Available online at www.sciencedirect.com
ScienceDirect
Applied Numerical Mathematics 57 (2007) 801–810
APPLIED NUMERICAL MATHEMATICS
www.elsevier.com/locate/apnum

An efficient semi-coarsening multigrid method for variable diffusion problems in cylindrical coordinates

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Available online 2 August 2006

Abstract

In this paper, we present an efficient multigrid (MG) algorithm for solving the three-dimensional variable coefficient diffusion equation in cylindrical coordinates. The multigrid V-cycle combines a semi-coarsening in azimuthal direction with the red-black Gauss–Seidel plane (radial-axial plane) relaxation. On each plane relaxation, we further semi-coarsen the axial direction with red-black line relaxation in the radial direction. We also prove the convergence of two-level MG with plane Jacobi relaxation. Numerical results show that the present multigrid method indeed is scalable with the mesh size.

Keywords: Multigrid method; V-cycle; Variable diffusion equation; Cylindrical coordinates

1. Introduction

The variable coefficient elliptic equation in three-dimensional domain arises in many physical applications. The heat transfer in heterogeneous material where the thermal conductivity depends on the position is one of the classical examples. The fluid flows with non-constant viscosity is another application. Very often, the geometry we consider is no longer Cartesian, say a circular cylinder instead. Thus, it is more convenient to write the equation in cylindrical coordinates.

In this paper, we consider the following variable coefficient diffusion problem written in cylindrical coordinates on a 3D domain $\Omega = \{(r,\theta,z) \mid 0 < r < 1,\ 0 \leqslant \theta < 2\pi,\ 0 < z < 1\}$ as

$$-\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\beta\frac{\partial u}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\frac{\beta}{r}\frac{\partial u}{\partial\theta}\right)+r\frac{\partial}{\partial z}\left(\beta\frac{\partial u}{\partial z}\right)\right]=f(r,\theta,z) \quad \text{in } \Omega, \tag{1}$$

$$u(1,\theta,z)=g(\theta,z), \qquad u(r,\theta,0)=p(r,\theta), \qquad u(r,\theta,1)=q(r,\theta). \tag{2}$$

Here the diffusion coefficient $\beta(r,\theta,z)$ is inhomogeneous and satisfies $0 < \mu \leqslant \beta(r,\theta,z) \leqslant 1$. The upper bound is not a restriction of the method discussed in this paper since we can re-scale both sides of Eq. (1) by the maximum of β.

When we solve Eq. (1) numerically, the first issue called the coordinate singularity arises. This is because the equation is not valid at $r = 0$ when it is written in cylindrical coordinates. In [8], Lai et. al. have developed a FFT-based fast direct solver for the Poisson equation (a special case of $\beta(r,\theta,z) = 1$ in Eq. (1)). The solver relies on the truncated

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0168-9274/$30.00
doi:10.1016/j.apnum.2006.07.019

Fourier series expansion, where the partial differential equations of Fourier coefficients are solved by the standard centered difference scheme under a shifted grid in the radial direction. The method handles the coordinate singularity without special treatment and the resultant matrix equations can be solved by the fast direct solver such as those provided in public software package—FISHPACK [1]. However, when the diffusion coefficient is inhomogeneous, solving the numerical solution for Eq. (1) is a different story. Since now the elliptic equation has a variable diffusion coefficient, we are unable to write the solution as Fourier series expansion, which means the fast Fourier transform (FFT) cannot be called directly. Furthermore, the equation is not a separable type, the resultant linear system after the discretization cannot be solved by the fast direct solver as mentioned before.

Multigrid (MG) methods are known to be very efficient for solving the elliptic problems. Its main idea consists of applying simple relaxations on the fine grid to smooth the error and correcting the smooth error on the coarse grid, on which the smooth error is relatively oscillatory and can be eliminated more effectively by relaxations. It has been shown that the convergence rates of traditional MG methods are independent of the mesh size for many elliptic problems [2,3]. However, for solving problems with strong anisotropic diffusion or solving problems on anisotropic meshes, the convergence of the MG methods is seriously deteriorated [4,7,13]. The reasons contributing to the slow multigrid convergence in such problems are, first, the traditional Jacobi and Gauss–Seidel relaxations fail to smooth the errors in some directions of the domain and, second, the standard coarse grids cannot represent the errors that are highly frequent in these directions. Many cures have been proposed to improve the convergence rate. For some two-dimensional anisotropic problems on the Cartesian coordinates, Brandt, Hackbush and Mulder [4,7,11] have obtained robust convergence of multigrid in which coarse grids from semi-coarsening are employed. For three-dimensional anisotropic problems, Dendy [6], Llorente and Melson [10], and Washio and Oosterlee [12] obtained robust convergence by using semi-coarsened grids and the alternating plane relaxations in their MG methods. Schaffer [14] also developed an efficient semi-coarsening multigrid method for symmetric and nonsymmetric elliptic PDEs with highly discontinuous and anisotropic coefficients in two- and three-dimensional Cartesian domains. In this paper, we extend the discretization used in [9] to the 3D variable diffusion equation in cylindrical coordinates (1)–(2), and present an efficient semi-coarsening MG algorithm to solve the resultant linear system.

The rest of this paper is as follows. In Section 2, we introduce the finite difference discretization to Eq. (1). In Section 3, we first review some elements of the multigrid method and then provide a two-level convergence analysis for the semi-coarsening MG with Jacobi plane relaxation. The numerical results including the accuracy check and the detailed performance comparison for the present multigrid algorithm are shown in Section 4.

