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碩 士 論 文

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Limit Theorems for the Cost of Splitting Random Cayley Trees

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前 言

在計算科學中，常常需要處理將集合合併的問題，而為了有效率的解決這個問題，需要建造適當的的資料結構和相對應的演算法，這類的問題又稱為"Union-Find problem"; 事實上，已經有許多種不同的資料結構和相對應的演算法被提出來以解決這個問題，為了了解各個方法的優劣，我們考慮最簡單的狀況，也就是將 n 個不同的元素兩兩合併，直到成為一個集合為止所需的複雜度。在這篇論文中，我們假設每個合併過程所需的複雜度是被合併的兩個集合的大小之和的冪次方，求其在隨機生成樹的機率模型底下，所需總複雜度的極限分布。

在這裡，我們使用一個近幾年才發展出來的新工具，稱為奇異點分析 (Singularity Analysis); 這個工具可以讓我們直接由生成函數的漸進展開式得到生成函數係數的漸進式。我們將使用這個方法來計算總複雜度的各階動差，進而再利用動差法推導出總複雜度的極限分布。

這篇論文的一開始，也就是第一章我們先介紹問題的背景並給出我們研究的結果; 在第二章我們介紹我們研究所使用的工具，也就是奇異點分析; 而第三章是這篇這篇論文最主要的部分，我們利用奇異點分析導出複雜度的期望值和各階動差並在第四章利用動差法證明了我們的結果; 最後在第五章我們對整篇論文做一個總結。

Preface

In computer science, the so-called "Union-Find problem" is concerned with establishing a data structure for maintaining a collection of disjoint sets such that the process of merging sets can be carried out efficiently. Indeed, several data structures and corresponding algorithms for merging sets have been proposed. For the purpose of comparing the complexity of these algorithms, it is naturally to consider the total cost incurred from merging n singleton sets into one set. In this thesis, we assume that the cost of each merging step is the power of the sum of the sizes of the sets being merged and then derive the expected value and the limiting distribution of the total cost under the random spanning tree model.

The main tool used in this thesis is singularity analysis, which is a method connecting the asymptotics of generating functions with the asymptotics of their coefficients. We will use it to derive the moments of each order. Then, with the method of moments, the limiting distribution of the total cost will follow.

In the first part of this thesis (Chapter 1), we introduce the problem of interest and state the results of our work. In Chapter 2, we give an introduction about our main tool, singularity analysis. The central part of this thesis, namely Chapter 3, is devoted to the derivation of the expected value and higher moments. These results will will then be used in Chapter 4 for prove our main result. Finally, we end the thesis with a conclusion in Chapter 5.

誌　謝

時光匆匆，三年的時間一眨眼就過了，回想這三年學習的日子，在工作和課業兩頭燒的情況下，總會有疲乏的時候想逃避的時候，但也多虧了身邊朋友同事們的鼓勵與幫助，我才能順利完成學業。當然，首先感謝的是我的指導老師--符麥克教授，不只在研究方面不厭其煩的給我鼓勵並引導我前進，他的教學態度與方法更讓我在數學的專業和教學上都有更深一層的體悟；再來要感謝大甲國中的夥伴，特別是美宇和玉珍，有妳們的支持與協助，我才能安心的到交大進修；還有光祥學長、鈺傑學長、憶奴和紀葳，謝謝你們帶著我熟悉這個交大這個陌生的環境；最後感謝兩位口試委員，對我的論文提出了許多寶貴的建議。

Contents

Chapter 1

Introduction

In computer science, some applications involve grouping n distinct elements into a collection of disjoint sets and then performing two operations $union(x, y)$ and $find(x)$ on the disjoint sets, with

(1) $union(x, y)$: unites the sets that contain x and y, say S_x and S_y, into a new set that is the union of these two sets.

(2) $find(x)$: returns the representative of the set containing x.

As an example where these operations are used consider the following algorithm for the construction of the minimum spanning tree of an undirected graph: initially, we set each vertex as a singleton set. Then, we find edges (m, n) by some strategy and check whether m and n are in the same set or not, i.e. whether $find(m)$ equals to $find(n)$. If they are not, we add (m, n) to the tree being constructed and merge the set containing m and n, i.e. $union(m, n)$. If m and n are already in the same set, just drop the edge (m, n) and find a new one.
The central problem for the above example is to establish a data structure for maintaining a collection of disjoint sets so that a sequence of union and find instructions can be

carried out efficiently. (this problem is called the "Union-Find problem" in [2, Cphater 4]). Indeed, several data structures and corresponding algorithms for $union(x, y)$ and $find(x)$ have been proposed. Following Yao [12], the data structure used to represent a set is a rooted tree and four algorithms for implementing $union(x, y)$ and $find(x)$ are mentioned, namely Quick-Find Algorithm, Quick-Merge Algorithm, Quick-Find with Weighting Rule, Quick-Merge with Weighting Rule. Moreover, the operations $union(x, y)$ and $find(x)$ take different cost under each algorithm.

Let us now return to our original problem. Suppose that initially there is a collection of n singleton sets, named $S = \{[1], [2], \cdots, [n]\}$, and that the sets are merged in some sequence of $n-1$ $union(x, y)$-instructions until all of the n elements are in one set. Here, the main problem of interest is the average-cost of this process. In general, a probabilistic model Γ_n indexed by n is assumed to reflect the nature of input instructions. Then, the cost function becomes a random variable X_n induced by Γ_n. Moreover, if the time is reversed, the process can be thought as the splitting of a random tree. That is, for a tree of n nodes that is chosen at random, we cut its edge also at random. This separates the tree into two smaller trees and the cost of incurred by splitting the tree is c_n. Then, we continue this process with each resulting trees until the completely disconnected graph is obtained. Thus, a distributional recurrence that relates the random variables X_n as follows

$$X_n \stackrel{d}{=} X_{S_n} + X^*_{n-S_n} + c_n, \quad \text{for } n \geq 2,\ X_1 = 0 \tag{1.1}$$

arises naturally, where c_n is a quantity, called toll function, that represents either the cost incurred by splitting a random tree of size n, or alternatively merging two sets into a set of size n; S_n is the (random) size of the first subtree and the second subtree has size $n - S_n$. X^*_n is an independent copy of X_n.

Two probabilistic models have been introduced in [12]: the random graph model and the random spanning tree model. In this thesis, we only focus on the random spanning tree model. In this model, a spanning tree of a complete graph with n vertices is chosen randomly and then the edges of this spanning tree are randomly ordered. This leads to a sequence of $n-1$ edges, named e_1, e_2,. . ., e_{n-1}. This sequence also gives us a sequence of $union(x, y)$-instructions if we take each edge $e_i = (x, y)$ as an $union(x, y)$-instruction. By a famous result due to Cayley, there are n^{n-2} such labeled unrooted trees on n nodes. Hence $n^{n-2}(n-1)!$ possible sequences of $union(x, y)$-instructions are equally likely. Thus, S_n is distributed as follows:

$$\mathbb{P}(S_n = k) = \binom{n}{k} \frac{k^{k-1}(n-k)^{n-k-1}}{2(n-1)n^{n-2}}, \tag{1.2}$$

Indeed, the distributional recurrence (1.1) has been already analyzed fully for the toll functions induced from Quick-Find Algorithm, Quick-Merge Algorithm, Quick-Find with Weighting Rule and Quick-Merge with Weighting Rule. But in [10], Knuth and Pittel considered the case when the toll function c_n equals n^a, i.e. the distributional recurrence

$$X_n \stackrel{d}{=} X_{S_n} + X^*_{n-S_n} + n^\alpha, \quad \text{for } n \geq 2 \tag{1.3}$$

and gave the order-of-growth of the excepted value of X_n by the approach of repertoire. Their result is the following theorem.

Theorem 1.1. *(Knuth and Pittel [10]). Under the random spanning tree model, the expected value of X_n which satisfies the distributional recurrence (1.3) with the initial condition $X_1 = 0$ is*

$$\mathbb{E}X_n = \begin{cases} \frac{\Gamma(\alpha-\frac{1}{2})}{\sqrt{2}\Gamma(\alpha)} n^{\alpha+\frac{1}{2}} + O(n^\alpha), & \text{if } \alpha > 1; \\ \frac{\Gamma(\alpha-\frac{1}{2})}{\sqrt{2}\Gamma(\alpha)} n^{\frac{3}{2}} + O(n\log n), & \text{if } \alpha = 1; \\ \frac{\Gamma(\alpha-\frac{1}{2})}{\sqrt{2}\Gamma(\alpha)} n^{\alpha+\frac{1}{2}} + O(n), & \text{if } \frac{1}{2} < \alpha < 1; \\ \frac{1}{\sqrt{2\pi}} n\ln n + O(n), & if\ \alpha = \frac{1}{2}; \\ O(n), & \text{if } 0 < \alpha < \frac{1}{2}. \end{cases}$$

Recently, Fill, Flajolet and Kapur used a new method, called singularity analysis which was developed by Flajolet and Odlyzkoto in [7] to give a more precise estimate of the excepted value of X_n (with the initial condition $X_1 = 1$). Their result is described in the following theorem.

Theorem 1.2. *(Fill, Flajolet and Kapur [4]) Under the random spanning tree model, the expected value of X_n which satisfies the distributional recurrence (1.3) with the initial condition $X_1 = 1$ is*

$$\mathbb{E}X_n = \begin{cases} \frac{\Gamma(\alpha-\frac{1}{2})}{\sqrt{2}\Gamma(\alpha)}n^{\alpha+\frac{1}{2}} + O(n^{\alpha-\frac{1}{2}}), & \text{if } \alpha > \frac{3}{2}; \\ \frac{1}{\sqrt{2}\Gamma(\frac{3}{2})}n^2 + O(n\log n), & \text{if } \alpha = \frac{3}{2}; \\ \frac{\Gamma(\alpha-\frac{1}{2})}{\sqrt{2}\Gamma(\alpha)}n^{\alpha+\frac{1}{2}} + O(n), & \text{if } \frac{1}{2} < \alpha < \frac{3}{2}; \\ \frac{1}{\sqrt{2\pi}}n\ln n + O(n), & \text{if } \alpha = \frac{1}{2}; \\ (1 + \frac{1}{2}K_\alpha)n + O(n^{\alpha+\frac{1}{2}}), & \text{if } 0 < \alpha < \frac{1}{2}. \end{cases}$$

Although the analysis of the excepted value of X_n is complete, higher moments and the limit distribution are still missing. The aim of this thesis is (1) to correct some mistakes of the excepted value of X_n in [4], i.e. Theorem 1.2, and (2) to extend their result by characterizing the higher moments and the limiting distribution of X_n.
We present our result by the following two theorems.

Theorem 1.3. *Under the random spanning tree model, the expected value of X_n which satisfies the distributional recurrence (1.3) with the initial condition $X_1 = 0$ is*

$$\mathbb{E}X_n = \begin{cases} \frac{\Gamma(\alpha-\frac{1}{2})}{\sqrt{2}\Gamma(\alpha)}n^{\alpha+\frac{1}{2}} + O(n^{\alpha}), & \text{if } \alpha \geq 1; \\ \frac{\Gamma(\alpha-\frac{1}{2})}{\sqrt{2}\Gamma(\alpha)}n^{\alpha+\frac{1}{2}} + O(n), & \text{if } \frac{1}{2} < \alpha < 1; \\ \frac{1}{\sqrt{2\pi}}n\ln n + O(n), & \text{if } \alpha = \frac{1}{2}; \\ \frac{1}{2}K_\alpha n + O(n^{\alpha+\frac{1}{2}}), & \text{if } 0 < \alpha < \frac{1}{2}. \end{cases}$$

Moreover, the error terms are optimal.

Comparing our result with Knuth and Pittel's result, we discover that Knuth and Pittel's result is almost optimal except for the case of $\alpha = 1$ and $0 < \alpha < \frac{1}{2}$. Moreover, there are indeed some mistakes in the error terms in Fill, Flajolet and Kapur's result as $\alpha > \frac{1}{2}$. Finally, our main result about the limiting distribution of X_n is described in the following theorem.

Theorem 1.4. *Let Y_n ba a random variable defined as follows:*

$$Y_n = \begin{cases} \frac{X_n - \frac{1}{2}K_\alpha n}{n^{\alpha+\frac{1}{2}}}, & \text{if } 0 < \alpha < \frac{1}{2}; \\ \frac{X_n}{n^{\alpha+\frac{1}{2}}}, & \text{if } \alpha > \frac{1}{2}, \end{cases}$$

Then, we have

$$Y_n \xrightarrow{d} Y,$$

where Y is a random variable whose distribution is unique and characterized by its moments:

$$\mathbb{E}Y^k = \frac{A_k\sqrt{\pi}}{\Gamma(k(\alpha+\frac{1}{2})-\frac{1}{2})}$$

with

$$A_k = \frac{1}{4}\sum_{j=1}^{k-1}\binom{k}{j}A_jA_{k-j} + \frac{\sqrt{2}}{2}kA_{k-1}\frac{\Gamma(k\alpha+\frac{k}{2}-1)}{\Gamma((k-1)\alpha+\frac{k}{2}-1)},$$

for $k \geq 2$, $A_1 = \frac{1}{\sqrt{2\pi}}\Gamma(\alpha-\frac{1}{2})$.

Remark 1.1. The K_α in the previous two theorems is a constant and it can be explicitly computed; for details see Chapter 3.

Chapter 2

Tools

2.1 Singularity Analysis

2.1.1 Introduction

Generating functions are a useful tool for counting in combinatorics. In a number of situations, the generating function is explicit and can be expanded such that an explicit formula results for the coefficients. For example, consider the enumeration of binary trees: let C_n denote the number of binary trees with n internal nodes and $C(z)$ be the ordinary generation of C_n. Then, by the recurrence

$$C_n = \sum_{k=0}^{n-1} C_n C_{n-1-k}, \quad \text{for } n \geq 1,\ C_0 = 1,$$

we can derive that $C(z)$ satisfies the following equation

$$C(z) = 1 + zC(z)^2.$$

Solving with the quadratic formula, we get

$$C(z) = \frac{1 - \sqrt{1 - 4z}}{2z}.$$

Then, expanding by Newton's binomial theorem yields

$$C_n = [z^n]C(z) = -\frac{1}{2}\binom{\frac{1}{2}}{n+1}(-4)^{n+1} = \frac{1}{n+1}\binom{2n}{n}.$$

But in a number of cases, either the generating function can not be obtained in an explicit way, or if an explicit formula of the generating function is available, we still can not find a closed form for its coefficients. For example, consider the enumeration of alternating permutations.

