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A Regularity Theorem for a Nonconvex Scalar Conservation Law*

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In this paper we study the regularity properties of solutions of a single conservation law. We prove that if the flux function $f(\cdot)$ is smooth and totally nonlinear in the sense that $f''(\cdot)$ vanishes at isolated points only, and if the initial data $u_0(\cdot)$ are bounded and measurable, then $f'(u(\cdot, t))$ is in the class of functions of locally bounded variation for all $t > 0$.

1. INTRODUCTION

In this paper we will study the regularity properties of solutions of a single conservation law

$$\begin{aligned} u_t + f(u)_x = 0, &\qquad t > 0,\ -\infty < x < \infty, \\ u(x, 0) = u_0(x), &\qquad -\infty < x < \infty. \end{aligned} \tag{1.1}$$

We assume that $f(\cdot)$ is smooth and totally *nonlinear* in the sense that $f''(\cdot)$ vanishes at isolated points only. The initial data $u_0(\cdot)$ are bounded and measurable.

It is well known that, in general, the initial value problem for (1.1) does not have global smooth solutions even if the initial data $u_0(\cdot)$ are smooth. Hence we look for weak solutions: a weak solution of (1.1) is a bounded and measurable function u such that for any C^∞ function $g: R \times R \to R$ with compact support

$$\int_{R \times R^+} (ug_t + f(u)\, g_x)\, dx\, dt + \int_R u_0(x)\, g(x, 0)\, dx = 0, \tag{1.2}$$

where $R^+ = \{t \in R: t > 0\}$.

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In general, (1.1) does not have a unique weak solution. In order to single out a unique physical solution of (1.1), one requires that $u: R \times R^+ \to R$ satisfies an additional condition called *entropy condition*. One of the formulations is as follows: Let u be piecewise smooth. Then across a discontinuity line $x = x(t)$, the solution u satisfies the *Rankine–Hugoniot condition* (R–H) and the *entropy condition* (E) [14]

$$x'(t) = \sigma(u_-, u_+), \tag{R–H}$$

$$\sigma(u_-, u_+) \leqslant \sigma(u_-, u) \qquad \text{for all } u\text{'s between } u_- \text{ and } u_+, \tag{E}$$

where $u_+ = u(x(t) + 0, t)$, $u_- = u(x(t) - 0, t)$, and $\sigma(u_1, u_2)$ is the shock speed defined by

$$\sigma(u_1, u_2) = \frac{f(u_1) - f(u_2)}{u_1 - u_2}. \tag{1.3}$$

In the case $f(\cdot)$ is strictly convex (or concave), Lax [12] discovers an explicit solution for (1.1). Using an explicit representation of solutions similar to Lax's, Oleinik [13] studies the structure of solutions of (1.1) and shows that solutions are continuous except on the union of an at most countable set of Lipschitz continuous curves (shocks). Dafermos [8], using a different approach, can also establish the above results. In the case $f(\cdot)$ is uniformly strictly convex ($f''(\cdot) \geqslant \varepsilon > 0$), Lax [12] establishes that $u(\cdot, t)$ is in the class of functions of locally bounded variation in the sense of Tonelli and Cesari (space BV) for bounded and measurable initial data $u_0(\cdot)$.

When we remove the convexity condition on $f(\cdot)$, we still have existence and uniqueness theorems (Kruzkov [11]). But we know little about the structure of solutions for (1.1). Furthermore, the solutions for (1.1) are much more complicated for nonconvex $f(\cdot)$. See, for example, Ballou [1], Cheng [2, 3], Conlon [7], and Greenberg and Tong [9]. The result of Lax [12] mentioned above naturally leads one to conjecture that it still holds even $f''(\cdot)$ changes sign. If this conjecture were true, then membership in the space BV would provide maximal information on the structure of solutions of (1.1). Unfortunately, Cheng [4] gives two examples to show that the space BV is not enough for hyperbolic conservation laws. In these two examples, although $u(\cdot, t)$ does not belong to the space BV, $f'(u(\cdot, t))$ does belong to the space BV. Thus it is hopefully true that $f'(u(\cdot, t))$ belongs to the space BV for all $t > 0$ for nonconvex $f(\cdot)$.

Recently, Cheng [6] proved that $f'(u(\cdot, t))$ belongs to BV for all $t > 0$ in a special case where $f''(\cdot)$ vanishes at a single point only and changes sign there. In this paper we shall give a complete proof of the above results for a function f with exactly two zero points of $f''(u)$. But we already can see that our proof can be generalized to cover the case for general totally nonlinear f.