2. Finite difference discretization

We define the same grid points in the radial, azimuthal and axial directions as in [8] by

$$r_i = (i-1/2)\Delta r, \qquad r_{i+1/2} = r_i + \Delta r/2, \qquad r_{i-1/2} = r_i - \Delta r/2, \tag{3}$$

$$\theta_j = j\Delta\theta, \qquad \theta_{j+1/2} = \theta_j + \Delta\theta/2, \qquad \theta_{j-1/2} = \theta_j - \Delta\theta/2, \tag{4}$$

$$z_k = k\Delta z, \qquad z_{k+1/2} = z_k + \Delta z/2, \qquad z_{k-1/2} = z_k - \Delta z/2, \tag{5}$$

where $\Delta r = 2/(2n_r+1)$, $\Delta\theta = 2\pi/n_\theta$ and $\Delta z = 1/(n_z+1)$. By the choice of the radial mesh width, the boundary values are defined on the grid points. Let the discrete values be denoted by $u_{ijk} \approx u(r_i,\theta_j,z_k)$, $f_{ijk} \approx f(r_i,\theta_j,z_k)$, $g_{jk} \approx g(\theta_j,z_k)$, $p_{ij} \approx p(r_i,\theta_j)$ and $q_{ij} \approx q(r_i,\theta_j)$. The values at the half-integered points are defined in a similar fashion, such as $\beta_{i+1/2,j,k} \approx \beta(r_{i+1/2},\theta_j,z_k)$. Using the centered difference method to discretize Eq. (1), we have

$$\begin{aligned}
-\Bigg[&\left(r_{i+1/2}\beta_{i+1/2,j,k}\frac{u_{i+1,j,k}-u_{i,j,k}}{\Delta r} - r_{i-1/2}\beta_{i-1/2,j,k}\frac{u_{i,j,k}-u_{i-1,j,k}}{\Delta r}\right)\Big/\Delta r \\
&+\left(\frac{\beta_{i,j+1/2,k}}{r_i}\frac{u_{i,j+1,k}-u_{i,j,k}}{\Delta\theta} - \frac{\beta_{i,j-1/2,k}}{r_i}\frac{u_{i,j,k}-u_{i,j-1,k}}{\Delta\theta}\right)\Big/\Delta\theta \\
&+r_i\left(\beta_{i,j,k+1/2}\frac{u_{i,j,k+1}-u_{i,j,k}}{\Delta z} - \beta_{i,j,k-1/2}\frac{u_{i,j,k}-u_{i,j,k-1}}{\Delta z}\right)\Big/\Delta z\Bigg] = r_i f_{i,j,k}.
\end{aligned} \tag{6}$$

Among the above representation, the numerical boundary values are given by $u_{n_r+1,j,k} = g_{j,k}$, $u_{i,j,0} = p_{i,j}$, $u_{i,j,n_z+1} = q_{i,j}$, and $u_{i,0,k} = u_{i,n_\theta,k}$, $u_{i,n_\theta+1,k} = u_{i,1,k}$ since u is 2π periodic in the azimuthal direction. At $i = 1$,

we have immediately observed from Eq. (3) that $r_{1/2} = 0$, so the coefficient of $u_{0,j,k}$ is zero. This implies that the scheme does not need any extrapolation for the inner numerical boundary value $u_{0,j,k}$ so that there is no pole condition needed.

Let us order the unknowns to a solution vector u, the resultant linear system of (6) can be written as $Au = f$. Here the matrix A is given by

$$A = I_\theta \otimes B_{rz} + C_\theta \otimes I_{rz}, \tag{7}$$

where $\otimes$ is the regular Kronecker product, B_{rz} is the matrix with a five-point stencil obtained from discretization of the operator $\frac{\partial}{\partial r}(r\beta\frac{\partial u}{\partial r}) + \frac{\partial}{\partial z}(r\beta\frac{\partial u}{\partial z})$, C_θ is the circulant matrix with a three-point stencil obtained from discretization of the operator $\frac{\partial}{\partial\theta}(\frac{\beta}{r}\frac{\partial u}{\partial\theta})$, and I_θ and I_{rz} are the identity matrix of size $n_\theta \times n_\theta$ and $n_r \cdot n_z \times n_r \cdot n_z$, respectively. Clearly, in the region close to the origin, the problem (1) is strongly anisotropic, namely, the diffusion in the azimuthal direction is much greater than that in other directions. As a result, the matrix A becomes indefinite as the grid space decreases. Therefore, a scalable and efficient solver for the linear system $Au = f$ is desired.

3. Semi-coarsening multigrid method

In this paper, we would like to solve the aforementioned linear system by the semi-coarsening MG with plane relaxations. In the following, firstly, the general multigrid V-cycle algorithm is introduced, and then the convergence of a two-level V-cycle of the MG method is proved under the assumption that the solution of Eq. (1) is smooth enough so that the discretization error of the scheme (6) is second-order accurate.

Given a sequence of meshes $\{G^\ell\}$, let V^ℓ be the vector space of the nodal variables on G^ℓ, A^ℓ the matrix obtained from the difference equation (6) on G^ℓ, and w^ℓ the initial guess. Let MG_1 represents the direct solver on the coarsest grid G^1. A typical multigrid V-cycle on ℓ-level is shown recursively in Algorithm 1. Here, f^ℓ is the right-hand side obtained from (6) on G^ℓ, $(M^\ell)^{-1}$ represents the smoothing operator and the operators $I_\ell^{\ell-1}$ (restriction) and $I_{\ell-1}^\ell$ (prolongation) represent the grid transfers between $G^{\ell-1}$ and G^ℓ. The elements of the multigrid V-cycle are described as follows.

