'_1, v) = t-1$. By induction there exists a down-path $u_1 = u'_1, u_2, \ldots, u_t = v$ from u'_1 to v. Now the path $u = u_0, u_1, u_2, \ldots, u_t = v$ is a down-path from u to v. □

Lemma 5.7. *Let G be a bipartite graph with radius 3 satisfying the forbidden condition and the enforcing condition. Then for any two vertices u, v of length two there exists a down-path $\hat{P}_{u,v}$.*

Proof. This is clear from Lemma 5.2(i). □

H.M. Mulder proved the following result in [6].

Lemma 5.8. *Let G be a connected triangle free graph. If $|I(u, v, w)| = 1$ for any three vertices u, v, w such that $d(u, v) = 2$ then G is a median graph.* □

In fact, we only proved a part of the sufficient condition of the above Conjecture as following theorem.

Theorem 5.9. *Let G be a bipartite graph with radius 3 satisfying the forbidden condition and the enforcing condition. Then $|I(u, v, w)| \leq 1$ for all $u, v, w \in V(G)$ with $d(v, w) = 2$.*

Proof. To the contrary suppose $|I(u, v, w)| > 1$ for some $u, v, w \in V(G)$ with $d(v, w) = 2$. Thus, there are two vertices $a, a' \in I(u, v, w)$. Note that $d(u, v) \in \{d(u, w), d(u, w) + 2, d(u, w) - 2\}$. By Lemma 3.1, we have $a, a' \neq u, v, w$ and $d(u, v) = d(u, w) = d(u, a) + 1 = d(u, a') + 1$. If $d(u, a) = 1$ then the subgraph induced on $\{u, v, w, a, a'\}$ is $K_{2,3}$, a contradiction to the forbidden condition. Suppose $2 \leq d(u, a) = d(u, v) - 1 \leq 5$ as G has diameter at most 6. Since $d(v, w) = 2$, we prove it in two situations.

Case (1) $\ell(v) = \ell(w) = i$: By condition Lemma 5.2(ii), $\ell(a) \neq \ell(a')$. Without loss of generality, suppose $\ell(a) = i - 1$ and $\ell(a') = i + 1$ as shown in Figure 2. Note that $i = 1, 2$.

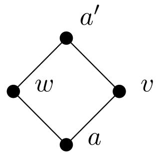

Figure 2. A diagram to Case (1).

Since $\ell(a) = 1$, we have $d(x, a) \leq 4$ for all $x \in V(G)$. Therefore, we only have to consider that $d(u, a)$ from 2 to 4.

- $d(u, a) = d(u, a') = 2$: It will cause $\ell(u) = i + 1$ or $i - 1$.

  1. Suppose $\ell(u) = i + 1$. Pick $y \in \hat{P}_{u,a} \cap N(a)$ and $y' \in \hat{P}_{u,a'} \cap N(a')$. As $|a^+ \cap\ a'^- \cap L_i| \leq 2$, $y \neq y'$ and by Lemma 5.7 there exists $x \in y^- \cap y'^- \cap L_{i-1}$ as shown in Figure 3-1. Since $d(a') \geq 3$, there is a cube Q_3 contains a', y', v, w by Lemma 5.2(i). Note that the cube also contains x, otherwise $|c^+ \cap\ y'^- \cap L_1| \geq 3$. Thus, $v \sim x$ or $w \sim x$. W.L.O.G., we suppose $v \sim x$, which make a contradiction to Lemma 5.2(ii) by $|a^+ \cap x^+ \cap L_2| \geq 2$.

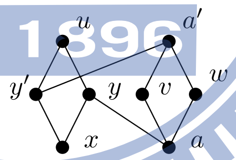

Figure 3-1. The case $d(u, a) = d(u, a') = 2$ and $\ell(x) = i - 1$.