We organize this paper as follows. In Section 2 we give the definition of generalized solutions of (1.1) and write down the existence and uniqueness theorems of Kruzkov [11]. In Section 3, we collect some well-known results for convex f which we need. In Section 4, we give the results for the case where $f''(\cdot)$ vanishes at a single point only and changes sign there. This is not only for the sake of completeness of this paper but also for the need and better understanding of Section 5. In Section 5 we give detailed information for the case where $f''(\cdot)$ vanishes at exactly two points and changes sign there. Finally, we give some remarks in Section 6 to discuss the case of totally nonlinear f.

2. Generalized Solutions

Following Kruzkov [11], we let

$$\Pi_T = \{(x, t): -\infty < x < \infty, 0 \leqslant t \leqslant T\}.$$

We assume that $f(u)$ is a smooth function in $-\infty < u < \infty$ and $u_0(\cdot)$ is a bounded and measurable function in $-\infty < x < \infty$.

Definition 2.1. A bounded measurable function $u(x, t)$ is called a generalized solution of problem (1.1) in the band Π_T if:

(1) for any constant k and any smooth function $g(x, t) \geqslant 0$ which has compact support strictly contained inside the interior of Π_T, the following inequality holds:

$$\iint_{\Pi_T} \{|u(x, t)-k|\, g_t + \operatorname{sign}(u(x, t)-k)[f(u(x, t))-f(k)]\, g_x\}\, dx\, dt \geqslant 0; \tag{2.1}$$

(2) there exists a set $E \subset [0, T]$ of measure zero such that for $t \in [0, T]-E$ the function $u(x, t)$ is defined almost everywhere in $-\infty < x < \infty$, and for any N

$$\lim_{\substack{t \to 0 \\ t \in [0,T]-E}} \int_{-N}^{N} |u(x, t) - u_0(x)|\, dx = 0. \tag{2.2}$$

We have

Theorem 2.2 (Kruzkov). *Let the functions $u(x, t)$ and $v(x, t)$ be* *generalized solutions of problem* (1.1) *with initial functions* $u_0(x)$ *and* $v_0(x)$, *respectively, where* $|u(x, t)| \leqslant M$ *and* $|v(x, t)| \leqslant M$ *almost everywhere in* Π_T. *Then for almost all* $t \in [0, T_0]$

$$\int_{a+Vt}^{b-Vt} |u(x, t) - v(x, t)|\, dx \leqslant \int_a^b |u_0(x) - v_0(x)|\, dx, \tag{2.3}$$

where $-\infty < a < b < \infty$, $V = \max\{|f'(u)|: |u| \leqslant M\}$, *and* $T_0 = \min\{T, (b-a)/2V\}$.

Hence we have the uniqueness of generalized solutions of (1.1).

THEOREM 2.3 (Kruzkov). *The generalized solution of problem* (1.1) *in the band* Π_T *is unique.*

We need also the following ordering principle.

THEOREM 2.4 (Kruzkov). *Let the functions* $u(x, t)$ *and* $v(x, t)$ *be the generalized solutions of* (1.1) *with initial functions* $u_0(x)$ *and* $v_0(x)$, *respectively. Let* $u_0(x) \leqslant v_0(x)$ *almost everywhere in* $(-\infty, \infty)$. *Then* $u(x, t) \leqslant v(x, t)$ *almost everywhere in* Π_T.

For the existence part of (1.1), we have

THEOREM 2.5 (Kruzkov). *Let* $|u_0(x)| \leqslant M$ *and* $u^\varepsilon(x, t)$ *be the solution of the following problem*:

$$u_t + f(u)_x = \varepsilon u_{xx}, \qquad t > 0,\ -\infty < x < \infty,$$

$$\lim_{t \to 0+} \int_a^b u(x, t)\, dx = \int_a^b u_0(x)\, dx \qquad \text{for all } -\infty < a < b < \infty.$$

Then $u^\varepsilon(x, t)$ *converges as* $\varepsilon \to 0+$ *almost everywhere in* Π_T *to a function* $u(x, t)$ *which is a generalized solution of* (1.1) *and* $|u(x, t)| \leqslant M$.

THEOREM 2.6 (Kruzkov). *A generalized solution of* (1.1) *exists.*

3. THE CASE $f''(u) \geqslant 0$

3.1. *Construction of Solutions*

Following Lax [12], we assume that f satisfies:

(i) $f(u)$ is a smooth function on $-\infty < u < \infty$;

(ii) $f''(u) \geqslant 0$ for all u and $f''(u) = 0$ at isolated points only;

(iii) $f(u) \to \infty$ as $|u| \to \infty$;

(iv) $f(u)/|u| \to \infty$ as $|u| \to \infty$.