Algorithm 1. Multigrid V-cycle $MG_\ell(w^\ell, f^\ell)$

(1) Set $u^\ell = w^\ell$;
(2) (Pre-smoothing) $u^\ell = u^\ell + (M^\ell)^{-1}(f^\ell - A^\ell u^\ell)$, ν_1 times;
(3) (Restriction) $\bar{f}^\ell = I_\ell^{\ell-1}(f^\ell - A^\ell u^\ell)$;
(4) (Coarser grid correction) $q_1 = MG_{\ell-1}(0, \bar{f}^\ell)$;
(5) (Prolongation) $\bar{q}_1 = I_{\ell-1}^\ell q_1$, and set $u^\ell = u^\ell + \bar{q}_1$;
(6) (Post-smoothing) $u^\ell = u^\ell + (M^\ell)^{-1}(f^\ell - A^\ell u^\ell)$ ν_2, times;
(7) Set $w^\ell = MG_\ell(w^\ell, f^\ell) = u^\ell$.

For the MG considered here, (i) the coarse grids $G^{\ell-1}$ are obtained from semi-coarsening the fine grid G^ℓ along the azimuthal direction, (ii) the prolongation operator $I_{\ell-1}^\ell$ is the linear interpolation along the azimuthal direction and the restriction operator $I_\ell^{\ell-1}$ is the transpose of the interpolation, (iii) the matrix $A^{\ell-1}$ is obtained from discretization scheme (6) on $(\ell-1)$-level grid $G^{\ell-1}$ and (iv) the smoothing operator M^ℓ consists of the Jacobi, or Gauss–Seidel plane relaxations. From now on, we use the notations $MG_J(\nu_1, \nu_2)$ and $MG_{\mathrm{GS}}(\nu_1, \nu_2)$ to represent the different versions of the above multigrid method in which ν_1 times of pre-smoothing and ν_2 times of post-smoothing by plane Jacobi and plane Gauss–Seidel smoothers, respectively, are applied.

3.1. The convergence of two-level MG

In the following, we present the convergence analysis for the two-level V-cycle MG method with Jacobi plane relaxations. To simplify our convergence proof, we assume the number of grid points used in r, θ and z directions are $n_r = n_\theta/2 = n_z = n$. Let $\|\cdot\|_2$ denote the standard vector 2-norm or matrix 2-norm depending on which quantities are considered. Let $\|w\|_Q = \sqrt{(Qw, w)}$ denote the Q-weighted Euclidean norm for $w \in V^\ell$, where Q is a symmetric

positive definite matrix and $(\cdot,\cdot)$ is the inner product on V^ℓ. For any linear transformation T from V^ℓ to V^ℓ, the matrix Q-norm, denoted by $\|T\|_Q$, is the matrix 2-norm associated with the Q-weighted Euclidean norm.

First, let us show that the Jacobi plane relaxation is convergent. Recall that the Jacobi plane relaxation is obtained from the matrix splitting

$$A^\ell = M^\ell - N^\ell, \tag{8}$$

where

$$M^\ell = I_\theta \otimes B_{rz} + D_\theta \otimes I_{rz} \quad \text{and} \quad N^\ell = (D_\theta - C_\theta) \otimes I_{rz}, \tag{9}$$

and the matrix D_θ is the diagonal part of C_θ. The matrices M^ℓ and N^ℓ can also be written in the following block form

$$M^\ell = \begin{bmatrix} M_1 & 0 & & & \\ 0 & M_2 & 0 & & \\ & \ddots & \ddots & \ddots & \\ & & 0 & M_{2n-1} & 0 \\ & & & 0 & M_{2n} \end{bmatrix}, \qquad N^\ell = \begin{bmatrix} 0 & N_1 & 0 & 0 & N_{2n} \\ N_1 & 0 & N_2 & 0 & 0 \\ 0 & N_2 & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & 0 & N_{2n-1} \\ N_{2n} & 0 & 0 & N_{2n-1} & 0 \end{bmatrix}, \tag{10}$$

where M_j are the block tridiagonal matrices with the stencils

$$\begin{cases} \Big[-\frac{r_{i-1/2}\beta_{i-1/2,j,k}}{\Delta r^2}, \frac{r_{i-1/2}\beta_{i-1/2,j,k}+r_{i+1/2}\beta_{i+1/2,j,k}}{\Delta r^2} + \frac{r_i\beta_{i,j,k-1/2}+r_i\beta_{i,j,k+1/2}}{\Delta z^2} + \frac{\beta_{i,j-1/2,k}+\beta_{i,j+1/2,k}}{r_i\Delta\theta^2}, \\ -\frac{r_{i+1/2}\beta_{i+1/2,j,k}}{\Delta r^2}\Big]_{i,k} & \text{in the main diagonal blocks,} \\ \Big[-\frac{r_i\beta_{i,j,k-1/2}}{\Delta z^2}\Big]_{i,k} & \text{in the upper off-diagonal blocks,} \\ \Big[-\frac{r_i\beta_{i,j,k+1/2}}{\Delta z^2}\Big]_{i,k} & \text{in the lower off-diagonal blocks,} \end{cases} \tag{11}$$

and the matrices N_j, $j = 1 \ldots 2n$, are the block diagonal matrices with stencils $[\frac{\beta_{i,j+1/2,k}}{r_i\Delta\theta^2}]_{i,k}$, respectively. Let $S_J^\ell = (M^\ell)^{-1}N^\ell$ be the error reduction operator of the Jacobi plane relaxation. It is clear that, due to the given Dirichlet boundary conditions, M^ℓ and A^ℓ are irreducible M-matrices [15]. Therefore, $(M^\ell)^{-1}$ and $(A^\ell)^{-1}$ are positive. Since N^ℓ is non-negative, the splitting (8) is a regular splitting. As a result, the Jacobi plane relaxation S_J is convergent since $\|S_J^\ell\|_2 < 1$ ([15] Theorem 3.29).