Definition 2.1. *A permutation $\sigma_1\sigma_2\cdots\sigma_n$ is an alternating permutation if $\sigma_1 > \sigma_2 < \sigma_3 > \sigma_4 < \cdots$,*

Let T_n denote the number of alternating permutations of odd size n and $T(z)$ be the ordinary generating function of T_n. To each alternating permutation we can associate bijectively a binary tree of special type called increasing binary tree. The correspondence is as follows: given an alternating permutation, $\sigma = \sigma_1\sigma_2\cdots\sigma_n$, factor it into the form $\sigma = \sigma_L \cdot \min(\sigma) \cdot \sigma_R$, with $\min(\sigma)$ the smallest label value in the permutation, and σ_L, σ_R the factors left and right of $\min(\sigma)$. Then, a labelled binary tree $\beta(\sigma)$ can be defined recursively in the format (left, root, right) by

$$\beta(\sigma) = (\beta(\sigma_L), \min(\sigma), \beta(\sigma_R)).$$

This labelled binary tree is called increasing binary tree since every sequence of labels from the root to any leave is increasing. Conversely, reading recursively the labels of an increasing binary tree (in the same way as above) gives back the original permutation. From this correspondence, we get the recurrence for T_n

$$T_n = \sum_{k=1}^{n-1}\binom{n-1}{k}T_kT_{n-1-k}, \quad \text{for } n \geq 1,\ T_0 = 0,\ T_1 = 1.$$

Thus, we derive that $T(z)$ satisfies the integral equation

$$T(z) - z = \int_0^z T(w)^2 dw.$$

Solving this integral equation, we get

$$T(z) = \tan z.$$

In this case, since

$$T(z) = \tan z = \frac{\sin z}{\cos z} \quad \Longrightarrow \quad T(z)\cos z = \sin z,$$

we get the following relation

$$T_n - \binom{n}{2}T_{n-2} + \binom{n}{4}T_{n-4} - \cdots = (-1)^{(n-1)/2},$$

for n odd. Thus, we can now compute an arbitrary number of terms of the counting sequence $\{T_n\}$ by a simple algorithm based on the above relation, but an explicit formula for them is still not available.

In this chapter, we introduce an approach, called singularity analysis, to the analysis of coefficients of generating functions. With this method, it is possible to estimate asymptotically the coefficients of virtually any generating function, even if they are complicated. Thus, we can easily interpret and compare the counting sequences according to their asymptotic formulas of coefficients.

2.1.2 Fundamentals of singularity analysis

From now on, we treat generating functions as analytic objects. We assign values to variables that appear in generating functions, in particular complex values. This will bring us more benefit than only assigning real values. Thus, a generating function becomes a complex-valued function. The theory of singularity analysis illustrates a correspondence between the asymptotic expansion of a function near its dominant singularity (the singularity closest to 0) and the asymptotic expansion of the function's coefficients. The development of this theory is based on Cauchy's coefficient formula, a technique of complex analysis for obtaining coefficients of a function:

$$f_n \equiv [z^n]f(z) = \frac{1}{2i\pi}\int_\gamma f(z)\frac{dz}{z^{n+1}}$$

and using a special contour known as Hankel contour. We only state results and refer the reader for proofs to Flajolet and Sedgewick [8].

Theorem 2.1. *Let α be an arbitrary complex number in $\mathbb{C}\backslash\mathbb{Z}_{\leq 0}$.*

(1) Consider the function

$$f(z) = (1-z)^{-\alpha}.$$

Then,

$$a_n = [z^n]f(z) \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)}\left(1 + \sum_{k=1}^{\infty}\frac{e_k(\alpha)}{n^k}\right),$$

where $e_k(\alpha)$ is a polynomial in α of degree $2k$.

(2) Consider the function

$$f(z) = (1-z)^{-\alpha}\left(\frac{1}{z}\log\frac{1}{1-z}\right)^{\beta}$$

where β is a complex number. Then,

$$a_n = [z^n]f(z) \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)}(\log n)^{\beta}\left(1 + \sum_{k=1}^{\infty}\frac{C_k(\alpha)}{\log^k n}\right),$$

where $C_k(\alpha) = \binom{\beta}{k}\Gamma(\alpha)\frac{d^k}{ds^k}\frac{1}{\Gamma(s)}|_{s=\alpha}$.

Remark 2.1. We use the following notations

$$L(z) = \log\frac{1}{1-z}$$

throughout this thesis.

Remark 2.2. When $\alpha \in \mathbb{Z}_{\leq 0}$, the coefficient of $f(z) = (1-z)^{-\alpha}$ will eventually vanish, so that the asymptotic expansion becomes trivial. Note that the formula in Theorem 2.1, part (1) actually remains valid with the convention $\frac{1}{\Gamma(0)} = \frac{1}{\Gamma(-1)} = \cdots = 0$. As for part (2), the formula also remains valid for $\alpha \in \mathbb{Z}_{\leq 0}$ (again with the same convention as before).

Definition 2.2. *Given two numbers R and ϕ, with $R > 1$ and $0 < \phi < \frac{\pi}{2}$. Then,*

$$\Delta := \Delta(R, \phi) := \{z||z| < R, z \neq 1, |arg(z-1)| > \phi\}$$

is called a Δ-domain. Moreover, a function is called Δ-analytic if it is analytic in some Δ-domain.

Theorem 2.2. *Let α, β be two arbitrary real numbers and $f(z)$ be a Δ-analytic function.*

(1) Assume that

$$f(z) = O((1-z)^{-\alpha}(L(z))^{\beta}), \quad \textit{as } z \to 1, z \in \Delta.$$

Then, $[z^n]f(z) = O(n^{\alpha-1}(\log n)^{\beta})$.

(2) Assume that

$$f(z) = o((1-z)^{-\alpha}(L(z))^{\beta}), \quad \textit{as } z \to 1, z \in \Delta.$$

Then, $[z^n]f(z) = o(n^{\alpha-1}(\log n)^{\beta})$.

Theorem 2.1 and Theorem 2.2 establish a correspondence between properties of a function $f(z)$ singular at an isolated point ($z = 1$) and the asymptotic behavior of its coefficients $f_n = [z^n]f(z)$. At this stage, we thus have enough tools to derive the term-by-term transfers from the expansion of a function at its singularity, also called singular expansion, to the asymptotic estimate of its coefficients. The process can be stated as follows:

Suppose that $f(z)$ is a function with the dominant singularity at $z = \zeta$ and that it is analytic in some domain of the form $\zeta\Delta$. Analyze $f(z)$ as $z \to \zeta$ in the domain $\zeta\Delta$ and determine an expansion of the form

$$f(z) \underset{z\to\zeta}{=} g(z) + O(h(z)) \quad \text{with } h(z) = o(g(z)), \tag{2.1}$$

where $g(z/\zeta)$ and $h(z/\zeta)$ should belong to a standard scale of functions of the form $(1-z)^{-\alpha}\lambda(z)^{\beta}$ with $\lambda(z) := z^{-1}\log(1-z)^{-1}$. Then, by taking Taylor coefficients in (2.1) and with Theorem 2.2, we have

$$\begin{aligned} f_n \equiv [z^n]f(z) =& [z^n]g(z) + [z^n]O(h(z)) \\ =& \zeta^{-n}[z^n]g(z/\zeta) + O(\zeta^{-n}[z^n]h(z/\zeta)) \\ =& \zeta^{-n}g_n + O(\zeta^{-n}h_n). \end{aligned}$$

We give a simple example. In Section 2.1.1, we have already derived that the number of binary tree with n internal nodes is

$$C_n = [z^n]C(z) = \frac{1}{n+1}\binom{2n}{n}.$$

Applying Stirling's formula to C_n, we obtain the asymptotic expansion of C_n,

$$C_n = \frac{1}{n+1}\frac{(2n)!}{(n!)^2} \sim \frac{1}{n}\frac{(2n)^{2n}e^{-2n}\sqrt{4\pi n}}{n^{2n}e^{-2n}2\pi n} = \frac{4^n}{\sqrt{\pi n^3}}.$$

But now, with the method of singularity analysis, we can obtain the the asymptotic behavior of C_n directly from its generating function $C(z)$ which is given by

$$C(z) = \frac{1-\sqrt{1-4z}}{2z}.$$

Note that $C(z)$ has the dominant singularity at $z = \frac{1}{4}$ and by rescaling we can obtain that

$$\begin{aligned}C_n = [z^n]C(z) = \left(\frac{1}{4}\right)^{-n}[z^n]C\left(\frac{1}{4}z\right)\\ =& 4^n[z^n]\left(\frac{2-2\sqrt{1-z}}{z}\right)\\ =& 4^n[z^n][2-(1-z)^{\frac{1}{2}}+O(|1-z|)] \qquad \text{(expanded at } z=1)\\ =& \frac{4^n}{\sqrt{\pi n^3}}+O(n^{-2}) \sim \frac{4^n}{\sqrt{\pi n^3}}.\end{aligned}$$

2.1.3 Differentiation and integration

Singularity analysis is robust because it is not only closed under several simple operations such as $\pm$, $\times$, $\div$ but also closed under differentiation and integration. Here, "closed" means that functions amenable to singularity analysis are still amenable to singularity analysis after applying operations. We describe this property through the following two theorems. In this subsection, we focus on the functions that are Δ-analytic and admit singular expansions of the form:

$$f(z) = \sum_{i=0}^{m} c_i(1-z)^{\alpha_i} + O(|1-z|^A), \tag{2.2}$$

for a sequence of complex numbers $\{c_i\}_{0\le i\le m}$ and an increasing sequence of real numbers $\{\alpha_i\}_{0\le i\le m}$ satisfying $\alpha_i < A$.

Theorem 2.3. *Let $f(z)$ be a Δ-analytic function having its singular expansion of the form (2.2). Then, the derivative of $f(z)$ is also Δ-analytic. Moreover,*

$$\frac{d}{dz}f(z) = -\sum_{i=0}^{k} c_i\alpha_i(1-z)^{\alpha_i-1} + O(|1-z|^{A-1}).$$

Remark 2.3. Theorem 2.3 can be extended to include logarithmic terms. For instance, if $f(z)$ satisfies

$$f(z) = O(|1-z|^A L(z)^k), \qquad \text{for } k \in \mathbb{Z}_{\ge 1},$$

then one has

$$\frac{d}{dz}f(z) = O(|1-z|^{A-1}L(z)^k).$$

Theorem 2.4. *Let $f(z)$ be a Δ-analytic function having its singular expansion of the form (2.2). Then, the integral of $f(z)$ is also Δ-analytic. Moreover,*

(1) If $A < -1$, then

$$\int_0^z f(w)dw = -\sum_{i=0}^{k} \frac{c_i}{\alpha_i + 1}(1-z)^{\alpha_i+1} + O(|1-z|^{A+1}).$$

(2) If $A > -1$, then

$$\int_0^z f(w)dw = -\sum_{i=0}^{k} \frac{c_i}{\alpha_i + 1}(1-z)^{\alpha_i+1} + L_0 + O(|1-z|^{A+1}),$$

where L_0 is a constant with the value

$$L_0 = \sum_{\alpha_i<-1} \frac{c_i}{\alpha_i + 1} + \int_0^1 [f(t) - \sum_{\alpha_i<-1} c_i(1-t)^{\alpha_i}]dt.$$

Remark 2.4. The case that either some α_i or A equals to -1 is treated by the following rules:

$$\int_0^z (1-w)^{-1}dw = L(z), \quad \int_0^z O((1-w)^{-1})dw = O(L(z)).$$

Moreover, the integration with powers of logarithms is done with the following rules (for $\alpha \neq -1$)

$$\int_0^z (1-w)^{\alpha} L^k(w)dw = (-1)^k \frac{\partial^k}{\partial \alpha^k} \int_0^z (1-w)^{\alpha}dw,$$

for k a positive integer.

2.2 Polylogarithms and Hadamard Products

2.2.1 Polylogarithms

Definition 2.3. *The generalized polylogarithm, commonly denoted by $Li_{\alpha,\gamma}$, is defined by a Taylor series as*

$$Li_{\alpha,\gamma}(z) := \sum_{n\geq 1}(\log n)^{\gamma}\frac{z^n}{n^{\alpha}}, \quad z\in\mathbb{C},\ \ |z|<1,\ \ \alpha\in\mathbb{R},\ \ \gamma\in\mathbb{Z}_{\geq 0}.$$

Moreover, we make use of the abbreviation $Li_{\alpha,0}\equiv Li_{\alpha}(z)$.

A good property, namely that polylogarithms are continuable to the whole of the complex plane slit along the ray $\mathbb{R}_{\geq 1}$, was established by Ford [9]. Thus, polylogarithms are amenable to singularity analysis and their singular expansions are described in the following theorem.

Theorem 2.5. *The function $Li_{\alpha,\gamma}(z)$ is Δ-analytic and for $\alpha\notin\{0,1,2,\cdots\}$ it satisfies the expansion*

$$Li_{\alpha,0}(z)\sim\Gamma(1-\alpha)w^{\alpha-1}+\sum_{j\geq 0}\frac{(-1)^j}{j!}\zeta(\alpha-j)w^j, \quad w=\sum_{k=1}^{\infty}\frac{(1-z)^k}{k}, \tag{2.3}$$

where $\zeta(z)$ denotes the Riemann's zeta function. For $\gamma>0$, the singular expansion of $Li_{\alpha,\gamma}(z)$ is obtained by

$$Li_{\alpha,\gamma}(z)=(-1)^{\gamma}\frac{\partial^{\gamma}}{\partial\alpha^{\gamma}}Li_{\alpha,0}(z)$$

For latter purpose, the following special case is required.

Corollary 2.1. *For $\epsilon>0$ and $\alpha<1$, we have the following singular expansion*

$$Li_{\alpha,\gamma}(z)=\sum_{k=0}^{\gamma}\lambda_k^{(\alpha,\gamma)}(1-z)^{\alpha-1}L^{\gamma-k}(z)+O(|1-z|^{\alpha-\epsilon})+(-1)^{\gamma}\zeta^{(\gamma)}(\alpha)[\alpha>0],$$

where $\lambda_k^{(\alpha,\gamma)} \equiv \binom{\gamma}{k}\Gamma^{(k)}(1-\alpha)$ *and* $[\alpha > 0]$ *has the value* 1 *if and only if* $\alpha > 0$.
Moreover, for $\alpha < 0$, *we have*

$$Li_{\alpha,0}(z) = \Gamma(1-\alpha)(1-z)^{\alpha-1} - \Gamma(1-\alpha)\frac{1-\alpha}{2}(1-z)^{\alpha} + O(|1-z|^{\alpha+1})$$
$$+\zeta(\alpha)[\alpha > -1].$$

The following lemma is the inverse of Corollary 2.1.

Lemma 2.1. *For any real number* $\alpha < 1$ *and* $\gamma \in \mathbb{Z}_{\geq 0}$, *there exist a* Δ*-domain such that*

$$(1-z)^{\alpha-1}L^{\gamma}(z) = \sum_{k=0}^{\gamma} \mu_k^{(\alpha,\gamma)} Li_{\alpha,\gamma-k}(z) + O(|1-z|^{\alpha-\epsilon}) + c_{\gamma}(\alpha)[\alpha > 0]$$

holds uniformly in the Δ*-domain, where* $\mu_k^{(\alpha,\gamma)}$, $c_\gamma(\alpha)$ *are constants with* $\mu_0^{(\alpha,0)} = \frac{1}{\Gamma(1-\alpha)}$ *and* $c_0(\alpha) = -\frac{\zeta(\alpha)}{\Gamma(1-\alpha)}$ *and* $\epsilon > 0$ *is arbitrarily small.*
Moreover, a special case which will be used extensively used below is

$$(1-z)^{\alpha-1} = \frac{1}{\Gamma(1-\alpha)} Li_{\alpha,0}(z) + O(|1-z|^{\alpha-\epsilon}) - \frac{\zeta(\alpha)}{\Gamma(1-\alpha)}[\alpha > 0].$$

2.2.2 Hadamard product

Definition 2.4. *Let* $f(z)$ *and* $g(z)$ *be two functions analytic at 0 with* $f(z) = \sum_{n\geq 0} f_n z^n$ *and* $g(z) = \sum_{n\geq 0} g_n z^n$. *Then, the Hadamard product of* $f(z)$ *and* $g(z)$ *is defined as*

$$f(z) \odot g(z) := \sum_{n\geq 0} f_n g_n z^n$$

Theorem 2.6. *Let* a *and* b *be two arbitrary complex numbers with neither* a,b *and* $a+b$ *is an integer. Then* $(1-z)^a \odot (1-z)^b$ *is also analytic in a* Δ*-domain, and admits an infinite expansion*

$$(1-z)^a \odot (1-z)^b \sim \sum_{k\geq 0} \lambda_k^{(a,b)} \frac{(1-z)^k}{k!} + \sum_{k\geq 0} \mu_k^{(a,b)} \frac{(1-z)^{a+b+1+k}}{k!},$$

where the coefficients $\lambda_k^{(a,b)}$ *and* $\mu_k^{(a,b)}$ *are given by*

$$\lambda_k^{(a,b)} = \frac{\Gamma(a+b+1)}{\Gamma(a+1)\Gamma(b+1)} \frac{(-a)^{\bar{k}}(-b)^{\bar{k}}}{(-a-b)^{\bar{k}}}, \ \mu_k^{(a,b)} = \frac{\Gamma(-a-b-1)}{\Gamma(-a)\Gamma(-b)} \frac{(a+1)^{\bar{k}}(b+1)^{\bar{k}}}{(a+b+2)^{\bar{k}}}.$$

Here, $x^{\bar{k}}$ *is defined as* $x^{\bar{k}} = x(x+1)\cdots(x+k-1)$ *for* $k \in \mathbb{Z}_{\geq 0}$.