  2. If $\ell(u) = i - 1$, then $i = 2$., pick $y' \in \hat{P}_{u,a'} \cap N(a')$ as shown in Figure 3-2. Since $d(a') \geq 3$, there is a cube Q_3 contains a', y', v, w by Lemma 5.2(i). Also, this cube contains u, otherwise $|c^+ \cap\ y'^- \cap L_1| \geq 3$. W.L.O.G., let $u \sim v$, which is a contradiction.

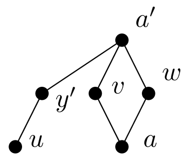

Figure 3-2. The case $d(u,a) = d(u,a') = 2$ and $\ell(y') = i$.

- $d(u,a) = d(u,a') = 3$: If $\ell(u) \leq 1$ then $d(u,a) \leq 2$. Thus, we have $\ell(u) = 2, 3$.

  1. If $\ell(u) = 3$, pick $z \in \hat{P}_{u,a'} \cap N(a')$ as shown in Figure 4-1. Note that v, w are not in $\hat{P}_{u,a'}$. In this case $a = c$ and $|a^+ \cap \ a'^{\,-} \cap L_1| \geq 3$.

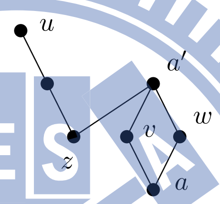

Figure 4-1. The case $d(u,a) = d(u,a') = 3$ and $\ell(z) = 1$.

  2. If $\ell(u) = 2$, pick $z \in \hat{P}_{u,a'} \cap N(u)$ as shown in 4-2. Since $d(a') \geq 3$, there is a cube Q_3 contains a', v, w by Lemma 5.2(i). Also, Q_3 contains z as above proof. W.L.O.G., let $v \sim z$, then $d(u,v) = 2$ which is a contradiction.

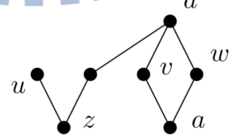

Figure 4-2. The case $d(u,a) = d(u,a') = 3$ and $\ell(z) = i - 1$.

- $d(u,a) = d(u,a') = 4$ If $\ell(u) \leq 2$, then $d(u,a) \leq 3$. Thus, $\ell(u) = 3$. Consider $\hat{P}_{u,a'}$ as shown in Figure 5. Since $d(a') \geq 3$, there is a cube Q_3 contains a', v, w by Lemma 5.2(i). Also, Q_3 contains z as above proof. W.L.O.G., let $v \sim z$, then $d(u,v) = 3$ which is a contradiction.

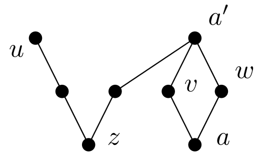

Figure 5. The case $d(u,a) = d(u,a') = 4$ and $\hat{P}_{u,a'}$.

Case (2) $\ell(v) = i+1, \ell(w) = i-1$: In this case, $\ell(a) = \ell(a') = i$ as shown in Figure 6. Note that $i = 1, 2$. Since $\ell(w) \leq 1$, we have $d(u,w) \leq 4$ and $d(u,a) = d(u,a') \leq 3$. Then we only have to consider that $d(u,a)$ from 2 to 3.

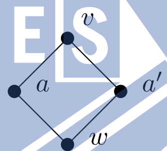

Figure 6. A diagram to Case (2).

- $d(u,a) = d(u,a') = 2$: If $\ell(u) \leq 1$, then $d(u,w) \leq 2$. Thus, $\ell(u) = 2, 3$.

  1. If $\ell(u) = 3$, pick $y \in \hat{P}_{u,a} \cap N(a)$ and $y' \in \hat{P}_{u,a'} \cap N(a')$. Note that $y \neq y'$ and $y \nsim a'$ and $y' \nsim a$, otherwise $|a^+ \cap \ a'^{\,+} \cap L_i| \geq 2$. Now the subgraph induced on $\{u, y', a', w, a, y\}$ is a C_6 as shown in Figure 7-1. Thus, there is a vertex $x \in L_2$ and $x' \in L_1$ such that $\{x, x', u, y', a', w, a, y\}$ is a Q_3 by Lemma 5.2(v). Observe that $v = x$, otherwise $|a^+ \cap \ a'^{\,+} \cap L_i| \geq 2$. Thus, we have $u \sim v$ which is a contradiction.