Throughout this section, we define the matrix R^ℓ as

$$R^\ell = I_\theta \otimes R_{rz}^\ell = I_\theta \otimes \big(I_z \otimes R_r^\ell\big), \tag{12}$$

where $R_r^\ell = \operatorname{diag}(r_i)$. Obviously, one also has $\|S_J^\ell\|_{R^\ell} < 1$.

Lemma 3.1. *Suppose M_j, $j = 1, \ldots, 2n$ and $N_l, l = 1, \ldots, 2n$ are the block matrices defined in* (10)*. The inequality*

$$\big\|M_j^{-1}N_l\big\|_{R_{rz}^\ell} < \frac{\kappa}{2\mu} \tag{13}$$

holds, where $\kappa < 1$ is a constant depending on $\beta(r,\theta,z)$ and μ is the lower bound of $\beta(r,\theta,z)$.

Proof. The matrix M_j can be rewritten as

$$M_j = \frac{1}{\Delta\theta^2}\big(R_{rz}^\ell\big)^{-1}\overline{B}_j(I_{rz} + C),$$

where $\overline{B}_j$ is a diagonal matrix whose diagonal entries $[\bar{\beta}_j]_{i,k}$ are defined by $\bar{\beta}_j = \beta_{i,j-1/2,k} + \beta_{i,j+1/2,k}$, and the matrix $C = \Delta\theta^2\overline{B}_j^{-1}R_{rz}^\ell M_j - I_{rz}$. Clearly, one has

$$\big\|M_j^{-1}N_l\big\|_{R_{rz}^\ell} \leqslant \big\|\big[I_{rz} + \big(R_{rz}^\ell\big)^{1/2}C\big(R_{rz}^\ell\big)^{-1/2}\big]^{-1}\Delta\theta^2\overline{B}_j^{-1}N_l\big\|_2 \tag{14}$$

$$= \left\|\big[I_{rz} + \big(R_{rz}^\ell\big)^{1/2}C\big(R_{rz}^\ell\big)^{-1/2}\big]^{-1}\left[\frac{\beta_{i,l+1/2,k}}{\bar{\beta}_j}\right]_{i,k}\right\|_2. \tag{15}$$

From stencils (11) and the given Dirichlet boundary conditions in (2), one can conclude that C is an irreducibly diagonal dominant matrix. Therefore, the matrix C is positive definitive. Moreover, $(R_{rz}^{\ell})^{1/2}C(R_{rz}^{\ell})^{-1/2}$ is also positive definite. Let $\Lambda_{\min}(\overline{C})$ denote the smallest eigenvalue of the matrix $\overline{C} = (R_{rz}^{\ell})^{1/2}C(R_{rz}^{\ell})^{-1/2}$. Obviously, the assumption that $\mu < \beta \leqslant 1$ and (14) imply

$$\left\| M_j^{-1} N_l \right\|_{R_{rz}^{\ell}} \leqslant \frac{1}{1+\Lambda_{\min}(\overline{C})} \left\| \left[\frac{\beta_{i,l+1/2,k}}{\bar{\beta}_j} \right]_{i,k} \right\|_2 \tag{16}$$

$$< \frac{\kappa}{2\mu}. \tag{17}$$

Hence, the inequality (13) holds. □

Corollary 3.2. *For the constant diffusion case, suppose the mesh width satisfies the relation as* $\Delta r = \eta_1 \Delta z = \eta_2 \Delta\theta \ll 1$, *where* $\eta_1 = \frac{2n_z+2}{2n_r+1}$ *and* $\eta_2 = \frac{n_\theta}{(2n_r+1)\pi}$ *are the grid aspect ratios of* Δz *and* $\Delta\theta$ *with respect to* Δr, *and* $\eta_1 \approx \eta_2 \approx 1$. *Then, we have*

$$\left\| M_j^{-1} N_l \right\|_{R_{rz}^{\ell}} \leqslant \frac{\kappa}{2}, \tag{18}$$

and $\kappa \approx (1 - c\Delta r^2)$ *for some constant* $c > 0$ *independent of mesh size. As a result of* (18)*, the plane Jacobi relaxation converges with*

$$\left\| S_J^{\ell} \right\|_{R^{\ell}} \leqslant 1 - c\Delta r^2. \tag{19}$$

Proof. First, the stencil of the matrix $\overline{C}$ in the proof of Lemma 3.1 can be rewritten as

$$\overline{C} = \left(\frac{\Delta\theta}{\Delta r}\right)^2 \widetilde{C}, \tag{20}$$

where $\widetilde{C}$ is a block tridiagonal matrix with $[-r_i\sqrt{r_i r_{i-1/2}}, (2+2\eta_1^2)r_i^2, -\eta_1^2 r_i\sqrt{r_i r_{i+1/2}}]$ in the diagonal blocks, and $[-r_i^2]$ and $[-\eta_1^2 r_i^2]$ in the lower and upper off-diagonal blocks, respectively. By the Gersgorin theorem, we obtain the following estimates as

$$\begin{aligned} \left|\Lambda_{\min}(\widetilde{C}) - \left(2+2\eta_1^2\right)r_i^2\right| &\leqslant r_i\left(\left(1+\eta_1^2\right)r_i + \sqrt{r_i r_{i-1/2}} + \eta_1^2\sqrt{r_i r_{i+1/2}}\right) \\ &\leqslant \left(2+2\eta_1^2\right)r_i^2 - c_0 r_i \Delta r, \quad \text{for some constant } c_0 > 0 \\ &\leqslant \left(2+2\eta_1^2\right)r_i^2 - 0.5c_0\Delta r^2. \end{aligned} \tag{21}$$