We define the conjugate function $g(s)$ by the relation

$$g(s) = \max_u \{us - f(u)\}. \tag{3.1}$$

It is easy to see that g also satisfies (i)–(iv). Let $u_0(\cdot)$ be a measurable function and $-\infty < a \leqslant u_0(x) \leqslant b < \infty$ for almost all $x \in (-\infty, \infty)$. Let

$$F(x, t; y) = tg\left(\frac{x-y}{t}\right) + \int_0^y u_0(\eta)\, d\eta \tag{3.2}$$

$$G(x, t) = \min_{-\infty < y < \infty} F(x, t; y). \tag{3.3}$$

We have

THEOREM 3.1 (Lax). (1) *For given $t > 0$, with the exception of a countable set of values of x, the function $F(x, t; y)$ assumes its minimum at a single point which we denote by $y_0(x, t)$.*

(2) *For all those (x, t) which $y_0(x, t)$ is defined, let*

$$u(x, t) = b\left(\frac{x - y_0(x, t)}{t}\right),$$

where $b(\cdot)$ is the inverse function of $f'(\cdot)$, that is, $b(f'(u)) = u$. Then $u(x, t)$ is a generalized solution of (1.1) *with $a \leqslant u(x, t) \leqslant b$ almost everywhere.*

(3) *$G(x, t)$ is a Lipschitz continuous function in $\{(x, t)$: $-\infty < x < \infty$, $0 \leqslant t < \infty\}$ and*

$$\partial G(x, t)/\partial x = u(x, t) \tag{3.4}$$

$$\partial G(x, t)/\partial t = -f(u(x, t)). \tag{3.5}$$

For our purposes we need another method of construction of $u(x, t)$. Let f satisfy: $f''(u) \geqslant 0$ for $u \in [a, b]$ and $f''(u) = 0$ at isolated points only. Let $u_0(\cdot)$ be a measurable function with $-\infty < a \leqslant u(x) \leqslant b < \infty$ almost everywhere and let $G(x, t)$ be

$$G(x, t) = \min_{a \leqslant u \leqslant b} \left\{ t(uf'(u) - f(u)) + \int_0^{x - f'(u)t} u_0(\eta)\, d\eta \right\}. \tag{3.6}$$

From the above results of Lax, we have

THEOREM 3.2. (1) *The $G(x, t)$ in (3.6) is a Lipschitz continuous function.*

(2) *For given $t > 0$, with the exception of a countable set of values of x, $\partial G(x, t)/\partial x$ and $\partial G(x, t)/\partial t$ exist.*

(3) *Let $u(x, t) \equiv \partial G(x, t)/\partial x$ when the latter exists. Then $u(x, t)$ is a generalized solution of (1.1).*

(4) $\partial G(x, t)/\partial t = -f(u(x, t))$.

Proof. Let $\tilde{f}(u)$ be a smooth extension of f to $(-\infty, \infty)$ and $\tilde{f}$ satisfies the assumptions (i)–(iv) indicated in the beginning of this subsection. Let $\tilde{g}$ be the conjugate function of $\tilde{f}$. Then from Theorem 3.1, we have

$$\begin{aligned}
&\min_{-\infty < y < \infty} \left\{ t\tilde{g}\left(\frac{x-y}{t}\right) + \int_0^y u_0(\eta)\, d\eta \right\} \\
&= \min_{x - f'(b)t \leqslant y \leqslant x - f'(a)t} \left\{ t\tilde{g}\left(\frac{x-y}{t}\right) + \int_0^y u_0(\eta)\, d\eta \right\} \\
&= \min_{a \leqslant u \leqslant b} \left\{ t\tilde{g}(\tilde{f}'(u)) + \int_0^{x - \tilde{f}'(u)t} u_0(\eta)\, d\eta \right\} \\
&= \min_{a \leqslant u \leqslant b} \left\{ t(uf'(u) - f(u)) + \int_0^{x - f'(u)t} u_0(\eta)\, d\eta \right\}.
\end{aligned}$$

Thus the $G(x, t)$ in (3.6) is the same $G(x, t)$ in (3.3). This completes the proof. Q.E.D.

Remark. If $f''(u) \leqslant 0$, then we use min in (3.1) and max in (3.3).