Remark 2.5. The case that either a or b is a integer is simple; we have

1. $(1-z)^a \odot g(z)$ is a polynomial if $a \in \mathbb{Z}_{\geq 0}$;

2. $(1-z)^{-a} \odot g(z)$ can be reduced to a derivative of $g(z)$ if $a \in \mathbb{Z}_{>0}$; more precisely, one has

$$(1-z)^{-a} \odot g(z) = \frac{1}{(m-1)!} \partial_z^{m-1}(z^{m-1} g(z)).$$

The case that $a+b$ is a integer is more complicated and it can be found in books by Abramowitz and Stegun [1, pp.559-560] and by Whittaker and Waston [11, Section 14.53].

Theorem 2.7. *Let* $f(z)$ *and* $g(z)$ *be two functions that are analytic in a* Δ*-domain* $\Delta(R, \phi)$*. Then,* $f(z) \odot g(z)$ *is also analytic in some* Δ*-domain. Moreover, if* $f(z) = O((1-z)^a)$ *and* $g(z) = O((1-z)^b)$ *for* $z \in \Delta(R, \phi)$*, then* $f(z) \odot g(z)$ *admits an expansion in some* Δ*-domain as follows:*

(1) If $a+b+1 < 0$*, then*

$$f(z) \odot g(z) = O((1-z)^{a+b+1}).$$

(2) If $k < a+b+1 < k+1$*, for some* $k \in \mathbb{Z}_{\geq -1}$*, then*

$$f(z) \odot g(z) = \sum_{j=0}^{k} \frac{(-1)^j}{j!} (f \odot g)^{(j)}(1)(1-z)^j + O((1-z)^{a+b+1}).$$

(3) If $a+b+1 \in \mathbb{Z}_{\geq 0}$, then

$$f(z) \odot g(z) = \sum_{j=0}^{k} \frac{(-1)^j}{j!}(f \odot g)^{(j)}(1)(1-z)^j + O((1-z)^{a+b+1}L(z)).$$

The following corollary is a consequence of Theorem 2.6 and Theorem 2.7.

Corollary 2.2. *Let $f(z)$ and $g(z)$ be two functions that are analytic in a Δ-domain $\Delta(R,\phi)$ with singular expansions of type (2.2):*

$$f(z) = \sum_{i=0}^{m} c_i(1-z)^{\alpha_i} + O(|1-z|^A) \text{ and } g(z) = \sum_{j=0}^{n} d_j(1-z)^{\beta_j} + O(|1-z|^B).$$

Then, the Hadamard product $(f \odot g)(z)$ is also Δ-analytic and admits the singular expansion of the form:

$$(f \odot g)(z) = \sum_{m,n} c_m d_n (1-z)^{\alpha_m} \odot (1-z)^{\beta_j} + P(1-z) + O(|1-z|^C),$$

where $C := 1 + \min(\alpha_0 + B, \beta_0 + A) \notin \mathbb{Z}_{\geq 0}$ and P is a polynomial of degree less than C.

The reason for the polynomial P is that the integral powers of $(1-z)$ do not leave a trace in the asymptotics of coefficients, since their contribution is zero.
In practice, this corollary allows us to establish the following algorithm, called Zigzag Algorithm, which is helpful in the computation of the singular expansions when composing function under Hadamard products.

Zigzag-algorithm (Fill, Flajolet and Kapur [4])
Input: two functions $f(z)$ and $g(z)$ that are Δ-analytic and have singular expansions of the form (2.2).
Output: the singular expansion of $h(z) := (f \odot g)(z)$

Step1. Use singularity analysis to determine the asymptotic expansions of $f_n = [z^n]f(z)$ and $g_n = [z^n]g(z)$.

Step2. Compute the asymptotic expansion of $h_n = [z^n]h(z)$ by multiplying the asymptotic expansions of f_n and g_n.

Step3. Construct a function $H(z)$ by using singularity analysis in the reverse direction such that the asymptotic expansion of its coefficients is compatible with the asymptotic expansion of h_n. By the construction, $H(z)$ is a sum of functions of the form $(1-z)^{\alpha}L(z)^{\beta}$, which are all singular at 1.

Step4. Output the singular expansion of $(f \odot g)$ as

$$(f \odot g)(z) = h(z) = H(z) + P(1-z) + O(|1-z|^{C}),$$

where C can be determined by the previous Corollary and P is a polynomial of degree δ, which is the largest integer less than C. Moreover, $P(z)$ can be determined as follows:

$$P(z) = \sum_{j=0}^{\delta} \frac{(-1)^j}{j!} \partial_z^j (h(z) - H(z))^{z=1} z^j.$$

Chapter 3

Moments by Singularity Analysis

So far, we have presented the tools which will be used in this thesis. From now on, we return to the analysis of the distributional recurrence:

$$X_n \overset{d}{=} X_{S_n} + X^*_{n-S_n} + n^\alpha, \quad \text{for } n \geq 2, \; X_1 = 0. \tag{3.1}$$

In this chapter, we will first give the generating functions of the moments and then compute all moments of the random variable X_n.

3.1 Expected Value -- Proof of Theorem 1.3

It is crucial for us to have an asymptotic expansion of the expectation $a_n = \mathbb{E}(X_n)$, because it is the initial case of mathematical induction to get all higher moments of X_n. As discussed in Chapter 1, such asymptotic expansions were derived by Fill, Flajolet and Kapur. However, there seems to be some imprecisions in their result. Consequently, we will re-derive their result (we closely follow the method in [4].) Now, starting from the distributional recurrence (3.1), conditioning on the size S_n yields,

$$
\begin{aligned}
a_n =& \mathbb{E}[\mathbb{E}_{S_n}(X_{S_n} + X^*_{n-S_n} + n^\alpha)] \\
=& \sum_{j=1}^{n-1} p_{n,j}(a_j + a_{n-j}) + n^\alpha.
\end{aligned}
$$

Here, $p_{n,j} = \mathbb{P}(S_n = j)$ is given by

$$p_{n,j} = \binom{n}{k} \frac{k^{k-1}(n-k)^{n-k-1}}{2(n-1)n^{n-2}}$$

as mentioned in (1.2) and we rewrite it into the form:

$$p_{n,j} = \frac{n}{2(n-1)} \frac{c_k c_{n-k}}{c_n},$$

where

$$c_k = \frac{k^{k-1}}{k!}, \text{ for } k \geq 1.$$

Thus, we have

$$
\begin{aligned}
a_n =& \sum_{j=1}^{n-1} \frac{n}{2(n-1)} \frac{c_j c_{n-j}}{c_n}(a_j + a_{n-j}) + n^\alpha \\
=& \sum_{j=1}^{n-1} \frac{n}{(n-1)} \frac{c_j c_{n-j}}{c_n} a_j + n^\alpha
\end{aligned}
$$

or

$$\frac{n-1}{n} c_n a_n = \sum_{j=1}^{n-1} c_j a_j c_{n-j} + \frac{n-1}{n} c_n n^\alpha. \tag{3.2}$$

Multiplying the equation by $\frac{z^n}{e^n}$ and summing over $n \geq 1$, we get

$$\sum_{n\geq 1} \frac{n-1}{n} a_n \frac{c_n}{e^n} z^n = \sum_{n\geq 1} \sum_{j=1}^{n-1} \frac{c_j}{e^j} a_j \frac{c_{n-j}}{e^{n-j}} z^n + \sum_{n\geq 1} \frac{n-1}{n} \frac{c_n}{e^n} n^\alpha z^n. \tag{3.3}$$

Let $A(z)$ and $C(z)$ be the ordinary generating functions of the sequences a_n and c_n, that is

$$A(z) = \sum_{n\geq 1} a_n z^n, \quad C(z) = \sum_{n\geq 1} c_n z^n.$$

Then the relation (3.3) can be reduced to

$$A(z)\odot C(z/e)-\int_0^z A(w)\odot C(w/e)\frac{dw}{w} \tag{3.4}$$
$$= (A(z)\odot C(z/e))C(z/e)+\sum_{n\geq 1}\frac{n-1}{n}\frac{c_n}{e^n}n^\alpha z^n.$$

Moreover, $C(z)$ which is known as Cayley function.satisfies the functional equation

$$C(z) = ze^{C(z)}. \tag{3.5}$$

and it admits the singular expansion at the dominant singularity $z = e^{-1}$ (see [6, Proposition 1])

$$C(z) = 1-\sqrt{2}(1-ez)^{1/2}-\frac{1}{3}(1-ez)+O(|1-ez|^{3/2}). \tag{3.6}$$

By differentiating equation (3.5) on z, we get

$$C'(z) = e^{C(z)} + ze^{C(z)}C'(z). \tag{3.7}$$

This yields

$$zC'(z) - C(z) = zC'(z)C(z).$$

Finally, by taking the coefficients on both size of the equation, we have

$$nc_n - c_n = \sum_{j=0}^{n} jc_jc_{n-j} \qquad (c_0 := 0)$$

and consequently

$$nc_n - c_n = \sum_{j=0}^{n}\frac{n}{2}c_jc_{n-j}.$$

Thus, we have

$$\frac{nc_n - c_n}{n} = \sum_{j=0}^{n}\frac{1}{2}c_jc_{n-j}.$$

Substituting this into (3.4) leads to

$$
\begin{aligned}
A(z)\odot C(z/e) - \int_0^z A(w) \odot C(w/e)\frac{dw}{w} \\
= (A(z) \odot C(z/e))C(z/e) + \sum_{n\geq 1}\sum_{j=0}^{n} \frac{1}{2}\frac{c_j c_{n-j}}{e^n} n^\alpha z^n.
\end{aligned}
$$

Next, let $B(z) = \sum_{n\geq 1} n^\alpha z^n$ be the ordinary generating function of the sequence n^α. Then we arrive at

$$
\begin{aligned}
A(z) \odot C(z/e) - \int_0^z A(w) \odot C(w/e)\frac{dw}{w} \\
= (A(z) \odot C(z/e))C(z/e) + \frac{1}{2}B(z) \odot C(z/e)^2.
\end{aligned}
$$

For convenience, putting $f(z) := A(z) \odot C(z/e)$ and $t(z) := \frac{1}{2}B(z) \odot C(z/e)^2$, we have

$$
f(z) - \int_0^z f(w)\frac{dw}{w} = f(z)C(z/e) + t(z). \tag{3.8}
$$

Then by differentiating, we transfer the above integral equation into a linear differential equation

$$
\frac{df(z)}{dz} - \frac{f(z)}{z} = \frac{df(z)}{dz}C(z/e) + f(z)\frac{dC(z/e)}{dz} + \frac{dt(z)}{dz}
$$

or

$$
(1 - C(z/e))\frac{df(z)}{dz} = \left(\frac{1}{z} + \frac{dC(z/e)}{dz}\right) f(z) + \frac{dt(z)}{dz}. \tag{3.9}
$$

Moreover, from (3.7) and (3.5), we have

$$
\frac{dC(z/e)}{dz} = \frac{\frac{1}{z}C(z/e)}{1 - C(z/e)}.
$$

Substituting this into (3.9) yields

$$
\frac{df(z)}{dz} = \frac{1}{z(1 - C(z/e))^2} f(z) + \frac{1}{1 - C(z/e)}\frac{dt(z)}{dz}. \tag{3.10}
$$

Now, we solve this differential equation by the method of variation-of-constants. First, we consider the homogenous part, that is

$$\frac{df(z)}{dz} = \frac{1}{z(1-C(z/e))^2} f(z).$$

Then, we have

$$\ln f(z) = \int_0^z \frac{dw}{w(1-C(w/e))^2} + K = -\ln(1-C(z/e)) + \ln C(z/e) + K,$$

where K is a constant. Thus, the solution of the homogenous part is

$$f(z) = \frac{C(z/e)}{1-C(z/e)} e^K.$$

Letting $f(z) = \frac{C(z/e)}{1-C(z/e)} e^{K(z)}$ and substituting it into (3.10), we obtain that

$$e^{K(z)} = \int_0^z \frac{\partial_w t(w)}{C(w/e)} dw + D,$$

where D is a constant. Consequently, this gives the solution of equation (3.8), that is

$$f(z) = D\frac{C(z/e)}{1-C(z/e)} + \frac{C(z/e)}{1-C(z/e)} \int_0^z \partial_w t(w) \frac{dw}{C(w/e)}.$$

Finally, with the initial condition $a_1 = 0$, we obtain

$$A(z) \odot C(z/e) = \frac{1}{2} \frac{C(z/e)}{1-C(z/e)} \int_0^z \partial_w [B(w) \odot C(w/e)^2] \frac{dw}{C(w/e)}. \qquad (3.11)$$

Theorem 3.1. *The generating function $A(z) \odot C(z/e)$ is analytic in some Δ-domain. Moreover, it admits the following singular expansions at $z = 1$.*

1. *For $\alpha > \frac{3}{2}$:*
   $A(z) \odot C(z/e) = \frac{1}{2\sqrt{\pi}} \Gamma\left(\alpha - \frac{1}{2}\right)(1-z)^{-\alpha} + \frac{1}{\sqrt{2\pi}} \Gamma\left(\alpha - \frac{1}{2}\right)\left(\frac{\alpha - \frac{1}{2}}{\alpha - 1} - \frac{7}{6}\right)(1-z)^{-\alpha+\frac{1}{2}} + O(|1-z|^{-\alpha+1})$.

2. *For $\alpha = \frac{3}{2}$:*
   $A(z) \odot C(z/e) = \frac{1}{2\sqrt{\pi}}(1-z)^{-\frac{3}{2}} + \frac{5}{12}\sqrt{\frac{2}{\pi}}(1-z)^{-1} + O(|1-z|^{-\frac{1}{2}} L(z)^2)$.