Clearly, from (20) and (21), there exists a constant $c_1 \approx 0.5c_0(\frac{\Delta\theta}{\Delta r})^2 = 0.5c_0(\frac{1}{\eta_2})^2 > 0$ such that $\Lambda_{\min}(\overline{C}) \geqslant c_1\Delta r^2$. Next, let $e_1 = [1, 0, \ldots, 0]$. Since

$$\begin{aligned} \|\overline{C}e_1\|_2 &= \left(\frac{\Delta\theta}{\Delta r}\right)^2 \left(\left((2+2\eta_1)r_1^2\right)^2 + \eta_1^2 r_2^2 r_2 r_{3/2} + r_1^4\right)^{1/2} \\ &= \frac{\sqrt{16\eta_1^2 + 60\eta_1 + 5}}{4\eta_2^2}\Delta r^2 = c_2\Delta r^2 \end{aligned}$$

from the positive definiteness of $\overline{C}$, one can conclude that $\Lambda_{\min}(\overline{C}) \leqslant c_2\Delta r^2$, for some constant $c_2 > 0$. By using the upper and lower bounds of $\Lambda_{\min}(\overline{C})$, the inequality (16) can now be shown easily as

$$\begin{aligned} \left\| M_j^{-1} N_l \right\|_{R_{rz}^{\ell}} &\leqslant \frac{1}{2(1+\Lambda_{\min}(\overline{C}))} \leqslant \frac{1}{2}\left(1 - \Lambda_{\min}(\overline{C}) + \Lambda_{\min}(\overline{C})^2\right) \\ &\leqslant \frac{1}{2}\left(1 - c_1\Delta r^2 + c_2^2\Delta r^4\right) \leqslant \frac{1}{2}\left(1 - c\Delta r^2\right), \end{aligned}$$

for some constant $c > 0$. The last inequality arises easily from the assumption that the radial mesh Δr is much smaller than one. The inequality (19) can now be derived directly from the Gersgorin theorem. □

Now, we are ready to prove the convergence of the 2-level MG. First, the error reduction operator of the 2-level MG for solving the linear system obtained from (6) can be written as

$$E_{\mathrm{MG}} = \big(\big(A^{\ell}\big)^{-1} - I_{\ell-1}^{\ell}\big(A^{\ell-1}\big)^{-1} I_{\ell}^{\ell-1}\big) R^{\ell} A^{\ell} \big(S_J^{\ell}\big)^{\nu},$$

where ν is the number of relaxations. To prove the convergence of MG is independent of mesh size, we need to show that there is some number $\nu > 0$ such that

$$\|E_{\mathrm{MG}}\|_{R^{\ell}} < 1. \tag{22}$$

In the following, we let e^{ℓ} be an arbitrary error vector in V^{ℓ} satisfying $\|e^{\ell}\|_{R^{\ell}} = 1$, and $(f^{\ell})^{s} = A^{\ell}(S_J^{\ell})^{\nu} e^{\ell}$ be the corresponding residual of the relaxed error. Clearly, one has

$$\begin{aligned}
&\big(R^{\ell}\big[\big(\big(A^{\ell}\big)^{-1} - I_{\ell-1}^{\ell}(A^{\ell-1})^{-1} I_{\ell}^{\ell-1}\big) R^{\ell} A^{\ell} (S_J^{\ell})^{\nu}\big] e^{\ell}, e^{\ell}\big) \\
&\quad = \big(R^{\ell}\big[\big(\big(R^{\ell}\big)^{-1} A^{\ell}\big)^{-1} - I_{\ell-1}^{\ell}\big(\big(R^{\ell-1}\big)^{-1} A^{\ell-1}\big)^{-1} I_{\ell}^{\ell-1}\big]\big(f^{\ell}\big)^{s}, e^{\ell}\big) \\
&\quad \leqslant \big\|\big[\big(\big(R^{\ell}\big)^{-1} A^{\ell}\big)^{-1} - I_{\ell-1}^{\ell}\big(\big(R^{\ell-1}\big)^{-1} A^{\ell-1}\big)^{-1} I_{\ell}^{\ell-1}\big]\big(f^{\ell}\big)^{s}\big\|_{R^{\ell}}.
\end{aligned}$$

Since $(R^{\ell})^{-1}A^{\ell}$ and $(R^{\ell-1})^{-1}A^{\ell-1}$ represent the discrete difference operators in (6) on grids G^{ℓ} and $G^{\ell-1}$, respectively, the truncation errors are $O(\Delta r^2)$ (the meshes $\Delta\theta$ and Δz are the same order as Δr) if the solution is smooth. Thus, there is a constant $c_3 > 0$ such that

$$\begin{aligned}
\|E_{\mathrm{MG}}\|_{R^{\ell}} &\leqslant c_3 \Delta r^2 \sup_{e^{\ell}} \big(R^{\ell} A^{\ell} \big(S_J^{\ell}\big)^{\nu} e^{\ell}, A^{\ell} \big(S_J^{\ell}\big)^{\nu} e^{\ell}\big)^{1/2} \\
&\leqslant c_3 \big\|\Delta r^2 A^{\ell} \big(S_J^{\ell}\big)^{\nu}\big\|_{R^{\ell}} \leqslant c_3 \big\|\Delta r^2 A^{\ell} S_J^{\ell}\big\|_{R^{\ell}} \big\|\big(S_J^{\ell}\big)^{\nu-1}\big\|_{R^{\ell}}.
\end{aligned} \tag{23}$$

