國 立 交 通 大 學

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碩 士 論 文

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On the Imperfection Ratio of Unit Disc Graphs

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摘　要

無線網路上的干擾問題是重要且困難的，該問題對應到單位圓盤圖的著色問題。在文獻[9]中，Mani和Petr對隨機網路的單位圓盤圖做了大量的模擬，並且觀察到單位圓盤圖的完全子圖數$\omega(G)$和著色數$\chi(G)$是非常接近的。為了估計$\omega(G)$與$\chi(G)$的接近程度，Mani和Petr採用了「不完美比例」$imp(G)=\sup_R \frac{\chi(G')}{\omega(G')}$。這裡的$G'$是由$G$轉換過來的圖，而這個sup是考慮所有權重向量$R$之後計算所得。已有學者證明出$imp(G)$的理論上界是2.155，$imp(G)=1$若且唯若$G$是完美圖。基於模擬所得之結果，Mani和Petr認為，對單位圓盤圖而言，一個切合實際的上界是$imp(G)=1.2079$，這個上界值遠小於猜測中的上界值1.5，也遠小於理論上界值2.155。本論文之目的在於證明確實存在單位圓盤圖其$imp(G)>1.2079$，且猜測中的上界值$imp(G)=1.5$是可達到的。特別地，我們證明了：若G是長度≥ 5的奇圈或者是Harary圖$H_{2m,3m+2}$（其中m是奇數），則$imp(G)=1.5$；若G是輪圖W_6，則$imp(G)=4/3$。

關鍵詞：無線干擾，完全子圖數，著色數，單位圓盤圖，不完美比例。

中 華 民 國 一 百 年 六 月

On the Imperfection Ratio of Unit Disc Graphs

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Abstract

The interference problem between nodes in a wireless network is important and difficult and it corresponds to the coloring problem in Unit Disc Graphs (UDGs). In [9], Mani and Petr performed extensive simulations with UDGs of random networks and observed that in a UDG G, the clique number $\omega(G)$ and the chromatic number $\chi(G)$ were typically very close to one another. To evaluate the closeness of $\chi(G)$ and $\omega(G)$, Mani and Petr used the measure "imperfection ratio" $imp(G) = \sup_R \frac{\chi(G')}{\omega(G')}$. Here G' is a graph transformed from G and the supremum is computed over all possible weight vectors R. It has been proven that the theoretical bound of $imp(G)$ is 2.155 and $imp(G) = 1$ if and only if G is perfect. Based on the simulation results, Mani and Petr concluded that a practical bound for UDGs is $imp(G) \leq 1.2079$, which is far less than the conjectured upper bound of 1.5 or the theoretic upper bound of 2.155. The purpose of this thesis is to show that there exist UDGs such that $imp(G) > 1.2079$ and the conjectured upper bound $imp(G) = 1.5$ is achievable. In particular, we show that: if G is an odd cycle of length ≥ 5 or is the Harary graph $H_{2m,3m+2}$ where m is odd, then $imp(G) = 1.5$, and if G is the wheel W_6, then $imp(G) = 4/3$.

Keywords: Wireless interference, clique number, chromatic number, unit disk graph, imperfection ratio.

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1 Introduction

A wireless ad hoc network (or simply a wireless network) consists of a set of nodes that communicate with each other without any physical infrastructure or centralized administration. The interference problem between nodes in a wireless network is important and difficult and it can be modeled using graph theoretic techniques, in particular, the theory associated with Unit Disc Graphs (UDGs). As we will see below, the chromatic number of a UDG model of a wireless network is directly related to the interference problem. The chromatic number is a graph invariant. The clique number is another graph invariant and is closely related to the chromatic number. In some special cases, the clique number is equal to the chromatic number.

Before going further, we give some definitions. Our graph terminology and notation are standard; see [2] and [11] except as indicated. All graphs in thesis are assumed simple. Let $G = (V, E)$ be a graph. We say that G is *k-colorable* if the vertices of G can be colored by using at most k colors such that no two adjacent vertices receive the same color. The *chromatic number* of G, denoted by $\chi(G)$, is defined to be the smallest k such that G is k-colorable. A *clique* of G is a complete subgraph in G. A *maximum clique* of G is a clique of the largest possible size in G. The *clique number* of G, denoted by $\omega(G)$, is the number of vertices in a maximum clique in G.

It is well known that the graph coloring problem is NP-complete and that even the problem of approximating the chromatic number within any constant ratio is NP-hard [6]. In [3], Clark et al. proved that the coloring problem remains NP-complete for UDGs. In fact, Clark et al. proved that the problem of determining, given a UDG G, whether G is 3-colorable is NP-complete. Notice that in [1], Breu and Kirkpatrick have proved that the problem of determining, given a graph G, whether G is a UDG is NP-hard. In [5], Graf et al. improved the result of Clark et al. by showing that the problem of determining, given a UDG G and a fixed integer k, whether G is k-colorable remains NP-complete for any fixed $k \geq 3$; they also proposed a 3-approximation algorithm for the coloring problem.

It is clear that for any graph G, the chromatic number is always lower bounded by the clique number, i.e., $\chi(G) \geq \omega(G)$. For the special case of "perfect graphs", the chromatic number and the clique number have equal values in every induced subgraph. While computing $\chi(G)$ is still NP-complete for UDGs, computing $\omega(G)$ can be done in polynomial time for UDGs [3].