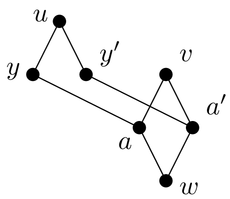

Figure 7-1. The case $d(u,a) = d(u,a') = 2$ and C_6.

2. If $\ell(u) = 2$, then $i = 2$. Pick $y \in \hat{P}_{u,a} \cap N(a)$ and $y' \in \hat{P}_{u,a'} \cap N(a')$. Note that $y \neq y'$ and $y \nsim a'$ and $y' \nsim a$, otherwise $|w^+ \cap \ y'^{\,+} \cap L_2| \geq 2$. Now the subgraph induced on $\{u, y', a', w, a, y\}$ is a C_6 as shown in Figure 7-2. Thus, there is a vertex $x \in L_3$ such that $\{x, c, u, y', a', w, a, y\}$ is a Q_3. Observe that $v = x$, otherwise it violate Lemma 5.2(ii) by $|a^+ \cap \ a'^{\,+} \cap L_3| \geq 2$. Thus, we have $u \sim v$ which is a contradiction to $d(u,v) = 3$.

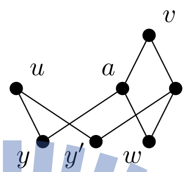

Figure 7-2. The diagram to illustrate above proof.

- $d(u,a) = d(u,a') = 3$: If $\ell(u) \leq 2$, then $d(u,w) \leq 3$. Thus, $\ell(u) = 3$. Pick $z \in \hat{P}_{u,a} \cap N(a)$ and $z' \in \hat{P}_{u,a'} \cap N(a')$. Note that $z \neq z'$ and $z \nsim a'$ and $z' \nsim a$, otherwise it will violate Lemma 5.2(ii) $|w^+ \cap z'^{\,+} \cap L_2| \geq 2$.

  Suppose there exists $y \in \hat{P}_{u,a'} \cap \hat{P}_{u,a}$ such that $y \sim z$ and $y \sim z'$, as Figure 8-1. Now the subgraph induced on $\{u, z', z, w, a, a'\}$ is a C_6. Thus, there is a vertex $x \in L_3$ such that $\{x, c, z, z', y, a', a, w\}$ is a Q_3. As $|a^+ \cap \ a'^{\,+} \cap L_3| \leq 2$, we have $x = v$. Therefore, $d(u,v) = 2$ which is a contradiction to $d(u,v) = 4$.

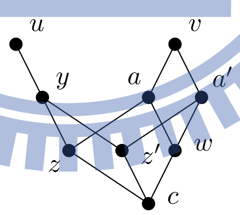

Figure 8-1. The case $d(u,a) = d(u,a') = 3$ and $\ell(y) = 2$.

If there does not exist $y \in \hat{P}_{u,a'} \cap \hat{P}_{u,a}$ such that $y \sim z$ and $y \sim z'$, then there exists $y, y' \in \hat{P}_{u,a'} \cap \hat{P}_{u,a}$ such that $y \sim z$ and $y' \sim z'$. Now the subgraph induced on $\{u, z', z, c, y, y'\}$ is a C_6 as shown in Figure 8-2. Thus, there is a cube contains $\{u, z', z, c, y, y'\}$ by the enforcing condition. Therefore, there exists a $x \in L_2$ such that $x \in \hat{P}_{u,a'} \cap \hat{P}_{u,a}$ and $x \sim z$ and $x \sim z'$. Now the situation is similar to above proof.

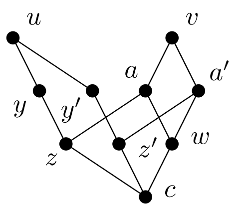

Figure 8-2. The case $d(u, a) = d(u, a') = 3$ and C_6.

Now we have proved this theorem. □

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