3. *For* $\frac{1}{2} < \alpha < \frac{3}{2}$*:*
$A(z) \odot C(z/e) = \frac{1}{2\sqrt{\pi}}\Gamma\left(\alpha - \frac{1}{2}\right)(1-z)^{-\alpha} + \frac{1}{\sqrt{2\pi}}\Gamma\left(\alpha-\frac{1}{2}\right)\left(\frac{\alpha-\frac{1}{2}}{\alpha-1} - \frac{7}{6}\right)(1-z)^{-\alpha+\frac{1}{2}} + \frac{\sqrt{2}}{4}L_0(1-z)^{-\frac{1}{2}} - \frac{7}{12}L_0 + O(|1-z|^{-\alpha+1})$.

4. *For* $\alpha = \frac{1}{2}$*:*
$A(z) \odot C(z/e) = \frac{1}{2\sqrt{\pi}}(1-z)^{-\frac{1}{2}}L(z) + \frac{7}{6\sqrt{2}}(1-z)^{-\frac{1}{2}} + O(L(z))$.

5. *For* $0 < \alpha < \frac{1}{2}$*:*
$A(z) \odot C(z/e) = \frac{\sqrt{2}}{4}K_\alpha(1-z)^{-\frac{1}{2}} + \frac{1}{2\sqrt{\pi}}\Gamma\left(\alpha-\frac{1}{2}\right)(1-z)^{-\alpha} - \frac{7}{12}K_\alpha + O(|1-z|^{-\alpha+\frac{1}{2}})$.

Proof.
First, we compute the singular expansion of $B(z) \odot C(z/e)^2$ for each case by the Zigzag-algorithm. Since

$$C(z/e) = 1 - \sqrt{2}(1-z)^{1/2} - \frac{1}{3}(1-z) + O(|1-z|^{3/2}),$$

we obtain

$$C(z/e)^2 = 1 - 2\sqrt{2}(1-z)^{1/2} + \frac{4}{3}(1-z) + O(|1-z|^{3/2}).$$

Moreover,

$$B(z) = \sum_{n\geq 2} n^\alpha z^n = \Gamma(1+\alpha)(1-z)^{-\alpha-1} + O(|1-z|^{-\alpha}),$$

Then, by the method of singularity analysis, we have

$$[z^n]C(z/e)^2 = \frac{-2\sqrt{2}}{\Gamma(-\frac{1}{2})}n^{-\frac{3}{2}} + O(n^{-\frac{5}{2}}) \quad \text{and} \quad [z^n]B(z) = n^\alpha$$

and this implies that

$$[z^n]B(z) \odot C(z/e)^2 = \sqrt{\frac{2}{\pi}}n^{\alpha-\frac{3}{2}} + O(n^{\alpha-\frac{5}{2}}).$$

Thus, converting back this information to the function by Zigzag-algorithm, we find the singular expansion of $B(z) \odot C(z/e)^2$.

1. For $\alpha > \frac{3}{2}$:
$B(z) \odot C(z/e)^2 = \sqrt{\frac{2}{\pi}}\Gamma\left(\alpha - \frac{1}{2}\right)(1-z)^{-\alpha+\frac{1}{2}} + O(|1-z|^{-\alpha+\frac{3}{2}})$.

2. For $\alpha = \frac{3}{2}$:
$B(z) \odot C(z/e)^2 = \sqrt{\frac{2}{\pi}}(1-z)^{-1} + O(L(z))$.

3. For $\frac{1}{2} < \alpha < \frac{3}{2}$:
$B(z) \odot C(z/e)^2 = \sqrt{\frac{2}{\pi}}\Gamma\left(\alpha - \frac{1}{2}\right)(1-z)^{-\alpha+\frac{1}{2}} + K + O(|1-z|^{-\alpha+\frac{3}{2}})$.

4. For $\alpha = \frac{1}{2}$:
$B(z) \odot C(z/e)^2 = \sqrt{\frac{2}{\pi}}L(z) + K + O(|1-z|L(z))$.

5. For $0 < \alpha < \frac{1}{2}$:
$B(z) \odot C(z/e)^2 = \sqrt{\frac{2}{\pi}}\Gamma\left(\alpha - \frac{1}{2}\right)(1-z)^{-\alpha+\frac{1}{2}} + K + K_1(1-z) + O(|1-z|^{-\alpha+\frac{3}{2}})$.

Here, K and K_1 are some constants. Then applying differential and integral rules for singularity analysis according to Theorem 2.3 and Theorem 2.4, we obtain the following.

1. For $\alpha > \frac{3}{2}$:

$$
\begin{aligned}
&\int_0^z \partial_w[B(w) \odot C(w/e)^2]\frac{dw}{C(w/e)} \\
&= \int_0^z \left[\sqrt{\frac{2}{\pi}}\Gamma\left(\alpha - \frac{1}{2}\right)\left(\alpha - \frac{1}{2}\right)(1-w)^{-\alpha-\frac{1}{2}} + O(|1-w|^{-\alpha+\frac{1}{2}})\right] \\
&\qquad\qquad\qquad \times \left[1 + \sqrt{2}(1-w)^{1/2} + \frac{7}{3}(1-w) + O(|1-w|^{3/2})\right] dw \\
&= \int_0^z \sqrt{\frac{2}{\pi}}\Gamma\left(\alpha - \frac{1}{2}\right)\left(\alpha - \frac{1}{2}\right)(1-w)^{-\alpha-\frac{1}{2}} \\
&\qquad\qquad\qquad + \frac{2}{\sqrt{\pi}}\Gamma(\alpha - \frac{1}{2})(\alpha - \frac{1}{2})(1-w)^{-\alpha} + O(|1-w|^{-\alpha+\frac{1}{2}})dw
\end{aligned}
$$

$$= \sqrt{\frac{2}{\pi}}\Gamma\left(\alpha - \frac{1}{2}\right)(1-z)^{-\alpha+\frac{1}{2}}$$

$$+ \frac{2}{\sqrt{\pi}}\Gamma\left(\alpha - \frac{1}{2}\right)\frac{\alpha - \frac{1}{2}}{\alpha - 1}(1-z)^{-\alpha+1} + O(|1-z|^{-\alpha+\frac{3}{2}}).$$

2. For $\alpha = \frac{3}{2}$:

$$\int_0^z \partial_w[B(w) \odot C(w/e)^2]\frac{dw}{C(w/e)}$$

$$= \int_0^z \left[\sqrt{\frac{2}{\pi}}(1-w)^{-2} + O(|1-w|^{-1}L(w))\right]$$

$$\times \left[1 + \sqrt{2}(1-w)^{1/2} + \frac{7}{3}(1-w) + O(|1-w|^{3/2})\right] dw$$

$$= \int_0^z \sqrt{\frac{2}{\pi}}(1-w)^{-2} + \frac{2}{\sqrt{\pi}}(1-w)^{-\frac{3}{2}} + O(|1-w|^{-1}L(w))dw$$

$$= \sqrt{\frac{2}{\pi}}(1-z)^{-1} + \frac{4}{\sqrt{\pi}}(1-z)^{-\frac{1}{2}} + O(L(z)^2).$$

3. For $\frac{1}{2} < \alpha < \frac{3}{2}$:

$$\int_0^z \partial_w[B(w) \odot C(w/e)^2]\frac{dw}{C(w/e)}$$

$$= \int_0^z \sqrt{\frac{2}{\pi}}\Gamma\left(\alpha - \frac{1}{2}\right)\left(\alpha - \frac{1}{2}\right)(1-w)^{-\alpha-\frac{1}{2}}$$

$$+ \frac{2}{\sqrt{\pi}}\Gamma\left(\alpha - \frac{1}{2}\right)\left(\alpha - \frac{1}{2}\right)(1-w)^{-\alpha} + O(|1-w|^{-\alpha+\frac{1}{2}})dw$$

$$= \sqrt{\frac{2}{\pi}}\Gamma\left(\alpha - \frac{1}{2}\right)(1-z)^{-\alpha+\frac{1}{2}}$$

$$+ \frac{2}{\sqrt{\pi}}\Gamma\left(\alpha - \frac{1}{2}\right)\frac{\alpha - \frac{1}{2}}{\alpha - 1}(1-z)^{-\alpha+1} + L_0 + O(|1-z|^{-\alpha+\frac{3}{2}}).$$

Where L_0 is the integration constant which can be computed by Theorem (2.4).

4. For $\alpha = \frac{1}{2}$:

$$\begin{aligned}
&\int_0^z \partial_w[B(w) \odot C(w/e)^2] \frac{dw}{C(w/e)} \\
&= \int_0^z \left[\sqrt{\frac{2}{\pi}}(1-w)^{-1} + O(L(w))\right] \\
&\qquad\qquad \times \left[1 + \sqrt{2}(1-w)^{1/2} + \frac{7}{3}(1-w) + O(|1-w|^{3/2})\right] dw \\
&= \int_0^z \sqrt{\frac{2}{\pi}}(1-w)^{-1} + \frac{2}{\sqrt{\pi}}(1-w)^{-\frac{1}{2}} + \frac{7\sqrt{2}}{3\sqrt{\pi}} + O(L(w))dw \\
&= \sqrt{\frac{2}{\pi}}L(z) + \frac{4}{\sqrt{\pi}}(1-z)^{\frac{1}{2}} + \frac{7\sqrt{2}}{3\sqrt{\pi}}z + O(|1-z|L(z)) \\
&= \sqrt{\frac{2}{\pi}}L(z) + \frac{7\sqrt{2}}{3\sqrt{\pi}} + \frac{4}{\sqrt{\pi}}(1-z)^{\frac{1}{2}} - \frac{7\sqrt{2}}{3\sqrt{\pi}}(1-z) + O(|1-z|L(z)).
\end{aligned}$$

5. For $0 < \alpha < \frac{1}{2}$:

$$\begin{aligned}
&\int_0^z \partial_w[B(w) \odot C(z/e)^2] \frac{dw}{C(z/e)} \\
&= \int_0^z \left[\sqrt{\frac{2}{\pi}}\Gamma\left(\alpha - \frac{1}{2}\right)\left(\alpha - \frac{1}{2}\right)(1-z)^{-\alpha-\frac{1}{2}} - K_1 + O(|1-w|^{-\alpha+\frac{1}{2}})\right] \\
&\qquad\qquad \times \left[1 + \sqrt{2}(1-w)^{1/2} + \frac{7}{3}(1-w) + O(|1-w|^{3/2})\right] dw \\
&= \int_0^z \sqrt{\frac{2}{\pi}}\Gamma\left(\alpha - \frac{1}{2}\right)\left(\alpha - \frac{1}{2}\right)(1-w)^{-\alpha-\frac{1}{2}} \\
&\qquad\qquad + \frac{2}{\sqrt{\pi}}\Gamma\left(\alpha - \frac{1}{2}\right)\left(\alpha - \frac{1}{2}\right)(1-w)^{-\alpha} - K_1 + O(|1-w|^{-\alpha+\frac{1}{2}}) \\
&= \sqrt{\frac{2}{\pi}}\Gamma\left(\alpha - \frac{1}{2}\right)(1-z)^{-\alpha+\frac{1}{2}} + \frac{2}{\sqrt{\pi}}\Gamma\left(\alpha - \frac{1}{2}\right)\frac{\alpha - \frac{1}{2}}{\alpha - 1}(1-z)^{-\alpha+1} \\
&\qquad\qquad + L_0 - K_1 + K_1(1-z) + O(|1-z|^{-\alpha+\frac{3}{2}}).
\end{aligned}$$

Where, L_0 is the integration constant which can be computed by Theorem 2.4.

Moreover, setting

$$L_0 - K_1 = K_\alpha = \int_0^1 \partial_w(B(w)\odot C(z/e)^2)\frac{dw}{C(w/e)},$$

we obtain

$$\begin{aligned}
&\int_0^z \partial_w[B(w)\odot C(z/e)^2]\frac{dw}{C(z/e)}\\
&= \sqrt{\frac{2}{\pi}}\Gamma(\alpha-\frac{1}{2})(1-z)^{-\alpha+\frac{1}{2}}\\
&\qquad + \frac{2}{\sqrt{\pi}}\Gamma(\alpha-\frac{1}{2})\frac{\alpha-\frac{1}{2}}{\alpha-1}(1-z)^{-\alpha+1} + K_\alpha + K_1(1-z) + O(|1-z|^{-\alpha+\frac{3}{2}}).
\end{aligned}$$

Finally, we multiply by $\frac{1}{2}\frac{C(z/e)}{1-C(z/e)}$ which is

$$\begin{aligned}
\frac{1}{2}\frac{C(z/e)}{1-C(z/e)} &= \frac{1}{2}\frac{1-\sqrt{2}(1-z)^{1/2}-\frac{1}{3}(1-z)+O(|1-z|^{3/2})}{\sqrt{2}(1-z)^{1/2}+\frac{1}{3}(1-z)+O(|1-z|^{3/2})}\\
&= \frac{1}{2}\frac{1}{\sqrt{2}(1-z)^{\frac{1}{2}}}\frac{1-\sqrt{2}(1-z)^{1/2}-\frac{1}{3}(1-z)+O(|1-z|^{3/2})}{1+\frac{\sqrt{2}}{6}(1-z)^{1/2}+O(|1-z|)}\\
&= \frac{\sqrt{2}}{4}(1-z)^{-\frac{1}{2}}(1-\sqrt{2}(1-z)^{1/2}-\frac{1}{3}(1-z)+O(|1-z|^{3/2}))\\
&\qquad\qquad (1-\frac{\sqrt{2}}{6}(1-z)^{1/2}+O(|1-z|))\\
&= \frac{\sqrt{2}}{4}(1-z)^{-\frac{1}{2}}-\frac{7}{12}+O((1-z)^{\frac{1}{2}}).
\end{aligned}$$

The end result is then as follows.