Next, since $A^{\ell} S_J^{\ell} = N^{\ell}(I - S_J^{\ell})$ and

$$\big\|\Delta r^2 N^{\ell}\big\|_{R^{\ell}} \leqslant 2 \left\| \left[\frac{\beta_{i,l+1/2,k}}{\pi^2} \right]_{i,k} \right\|_2 \leqslant \frac{2}{\pi^2},$$

clearly, the inequality (23) implies

$$\|E_{\mathrm{MG}}\|_{R^{\ell}} \leqslant \frac{2c_3}{\pi^2} \big\|I - S_J^{\ell}\big\|_{R^{\ell}} \big\|\big(S_J^{\ell}\big)^{\nu-1}\big\|_{R^{\ell}} \leqslant \frac{4c_3}{\pi^2} \big\|S_J^{\ell}\big\|_{R^{\ell}}^{\nu-1}. \tag{24}$$

Therefore, from the fact $\|S_J^{\ell}\|_{R^{\ell}} < 1$, the inequality (22) holds when the number of Jacobi plane relaxations is large enough. We conclude our result of the multigrid convergence in the following theorem.

Theorem 3.3. *For the variable diffusion problem* (1) *discretized by the finite difference scheme* (6) *on the cylindrical uniform grids G^{ℓ}, let S_J^{ℓ} and E_{MG} be the error reduction operator of plane Jacobi relaxation and the present* 2*-level MG, respectively. There exists a positive constant c_4 independent of the grid size such that*

$$\|E_{\mathrm{MG}}\|_{R^{\ell}} < c_4 \big\|S_J^{\ell}\big\|_{R^{\ell}}^{\nu-1}, \tag{25}$$

where ν is the number of smoothing steps in the MG. If ν is large enough, the 2*-level MG is convergent independent of the mesh size.*

Remark 3.4. From the convergence rate of the plane Jacobi relaxation estimated by the inequality (19) of Corollary 3.2, Theorem 3.3 suggests that for the constant diffusion case, the number of Jacobi plane relaxations should increase four times when the mesh is refined to achieve grid-independent convergence of MG. On the other hand, since the plane Jacobi matrix $S_J^{\ell} > 0$ with zero diagonal blocks and $(R^{\ell})^{1/2} S_J^{\ell} (R^{\ell})^{-1/2}$ is block-symmetric, irreducible and convergent, by the Perron–Frobenius theorem, and Theorem 4.15 of the Varga's book [15], the relation of the asymptotic convergence rates between the plane Jacobi S_J^{ℓ} and plane Gauss–Seidel S_{GS}^{ℓ} can be derived as follows.

$$\big\|S_{\mathrm{GS}}^{\ell}\big\|_{R^{\ell}} \leqslant \frac{\|S_J^{\ell}\|_{R^{\ell}}}{2 - \|S_J^{\ell}\|_{R^{\ell}}} \leqslant \frac{1 - c\Delta r^2}{1 + c\Delta r^2} \approx 1 - 2c\Delta r^2.$$

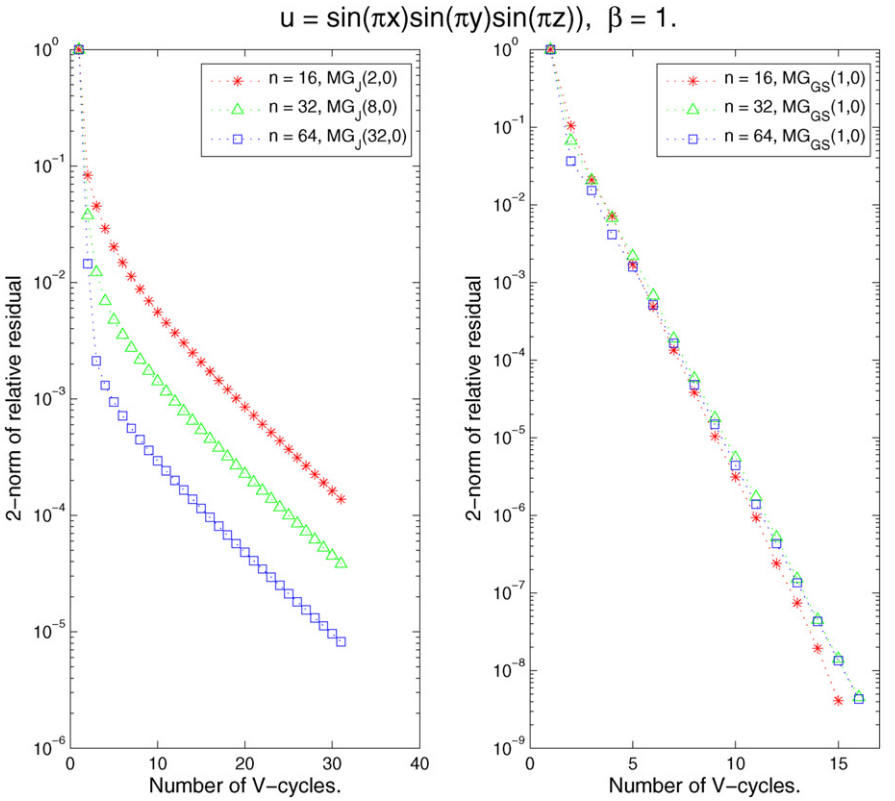

Fig. 1. The 2-norm of relative residuals with grid sizes $n = 16(*), 32(\triangle), 64(\square)$ for MG_{J} (left) and MG_{GS} (right).

Moreover, the estimate of the error reduction rate of MG with S^{ℓ}_{GS} smoother in (24) becomes $\|E_{\mathrm{MG}}\|_{R^{\ell}} \leqslant \frac{\sqrt{10}c}{\pi^2}\|S^{\ell}_{\mathrm{GS}}\|^{\nu-1}_{R^{\ell}}$. Hence, the grid-independent convergence of MG should be easier to achieve when plane Gauss–Seidel relaxations are employed in the pre-smoothing and post-smoothing steps of the MG.