We assume that the given wireless network has n nodes and their respective position coordinates is in 2D. The *transmission range* (TR) of a given node is defined as the maximum distance at which the nodes transmission can be successfully received, and all nodes that lie within the transmission range of a given node are called its *communicating neighbors*. The *interference range* (IR) is defined as the maximum distance at which a given node's transmission can interfere with or corrupt a simultaneous transmission or reception attempt by another node, and all nodes that lie within interference range of a given node are called its *interfering neighbors*. Clearly, all communicating neighbors are interfering neighbors and vise versa.

Recently, in [9], Mani and Petr treated the case in which IR is the same for all nodes. The graph model is a UDG and is called an *interference graph*. More precisely, a UDG G is formed by taking the nodes in the wireless network as its vertices, and there is an edge between vertices u and v if and only if the Euclidean distance between u and v, denoted by $d(u, v)$, is less than or equal to 1. Notice that we will use the terms node and vertex interchangeably. If two nodes share an edge, then it means that they are mutually interfering and hence cannot transmit simultaneously in the same timeslot. There are two possible scenarios: balanced load scenario and unbalanced load scenario. In the former case, each node require the identical number of transmission timeslots per second to suit their traffic requirements; as a result, the chromatic number gives the minimum number of timeslots per second. See Figure 1. However, a balanced load scenario rarely occurs in the real world.

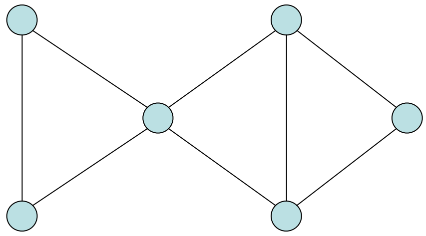

Figure 1: This graph has chromatic number 3. If a balanced load scenario occurs, then three timeslots are required.

In [9], Mani and Petr considered the unbalanced load scenario, wherein the traffic rates of each node need not be identical. In particular, for each node v_i, let r_i be the number of timeslots required per second by v_i to satisfy its traffic needs. The UDG now becomes a weighted UDG such that each vertex v_i has a weight r_i associated with it. To find out the optimal (i.e., minimum) number of timeslots required per second for a weighted UDG $G = (V, E)$, Mani and Petr used weighted vertex coloring [4] algorithms, which is simply normal (un-weighted) coloring done on a transformed graph G'. The graph $G' = (V', E')$ is obtained from G by replacing each vertex v_i in G by a clique of size r_i and the edge set E is augmented to obtain E' in such a way that if two vertices u, v are neighbors in G, then in G' every node in the clique corresponding to u is also a neighbor of every node in the clique corresponding to v. (See Figure 2.)

As was mentioned above, the chromatic number $\chi(G)$ of a UDG model of a wireless network is directly related to interference. Closely related to $\chi(G)$ is the clique number $\omega(G)$. For most classes of graphs, computing $\chi(G)$ and $\omega(G)$ are both NP-complete. But for UDGs, while computing $\chi(G)$ is still NP-complete, computing $\omega(G)$ can be done in polynomial time. This raises the question: How close is $\omega(G)$ to $\chi(G)$? For general

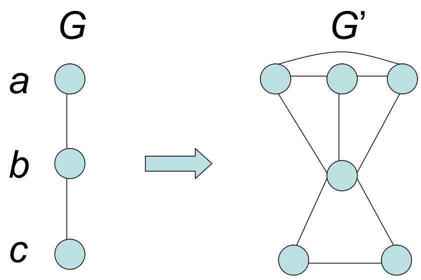

Figure 2: Transforming a weighted graph G into an un-weighted graph G'; the weights on vertices a, b, c in G are 3, 1, 2, respectively.

graphs, $\chi(G)/\omega(G)$ can be very large. In [10], Peeters has observed that

$$\chi(G) \leq 3\omega(G) - 2 \text{ if } G \text{ is a UDG.}$$

See also [7]. In [9], Mani and Petr performed extensive simulations with UDGs of random networks and observed that in a UDG G, the clique number $\omega(G)$ and the chromatic number $\chi(G)$ were typically very close to one another. To evaluate the closeness of $\chi(G)$ and $\omega(G)$, Mani and Petr used the measure "imperfection ratio"

$$imp(G) = \sup_R \frac{\chi(G')}{\omega(G')}$$

of a transformed weighted graph, defined as the supremum of the ratio of its chromatic number to its clique number. Here the supremum is computed over all possible weight vectors R.

It has been proven that the theoretical bound of $imp(G)$ is 2.155 and $imp(G) = 1$ if and only if G is perfect [4]. Based on the simulation results, Mani and Petr concluded that a practical bound for UDGs is $imp(G) \leq 1.2079$, which is far less than the conjectured upper bound of 1.5 or the theoretic upper bound of 2.155. The following is Mani and Petr's simulation scenario: they assume the simulation area is a disk of radius 1 and place n nodes in randomly chosen locations within the disc. Node v_i is assigned an integer weight r_i that corresponds to its traffic requirements. The weights are chosen randomly in $1, 2, \ldots, K$, where K corresponds to the maximum weight. They varied n as 10, 25, 50, 75 and 100 and independently varied K as 1, 5 10, 20, 30, 40, and 50. The smallest mean size of the UDG (in terms of number of vertices) is 10 and the largest is 2550. They observed that in a UDG G, $\omega(G)$ and $\chi(G)$ were typically very close to one another. Based on the simulation results, Mani and Petr concluded that a practical bound for UDGs is $imp(G) \leq 1.2079$ and $\omega(G)$ can be used as a very good approximation to $\chi(G)$; in particular, they said that a practical bound of $\chi(G) \leq 1.21\omega(G)$ could be used if G is a UDG.