1. For $\alpha > \frac{3}{2}$:

$$\begin{aligned}
&A(z)\odot C(z/e)\\
&= \left[\frac{\sqrt{2}}{4}(1-z)^{-\frac{1}{2}}-\frac{7}{12}+O((1-z)^{\frac{1}{2}})\right]\times\left[\sqrt{\frac{2}{\pi}}\Gamma\left(\alpha-\frac{1}{2}\right)(1-z)^{-\alpha+\frac{1}{2}}\right.\\
&\qquad\qquad\left.+\frac{2}{\sqrt{\pi}}\Gamma\left(\alpha-\frac{1}{2}\right)\frac{\alpha-\frac{1}{2}}{\alpha-1}(1-z)^{-\alpha+1}+O(|1-z|^{-\alpha+\frac{3}{2}})\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\sqrt{\pi}}\Gamma\left(\alpha-\frac{1}{2}\right)(1-z)^{-\alpha} \\
&\qquad\qquad + \frac{1}{\sqrt{2\pi}}\Gamma\left(\alpha-\frac{1}{2}\right)\left(\frac{\alpha-\frac{1}{2}}{\alpha-1}-\frac{7}{6}\right)(1-z)^{-\alpha+\frac{1}{2}} + O(|1-z|^{-\alpha+1}).
\end{aligned}
$$

2. For $\alpha = \frac{3}{2}$:

$$
\begin{aligned}
&A(z)\odot C(z/e) \\
&= \left[\frac{\sqrt{2}}{4}(1-z)^{-\frac{1}{2}} - \frac{7}{12} + O((1-z)^{\frac{1}{2}})\right] \\
&\qquad\qquad \times \left[\sqrt{\frac{2}{\pi}}(1-z)^{-1} + \frac{4}{\sqrt{\pi}}(1-z)^{-\frac{1}{2}} + O(L(z)^2)\right] \\
&= \frac{1}{2\sqrt{\pi}}(1-z)^{-\frac{3}{2}} + \frac{5}{12}\sqrt{\frac{2}{\pi}}(1-z)^{-1} + O(|1-z|^{-\frac{1}{2}}L(z)^2).
\end{aligned}
$$

3. For $\frac{1}{2} < \alpha < \frac{3}{2}$:

$$
\begin{aligned}
&A(z)\odot C(z/e) \\
&= \left[\frac{\sqrt{2}}{4}(1-z)^{-\frac{1}{2}} - \frac{7}{12} + O((1-z)^{\frac{1}{2}})\right] \times \left[\sqrt{\frac{2}{\pi}}\Gamma\left(\alpha-\frac{1}{2}\right)(1-z)^{-\alpha+\frac{1}{2}}\right. \\
&\qquad\qquad \left. + \frac{2}{\sqrt{\pi}}\Gamma\left(\alpha-\frac{1}{2}\right)\frac{\alpha-\frac{1}{2}}{\alpha-1}(1-z)^{-\alpha+1} + L_0 + O(|1-z|^{-\alpha+\frac{3}{2}})\right] \\
&= \frac{1}{2\sqrt{\pi}}\Gamma\left(\alpha-\frac{1}{2}\right)(1-z)^{-\alpha} + \frac{\sqrt{2}}{4}L_0(1-z)^{-\frac{1}{2}} \\
&\qquad + \frac{1}{\sqrt{2\pi}}\Gamma\left(\alpha-\frac{1}{2}\right)\left(\frac{\alpha-\frac{1}{2}}{\alpha-1}-\frac{7}{6}\right)(1-z)^{-\alpha+\frac{1}{2}} - \frac{7}{12}L_0 + O(|1-z|^{-\alpha+1}) \\
&= \begin{cases} \frac{1}{2\sqrt{\pi}}\Gamma(\alpha-\frac{1}{2})(1-z)^{-\alpha} + O(|1-z|^{-\frac{1}{2}}), & \text{if } \frac{1}{2} < \alpha < 1; \\ \frac{1}{2\sqrt{\pi}}\Gamma(\alpha-\frac{1}{2})(1-z)^{-\alpha} + O(|1-z|^{-\alpha+\frac{1}{2}}), & \text{if } 1 \le \alpha < \frac{3}{2}. \end{cases}
\end{aligned}
$$

4. For $\alpha = \frac{1}{2}$:

$$
\begin{aligned}
&A(z) \odot C(z/e) \\
&= \left[\frac{\sqrt{2}}{4}(1-z)^{-\frac{1}{2}} - \frac{7}{12} + O((1-z)^{\frac{1}{2}}) \right] \\
&\qquad \times \left[\sqrt{\frac{2}{\pi}} L(z) + \frac{7\sqrt{2}}{3\sqrt{\pi}} + \frac{4}{\sqrt{\pi}}(1-z)^{\frac{1}{2}} - \frac{7\sqrt{2}}{3\sqrt{\pi}}(1-z) + O(|1-z|L(z)) \right] \\
&= \frac{1}{2\sqrt{\pi}}(1-z)^{-\frac{1}{2}} L(z) + \frac{7}{6\sqrt{\pi}}(1-z)^{-\frac{1}{2}} + O(L(z)).
\end{aligned}
$$

5. For $0 < \alpha < \frac{1}{2}$

$$
\begin{aligned}
&A(z) \odot C(z/e) \\
&= \left[\frac{\sqrt{2}}{4}(1-z)^{-\frac{1}{2}} - \frac{7}{12} + O((1-z)^{\frac{1}{2}}) \right] \times \left[\sqrt{\frac{2}{\pi}} \Gamma\left(\alpha - \frac{1}{2}\right)(1-z)^{-\alpha+\frac{1}{2}} \right. \\
&\qquad \left. + \frac{2}{\sqrt{\pi}} \Gamma\left(\alpha - \frac{1}{2}\right) \frac{\alpha - \frac{1}{2}}{\alpha - 1}(1-z)^{-\alpha+1} + K_\alpha + K_1(1-z) + O(|1-z|^{-\alpha+\frac{3}{2}}) \right] \\
&= \frac{\sqrt{2}}{4} K_\alpha (1-z)^{-\frac{1}{2}} + \frac{1}{2\sqrt{\pi}} \Gamma\left(\alpha - \frac{1}{2}\right)(1-z)^{-\alpha} - \frac{7}{12} K_\alpha + O(|1-z|^{-\alpha+\frac{1}{2}}).
\end{aligned}
$$

This complete the proof of Theorem 3.1. ∎

Moreover, applying the result in Theorem 2.1 and Theorem 2.2 to Theorem 3.1 gives the excepted value of X_n. This establishes Theorem 1.3.

3.2 Higher Moments

Now, we turn back to the distributional recurrence (1.3) to estimate the moments of higher orders. Here, we will analyze separately the cases $0 < \alpha < \frac{1}{2}$ in Section 3.2.1, and $\frac{1}{2} < \alpha$ in Section 3.2.2. The technique used to derive the asymptotics is induction.

3.2.1 $\frac{1}{2} < \alpha$

Raising both side of (1.3) to the integral power k yields

$$X_n{}^k = \sum_{k_1+k_2+k_3=k} \binom{k}{k_1, k_2, k_3} X_{S_n}{}^{k_1} X^*_{n-S_n}{}^{k_2} (n^\alpha)^{k_3}.$$

Taking expectations and conditioning on the size S_n, we obtain

$$\mathbb{E}(X_n{}^k) = \mathbb{E}[\mathbb{E}_{S_n}(\sum_{k_1+k_2+k_3=k} \binom{k}{k_1, k_2, k_3} X_{S_n}{}^{k_1} X^*_{n-S_n}{}^{k_2} (n^\alpha)^{k_3})].$$

Set $\hat{m}_n(k) := \mathbb{E}(X_n{}^k)$. Then, the recurrence becomes

$$\hat{m}_n(k) = \mathbb{E}_{S_n} \left(\sum_{k_1+k_2+k_3=k} \binom{k}{k_1, k_2, k_3} \hat{m}_{S_n}(k_1) \hat{m}_{n-S_n}(k_2)(n^\alpha)^{k_3} \right)$$

and consequently,

$$\hat{m}_n(k) = \sum_{j=1}^{n-1} p_{n,j} \sum_{k_1+k_2+k_3=k} \binom{k}{k_1, k_2, k_3} \hat{m}_j(k_1) \hat{m}_{n-j}(k_2)(n^\alpha)^{k_3}.$$

Isolating the two k-th powers yields

$$\begin{aligned}\hat{m}_n(k) =& \sum_{j=1}^{n-1} \frac{n}{2(n-1)} \frac{c_j c_{n-j}}{c_n} (\hat{m}_j(k) + \hat{m}_{n-j}(k)) \\ &+ \sum_{\substack{k_1+k_2+k_3=k \\ k_1,k_2<k}} \binom{k}{k_1, k_2, k_3} (n^\alpha)^{k_3} \sum_{j=1}^{n-1} \frac{n}{2(n-1)} \frac{c_j c_{n-j}}{c_n} \hat{m}_j(k_1) \hat{m}_{n-j}(k_2).\end{aligned}$$

Multiply both sides by $\frac{(n-1)c_n}{ne^n}$, we obtain

$$\begin{aligned}\frac{n-1}{n} \frac{\hat{m}_n(k) c_n}{e^n} =& \sum_{j=1}^{n-1} \frac{c_j c_{n-j}}{e^n} \hat{m}_j(k) \\ &+ \sum_{\substack{k_1+k_2+k_3=k \\ k_1,k_2<k}} \binom{k}{k_1, k_2, k_3} (n^\alpha)^{k_3} \frac{1}{2} \sum_{j=1}^{n-1} \frac{c_j \hat{m}_j(k_1)}{e^j} \frac{c_{n-j} \hat{m}_{n-j}(k_2)}{e^{n-j}}.\end{aligned}$$

Let $m_n(k) = \frac{c_n \hat{m}_n(k)}{e^n}$. Then

$$\frac{n-1}{n} m_n(k) = \sum_{j=1}^{n-1} \frac{c_{n-j}}{e^{n-j}} m_j(k) + \sum_{\substack{k_1+k_2+k_3=k \\ k_1,k_2<k}} \binom{k}{k_1, k_2, k_3} (n^\alpha)^{k_3} \frac{1}{2} \sum_{j=1}^{n-1} m_j(k_1) m_{n-j}(k_2).$$

Multiplying by z^n and summing over $n \geq 1$,

$$\sum_{n\geq 1} \frac{n-1}{n} m_n(k) z^n = \sum_{n\geq 1} \sum_{j=1}^{n-1} \frac{c_{n-j}}{e^{n-j}} m_j(k) z^n \tag{3.12}$$
$$+ \sum_{\substack{k_1+k_2+k_3=k \\ k_1,k_2<k}} (n^\alpha)^{k_3} \frac{1}{2} \sum_{j=1}^{n-1} \binom{k}{k_1, k_2, k_3} m_j(k_1) m_{n-j}(k_2) z^n.$$

Let $M_k(z)$ denote the ordinary generating function of $m_n(k)$, i.e.,

$$M_k(z) = \sum_{n\geq 1} m_n(k) z^n.$$

Then, the relation (3.12) becomes

$$M_k(z) - \int_0^z M_k(w) \frac{dw}{w} = M_k(z) C\left(\frac{z}{e}\right) + R_k(z), \tag{3.13}$$

where

$$\begin{aligned} R_k(z) &= \sum_{n\geq 1} \sum_{\substack{k_1+k_2+k_3=k \\ k_1,k_2<k}} \frac{1}{2} (n^\alpha)^{k_3} \sum_{j=1}^{n-1} \binom{k}{k_1, k_2, k_3} m_j(k_1) m_{n-j}(k_2) z^n \\ &= \sum_{\substack{k_1+k_2+k_3=k \\ k_1,k_2<k}} \binom{k}{k_1, k_2, k_3} \sum_{n\geq 1} (n^\alpha)^{k_3} \frac{1}{2} \sum_{j=1}^{n-1} m_j(k_1) m_{n-j}(k_2) z^n \\ &= \sum_{\substack{k_1+k_2+k_3=k \\ k_1,k_2<k}} \binom{k}{k_1, k_2, k_3} (B(z)^{\odot k_3}) \odot [\frac{1}{2} M_{k_1}(z) M_{k_2}(z)] \end{aligned}$$

with

$$B(z)^{\odot k_3} = \underbrace{B(z) \odot \cdots \odot B(z)}_{k_3 \text{ times}}.$$

Note that the equation (3.13) is the same as the equation (3.8), so the solution of this equation (3.13) is given by

$$M_k(z) = \frac{C(z/e)}{1 - C(z/e)} \int_0^z \partial_w R_k(w) \frac{dw}{C(w/e)} \tag{3.14}$$

with $M_0(z) = C(z/e)$ and $M_1(z) = A(z) \odot C(z/e)$.

Now, we can state the result about the singular expansion of the generating function $M_k(z)$ at $z = 1$ when $\alpha > \frac{1}{2}$.

Proposition 3.1. *Let $\varepsilon > 0$ be given. Then, the generating function $M_k(z)$ is analytic in some Δ-domain. Moreover, it admits the following singular expansions at $z = 1$*

$$M_k(z) = \frac{\sqrt{2}}{2} A_k (1-z)^{-k(\alpha+\frac{1}{2})+\frac{1}{2}} + O(|1-z|^{-k(\alpha+\frac{1}{2})+\frac{1}{2}+c}), \ k \geq 1,$$

where

$$c := \begin{cases} \alpha - \frac{1}{2}, & for\ \frac{1}{2} < \alpha < 1 \\ \frac{1}{2} - \epsilon, & for\ \alpha = 1 \\ \frac{1}{2}, & for\ \alpha > 1 \end{cases}$$

and the coefficients are defined by following recurrence:

$$A_k = \frac{1}{4} \sum_{j=1}^{k-1} \binom{k}{j} A_j A_{k-j} + \frac{\sqrt{2}}{2} k A_{k-1} \frac{\Gamma(k\alpha + \frac{k}{2} - 1)}{\Gamma((k-1)\alpha + \frac{k}{2} - 1)}, \tag{3.15}$$

for $k \geq 2$, $A_1 = \frac{1}{\sqrt{2\pi}} \Gamma(\alpha - \frac{1}{2})$

Proof.

We prove this proposition by induction.

For $k = 1$, the proposition has been established in Theorem (3.1) with $M_1(z) = A(z) \odot C(z/e)$.

For $k \geq 2$, we claim that $R_k(z)$ admits a singular expansion

$$R_k(z) = A_k (1-z)^{-k(\alpha+\frac{1}{2})+1} + O(|1-z|^{-k(\alpha+\frac{1}{2})+1+c}),$$

and this will give us the desired result.

For the proof, we divide $R_k(z)$ into five parts as (1) $k_3 = 0$, $k_1 \neq 0$, $k_2 \neq 0$; (2) $k_3 \neq 0$, $k_1 \neq 0$, $k_2 \neq 0$; (3) $k_3 \neq 0$, $k_1 \neq 0$, $k_2 = 0$; (4) $k_3 \neq 0$, $k_1 = 0$, $k_2 \neq 0$; (5) $k_3 \neq 0$, $k_1 = 0$, $k_2 = 0$, and analyze them separately.