3.2. *Properties of Solutions*

From the results of Hopf [10], Lax [12], Oleinik [13], and Dafermos [8] we have

THEOREM 3.3. *The generalized solution $u(x, t)$ of Theorem 3.2 has the following properties:*

(1) *$u(x, t)$ is continuous on $\{(x, t)\colon -\infty < x < \infty, 0 < t < \infty\}$ except on Γ, which is the union of an at most countable set of Lipschitz continuous curves.*

(2) *$u(x \pm 0, t)$ exists for all $t > 0$ and for all x and*

$$f'(u(x-0, t)) \geqslant f'(u(x+0, t)).$$

(3) *For a fixed point* (x_0, t_0), $t_0 > 0$,

$$u(x, t) = u(x_0 \pm 0, t_0)$$

for all $(x, t) \in L^{\pm}$, *where*

$$L^{\pm} = \{(x, t): x = x_0 - f'(u(x_0 \pm 0, t))(t_0 - t), 0 < t < t_0\}.$$

(4) *If* $u_0(\cdot)$ *is piecewise monotone, then* $u(\cdot, t)$ *is also piecewise monotone.*

3.3. *Estimates of Total Variation of* $f'(u(\cdot, T))$ *on* $[a, b]$

The following results are well known (see, for example, Dafermos [8]).

THEOREM 3.4. *If* $u(x, t)$ *is a generalized solution of* (1.1), *then for* $-\infty < x < y < \infty$ *and* $t > 0$,

$$\frac{f'(u(y \pm 0, t)) - f'(u(x \pm 0, t))}{y - x} \leqslant \frac{1}{t}. \tag{3.7}$$

In particular, the increasing (*and consequently also the total*) *variation of* $f'(u(\cdot, t))$ *is locally bounded on* $(-\infty, \infty)$.

We give another estimate of the total variation of $f'(u(\cdot, t))$. Although the following estimate is not sharper than (3.7), the method used to obtain this result can be generalized to the nonconvex case.

THEOREM 3.5. *If* $u(x, t)$ *is a generalized solution of* (1.1), *then there exist constants* C_1 *and* C_2 *depending only on* f *and the bound* M *of* $u(x, t)$, *such that for all* $-\infty < a < b < \infty$ *and* $T > 0$,

$$V^{+}(f'(u(\cdot, T)); [a, b]) \leqslant \frac{C_1(b-a)}{T} + C_2, \tag{3.8}$$

where $V^{+}(f'(u(\cdot, T)); [a, b])$ *is the increasing variation of* $f'(u(\cdot, T))$ *on* $[a, b]$.

Proof. Let $a \leqslant c < d \leqslant b$ and $f'(u(\cdot, T))$ be increasing in the interval (c, d). From (c, T) and (d, T) we draw two backward characteristics L_c and L_d, respectively, where

$$L_c = \{(x, t): x = c - f'(u(c + 0, T))(T - t), 0 \leqslant t \leqslant T\},$$
$$L_d = \{(x, t): x = d - f'(u(d - 0, T))(T - t), 0 \leqslant t \leqslant T\}.$$

From Theorem 3.3,

$$c_0 \equiv c - f'(u(c+0, T))\, T \leqslant d - f'(u(d-0, T))\, T \equiv d_0. \tag{3.9}$$

Now let $A_{[c,d]}$ be the area of the region bounded by lines L_c, L_d, $t=0$ and $t=T$. Then

$$\begin{aligned} A_{[c,d]} &= \int_0^T \{[d - f'(u(d-0, T))(T-t)] \\ &\quad - [c - f'(u(c+0, T))(T-t)]\}\, dt \\ &= \int_0^T \{(d_0 - c_0) + [f'(u(d-0, T)) - f'(u(c+0, T))]\, t\}\, dt \\ &= (d_0 - c_0)\, T + \tfrac{1}{2}T^2[f'(u(d-0, T)) - f'(u(c+0, T))]. \end{aligned} \tag{3.10}$$

From (3.9) and (3.10) we have

$$[f'(u(d-0, T)) - f'(u(c+0, T))] \leqslant \frac{2A_{[c,d]}}{T^2}. \tag{3.11}$$

Since $f'(u(\cdot, T))$ decreases across shocks, we have

$$V^+(f'(u(\cdot, T)); [a, b]) = \sum_{(c,d)} [f'(u(d-0, T)) - f'(u(c+0, T))],$$

where the sum is over all disjoint open intervals $(c, d) \subset [a, b]$ such that $f'(u(\cdot, T))$ is increasing on (c, d). This sum is at most a countable sum. Using (3.11) we have

$$V^+(f'(u(\cdot, T)); [a, b]) \leqslant \frac{2}{T^2} \sum_{(c,d)} A_{[c,d]}. \tag{3.12}$$

Now let $V = \max\{|f'(u)|: |u| \leqslant M\}$, we have

$$\sum_{(c,d)} A_{[c,d]} \leqslant T(b-a) + VT^2. \tag{3.13}$$

Thus we have

$$\begin{aligned} V^+(f'(u(\cdot, T)); [a, b]) &\leqslant \frac{2}{T^2}\{T(b-a) + VT^2\} \\ &= \frac{C_1(b-a)}{T} + C_2 \end{aligned} \tag{3.14}$$

where $C_1 = 2$ and $C_2 = 2V$. This completes the proof of Theorem 3.5.