The purpose of this thesis is to show that there exist UDGs such that $imp(G) > 1.2079$, and moreover, the conjectured upper bound $imp(G) = 1.5$ is achievable. In particular, we show that: if G is an odd cycle of length ≥ 5 or is the Harary graph $H_{2m,3m+2}$ where m is odd, then $imp(G) = 1.5$, and if G is the wheel W_6 (see Figure 3), then $imp(G) = 4/3$. We also propose an algorithm to color the nodes of a UDG and perform simulations to compare the number of colors used by our algorithm and that used by the First-Fit coloring algorithm.

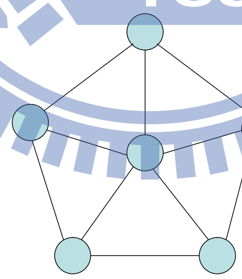

Figure 3: W_6, the wheel graph with 6 vertices.

This thesis is organized as follows. In Section 2, we gives UDGs with $imp(G) > 1.2079$. In Section 3, we propose a coloring algorithm and compare the results of our algorithm with the classical First-Fit coloring algorithm. Concluding remarks are given in the final section.

2 UDGs with $imp(G) > 1.2079$

The fact that $imp(G) \geq \chi(G)/\omega(G)$ will be used throughout this section.

Lemma 2.1. *There exists a general graph G such that $imp(G) \rightarrow \infty$.*

Proof. This lemma follows from the fact that we can use Mycielski construction [11] to obtain a new triangle-free graph G^* from a given triangle-free graph G such that $\chi(G^*) = \chi(G) + 1$ and $\omega(G^*) = \omega(G) = 2$. ∎

Before going further, we introduce some notations. Let G be a weighted graph. We use $w_G(v)$ to denote the weight of a vertex v in G and use G' to denote the un-weighted transformed graph of G; see Section 1 and Figure 2 for an illustration of G'. For convenience, we use S_v to denote the set of vertices in G' that correspond to a vertex v in G. Let C_n denote a cycle of length n. It is not difficult to verify that C_n is a UDG. C_n is called an *odd cycle* if it is of odd length. We have the following theorem.

Theorem 2.2. *If G is an odd cycle of length ≥ 5, then $imp(G) = 1.5$.*

Proof. Since $\chi(G) = 3$ and $\omega(G) = 2$, we have $imp(G) \geq 1.5$. On the other hand, let $V(G) = \{v_1, v_2, \ldots, v_n\}$, $w_G(v_i) = r_i$ for each i, and $E(G) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}$. Without loss of generality, we may assume that $r_1 + r_2 = \omega(G')$. Clearly,

$$\begin{cases} r_2 + r_3 \leq r_1 + r_2 \\ r_n + r_1 \leq r_1 + r_2. \end{cases}$$

Thus $r_3 + r_n \leq r_1 + r_2 = \omega(G')$. By the pigeonhole principle, $r_3 \leq 0.5\omega(G')$ or $r_n \leq 0.5\omega(G')$ must occur. Suppose $r_n \leq 0.5\omega(G')$ occurs; the case that $r_3 \leq 0.5\omega(G')$ occurs can be proven in a similar way and we omit its proof. To prove this theorem, it suffices to prove that G' is $1.5\omega(G')$-colorable.

Since $|S_{v_1} \cup S_{v_2} \cup S_{v_n}| \leq 1.5\omega(G')$, the subgraph induced by $S_{v_1} \cup S_{v_2} \cup S_{v_n}$ is $1.5\omega(G')$-colorable. Denote the set of colors on $S_{v_1} \cup S_{v_2}$ by $\mathcal{C}^{1,2}$ and denote the set of colors on S_{v_n}

by $\mathcal{C}^n$. Notice that $\mathcal{C}^{1,2} \cap \mathcal{C}^n = \emptyset$. Then, color the vertices in S_{v_i} according to the ordering $i = 3, 4, \ldots, n-1$. For each i, before S_{v_i} is to be colored, the vertices in $S_{v_{i-2}} \cup S_{v_{i-1}}$ have already been colored; denote the set of colors on $S_{v_{i-1}}$ by $\mathcal{C}^{i-1}$. Since G is an odd cycle of length ≥ 5, in G' there is no edge joining a vertex in S_{v_i} and a vertex in $S_{v_{i-2}}$. Hence it is possible to colors the vertices in S_{v_i} by using the colors in $\mathcal{C}^{1,2} \setminus \mathcal{C}^{i-1}$. The above process proves that G' is $1.5\omega(G')$-colorable. ∎

A wheel graph with n vertices, denoted by W_n, is the graph obtained by adding a new vertex to the cycle C_{n-1} and making this new vertex joining each vertex of C_{n-1}. It is not difficult to see that W_6 is a UDG. We have the following theorem for W_6.

Theorem 2.3. *If $G = W_6$, then $imp(G) = 4/3$.*

Proof. Clearly, $imp(G) \geq \frac{\chi(G)}{\omega(G)} = 4/3$. Suppose $V(G) = \{v_1, v_2, \ldots, v_6\}$, $w_G(v_i) = r_i$ for each i, and $E(G) = \{v_1v_2, v_2v_3, \ldots, v_4v_5, v_5v_1\} \cup \{v_iv_6 | 1 \leq i \leq 5\}$. Let H be the subgraph of G induced by $\{v_1, v_2, \ldots, v_5\}$. Then $\frac{\chi(G')}{\omega(G')} = \frac{\chi(H')+w_G(v_6)}{\omega(H')+w_G(v_6)}$ and if we want to make $\frac{\chi(G')}{\omega(G')}$ as large as possible, then we must have $w_G(v_6) = 1$. Without loss of generality, we may assume that $r_1 + r_2 + 1 = \omega(G')$. Set $\omega = r_1 + r_2 + 1$ for easy writing. There are two cases.