1. For $k_3 = 0$, $k_1 \neq 0$, $k_2 \neq 0$:

Since k_1,k_2 are both nonzero, it follows from the induction hypothesis that

$$\begin{aligned}
&\frac{1}{2}M_{k_1}(z)M_{k_2}(z)\\
=&\frac{1}{2}\left[\frac{\sqrt{2}}{2}A_{k_1}(1-z)^{-k_1(\alpha+\frac{1}{2})+\frac{1}{2}} + O(|1-z|^{-k_1(\alpha+\frac{1}{2})+\frac{1}{2}+c})\right]\\
&\qquad\qquad \times\left[\frac{\sqrt{2}}{2}A_{k_2}(1-z)^{-k_2(\alpha+\frac{1}{2})+\frac{1}{2}} + O(|1-z|^{-k_2(\alpha+\frac{1}{2})+\frac{1}{2}+c})\right]\\
=&\frac{1}{4}A_{k_1}A_{k_2}(1-z)^{-(k_1+k_2)(\alpha+\frac{1}{2})+1} + O(|1-z|^{-(k_1+k_2)(\alpha+\frac{1}{2})+1+c}).
\end{aligned}$$

Thus, the contribution to $R_k(z)$ in this case is

$$\frac{1}{4}\sum_{j=1}^{k-1}\binom{k}{j}A_jA_{k-j}(1-z)^{-k(\alpha+\frac{1}{2})+1} + O(|1-z|^{-k(\alpha+\frac{1}{2})+1+c}).$$

2. For $k_3 \neq 0$, $k_1 \neq 0$, $k_2 \neq 0$:

Here, we make use of Lemma (2.1) to express

$$\begin{aligned}
&\frac{1}{2}M_{k_1}(z)M_{k_2}(z)\\
=&\frac{1}{4}A_{k_1}A_{k_2}(1-z)^{-(k_1+k_2)(\alpha+\frac{1}{2})+1} + O(|1-z|^{-(k_1+k_2)(\alpha+\frac{1}{2})+1+c})\\
=&\frac{A_{k_1}A_{k_2}}{4\Gamma((k_1+k_2)(\alpha+\frac{1}{2})-1)}Li_{-(k_1+k_2)(\alpha+\frac{1}{2})+2,0}(z) + O(|1-z|^{-(k_1+k_2)(\alpha+\frac{1}{2})+1+c})\\
&-\frac{A_{k_1}A_{k_2}}{4}\frac{\zeta(-(k_1+k_2)(\alpha+\frac{1}{2})+2)}{\Gamma((k_1+k_2)(\alpha+\frac{1}{2})-1)}[(k_1+k_2)(\alpha+\frac{1}{2})<2].
\end{aligned}$$

Then,

$$
\begin{aligned}
&(B(z)^{\odot k_3}) \odot \left[\frac{1}{2}M_{k_1}(z)M_{k_2}(z)\right] \\
=&\frac{A_{k_1}A_{k_2}}{4\Gamma((k_1+k_2)(\alpha+\frac{1}{2})-1)}Li_{-(k_1+k_2)(\alpha+\frac{1}{2})-k_3\alpha+2,0}(z) \\
&\qquad\qquad\qquad\qquad + Li_{-k_3\alpha,0}(z)\odot O(|1-z|^{-(k_1+k_2)(\alpha+\frac{1}{2})+1+c}) \\
=&\frac{A_{k_1}A_{k_2}}{4\Gamma((k_1+k_2)(\alpha+\frac{1}{2})-1)}Li_{-k(\alpha+\frac{1}{2})+\frac{k_3}{2}+2,0}(z) + O(|1-z|^{-k(\alpha+\frac{1}{2})+1+c}) \\
=&O(|1-z|^{-k(\alpha+\frac{1}{2})+\frac{k_3}{2}+1}) + O(|1-z|^{-k(\alpha+\frac{1}{2})+1+c}) \\
=&O(|1-z|^{-k(\alpha+\frac{1}{2})+1+c}), \quad \text{since } k_3 \geq 1.
\end{aligned}
$$

Thus, the contribution to $R_k(z)$ in this case is

$$O(|1-z|^{-k(\alpha+\frac{1}{2})+1+c}).$$

3. For $k_3 \neq 0$, $k_1 \neq 0$, $k_2 = 0$:

First, we have

$$
\begin{aligned}
\frac{1}{2}M_{k_1}(z)M_{k_2}(z) =&\frac{1}{2}M_{k_1}(z)C(z/e) \\
=&\frac{1}{2}\left[\frac{\sqrt{2}}{2}A_{k_1}(1-z)^{-k_1(\alpha+\frac{1}{2})+\frac{1}{2}} + O((1-z)^{-k_1(\alpha+\frac{1}{2})+\frac{1}{2}+c})\right] \\
&\qquad\qquad\qquad \times [1-\sqrt{2}(1-z)^{1/2} + O(|1-z|)] \\
=&\frac{\sqrt{2}}{4}A_{k_1}(1-z)^{-k_1(\alpha+\frac{1}{2})+\frac{1}{2}} + O((1-z)^{-k_1(\alpha+\frac{1}{2})+\frac{1}{2}+c}) \\
=&\frac{\sqrt{2}A_{k_1}}{4\Gamma(k_1(\alpha+\frac{1}{2})-\frac{1}{2})}Li_{-k_1(\alpha+\frac{1}{2})+\frac{3}{2},0}(z) + O((1-z)^{-k_1(\alpha+\frac{1}{2})+\frac{1}{2}+c}) \\
&\qquad - \frac{\sqrt{2}A_{k_1}\zeta(-k_1(\alpha+\frac{1}{2})+\frac{3}{2})}{4\Gamma(k_1(\alpha+\frac{1}{2})-\frac{1}{2})}[k_1(\alpha+\frac{1}{2}) < \frac{3}{2}].
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
&(B(z)^{\odot k_3}) \odot \left[\frac{1}{2}M_{k_1}(z)C(\frac{z}{e})\right] \\
=&\frac{\sqrt{2}A_{k_1}}{4\Gamma(k_1(\alpha+\frac{1}{2})-\frac{1}{2})}Li_{-k_1(\alpha+\frac{1}{2})-k_3\alpha+\frac{3}{2},0}(z)+Li_{-k_3\alpha,0}(z)\odot O((1-z)^{-k_1(\alpha+\frac{1}{2})+\frac{1}{2}+c}) \\
=&\frac{\sqrt{2}A_{k_1}}{4\Gamma(k_1(\alpha+\frac{1}{2})-\frac{1}{2})}Li_{-k(\alpha+\frac{1}{2})+\frac{k_3}{2}+\frac{3}{2},0}(z)+O((1-z)^{-k(\alpha+\frac{1}{2})+\frac{k_3}{2}+\frac{1}{2}+c}) \\
=&\frac{\sqrt{2}}{4}A_{k_1}\frac{\Gamma(k\alpha+\frac{k_1}{2}-\frac{1}{2})}{\Gamma(k_1\alpha+\frac{k_1}{2}-\frac{1}{2})}(1-z)^{-k(\alpha+\frac{1}{2})+\frac{k_3}{2}+\frac{1}{2}}+O(|1-z|^{-k\alpha-\frac{k}{2}+\frac{k_3}{2}+\frac{1}{2}+c}) \\
=&\begin{cases} O(|1-z|^{-k\alpha-\frac{k}{2}+1+c}), & \text{if } k_3\geq 2; \\ \frac{\sqrt{2}}{4}A_{k-1}\frac{\Gamma(k\alpha+\frac{k}{2}-1)}{\Gamma((k-1)\alpha+\frac{k}{2}-1)}(1-z)^{-k\alpha-\frac{k}{2}+1}+O(|1-z|^{-k\alpha-\frac{k}{2}+1+c}), & \text{if } k_3=1. \end{cases}
\end{aligned}
$$

Thus, the contribution to $R_k(z)$ in this case is

$$\frac{\sqrt{2}}{4}kA_{k-1}\frac{\Gamma(k\alpha+\frac{k}{2}-1)}{\Gamma((k-1)\alpha+\frac{k}{2}-1)}(1-z)^{-k\alpha-\frac{k}{2}+1}+O(|1-z|^{-k\alpha-\frac{k}{2}+1+c}).$$

4. For $k_3\neq 0$, $k_1=0$, $k_2\neq 0$:

This case is the same as the previous one. Hence, we also have the contribution

$$\frac{\sqrt{2}}{4}kA_{k-1}\frac{\Gamma(k\alpha+\frac{k}{2}-1)}{\Gamma((k-1)\alpha+\frac{k}{2}-1)}(1-z)^{-k\alpha-\frac{k}{2}+1}+O(|1-z|^{-k\alpha-\frac{k}{2}+1+c}).$$

5. For $k_3\neq 0$, $k_1=0$, $k_2=0$:

Since

$$
\begin{aligned}
\frac{1}{2}M_{k_1}(z)M_{k_2}(z)=\frac{1}{2}C(z/e)^2&=\frac{1}{2}-\sqrt{2}(1-z)^{\frac{1}{2}}+O(|1-z|) \\
&=\frac{1}{2}+\frac{\sqrt{2}\zeta(\frac{3}{2})}{\Gamma(-\frac{1}{2})}-\frac{\sqrt{2}}{\Gamma(-\frac{1}{2})}Li_{\frac{3}{2},0}(z)+O(|1-z|),
\end{aligned}
$$

then

$$
\begin{aligned}
(B(z)^{\odot k_3})\odot\left[\frac{1}{2}M_{k_1}(z)M_{k_2}(z)\right] =&(B(z)^{\odot k})\odot\left[\frac{1}{2}C(z/e)^2\right]\\
=&-\frac{\sqrt{2}}{\Gamma(-\frac{1}{2})}Li_{-k\alpha+\frac{3}{2},0}(z)+Li_{-k\alpha,0}(z)\odot O(|1-z|)\\
=&O(|1-z|^{-k\alpha+\frac{1}{2}})\\
=&O(|1-z|^{-k(\alpha+\frac{1}{2})+1+c}) \quad \text{as } k\geq 2.
\end{aligned}
$$

Thus, the contribution to $R_k(z)$ is

$$
O(|1-z|^{-k(\alpha+\frac{1}{2})+1+c}).
$$

Adding all these five cases, we get

$$
\begin{aligned}
R_k(z) = &\left[\frac{1}{4}\sum_{j=1}^{k-1}\binom{k}{j}A_jA_{k-j}+\frac{\sqrt{2}}{4}kA_{k-1}\frac{\Gamma(k\alpha+\frac{k}{2}-1)}{\Gamma((k-1)\alpha+\frac{k}{2}-1)}\right](1-z)^{-k\alpha-\frac{k}{2}+1}\\
&+O(|1-z|^{-k\alpha-\frac{k}{2}+1+c})\\
= &A_k(1-z)^{-k(\alpha+\frac{1}{2})+1}+O(|1-z|^{-k(\alpha+\frac{1}{2})+1+c})
\end{aligned}
$$

Finally, with the relation (3.14), we get

$$
M_k(z)=\frac{\sqrt{2}}{2}A_k(1-z)^{-k(\alpha+\frac{1}{2})+\frac{1}{2}}+O(|1-z|^{-k(\alpha+\frac{1}{2})+\frac{1}{2}+c})
$$

and this completes the proof. ∎

Corollary 3.1. *Under the random spanning tree model, the k-th moment of X_n which satisfies the distributional recurrence (1.3) with the initial condition $X_1 = 0$ is*

$$
\mathbb{E}(X_n^k)=\frac{A_k\sqrt{\pi}}{\Gamma(k(\alpha+\frac{1}{2})-\frac{1}{2})}n^{k(\alpha+\frac{1}{2})}+O(n^{k(\alpha+\frac{1}{2})-c}),
$$

where

$$
c:=\begin{cases}\alpha-\frac{1}{2}, & as\ \frac{1}{2}<\alpha<1\\ \frac{1}{2}-\epsilon, & as\ \alpha=1\\ \frac{1}{2}, & as\ \alpha>1\end{cases}
$$

Proof.

Since the generating function $M_k(z)$ of $m_n(k) = \frac{c_n\hat{m}_n(k)}{e^n} = \frac{c_n\mathbb{E}(X_n^k)}{e^n}$ for $\alpha > \frac{1}{2}$ satisfies

$$M_k(z) = \frac{\sqrt{2}}{2}A_k(1-z)^{-k(\alpha+\frac{1}{2})+\frac{1}{2}} + O(|1-z|^{-k(\alpha+\frac{1}{2})+\frac{1}{2}+c}),$$

By Theorem (2.1) and Theorem (2.2), we have

$$\frac{c_n\mathbb{E}(X_n^k)}{e^n} = \frac{\sqrt{2}}{2}A_k\frac{n^{k(\alpha+\frac{1}{2})-\frac{3}{2}}}{\Gamma(k(\alpha+\frac{1}{2})-\frac{1}{2})} + O(n^{k(\alpha+\frac{1}{2})-\frac{3}{2}-c}).$$

Moreover, with

$$c_n = \frac{e^n}{\sqrt{2\pi}n^{3/2}}(1+O(\frac{1}{n})),$$

we obtain the desired result. ∎

3.2.2 $0 < \alpha < \frac{1}{2}$

Now, we discuss the case where $\alpha < \frac{1}{2}$. Since the mean $a_n = \mathbb{E}(X_n) = \frac{1}{2}K_\alpha n + O(n^{\alpha+\frac{1}{2}})$, the main term is of order n, irrespective of the value of α. Thus, we need to apply the centering technique to get dependence on α. Consider the original distributional recurrence (3.1). Defining $\overline{X}_n := X_n - \frac{1}{2}K_\alpha n$, then we obtain the distributional recurrence of $\overline{X}_n$ as follows:

$$\overline{X}_n \overset{d}{=} \overline{X}_{S_n} + \overline{X}^*_{n-S_n} + n^\alpha, \quad \text{for } n\geq 2,\ \overline{X}_1 = -\frac{K_\alpha}{2}. \tag{3.16}$$

This recurrence is the same as recurrence (3.1) except for the initial value. Thus, we can derive the generating function of the moments of $\overline{X}_n$ as before. Define $\widetilde{m}_n(k) := \mathbb{E}(\overline{X}_n^k)$ and $\overline{m}_n(k) := \frac{c_n\widetilde{m}_n(k)}{e^n}$. Let $\overline{M}_k(z)$ denote the ordinary generating function of $\overline{m}_n(k)$ in n. Then we have

$$\overline{M}_1(z) = -\frac{K_\alpha}{2}\frac{C(z/e)}{1-C(z/e)} + \frac{1}{2}\frac{C(z/e)}{1-C(z/e)}\int_0^z \partial_w[B(w)\odot C(w/e)^2]\frac{dw}{C(w/e)}, \tag{3.17}$$

and consequently,

$$\overline{M}_1(z) = \frac{1}{2\sqrt{\pi}}\Gamma\left(\alpha - \frac{1}{2}\right)(1-z)^{-\alpha} - \frac{3}{4}K_\alpha + O(|1-z|^{-\alpha+\frac{1}{2}})$$

and

$$\overline{M}_k(z) = \left(-\frac{K_\alpha}{2}\right)^k \frac{C(z/e)}{1-C(z/e)} + \frac{C(z/e)}{1-C(z/e)}\int_0^z \partial_w \overline{R}_k(w)\frac{dw}{C(w/e)}, \qquad (3.18)$$

for $k \geq 2$, where

$$\overline{R}_k(z) = \sum_{\substack{k_1+k_2+k_3=k \\ k_1,k_2<k}} \binom{k}{k_1, k_2, k_3}(B(z)^{\odot k_3})\odot\left[\frac{1}{2}\overline{M}_{k_1}(z)\overline{M}_{k_2}(z)\right].$$

Lemma 3.1. *The generating function $\overline{M}_2(z)$ admits the following singular expansions at $z = 1$:*

$$\overline{M}_2(z) = \frac{\sqrt{2}}{2}A_2(1-z)^{-2\alpha-\frac{1}{2}} + \begin{cases} O(|1-z|^{-\frac{1}{2}}), & \text{for } 0 < \alpha < \frac{1}{4}; \\ O(|1-z|^{-\frac{1}{2}}L(z)^2), & \text{for } \alpha = \frac{1}{4}; \\ O(|1-z|^{-2\alpha}), & \text{for } \frac{1}{4} < \alpha < \frac{1}{2}, \end{cases}$$

where $A_2 = \frac{1}{2}{A_1}^2 + \frac{\sqrt{2}A_1\Gamma(2\alpha)}{\Gamma(\alpha)}$ and $A_1 = \frac{\Gamma(\alpha-\frac{1}{2})}{\sqrt{2\pi}}$.

Proof.
With the relation (3.18), we have

$$\overline{M}_2(z) = \left(-\frac{K_\alpha}{2}\right)^2 \frac{C(z/e)}{1-C(z/e)} + \frac{C(z/e)}{1-C(z/e)}\int_0^z \partial_w \overline{R}_2(w)\frac{dw}{C(w/e)},$$

where

$$\overline{R}_2(z) = \frac{1}{2}B(z)^{\odot 2}\odot C(z/e)^2 + \overline{M}_1(z)^2 + 2B(z)\odot\left(\overline{M}_1(z)C(z/e)\right).$$

Here, we will compute $\frac{1}{2}B(z)^{\odot 2}\odot C(z/e)^2$, $\overline{M}_1(z)^2$ and $2B(z)\odot\left(\overline{M}_1(z)C(z/e)\right)$ separately to obtain the singular expansion of $\overline{M}_2(z)$.