4. The Case $f''(u)$ Vanishes at a Single Point Only and Changes Sign There

4.1. *Construction of Solution for Piecewise Monotone Initial Data*

Without loss of generality we assume that

(i) $f(0)=0$,

(ii) $f''(u)<0$ for $u<0$, $f''(0)=0$, and $f''(u)>0$ for $u>0$,

(iii) $f(u) \cong f'(0)\, u + Ku^k$ for $|u|$ small, where K is a positive constant and $k \geqslant 3$ is an odd integer.

Now assume that $u_0(\cdot)$ is piecewise monotone and bounded by M. We can construct a generalized solution of (1.1). Details of the construction method can be found in Cheng [5]. For our purposes and for the sake of completeness, we shall outline the construction method in the following.

Let $\eta<0$ be given. We define $\eta^*>0$ to be the unique number which satisfies

$$f'(\eta^*)=[f(\eta^*)-f(\eta)]/(\eta^*-\eta). \tag{4.1}$$

Similarly, let $\eta>0$ be given. We define $\eta_*<0$ to be the unique number which satisfies

$$f'(\eta_*)=[f(\eta_*)-f(\eta)]/(\eta_*-\eta). \tag{4.2}$$

Definition 4.1. Let $x(\cdot)\colon (a, b) \to R$ be a shock of the generalized solution of (1.1), that is, $u(x(t)+0, t) \neq u(x(t)-0, t)$ for all $t \in (a, b)$ and $x(\cdot)$ is Lipschitz continuous in (a, b). We call this shock

(i) a type-I shock if

$$f'(u(x(t)-0, t))>x'(t)>f'(u(x(t)+0, t))$$

for almost all $t \in (a, b)$.

(ii) a type-II shock if

$$f'(u(x(t)-0, t))>x'(t)=f'(u(x(t)+0, t))$$

for almost all $t \in (a, b)$.

For the convenience of the reader, we give some examples of these shocks. Consider first the case

$$\begin{aligned} u(x, t) &= u_l \quad \text{for} \quad x \leqslant \sigma t, \\ &= u_r \quad \text{for} \quad x > \sigma t, \end{aligned} \qquad t \geqslant 0$$

where $\sigma = (f(u_l) - f(u_r))/(u_l - u_r)$ and $u_l < 0$, $u_l < u_r < u_l^*$. It is easy to see that u is a generalized solution and

$$f'(u_r) < \sigma < f'(u_l).$$

Hence $x = \sigma t$ is a type-I shock.

Now consider the initial data $u_0(x)$. $u_0(x)$ is negative and monotonically decreasing in $(-\infty, 0]$ and $u_0(x) = u_r > 0$ for $x > 0$. We assume that $u_0(0-0) = u_l$ and $u_r = (u_l)^*$. We shall briefly describe how to construct the generalized solution for this initial data. First of all, since $u_0(x)$ is monotonically decreasing in $(-\infty, 0]$, is negative, and $f''(u) < 0$ for $u < 0$, we have $f'(u_0(x))$ is monotonically increasing in $(-\infty, 0]$. For given $y < 0$, we draw every possible line from $(y, 0)$ with speed v between $f'(u_0(y-0))$ and $f'(u_0(y+0))$. Along the line with speed v we assign $u = h_1(v)$, where h_1 is the inverse function of f' in $(-\infty, 0]$. It is easy to see that these lines can be extended to $t = \infty$ without intersections and cover a region $\{(x, t)\colon x \leqslant f'(u_l)\, t, 0 < t < \infty\} \equiv D$. It is also easy to see that the u defined above is a locally Lipschitz continuous function on D. Now consider another function u^* defined on D,

$$u^*(x, t) = (u(x, t))^* \qquad \text{for all } (x, t) \in D.$$

Consider the ordinary differential equation

$$\frac{dx(t)}{dt} = \frac{f(u(x(t), t)) - f(u^*(x(t), t))}{u(x(t), t) - u^*(x(t), t)} = f'(u^*(x(t), t))$$

$$x(0) = 0.$$

It is easy to see that there is a unique solution $x = x(t)$ such that $(x(t), t) \in D$ for all $0 < t < \infty$. Furthermore $x'(t)$ is Lipschitz continuous with $x''(t) \leqslant 0$. Now at every point of $(x(t), t)$, we draw a line L_t with speed $f'(u^*(x(t), t))$ in the positive time direction. These lines L_t will cover a fan-like region H without intersection with each other, where

$$H = \{(x, t)\colon x(t) < x < f'(u_r)\, t, 0 < t < \infty\}.$$

Now let $\bar{u}(x, t)$ be defined by

$$\begin{aligned} \bar{u}(x', t') &= u(x', t') && \text{if } x' \leqslant x(t'), \\ &= u^*(x(t), t) && \text{if } (x', t') \in L_t \subset H, \\ &= u_r && \text{if } f'(u_r)\, t' \leqslant x'. \end{aligned}$$