Case 1: $r_3 \leq \omega/3$ or $r_5 \leq \omega/3$. Suppose $r_5 \leq \omega/3$ occurs; the case that $r_3 \leq \omega/3$ occurs can be proven in a similar way and we omit its proof. To prove this theorem, it suffices to prove that G' is $\frac{4}{3}\omega$-colorable. Since $|S_{v_1} \cup S_{v_2} \cup S_{v_5}| \leq \frac{4}{3}\omega - 1$, the subgraph induced by $S_{v_1} \cup S_{v_2} \cup S_{v_5}$ is $(\frac{4}{3}\omega - 1)$-colorable. Denote the set of colors on $S_{v_1} \cup S_{v_2}$ by $\mathcal{C}^{1,2}$ and denote the set of colors on S_{v_5} by $\mathcal{C}^5$. Notice that $\mathcal{C}^{1,2} \cap \mathcal{C}^5 = \emptyset$. Then, color the vertices in S_{v_i} according to the ordering $i = 3, 4$. For each i, before S_{v_i} is to be colored, the vertices in $S_{v_{i-2}} \cup S_{v_{i-1}}$ have already been colored; denote the set of colors on $S_{v_{i-1}}$ by $\mathcal{C}^{i-1}$. Since H is an odd cycle of length 5, in G' there is no edge joining a vertex in S_{v_i} and a vertex in $S_{v_{i-2}}$. Hence it is possible to colors the vertices in S_{v_i} by using the colors in $\mathcal{C}^{1,2} \setminus \mathcal{C}^{i-1}$.

Assign v_6 any color that is not used on $S_{v_1} \cup S_{v_2} \cup S_{v_3} \cup S_{v_4} \cup S_{v_5}$. The above process proves that G' is $\frac{4}{3}\omega$-colorable.

Case 2: $r_3 > \omega/3$ and $r_5 > \omega/3$. It suffices to prove that G' is $\frac{4}{3}\omega$-colorable. Since $|S_{v_1} \cup S_{v_2}| = \omega - 1$, the subgraph induced by $S_{v_1} \cup S_{v_2}$ is $(\omega-1)$-colorable. Denote the set of colors on S_{v_1} and S_{v_2} by $\mathcal{C}^1$ and $\mathcal{C}^2$, respectively. Let $\mathcal{C}^0$ be a set of $\omega/3$ colors such that $\mathcal{C}^0 \cap (\mathcal{C}^1 \cup \mathcal{C}^2) = \emptyset$. Then, color the vertices in S_{v_3} by using the colors in $\mathcal{C}^0$ as their first choice and the colors in $\mathcal{C}^1$ as their secondary choice. Since $|S_{v_3}| = r_3$, the number of colors used on S_{v_3} in the set $\mathcal{C}^1$ equals $r_3 - \omega/3$. Let $\mathcal{C}_3$ denote the set of colors on S_{v_3}. Then, color the vertices in S_{v_5} by using the colors in $\mathcal{C}^0$ as their first choice and the colors in $\mathcal{C}^2$ as their secondary choice. Since $|S_{v_5}| = r_5$, the number of colors used on S_{v_5} in the set $\mathcal{C}^2$ equals $r_5 - \omega/3$. Let $\mathcal{C}_5$ denote the set of colors on S_{v_5}. Then, color the vertices in S_{v_4} by using the colors in $\mathcal{C}^1 \cup \mathcal{C}^2$. Since $r_1 \geq r_3 > \omega/3$ and $r_1 + r_5 < \omega$ together imply that $r_5 < 2\omega/3$, we have

$$r_1 + r_2 \geq r_3 + r_4 > r_3 + r_4 + r_5 - 2\omega/3 = (r_3 - \omega/3) + r_4 + (r_5 - \omega/3)$$

and therefore it is possible to color the vertices in S_{v_4} by using the colors in $(\mathcal{C}^1 \cup \mathcal{C}^2) \setminus (\mathcal{C}^3 \cup \mathcal{C}^5)$. Assign v_6 any color that is not used on $S_{v_1} \cup S_{v_2} \cup S_{v_3} \cup S_{v_4} \cup S_{v_5}$. The above process proves that G' is $\frac{4}{3}\omega$-colorable. ∎

Let H be an induced subgraph of G. We now show that it is possible that neither $imp(G) > imp(H)$ nor $imp(G) < imp(H)$ holds.

Observation 2.4. *There exists a UDG G such that $imp(G) > imp(H_1)$ and $imp(G) < imp(H_2)$ for some induced subgraph H_1 and H_2 of G.*

Proof. Let $G = W_6$, $H_1 = C_3$, and $H_2 = C_5$. Clearly, $imp(C_3) = 1$. By Theorems 2.3 and 2.2, $imp(W_6) = 4/3$ and $imp(C_5) = 1.5$. Thus we have $imp(W_6) > imp(C_3)$ and $imp(W_6) < imp(C_5)$. ∎

We now define the Harary graph $H_{k,n}$; see also [11]. Given $k < n$, place n vertices around a circle, equally spaces. Let the vertices be $0, 1, \dots, n-1$; the edges are added in the following ways.