For $\alpha \neq \frac{1}{4}$, we have

$$
\begin{aligned}
\frac{1}{2}B(z)^{\odot 2}\odot C(z/e)^2 =& Li_{-2\alpha}(z)\odot\left(\frac{1}{2}+\frac{\sqrt{2}\zeta(\frac{3}{2})}{\Gamma(-\frac{1}{2})}-\frac{\sqrt{2}}{\Gamma(-\frac{1}{2})}Li_{\frac{3}{2},0}(z)+O(|1-z|)\right)\\
=& -\frac{\sqrt{2}}{\Gamma(-\frac{1}{2})}Li_{-2\alpha+\frac{3}{2},0}(z)+Li_{-2\alpha}(z)\odot O(|1-z|)\\
=& C_0+O(|1-z|^{-2\alpha+\frac{1}{2}})
\end{aligned}
$$

with C_0 a constant.

Moreover,

$$
\begin{aligned}
\overline{M}_1(z)^2 =& \left(\frac{\sqrt{2}}{2}A_1(1-z)^{-\alpha}-\frac{3}{4}K_\alpha+O(|1-z|^{-\alpha+\frac{1}{2}})\right)^2\\
=& \frac{1}{2}{A_1}^2(1-z)^{-2\alpha}+O(|1-z|^{-\alpha}).
\end{aligned}
$$

Finally, with

$$
\begin{aligned}
&\overline{M}_1(z)C(z/e)\\
=& \left(\frac{\sqrt{2}}{2}A_1(1-z)^{-\alpha}-\frac{3}{4}K_\alpha+O(|1-z|^{-\alpha+\frac{1}{2}})\right)(1-\sqrt{2}(1-z)^{\frac{1}{2}}+O(|1-z|))\\
=& \frac{\sqrt{2}}{2}A_1(1-z)^{-\alpha}-\frac{3}{4}K_\alpha+O(|1-z|^{-\alpha+\frac{1}{2}})
\end{aligned}
$$

we have

$$
\begin{aligned}
&2B(z)\odot(\overline{M}_1(z)C(z/e))\\
=& Li_{-\alpha,0}(z)\odot\left(\frac{\sqrt{2}A_1}{\Gamma(\alpha)}Li_{-\alpha+1,0}(z)+O(|1-z|^{-\alpha+\frac{1}{2}})-\frac{3}{2}K_\alpha-\sqrt{2}A_1\frac{\zeta(-\alpha+1)}{\Gamma(\alpha)}\right)\\
=& \frac{\sqrt{2}A_1}{\Gamma(\alpha)}Li_{-2\alpha+1,0}(z)+Li_{-\alpha,0}(z)\odot O(|1-z|^{-\alpha+\frac{1}{2}})\\
=& C_1+\frac{\sqrt{2}A_1\Gamma(2\alpha)}{\Gamma(\alpha)}(1-z)^{-2\alpha}+O(|1-z|^{-2\alpha+\frac{1}{2}})
\end{aligned}
$$

where C_1 is a constant.

Combining the above three parts, we get

$$\overline{R}_2(z) = C_2 + A_2(1-z)^{-2\alpha} + O(|1-z|^{-2\alpha+\frac{1}{2}}),$$

where C_2 is a constant and $A_2 = \frac{1}{2}{A_1}^2 + \frac{\sqrt{2}A_1\Gamma(2\alpha)}{\Gamma(\alpha)}$.

Thus,

$$\begin{aligned}
&\int_0^z \partial_w \overline{R}_2(w)\frac{dw}{C(w/e)}\\
=&\int_0^z (-2\alpha A_2(1-z)^{-2\alpha-1} + O(|1-z|^{-2\alpha-\frac{1}{2}}))(1+\sqrt{2}(1-z)^{\frac{1}{2}} + O(|1-z|))dw\\
=&\int_0^z -2\alpha A_2(1-z)^{-2\alpha-1} + O(|1-z|^{-2\alpha-\frac{1}{2}})dw\\
=&\begin{cases} L_0 + A_2(1-z)^{-2\alpha} + O(|1-z|^{-2\alpha+\frac{1}{2}}), & \text{if } 0<\alpha<\frac{1}{4}\\ A_2(1-z)^{-2\alpha} + O(|1-z|^{-2\alpha+\frac{1}{2}}), & \text{if } \frac{1}{4}<\alpha<\frac{1}{2}\end{cases}
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
\overline{M}_2(z) =& \left(-\frac{K_\alpha}{2}\right)^2 \frac{C(z/e)}{1-C(z/e)} + \frac{C(z/e)}{1-C(z/e)}\int_0^z \partial_w \overline{R}_2(w)\frac{dw}{C(w/e)}\\
=&\frac{\sqrt{2}}{2}A_2(1-z)^{-2\alpha-\frac{1}{2}} + O(|1-z|^{-2\alpha}) + O(|1-z|^{-\frac{1}{2}})\\
=&\begin{cases} \frac{\sqrt{2}}{2}A_2(1-z)^{-2\alpha-\frac{1}{2}} + O(|1-z|^{-\frac{1}{2}}), & \text{if } 0<\alpha<\frac{1}{4};\\ \frac{\sqrt{2}}{2}A_2(1-z)^{-2\alpha-\frac{1}{2}} + O(|1-z|^{-2\alpha}), & \text{if } \frac{1}{4}<\alpha<\frac{1}{2}.\end{cases}
\end{aligned}$$

For $\alpha = \frac{1}{4}$, the proof is almost the same as the case of $\alpha\neq\frac{1}{4}$. First, we have

$$\begin{aligned}
\frac{1}{2}B(z)^{\odot 2}\odot C(z/e)^2 =& Li_{-\frac{1}{2}}(z)\odot\left(\frac{1}{2} + \frac{\sqrt{2}\zeta(\frac{3}{2})}{\Gamma(-\frac{1}{2})} - \frac{\sqrt{2}}{\Gamma(-\frac{1}{2})}Li_{\frac{3}{2},0}(z) + O(|1-z|)\right)\\
=& -\frac{\sqrt{2}}{\Gamma(-\frac{1}{2})}Li_{1,0}(z) + Li_{-\frac{1}{2}}(z)\odot O(|1-z|)\\
=& O(L(z))
\end{aligned}$$

And,

$$\begin{aligned}\overline{M}_1(z)^2 =& \left(\frac{\sqrt{2}}{2}A_1(1-z)^{-\frac{1}{4}} - \frac{3}{4}K_{\frac{1}{4}} + O(|1-z|^{\frac{1}{4}})\right)^2 \\ =& \frac{1}{2}{A_1}^2(1-z)^{-\frac{1}{2}} + O(|1-z|^{-\frac{1}{4}}).\end{aligned}$$

Moreover,

$$\begin{aligned}2B(z)\odot(\overline{M}_1(z)C(z/e)) =& \frac{\sqrt{2}A_1}{\Gamma(\frac{1}{4})}Li_{\frac{1}{2},0}(z) + Li_{-\frac{1}{4},0}(z)\odot O(|1-z|^{\frac{1}{4}}) \\ =& \frac{\sqrt{2}A_1\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{4})}(1-z)^{-\frac{1}{2}} + O(L(z))\end{aligned}$$

Combining the above three parts as usual, we get

$$\overline{R}_2(z) = A_2(1-z)^{-2\alpha} + O(L(z)),$$

where $A_2 = \frac{1}{2}{A_1}^2 + \frac{\sqrt{2}A_1\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{4})}$.

Thus,

$$\begin{aligned}&\int_0^z \partial_w \overline{R}_2(w)\frac{dw}{C(w/e)} \\ =& \int_0^z (-2\alpha A_2(1-z)^{-\frac{3}{2}} + O(|1-z|^{-1}L(z)))(1+\sqrt{2}(1-z)^{\frac{1}{2}} + O(|1-z|))dw \\ =& \int_0^z -2\alpha A_2(1-z)^{-\frac{3}{2}} + O(|1-z|^{-1}L(z))dw \\ =& A_2(1-z)^{-\frac{1}{2}} + O(L(z)^2).\end{aligned}$$

Finally, we obtain

$$\begin{aligned}\overline{M}_2(z) =& \left(-\frac{K_\alpha}{2}\right)^2 \frac{C(z/e)}{1-C(z/e)} + \frac{C(z/e)}{1-C(z/e)}\int_0^z \partial_w \overline{R}_2(w)\frac{dw}{C(w/e)} \\ =& \frac{\sqrt{2}}{2}A_2(1-z)^{-1} + O(|1-z|^{-\frac{1}{2}}L(z)^2).\end{aligned}$$

This completes the proof. ∎

Proposition 3.2. *The generating function $\overline{M}_k(z)$ is analytic in some Δ-domain. Moreover, it admits the following singular expansions at $z = 1$*

$$\overline{M}_k(z) = \frac{\sqrt{2}}{2} A_k (1-z)^{-k(\alpha+\frac{1}{2})+\frac{1}{2}} + O(|1-z|^{-k(\alpha+\frac{1}{2})+\frac{1}{2}+\alpha}).$$

The coefficients are defined by following recurrence:

$$A_k = \frac{1}{4}\sum_{j=1}^{k-1}\binom{k}{j} A_j A_{k-j} + \frac{\sqrt{2}}{2} k A_{k-1} \frac{\Gamma(k\alpha+\frac{k}{2}-1)}{\Gamma((k-1)\alpha+\frac{k}{2}-1)},$$

for $k \geq 2$, $A_1 = \frac{1}{\sqrt{2\pi}}\Gamma(\alpha-\frac{1}{2})$

Proof.

For $k = 2$, the result has been established in Lemma (3.1).

Now, we analyze the case for $k \geq 2$.

First, we consider $\overline{M}_{k_1}(z)\overline{M}_{k_2}(z)$ for $k_1, k_2 \neq 0$. When $k_1, k_2 \geq 2$, it follows from the induction hypothesis that

$$\begin{aligned}
\frac{1}{2}\overline{M}_{k_1}(z)\overline{M}_{k_2}(z) =& \frac{1}{2}\left[\frac{\sqrt{2}}{2} A_{k_1}(1-z)^{-k_1(\alpha+\frac{1}{2})+\frac{1}{2}} + O(|1-z|^{-k_1(\alpha+\frac{1}{2})+\frac{1}{2}+\alpha})\right] \\
&\times \left[\frac{\sqrt{2}}{2} A_{k_2}(1-z)^{-k_2(\alpha+\frac{1}{2})+\frac{1}{2}} + O(|1-z|^{-k_2(\alpha+\frac{1}{2})+\frac{1}{2}+\alpha})\right] \\
=& \frac{1}{4} A_{k_1} A_{k_2} (1-z)^{-(k_1+k_2)(\alpha+\frac{1}{2})+1} + O(|1-z|^{-(k_1+k_2)(\alpha+\frac{1}{2})+1+\alpha}).
\end{aligned}$$

When either k_1 or k_2 equals to 1, we also have

$$\begin{aligned}
\frac{1}{2}\overline{M}_{k_1}(z)\overline{M}_{1}(z) =& \frac{1}{2}\left[\frac{\sqrt{2}}{2} A_{k_1}(1-z)^{-k_1(\alpha+\frac{1}{2})+\frac{1}{2}} + O(|1-z|^{-k_1(\alpha+\frac{1}{2})+\frac{1}{2}+\alpha})\right] \\
&\times \left[\frac{\sqrt{2}}{2} A_1 (1-z)^{-\alpha} - \frac{3}{4}K_\alpha + O(|1-z|^{-\alpha+\frac{1}{2}})\right] \\
=& \frac{1}{4} A_1 A_{k_1} (1-z)^{-(k_1+1)(\alpha+\frac{1}{2})+1} + O(|1-z|^{-(k_1+1)(\alpha+\frac{1}{2})+1+\alpha}),
\end{aligned}$$

and

$$\frac{1}{2}\overline{M}_1(z)^2 = \frac{1}{4}{A_1}^2(1-z)^{-2\alpha} + O(|1-z|^{-\alpha}).$$

Thus,

$$\frac{1}{2}\overline{M}_{k_1}(z)\overline{M}_{k_2}(z) = \frac{1}{4}A_{k_1}A_{k_2}(1-z)^{-(k_1+k_2)(\alpha+\frac{1}{2})+1} + O(|1-z|^{-(k_1+k_2)(\alpha+\frac{1}{2})+1+\alpha}),$$

for $k_1, k_2 \neq 0$.

As in the previous section, we divide $R_k(z)$ into five parts as (1) $k_3 = 0$, $k_1 \neq 0$, $k_2 \neq 0$; (2) $k_3 \neq 0$, $k_1 \neq 0$, $k_2 \neq 0$; (3) $k_3 \neq 0$, $k_1 \neq 0$, $k_2 = 0$; (4) $k_3 \neq 0$, $k_1 = 0$, $k_2 \neq 0$; (5) $k_3 \neq 0$, $k_1 = 0$, $k_2 = 0$, and analyze them separately.

1. For $k_3 = 0$, $k_1 \neq 0$, $k_2 \neq 0$:

The contribution to $\overline{R}_k(z)$ is:

$$\begin{aligned}&\sum_{\substack{k_1+k_2=k\\k_1,k_2<k}}\binom{k}{k_1,k_2}\frac{1}{2}\overline{M}_{k_1}(z)\overline{M}_{k_2}(z)\\ =&\frac{1}{4}\sum_{j=1}^{k-1}\binom{k}{j}A_jA_{k-j}(1-z)^{-k(\alpha+\frac{1}{2})+1} + O(|1-z|^{-k(\alpha+\frac{1}{2})+1+\alpha}).\end{aligned}$$

2. For $k_3 \neq 0$, $k_1 \neq 0$, $k_2 \neq 0$:

First, as in Proposition 3.1, we have

$$\begin{aligned}&\frac{1}{2}\overline{M}_{k_1}(z)\overline{M}_{k_2}(z)\\ =&\frac{A_{k_1}A_{k_2}}{4\Gamma((k_1+k_2)(\alpha+\frac{1}{2})-1)}Li_{-(k_1+k_2)(\alpha+\frac{1}{2})+2,0}(z) + O(|1-z|^{-(k_1+k_2)(\alpha+\frac{1}{2})+1+\alpha})\\ &-\frac{A_{k_1}A_{k_2}}{4}\frac{\zeta(-(k_1+k_2)(\alpha+\frac{1}{2})+2)}{\Gamma((k_1+k_2)(\alpha+\frac{1}{2})-1)}[(k_1+k_2)(\alpha+\frac{1}{2})<2].\end{aligned}$$

Then,

$$
\begin{aligned}
&(B(z)^{\odot k_3}) \odot \left[\frac{1}{2}\overline{M}_{k_1}(z)\overline{M}_{k_2}(z)\right] \\
=&\frac{A_{k_1}A_{k_2}}{4\Gamma((k_1+k_2)(\alpha+\frac{1}{2})-1)} Li_{-(k_1+k_2)(\alpha+\frac{1}{2})-k_3\alpha+2,0}(z) \\
&\qquad\qquad + Li_{-k_3\alpha,0}(z) \odot O(|1-z|^{-(k_1+k_2)(\alpha+\frac{1}{2})+\frac{k_3}{2}+1+\alpha}) \\
=&\frac{A_{k_1}A_{k_2}}{4\Gamma((k_1+k_2)(\alpha+\frac{1}{2})-1)} Li_{-k(\alpha+\frac{1}{2})+\frac{k_3}{2}+2,0}(z) + O(|1-z|^{-k(\alpha+\frac{1}{2})+1+\alpha}).
\end{aligned}
$$