It is easy to see that $\bar{u}(x, t)$ is a generalized solution of (1.1) and $x = x(t)$ is a type-II shock. For detailed construction we refer to Ballou [1] and Cheng [5].

We first consider the case that the initial data $u_0(\cdot)$ are piecewise monotone and $u_0(x) \leqslant 0$ for $x < 0$ and $u_0(x) \geqslant 0$ for $x > 0$. We define

$$u_0^i(x) = \min(\max)\{u_0(x), 0\} \qquad \text{for} \quad i = 1(2), \tag{4.3}$$

$$F_i(x, t; u) = t(uf'(u) - f(u)) + \int_0^{x - f'(u)t} u_0^i(y)\, dy, \qquad i = 1, 2, \tag{4.4}$$

and

$$G_1(x, t) = \max_{u \leqslant 0} F_1(x, t; u), \tag{4.5}$$

$$G_2(x, t) = \min_{u \geqslant 0} F_2(x, t; u). \tag{4.6}$$

Now $G_i(x, t)$, $i = 1, 2$, are Lipschitz continuous functions. Hence the equation

$$G_1(x, t) = G_2(x, t) \tag{4.7}$$

determines a Lipschitz continuous curve $x = \gamma(t)$. It is easy to see that

$$\frac{\partial G_1}{\partial x}(\gamma(t), t)\,\gamma'(t) + \frac{\partial G_1}{\partial t}(\gamma(t), t) = \frac{\partial G_2}{\partial x}(\gamma(t), t)\,\gamma'(t) + \frac{\partial G_2}{\partial t}(\gamma(t), t). \tag{4.8}$$

We let $G(x, t) = G_1(x, t)$ if $x \leqslant \gamma(t)$ and $G(x, t) = G_2(x, t)$ if $x > \gamma(t)$. Then $G(x, t)$ is a Lipschitz continuous function. Using Theorem 3.2 and Eq. (4.8), it is easy to see that $u(x, t) \equiv \partial G(x, t)/\partial x$ is a weak solution of (1.1) in the sense of (1.2). But $u(x, t)$ is not necessarily a generalized solution of (1.1) in the sense of Definition 2.1. The reason is that $u(x, t)$ may be discontinuous across the curve $\gamma(t)$. In that case, (4.8) assures us that this discontinuity line $\gamma(t)$ satisfies the Rankine–Hugoniot condition (R–H). But $\gamma(t)$ may not satisfy the entropy condition (E). In Cheng [5], we give a detailed method to modify $\gamma(t)$ in order to obtain a generalized solution in the sense of Definition 2.1. We write the result into the following theorem.

THEOREM 4.2. *Let $u_0(\cdot)$ be bounded and piecewise monotone and $u_0(x) \leqslant 0$ for $x < 0$ and $u_0(x) \geqslant 0$ for $x > 0$. Then we can construct a Lipschitz continuous function $G(x, t)$ and a curve $x = \gamma(t)$, such that $u(x, t) \equiv$*

$\partial G(x, t)/\partial x$ is a generalized solution of (1.1) *and $u(x, t) \leqslant 0$ for $x < \gamma(t)$, $u(x, t) \geqslant 0$ for $x > \gamma(t)$. Furthermore, if $\{\gamma(t): a < t < b\}$ is a discontinuity line of $u(x, t)$, then it satisfies the entropy condition* (E) *and it is a type-*I *shock or type-*II *shock* (*Definition* 4.1).