Case 1: $k = 2m$. Add an edge between i and j whenever $i - m \le j \le i + m \pmod{n}$.

Case 2: $k = 2m+1$ *and* n *is even.* Construct $H_{k,n}$ from $H_{k-1,n}$ by adding an edge between i and $i + n/2$ for each $1 \le i \le n/2$.

Case 3: $k = 2m+1$ *and* n *is odd.* Construct $H_{k,n}$ from $H_{k-1,n}$ by adding an edge between 0 and $(n-1)/2$, 0 and $(n+1)/2$, i and $i + (n+1)/2$ for each $1 \le i \le (n-1)/2$.

Theorem 2.5. *Consider the Harary graph G of the form $H_{2m,3m+2}$. Then $imp(G) = 1.5$ if m is odd, $\frac{3m+2}{2m+2} \le imp(G) \le 1.5$ if m is even.*

Proof. Let $V(G) = \{v_1, v_2, \dots, v_{3m+2}\}$. This theorem holds if we can prove that $imp(G) = \frac{\lceil \frac{3m+2}{2} \rceil}{m+1}$. We first prove that $imp(G) \ge \frac{\lceil \frac{3m+2}{2} \rceil}{m+1}$. An *independent set* of a graph is a subset of its vertex set such that each pair of vertices in this subset are not adjacent. Since an independent set of a Harary graph is of size at most 2, each color can be used at most twice. Consequently, $\chi(G) \ge \left\lceil \frac{|V(G)|}{2} \right\rceil = \lceil \frac{3m+2}{2} \rceil$. Define a coloring

$$f : V(G) \to \{1, 2, \dots, \lceil \tfrac{3m+2}{2} \rceil\}$$

as follows:

(i) If m is odd, let $f(v_i) = f(v_{i+\frac{3m+1}{2}}) = i$ for each $1 \le i \le \frac{3m+1}{2}$ and let $f(v_{3m+2}) = \lceil \frac{3m+2}{2} \rceil$.

(ii) If m is even, let $f(v_i) = f(v_{i+\frac{3m+2}{2}}) = i$ for each $1 \le i \le \frac{3m+2}{2}$.

Hence $\chi(G) \le \lceil \frac{3m+2}{2} \rceil$. Therefore $\chi(G) = \lceil \frac{3m+2}{2} \rceil$. Notice that $\omega(G) = m + 1$. Thus we have $imp(G) \ge \frac{\lceil \frac{3m+2}{2} \rceil}{m+1}$.

We now prove that $imp(G) \le 1.5$. For each i, suppose $w_G(v_i)$ has weight r_i. Without loss of generality, we may assume that $r_1 + r_2 + \dots + r_{m+1} = \omega(G')$. Set $\omega = \omega(G')$ for easy writing. To prove this theorem, it suffices to prove that G' is 1.5ω-colorable. Let $k = \min\{0 \leq k \leq m | \sum_{i=1}^{k+1} r_i > 0.5\omega\}$. Then

$$\begin{cases} r_{m+2} + r_{m+3} + \cdots + r_{m+k+1} \leq r_1 + r_2 + \cdots + r_k \leq 0.5\omega \\ r_{2m+k+3} + r_{2m+k+4} + \cdots + r_{3m+2} \leq r_{k+2} + r_{k+3} + \cdots + r_{m+1} < 0.5\omega. \end{cases}$$

Since there is no edge joining a vertex in S_{v_i} and a vertex in S_{v_j} for all $m+2 \leq i \leq m+k+1$ and $2m+k+3 \leq j \leq 3m+2$, $\{\cup_{i=1}^{m+k+1} S_{v_i}\} \cup \{\cup_{i=2m+k+3}^{3m+2} S_{v_i}\}$ is 1.5ω-colorable. Denote the set of colors on S_{v_i} by $\mathcal{C}^i$ for $1 \leq i \leq m+1$. Then, $\sum_{j=1}^{i} |\mathcal{C}^j| > \sum_{j=m+k+2}^{m+i+1} |S_{v_j}|$ for $i = k+1, k+2, \ldots, m$, $\sum_{j=i}^{m+1} |\mathcal{C}^j| > \sum_{j=2m+i+1}^{2m+k+2} |S_{v_j}|$ for $i = k+1, k, \ldots, 1$ and $\sum_{j=1}^{m+1} |\mathcal{C}^j| \geq \sum_{j=m+k+2}^{2m+k+2} |S_{v_j}|$. This implies that we can color the vertices in S_{v_i} according to the ordering $i = m+k+2, m+k+3, \ldots, 2m+1, 2m+k+2, 2m+k+1, \ldots, 2m+2$ by using the colors in $\cup_{i=1}^{m+1} \mathcal{C}^i$. Thus G' is 1.5ω-colorable. ∎

We now list some imperfection ratios of Harary graphs $H_{2m,3m+2}$. It is not difficult to see that $imp(H_{2m,3m+2}) \to 1.5$ when m is even.

m	$3m+2$	$\chi(G)$	$\omega(G)$	lower bound of $imp(G)$	upper bound of $imp(G)$
1	5	3	2	1.5	1.5
2	8	4	3	1.333	1.5
3	11	6	4	1.5	1.5
4	14	7	5	1.4	1.5
5	17	9	6	1.5	1.5
6	20	10	7	1.429	1.5
7	23	12	8	1.5	1.5

Table 1: The imperfection ratios of Harary graphs $H_{2m,3m+2}$.