Fig. 1 shows the convergence behaviors of MG for the constant diffusion case $\beta = 1$. One can see that when the plane Jacobi smoother is employed in MG, in order to have grid-independent convergence rate, the number of smoothing steps must increase four times as the grid number n doubles. This is exactly what we expect from the previous remark. In practice, only one step of the plane Gauss–Seidel relaxation is required for convergence and the convergence rate is independent of the problem size as shown in Fig. 1.

4. Numerical results

In this section, we report several numerical tests for the multigrid method. On a given $n_r \times n_z \times n_\theta$ mesh size with $n_r = n_z = n_\theta/2$, the present multigrid algorithm is a complete $\log_2 n_\theta$-level V-cycle which employs semi-coarsening in the azimuthal (θ) direction, and one step of red-black plane (r–z plane) Gauss–Seidel relaxation for both the pre-smoothing and the post-smoothing. It is important to note that the present MG method semi-coarsens the grid in the θ direction instead of the r direction. This is because if the radial grid is coarsened, then the coarser grid would not coincide with the fine grid based on our radial grid arrangement (3). The other reason is that Eq. (1) is on r, θ, z rectangular domain, where the strongly anisotropic diffusion occurs along the θ direction near the z-axis. It is well known that the semi-coarsening direction is determined by which direction is strongly anisotropic [5].

In order to save the computation time, each plane relaxation step is further approximated by one step of $\log_2 n_z$-level multigrid V-cycle iteration where coarse grids are obtained by semi-coarsening in the axial (z) direction on each plane and employing a red-black line Gauss–Seidel relaxation in the radial (r) direction. The restriction (fine to coarse)

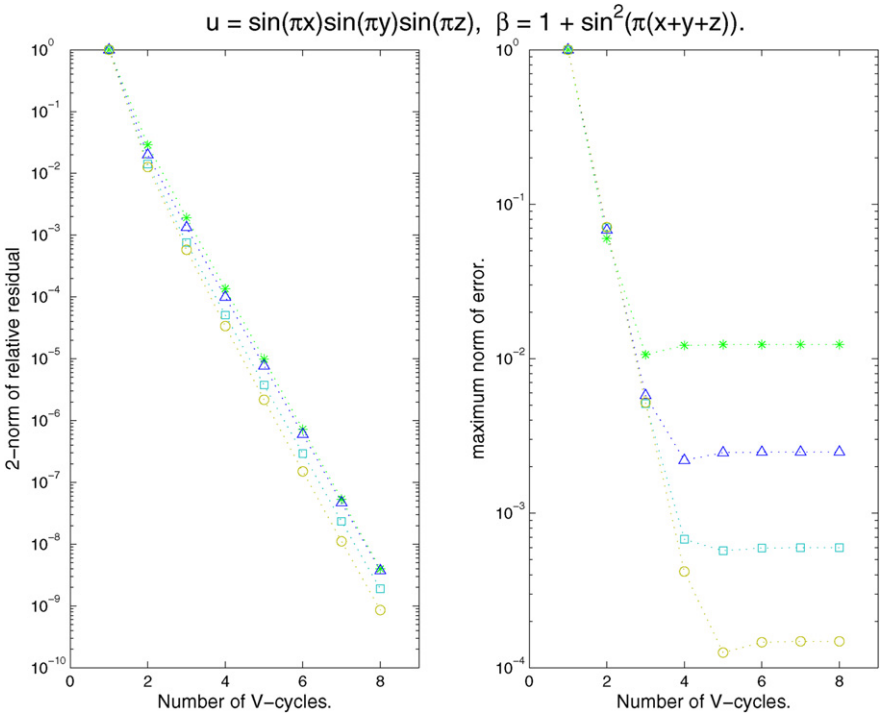

Fig. 2. The 2-norm of relative residuals of $MG_{\mathrm{RB}}(1,1)$ and the maximum norm of errors for Example 1 (smooth diffusion coefficient) with grid sizes $n = 16(*), 32(\triangle), 64(\square), 128(\circ)$.

and prolongation (coarse to fine) operators are the conventional full weighting and linear interpolation, respectively. Hereafter, we denote this version of the multigrid method as $MG_{\mathrm{RB}}(1,1)$.

In this paper, we have tested the convergence behavior and the numerical accuracy of the present MG algorithm by the following examples. In those examples, we choose the analytical solution $u(x, y, z) = \sin(\pi x)\sin(\pi y)\sin(\pi z)$ and vary the diffusion coefficients based on different smoothing characteristics as

(1) $\beta = 1 + \sin^2(\pi(x + y + z))$ (smooth case);
(2) $\beta = 1 + \sin^2(10\theta)$ (moderately oscillatory in θ direction);
(3) $\beta = 1 + \sin^2(10\theta)\sin^2(20\pi z)$ (moderately oscillatory in θ and z directions).

The right-hand side functions are obtained by substituting the solutions into Eq. (1).