Now, $k_3 \leq k-2$, so $-k(\alpha+\frac{1}{2})+\frac{k_3}{2}+1<0$. Thus,

$$
\begin{aligned}
&(B(z)^{\odot k_3}) \odot \left[\frac{1}{2}\overline{M}_{k_1}(z)\overline{M}_{k_2}(z)\right] \\
=&O(|1-z|^{-k(\alpha+\frac{1}{2})+\frac{k_3}{2}+1}) + O(|1-z|^{-k(\alpha+\frac{1}{2})+1+\frac{k_3}{2}+\alpha}) \\
=&O(|1-z|^{-k(\alpha+\frac{1}{2})+1+\alpha}) \quad \text{since } k_3 \geq 1.
\end{aligned}
$$

Thus, the contribution to $\overline{R}_k(z)$ is

$$
O(|1-z|^{-k(\alpha+\frac{1}{2})+1+\alpha}).
$$

3. For $k_3 \neq 0$, $k_1 \neq 0$, $k_2 = 0$:

As in Proposition 3.1, we have

$$
\begin{aligned}
\frac{1}{2}\overline{M}_{k_1}(z)\overline{M}_{k_2}(z) =& \frac{1}{2}\overline{M}_{k_1}(z)C(z/e) \\
=& \frac{\sqrt{2}}{4}A_{k_1}(1-z)^{-k_1(\alpha+\frac{1}{2})+\frac{1}{2}} + O((1-z)^{-k_1(\alpha+\frac{1}{2})+\frac{1}{2}+\alpha}) \\
=& \frac{\sqrt{2}A_{k_1}}{4\Gamma(k_1(\alpha+\frac{1}{2})-\frac{1}{2})} Li_{-k_1(\alpha+\frac{1}{2})+\frac{3}{2},0}(z) + O((1-z)^{-k_1(\alpha+\frac{1}{2})+\frac{1}{2}+\alpha}) \\
&- \frac{\sqrt{2}A_{k_1}\zeta(-k_1(\alpha+\frac{1}{2})+\frac{3}{2})}{4\Gamma(k_1(\alpha+\frac{1}{2})-\frac{1}{2})}[k_1(\alpha+\frac{1}{2})<\frac{3}{2}].
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
&(B(z)^{\odot k_3}) \odot \left[\frac{1}{2}M_{k_1}(z)C(\frac{z}{e})\right] \\
=&\frac{\sqrt{2}A_{k_1}}{4\Gamma(k_1(\alpha+\frac{1}{2})-\frac{1}{2})}Li_{-k_1(\alpha+\frac{1}{2})-k_3\alpha+\frac{3}{2},0}(z) + Li_{-k_3\alpha,0}(z)\odot O((1-z)^{-k_1(\alpha+\frac{1}{2})+\frac{1}{2}+\alpha}).
\end{aligned}
$$

Now, $k_3 \leq k-1$, so $-k(\alpha+\frac{1}{2})+\frac{k_3}{2}+1<0$. Thus,

$$
\begin{aligned}
&(B(z)^{\odot k_3}) \odot \left[\frac{1}{2}M_{k_1}(z)C(\frac{z}{e})\right] \\
=&\frac{\sqrt{2}A_{k_1}}{4\Gamma(k_1(\alpha+\frac{1}{2})-\frac{1}{2})}Li_{-k(\alpha+\frac{1}{2})+\frac{k_3}{2}+\frac{3}{2},0}(z) + O((1-z)^{-k(\alpha+\frac{1}{2})+\frac{k_3}{2}+\frac{1}{2}+\alpha}) \\
=&\frac{\sqrt{2}}{4}A_{k_1}\frac{\Gamma(k\alpha+\frac{k_1}{2}-\frac{1}{2})}{\Gamma(k_1\alpha+\frac{k_1}{2}-\frac{1}{2})}(1-z)^{-k(\alpha-\frac{1}{2})+\frac{k_3}{2}+\frac{1}{2}}+O(|1-z|^{-k\alpha-\frac{k}{2}+\frac{k_3}{2}+\frac{1}{2}+\alpha}) \\
=&\begin{cases} O(|1-z|^{-k\alpha-\frac{k}{2}+1+\alpha}), & \text{if } k_3 \geq 2; \\ \frac{\sqrt{2}}{4}A_{k-1}\frac{\Gamma(k\alpha+\frac{k}{2}-1)}{\Gamma((k-1)\alpha+\frac{k}{2}-1)}(1-z)^{-k\alpha-\frac{k}{2}+1}+O(|1-z|^{-k\alpha-\frac{k}{2}+1+\alpha}), & \text{if } k_3=1. \end{cases}
\end{aligned}
$$

Thus, the contribution to $\overline{R}_k(z)$ is

$$
\frac{\sqrt{2}}{4}kA_{k-1}\frac{\Gamma(k\alpha+\frac{k}{2}-1)}{\Gamma((k-1)\alpha+\frac{k}{2}-1)}(1-z)^{-k\alpha-\frac{k}{2}+1}+O(|1-z|^{-k\alpha-\frac{k}{2}+1+\alpha}).
$$

4. For $k_3 \neq 0$, $k_1=0$, $k_2 \neq 0$:

This case is the same as the previous one. Hence, we also have a contribution

$$
\frac{\sqrt{2}}{4}kA_{k-1}\frac{\Gamma(k\alpha+\frac{k}{2}-1)}{\Gamma((k-1)\alpha+\frac{k}{2}-1)}(1-z)^{-k\alpha-\frac{k}{2}+1}+O(|1-z|^{-k\alpha-\frac{k}{2}+1+\alpha})
$$

5. For $k_3 \neq 0$, $k_1=0$, $k_2=0$:

The computation is the same as in Proposition 3.1. Thus, we obtain

$$
\frac{1}{2}\overline{M}_{k_1}(z)\overline{M}_{k_2}(z)=\frac{1}{2}C(z/e)^2=O(|1-z|^{-k(\alpha+\frac{1}{2})+1+\alpha}).
$$

Adding all these five cases, we get

$$\begin{aligned}\overline{R}_k(z) &= \left[\frac{1}{4}\sum_{j=1}^{k-1}\binom{k}{j}A_jA_{k-j} + \frac{\sqrt{2}}{2}kA_{k-1}\frac{\Gamma(k\alpha+\frac{k}{2}-1)}{\Gamma((k-1)\alpha+\frac{k}{2}-1)}\right](1-z)^{-k\alpha-\frac{k}{2}+1} \\ &\qquad + O(|1-z|^{-k\alpha-\frac{k}{2}+1+\alpha}) \\ &= A_k(1-z)^{-k(\alpha+\frac{1}{2})+1} + O(|1-z|^{-k(\alpha+\frac{1}{2})+1+\alpha})\end{aligned}$$

Finally, with the relation (3.18), we get

$$\begin{aligned}\overline{M}_k(z) =& \frac{\sqrt{2}}{2}A_k(1-z)^{-k(\alpha+\frac{1}{2})+\frac{1}{2}} + O(|1-z|^{-k(\alpha+\frac{1}{2})+\frac{1}{2}+\alpha}) + O(|1-z|^{-\frac{1}{2}}) \\ =& \frac{\sqrt{2}}{2}A_k(1-z)^{-k(\alpha+\frac{1}{2})+\frac{1}{2}} + O(|1-z|^{-k(\alpha+\frac{1}{2})+\frac{1}{2}+\alpha}),\end{aligned}$$

as $k \geq 2$. This concludes the proof. ▮

Corollary 3.2. *Under the random spanning tree model, the k-th moment of $\overline{X}_n$ which satisfies the distributional recurrence (3.16) is*

$$\mathbb{E}(\overline{X}_n^k) = \frac{A_k\sqrt{\pi}}{\Gamma(k(\alpha+\frac{1}{2})-\frac{1}{2})}n^{k(\alpha+\frac{1}{2})} + O(n^{k(\alpha+\frac{1}{2})-\alpha}),$$

for $0 < \alpha < \frac{1}{2}$.

Chapter 4

Limiting Distributions

4.1 The Method of Moments

In this chapter, we will use our moment estimates with the method of moments to derive the limiting distributions for our distributional recurrence (3.1). Now, we introduce this method (for the proof see [3]).

Definition 4.1. *Let F, F_n be distribution functions. F_n converge weakly to F if*

$$\lim_n F_n(x) = F(x)$$

for every continuity point x of F; this will be denoted by $F_n \xrightarrow{d} F$.

Theorem 4.1. *Let μ be probability measure on the line having finite moments $\alpha_k = \int_{-\infty}^{\infty} x^k \mu(dx)$ of all orders. If the power series $\sum_k \alpha_k r^k/k!$ has a positive radius of convergence, then μ is the only probability measure with the moments α_1, α_2, $\ldots$.*

Theorem 4.2. *Suppose that the distribution of X is determined by its moments, that the X_n have moments of all orders, and that $\lim_n \mathbb{E}[X_n^r] = \mathbb{E}[X^r]$ for $r = 1,\ 2,\ \cdots$. Then $X_n \xrightarrow{d} X$.*

4.2 Proof of Theorem 1.4

So far, we have obtained the asympotics of moments. For $0 < \alpha < \frac{1}{2}$, we have

$$\mathbb{E}(\overline{X}_n^k) = \frac{A_k\sqrt{\pi}}{\Gamma(k(\alpha+\frac{1}{2})-\frac{1}{2})} n^{k(\alpha+\frac{1}{2})} + O(n^{k(\alpha+\frac{1}{2})-\alpha}).$$

Hence,

$$\mathbb{E}\left[\frac{\overline{X}_n}{n^{\alpha+\frac{1}{2}}}\right]^k = \frac{A_k\sqrt{\pi}}{\Gamma(k(\alpha+\frac{1}{2})-\frac{1}{2})} + O(n^{-\alpha}).$$

Similarly, for $\alpha > \frac{1}{2}$ we have

$$\mathbb{E}\left[\frac{X_n}{n^{\alpha+\frac{1}{2}}}\right]^k = \frac{A_k\sqrt{\pi}}{\Gamma(k(\alpha+\frac{1}{2})-\frac{1}{2})} + O(n^{-c}).$$

Where A_k is defined in recurrence (3.15) and c is defined in Theorem 3.1.

Lemma 4.1. *Let $\alpha' = \alpha + \frac{1}{2}$. There exists a constant $D < \infty$ depending only on α such that*

$$\left|\frac{A_k}{k!}\right| \leq D^k k^{\alpha' k}$$

for $k \geq 1$.

Proof.
We prove this Lemma by induction.
For each $\alpha > 0$ be given, choose k_0 such that $4^{\alpha'-1}(k^{-\alpha'} + k^{-2\alpha'+1}) < 1/2$ for all $k \geq k_0$.
Then, for $k \leq k_0$, the inequality is satisfied if we choose D large enough.
For $k \geq k_0$, setting $s_k := \frac{A_k}{k!}$ and dividing (3.15) by $k!$, we obtain

$$s_k = \frac{1}{4}\sum_{j=1}^{k-1} s_j s_{k-j} + \frac{\sqrt{2}}{2} s_{k-1} \frac{\Gamma(k\alpha'-1)}{\Gamma(k\alpha'-1-\alpha)}. \tag{4.1}$$

By Stirling's formula, we can find a constant $\gamma' < \infty$ depending only on α such that, for $k \geq 2$,

$$\left|\frac{\Gamma(k\alpha'-1)}{\Gamma(k\alpha'-1-\alpha)}\right| \leq \gamma' k^{\alpha}.$$

Moreover, define $\gamma = \frac{\sqrt{2}}{2}\gamma'$. Then, the recurrence (4.1) becomes

$$|s_k| \leq \frac{1}{4}\sum_{j=1}^{k-1}|s_j||s_{k-j}| + \gamma k^{\alpha}|s_{k-1}|.$$

Thus, by the induction hypothesis, we have

$$|s_k| \leq \frac{D^k}{4}\sum_{j=1}^{k-1}\left(j^j(k-j)^{k-j}\right)^{\alpha'} + \gamma k^{\alpha} D^{k-1}(k-1)^{(k-1)\alpha'}.$$

Since $j^j(k-j)^{k-j}$ decreases as j increases for $0 < j < \frac{k}{2}$, we can bound the sum by the $j = 1$ term, the $j = k-1$ term and $k-3$ times $j = 2$ term. Then, for $k \geq 2$

$$\begin{aligned}
|s_k| &\leq \frac{D^k}{4}\left(2((k-1)^{k-1})^{\alpha'} + (k-3)(4(k-2)^{k-2})^{\alpha'}\right) + \gamma k^{\alpha} D^{k-1} k^{(k-1)\alpha'} \\
&\leq \frac{D^k}{4}\left(4^{\alpha'}(k-1)^{(k-1)\alpha'} + 4^{\alpha'}(k-2)^{(k-2)\alpha'+1}\right) + \gamma D^{k-1} k^{\alpha' k - \frac{1}{2}} \\
&\leq D^k 4^{\alpha'-1}\left(k^{(k-1)\alpha'} + k^{(k-2)\alpha'+1}\right) + \gamma D^{k-1} k^{\alpha' k - \frac{1}{2}} \\
&= \left[4^{\alpha'-1}(k^{-\alpha'} + k^{-2\alpha'+1}) + \frac{\gamma}{D}k^{-\frac{1}{2}}\right] D^k k^{k\alpha'}
\end{aligned}$$

Now, choosing D even larger such that $\frac{\gamma}{D}k^{-\frac{1}{2}} < \frac{1}{2}$, we can obtain that

$$\left[4^{\alpha'-1}(k^{-\alpha'} + k^{-2\alpha'+1}) + \frac{\gamma}{D}k^{-\frac{1}{2}}\right] < 1,$$

for $k \geq k_0$, and this prove the lemma. ▮

Following Lemma (4.1), we conclude that

$$\frac{A_k\sqrt{\pi}}{\Gamma(k(\alpha+\frac{1}{2})-\frac{1}{2})} \leq R^k$$

for large enough D depending on α. Now, following Theorem 4.1, the lemma implies that X_n and $\overline{X}_n$ suitably normalized have limiting distributions that is characterized by their moments and this completes Theorem 1.4.

Chapter 5

Conclusion

We conclude this thesis with some remarks. The recurrence that we studied in this thesis is a "divide-and-conquer" recurrence. More precisely, it is a stochastic divide-and-conquer recurrence, that is the splitting size S_n is a random variable (depending on n) with support spread over a whole subinterval in $(0, n)$. Moreover, our recurrence is one example of a so-called "tree recurrences". In [4], Fill, Flajolet and Kapur introduced three kinds of tree recurrences that are of special interest in combinatorial mathematics and analysis of algorithms: the binary search tree recurrence, the union-find tree recurrence (the recurrence studied in this thesis) and the uniform binary tree recurrence. They gave the expected value of the cost of these recurrences and discussed how to find the idea of the derivation of the higher moments. In [5], Fill and Kapur gave a full analysis of the uniform binary tree recurrence, including the expected value, higher moments and the limiting distribution. In this thesis, we proved similar results for the union-find tree recurrence, except for the limiting distribution of the case $\alpha = \frac{1}{2}$ and the case of the toll function $c_n = \log n$. Indeed, these cases and, more generally, toll functions of the form $n^\alpha(\log n)^\beta$ are feasible as well. However, one needs to extend the result for the Hadamard products slightly (to include $L(z)$ terms).

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