3 Our coloring algorithm and simulation results

For convenience, let $c(G)$ by the number of colors used by a given algorithm. The performance of an algorithm is defined by $PR(G) = c(G)/\omega(G)$. Let *First-Fit* (also called a greedy coloring algorithm) denote the coloring algorithm that examines the vertices of a graph in an arbitrary order and assigns each vertex the smallest-indexed color not already used on its examined neighbors. To improve First-Fit, we examines the vertices of a graph in the order obtained by breadth-first search (BFS) form any node and we call our algorithm *BFS-First-Fit*. Since Mani and Petr [9] mentioned that $\chi(G)/\omega(G)$ is at most 1.2079 in their simulation results, if BFS-First-Fit obtains $PR(G) > 1.2$, then we will run BFS-First-Fit again by choosing another vertex as the root of the BFS until $PR(G) \leq 1.2$ or the above process has been repeated too many times (in this thesis, the threshold value of 10 is chosen). Notice that we can adjust the value 1.2 in $PR(G) > 1.2$ and the number of times that the root of BFS-First-Fit is changed to get a better performance.

This section is divided into four subsections. In Subsection 3.1, we consider randomly generated UDGs. In Subsection 3.2, we consider randomly generated weighted UDGs. In Subsection 3.3, we consider randomly generated UDGs in which the nodes are not evenly distributed. And in Subsection 3.4, we consider randomly generated UDGs that allow the addition of new nodes. For each subsection, simulations results for First-Fit and BFS-First-Fit are obtained.

3.1 Randomly generated unit disk graphs

To perform the simulations, we randomly construct 500 connected UDGs with n nodes in a $100\text{m} \times 100\text{m}$ area, where n is ranged from 100 to 500, with an increment of 50. The interference range of each node is assumed to be 25m.

Figure 4 shows the average $PR(G)$ obtained by First-Fit and BFS-First-Fit. Both of them increase as the number of nodes increases. The performance of BFS-First-Fit is better than that of First-Fit in all cases and the difference between them increases as the number of nodes increases.

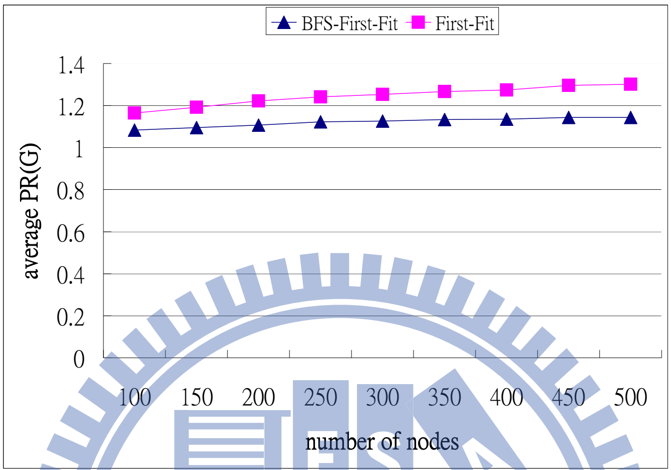

Figure 4: The average $PR(G)$.

Figure 5 shows the maximum $PR(G)$ obtained by First-Fit and BFS-First-Fit. We observe that maximum $PR(G)$ is about 1.2 if BFS-First-Fit is used and about 1.5 if First-Fit is used.

Among our 4500 simulations, $PR(G) \leq 1.2$ occurs for almost all cases, and there are only 7 simulations with $PR(G) > 1.2$. Among the 7 simulations with $PR(G) > 1.2$, the maximum $PR(G)$ of them is 1.2381, which is larger than 1.2079. There is an example of $PR(G) = 1.211$ in Figure 6; this UDG has 200 nodes and we color the edges of its maximum clique in color red. Based on our simulation results, we conclude that BFS-First-Fit has $PR(G) \leq 1.2$ for most of the cases.

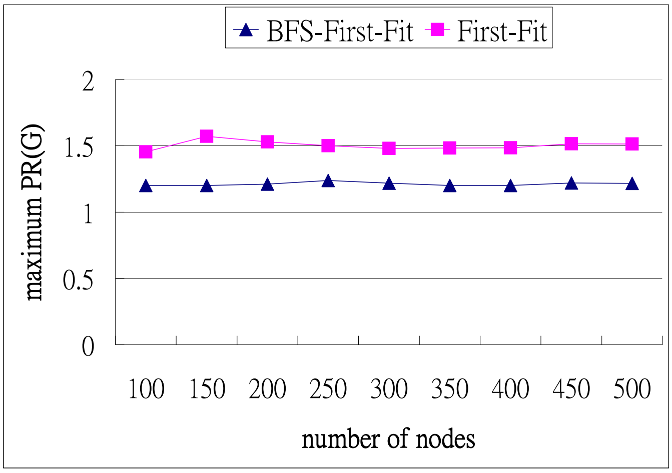

Figure 5: The maximum $PR(G)$.

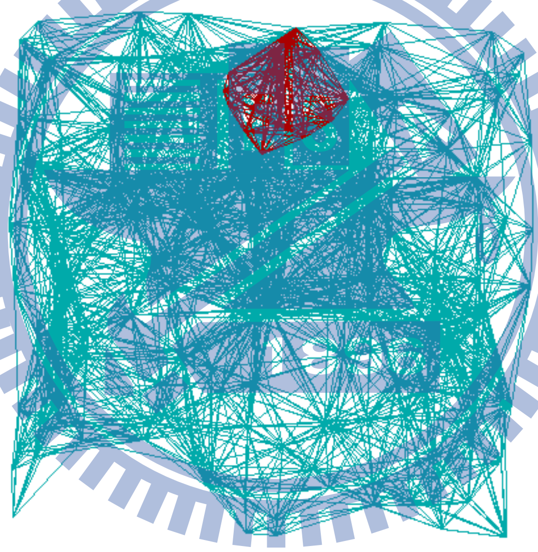

Figure 6: An example of $PR(G) = 1.211$.