The stopping tolerance of the iterations is set to be 10^{-8} for all of our test cases. All numerical runs are carried out on a PC with 1 GB RAM and 1.3 GHz CPU, and the program is written in MATLAB. All the following figures show the 2-norm of the relative residuals and the maximum norm of the errors versus the number of V-cycles. From Fig. 2, we can see that the residual norms decrease as the same rate for all mesh sizes and it takes about 8 V-cycles to reach the stopping criterion. However, it takes only 5 V-cycles to reach the discretization error for all meshes. Furthermore, the maximum norm of errors decrease by a factor of four as the resolution doubles which indicates that our finite difference discretization for Eq. (1) is indeed second-order accurate. Fig. 3 shows the convergence and accuracy performance for the case of moderately oscillatory diffusion in θ direction. This example takes about 3–4 more V-cycles than the smooth case to reach the residual stopping criterion and the discretization error. It leads to the same convergence rate for all meshes as well. Note that, since the diffusion coefficient is moderately oscillatory, the mesh $n = 16$ is under-resolving for this case. Fig. 4 shows the performance of our multigrid method for the case of moderately oscillatory diffusion in θ and z directions. Although the meshes $n = 16, 32$ show under-resolution accuracy, the residual convergence rates are still the same for all meshes. All three examples show our present multigrid V-cycle is scalable with the mesh size perfectly.

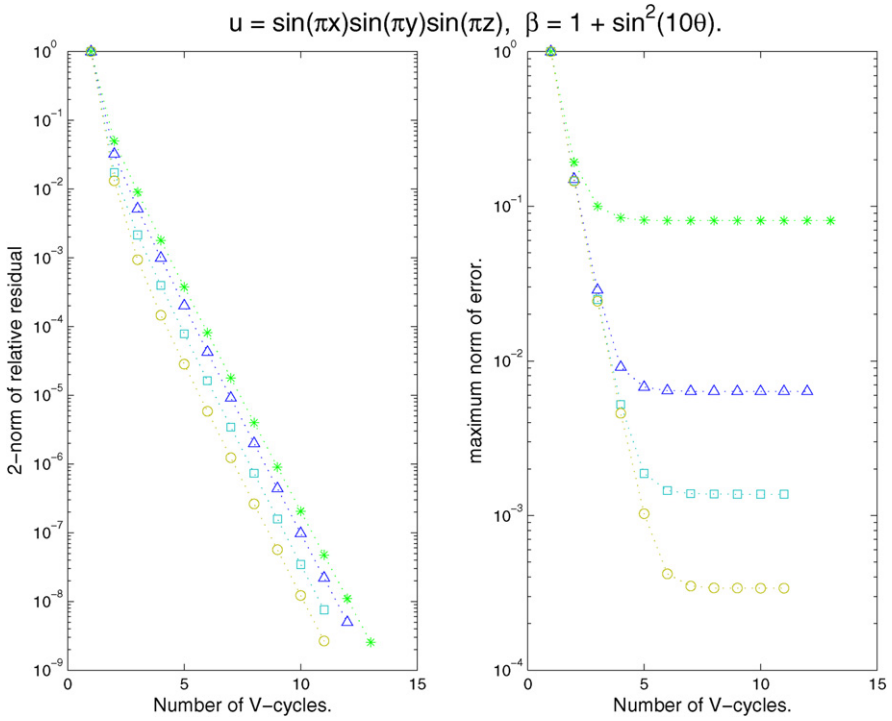

Fig. 3. The 2-norm of relative residuals of $MG_{\mathrm{RB}}(1,1)$ and the maximum norm of errors for Example 2 (diffusion coefficient moderately oscillatory in θ direction) with grid sizes $n = 16(*), 32(\triangle), 64(\square), 128(\circ)$.

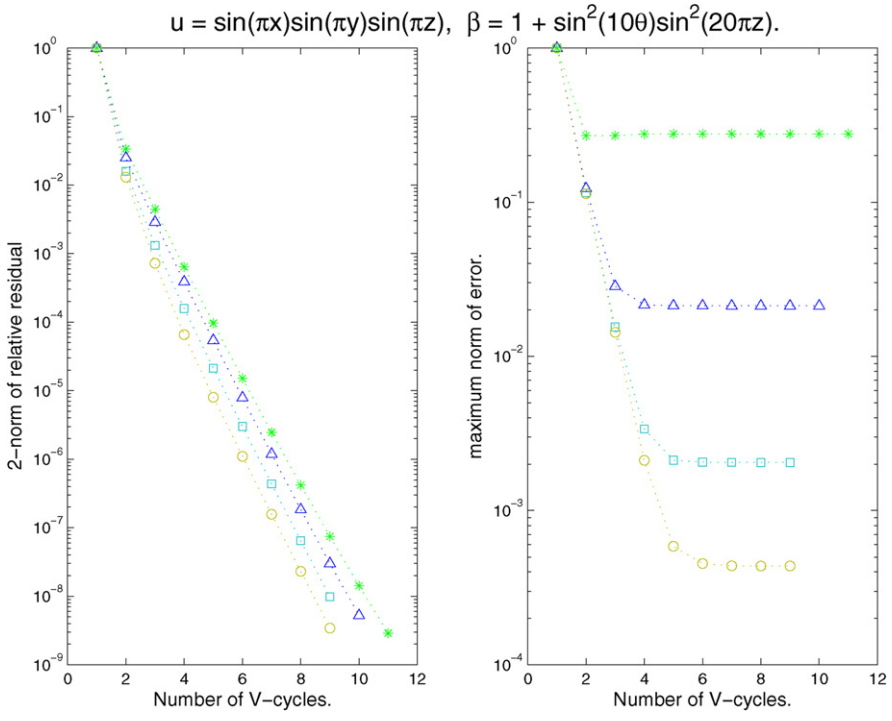

Fig. 4. The 2-norm of relative residuals of $MG_{\mathrm{RB}}(1,1)$ and the maximum norm of errors for Example 3 (diffusion coefficient moderately oscillatory in θ and z directions) with grid sizes $n = 16(*), 32(\triangle), 64(\square), 128(\circ)$.

Acknowledgements

The first author was supported by National Science Council of Taiwan under research grant NSC-94-2115-M-009-004. The second author was supported by National Science Council of Taiwan under research grant NSC-94-2119-M-009-003.

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