Now assume that $u_0(\cdot)$ is bounded and piecewise monotone. Then there exist $y_1 < y_2 < \cdots < y_N$, such that $u_0(x) \in (-\infty, 0]$ for $x \in (y_i, y_{i+1})$ if i is even and $u_0(x) \in [0, \infty)$ for $x \in (y_i, y_{i+1})$ if i is odd. We already set $y_0 = -\infty$ and $y_{N+1} = \infty$ for convenience in the above expression. Of course it is equally possible that $u_0(x) \in [0, \infty)$ for $x \in (y_i, y_{i+1})$ if i is even. But we consider only the first case. To construct the solution, we first use Theorem 4.2 to construct a $G_{12}(x, t)$ for the initial data $u_0^{12}(x)$, where $u_0^{12}(x) = u_0(x)$ for $x < y_2$ and $u_0^{12}(x) = 0$ for $y_2 < x$. Now let $u_0^3(x) = u_0(x)$ for $y_2 < x < y_3$ and $u_0^3(x) = 0$ otherwise. Define

$$F_3(x, t; u) = t(uf'(u) - f(u)) + \int_{y_2}^{x - f'(u)t} u_0^3(y)\, dy \tag{4.9}$$

and

$$G_3(x, t) = \max_{u \leqslant 0} F_3(x, t; u). \tag{4.10}$$

Now set $G_{12}(x, t) = G_3(x, t) + \int_{y_1}^{y_2} u_0(y)\, dy$ to determine a Lipschitz continuous curve $x = \gamma_2(t)$ with $\gamma_2(0) = y_2$, modifying $\gamma_2(t)$ if $\gamma_2(t)$ does not satisfy the entropy condition (E). Thus we can find a Lipschitz continuous $G_{123}(x, t)$ such that $u_{123}(x, t) = \partial G_{123}(x, t)/\partial x$ is a generalized solution for (1.1) with initial data $u_0^{123}(x)$, where $u_0^{123}(x) = u_0(x)$ for $x < y_3$ and $u_0^{123}(x) = 0$ otherwise. Repeating this process, we finally can construct $G(x, t) \equiv G_{12\cdots(N+1)}(x, t)$ such that $\partial G(x, t)/\partial x \equiv u(x, t)$ is the unique generalized solution of (1.1).

4.2. *Properties of Solutions for Piecewise Monotone Initial Data*

From the construction method of subsection 4.1, we have

THEOREM 4.3. *Assume that f satisfies assumptions* (i)–(iii) *at the beginning of subsection* 4.1 *and $u_0(\cdot)$ is bounded and piecewise monotone. Then the generalized solution of* (1.1), *$u(x, t)$, has the following properties*:

(i) *Regions $\{(x, t): u(x, t) \geqslant 0\}$ are separated from regions $\{(x, t): u(x, t) \leqslant 0\}$ by a finite number of Lipschitz continuous curves which are genuine characteristics, type-*I *shocks, or type-*II *shocks.*

(ii) *$u(x \pm 0, t)$ exists for all $t > 0$ and for all $x \in (-\infty, \infty)$ and*

$$f'(u(x - 0, t)) \geqslant f'(u(x + 0, t)).$$

(iii) *$u(x, t)$ is continuous except on the union of an at most countable set of Lipschitz continuous curves.*

(iv) *For every fixed (z, τ) with $\tau > 0$, there exists a backward generalized characteristic which is the union of a finite number of genuine characteristics, that is, there exist a nonnegative integer n, times $t_1, t_2, \ldots, t_n$, $0 < t_1 < \cdots < t_n < \tau$, $u_0, u_1, \ldots, u_n$, and $x_0(t), x_1(t), \ldots, x_n(t)$ such that*

(a) $x_n(t) = z + f'(u_n)(t-\tau)$, $\quad t_n \leqslant t \leqslant \tau$, $\quad x_k(t) = x_{k+1}(t_{k+1}) + f'(u_k)(t - t_{k+1})$, $t_k \leqslant t \leqslant t_{k+1}$, $k = 0, 1, \ldots, (n-1)$;

(b) $u(x_k(t) \pm 0, t) = u(x_k(t), t) = u_k$ *for* $t_k < t < t_{k+1}$, $k = 0, 1, \ldots, n$;

(c) $u_n = u(z-0, \tau)$ *and* u_k, $k = 0, 1, \ldots, (n-1)$, *satisfy*

$$f'(u_{k+1}) = \frac{f(u_{k+1}) - f(u_k)}{u_{k+1} - u_k},$$

that is, using (4.1) *and* (4.2)

$$u_{k+1} = u_k^* \qquad or \qquad u_{k+1} = (u_k)_*;$$

(d) *there is a type-*II *shock passing through $(x_k(t_k), t_k)$ for each $k = 1, 2, \ldots, n$ with shock speed $f'(u_k)$.*