3.2 Randomly generated weighted unit disk graph

In this subsection, we consider weighted UDGs and we use the same parameters as in [9] to compare the $PR(G)$ obtained by BFS-First-Fit with the simulation results in [9].

More precisely, we assume that n nodes are chosen randomly from a disk of radius 1 and each node has interference range 1. Node i has weight r_i. The weights are chosen randomly from $1, 2, \ldots, K$, where K corresponds to the maximum weight. We vary the number of nodes n as $10, 25, 50, 75$ and 100. For each n, we also vary the maximum weight K as $1, 5, 10, 20, 30, 40$ and 50. For each (n, K) pair, we perform 500 simulations. Table 2 shows the average value of $PR(G)$, and Table 3 shows the maximum value of $PR(G)$. From Table 2, we observe that $PR(G)$ increases as the number of nodes increases, but there are no obvious relation between $PR(G)$ and the maximum weight K. In Table 3, the maximum value of $PR(G)$ is 1.222.

n	$K = 1$	$K = 5$	$K = 10$	$K = 20$	$K = 30$	$K = 40$	$K = 50$
10	1.007	1.007	1.005	1.005	1.005	1.006	1.005
25	1.039	1.033	1.033	1.034	1.032	1.032	1.034
50	1.063	1.060	1.055	1.059	1.056	1.056	1.056
75	1.075	1.072	1.074	1.070	1.071	1.068	1.070
100	1.086	1.082	1.081	1.079	1.079	1.080	1.081

Table 2: The average value of $PR(G)$.

n	$K = 1$	$K = 5$	$K = 10$	$K = 20$	$K = 30$	$K = 40$	$K = 50$
10	1.167	1.111	1.138	1.145	1.135	1.213	1.153
25	1.222	1.175	1.138	1.146	1.149	1.126	1.166
50	1.176	1.169	1.181	1.136	1.172	1.153	1.158
75	1.160	1.167	1.188	1.195	1.159	1.185	1.187
100	1.182	1.208	1.186	1.179	1.186	1.166	1.182

Table 3: The maximum value of $PR(G)$.

3.3 Randomly generated unit disk graphs with different density of nodes

In this subsection, we consider UDGs with different density of nodes. We randomly construct 500 connected UDGs with n nodes in a 100m $\times$ 100m area (for convenience, denote this area by $\mathcal{A}$), where n is ranged from 100 to 500, with an increment of 50. The interference range of each node is assumed to be 25m. We consider four scenarios as follows.

(a) One of the four corner areas of $\mathcal{A}$ has more nodes. (See Figure 7(a).)

(b) The center area of $\mathcal{A}$ has more nodes. (See Figure 7(b).)

(c) The area near one side of $\mathcal{A}$ has more nodes. (See Figure 7(c).)

(d) The area near the middle line of $\mathcal{A}$ has more nodes. (See Figure 7(d).)

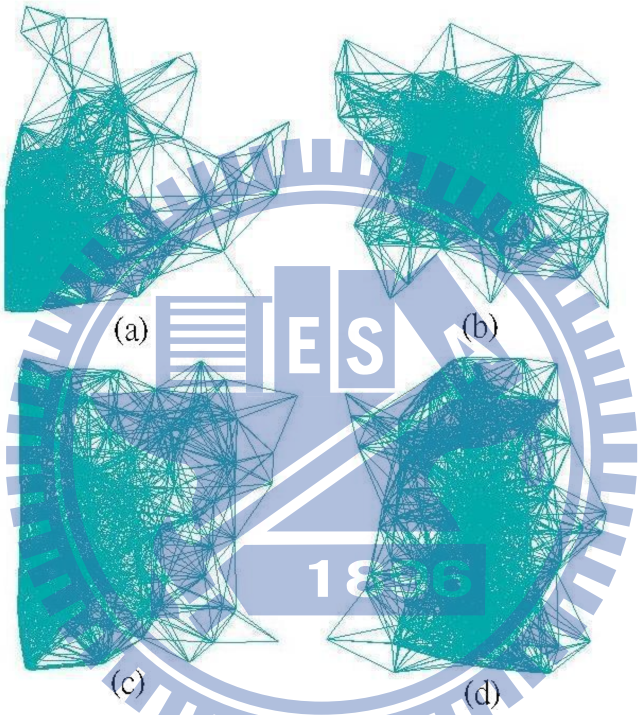

Figure 7: An example of different density of nodes.

From our simulations, we observe that the selection of the root of the BFS will effect the performance of BFS-First-Fit. Choosing the root in the area with high density of nodes will make BFS-First-Fit have a better performance but the difference is not big; thus we omit the details of these simulation results.

Figure 8 shows the average $PR(G)$ obtained by First-Fit and BFS-First-Fit. We can observe that the average $PR(G)$ obtained by First-Fit and BFS-First-Fit are very close in (a) and (b). The difference between them decreases as the number of nodes increases and this is because the chromatic number is very close to the clique number; so there is no big improvement. In (c) and (d), the improvement is larger.

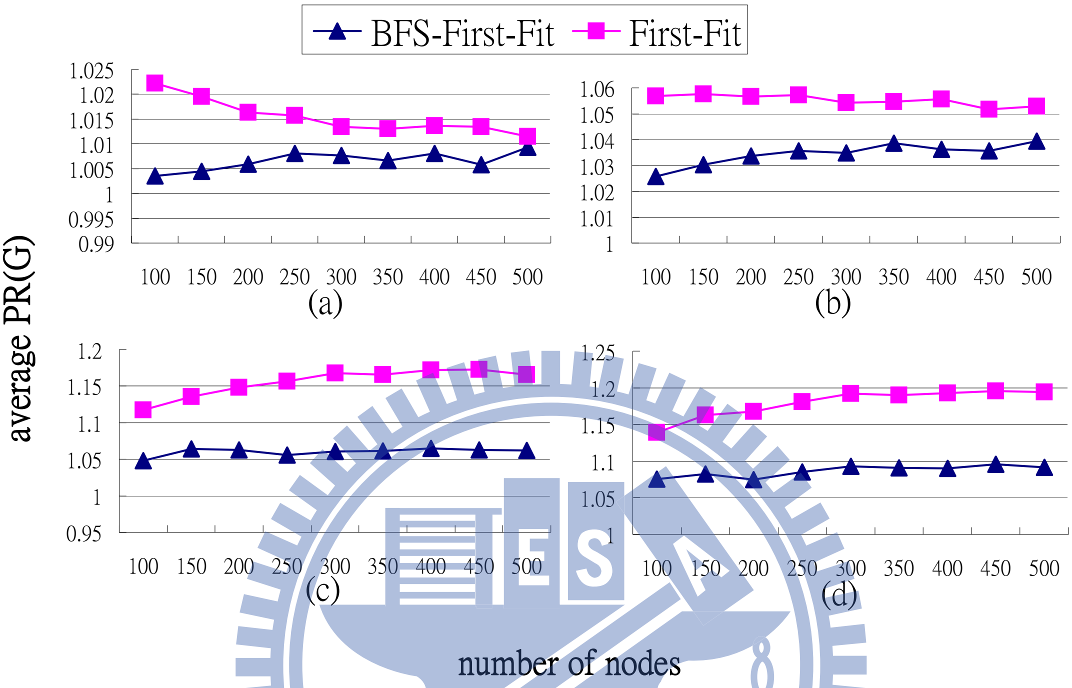

Figure 8: The average $PR(G)$ in UDGs with different density of nodes.

3.4 Randomly generated unit disk graphs that allow the addition of nodes

In real world, many networks have some nodes that do not exist initially but are added later. In this case, BFS-First-Fit can only be applied on initial nodes and when there are addition nodes, BFS-First-Fit must be restart for all nodes if we want to use it.

In this section, we simulate the UDGs which allows nodes to be added. We randomly construct 500 connected UDGs with initial n nodes in a 100m $\times$ 100m area, where n is 100 or 200. And each time we add $n/4$ nodes and totally we add the nodes for five times. The interference range of each node is assumed to be 25m. We use BFS-First-Fit to color the initial n nodes and use First-Fit to color the added. We compare results of such a BFS-First-Fit plus First-Fit manner with results that only use First-Fit.

Figure 9 shows the average $PR(G)$ of BFS-First-Fit and First-Fit. We observe that when the number of added nodes is more than $3n/4$, the $PR(G)$ of BFS-First-Fit and First-Fit become very close. Thus we suggest that when the wireless network allows the addition of nodes, BFS-First-Fit must be restarted if the number of added nodes is more than $3n/4$.

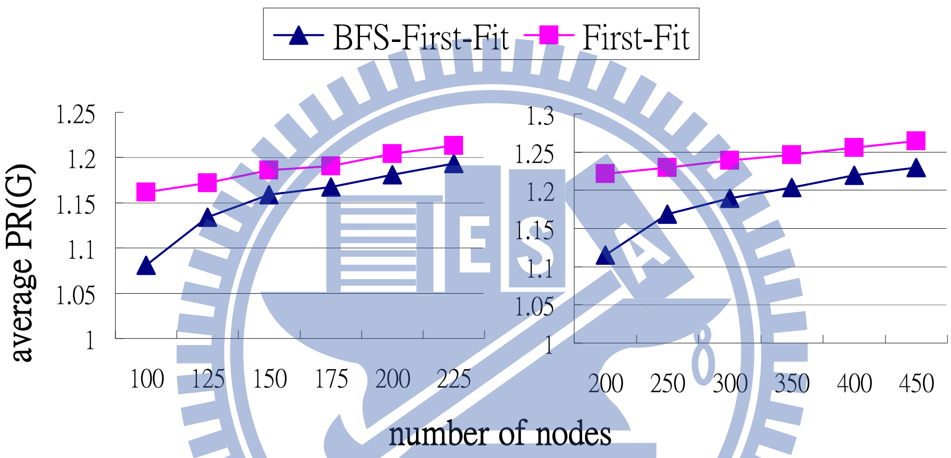

Figure 9: The average $PR(G)$ in UDGs that allow the addition of nodes.

4 Concluding remarks

In this thesis, we propose UDGs with $imp(G) > 1.2079$ and we propose simulations to compare the difference between the clique number and the chromatic number obtained by our algorithm and by First-Fit; some different types of random UDGs have been considered. We find that in almost all cases, our algorithm can color the graph G with $\chi(G) < 1.2\omega(G)$ colors. In the future, the theoretical bound of our algorithm will be considered.

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