Proof. Parts (i), (ii), (iii) are easy consequences of the construction method. To prove (iv), consider (z, τ) fixed with $\tau > 0$. If $u(z+0, \tau) = u(z-0, \tau)$, then u is continuous at (z, τ) and there exists a genuine characteristic (backward) passing through (z, τ) with speed $f'(u(z, \tau))$. This backward genuine characteristic can terminate only at the line $t = 0$ or at a type-II shock $x = \gamma(t)$. If it terminates at the line $t = 0$, then we are done. If it terminates at a type-II shock $x = \gamma(t)$ at $t' < \tau$, then $u(\gamma(t') + 0, t') = u(z, \tau)$ and $(u(\gamma(t') - 0, t'))^* = u(z, \tau)$ or $(u(\gamma(t') - 0, t'))_* = u(z, \tau)$. Now from $(\gamma(t'), t')$ we draw a backward genuine characteristic with speed $f'(u(\gamma(t') - 0, t'))$. Continuing this process, taking into consideration the piecewise monotonicity of $u_0(\cdot)$, we prove (iv). If $u(z+0, \tau) \neq u(z-0, \tau)$ and the shock passing through (z, τ) is a type-I shock, then we can draw two backward genuine characteristics from (z, τ) with speeds $f'(u(z+0, \tau))$ and $f'(u(z-0, \tau))$, respectively. If $u(z+0, \tau) \neq u(z-0, \tau)$ and the shock passing through (z, τ) is a type-II shock, then we can draw only a backward characteristic from (z, τ) with speed $f'(u(z-0, \tau))$. This completes the proof of this theorem. Q.E.D.

Of these properties, the last one is the most important one. We shall call this backward generalized characteristic $C_b(z, \tau)$. It is easy to see that $C_b(z, \tau)$ is not uniquely determined by (z, τ) in general. But $C_b(z, \tau)$ and $C_b(z', \tau)$ never intersect each other except at $t = 0$ for $z \neq z'$.

4.3. *Estimates of Total Variation of* $f'(u(\cdot, t))$

Now assume that $u_0(\cdot)$ is piecewise monotone and $|u_0(x)| \leqslant M$ for almost all $x \in (-\infty, \infty)$. Let $u(x, t)$ be the generalized solution corresponding to the initial data $u_0(\cdot)$. We have the following fundamental theorem.

THEOREM 4.4. *There exist constants* C_1 *and* C_2 *depending only on* f *and* M, *such that for all* $-\infty < a < b < \infty$ *and* $T > 0$,

$$V^+(f'(u(\cdot, T)); [a, b]) \leqslant \frac{C_1(b-a)}{T} + C_2, \tag{4.11}$$

where $V^+(g(\cdot); [a, b])$ *is the increasing variation of function* g *on* $[a, b]$.

To prove this theorem we need some lemmas.

LEMMA 4.5. *If* $x(\cdot)\colon (t_1, t_2) \to R$ *is a type-*II *shock for* $u(x, t)$, *then*

(i) $U(t) = u(x(t)-0, t)$ *exists for* $t \in (t_1, t_2)$ *and* $f'(U(t))$ *is a monotone decreasing function,*

(ii) $u(x(t)+0, t) = (u(x(t)-0, t))^*$ *or* $(u(x(t)-0, t))_*$,

(iii) $$x'(t) = \frac{f'(u(x(t)-0, t)) - f'(u(x(t)+0, t))}{u(x(t)-0, t) - u(x(t)+0, t)} = f'(u(x(t)+0, t)),$$

(iv) $x'(t)$ *is Lipschitz continuous on* (t_1, t_2) *and* $x''(t) \leqslant 0$.

Proof. This is a basic property for type-II shock. For a detailed proof, see Ballou [1]. We omit it. Q.E.D.

LEMMA 4.6. *Let* $z(\cdot)\colon [t_1, t_2] \to R$ *be a type-*II *shock for* $u(x, t)$. *Let* (y, τ) *be the intersection point of tangent lines of* $z(t)$ *from points* $(z(t_1), t_1)$ *and* $(z(t_2), t_2)$, *where* $t_1 < t_2$. *Then there exists a constant* $\delta > 0$ *depending only on* f *and* M, *such that*

$$(t_2 - \tau) \geqslant \tfrac{1}{4}(t_2 - t_1) \tag{4.12}$$

for all $0 < (t_2 - t_1) \leqslant \delta$.

Proof. The number y and τ satisfy the following equations:

$$y = z(t_1) + z'(t_1)(\tau - t_1),$$

$$y = z(t_2) + z'(t_2)(\tau - t_2).$$

Hence we have

$$\tau = \frac{z(t_2) - z(t_1) + z'(t_1)\, t_1 - z'(t_2)\, t_2}{z'(t_1) - z'(t_2)} \qquad \text{if} \quad z'(t_1) \neq z'(t_2).$$

A simple calculation gives us

$$\frac{t_2 - \tau}{t_2 - t_1} = \frac{z'(t_1)(t_2 - t_1) - (z(t_2) - z(t_1))}{(z'(t_1) - z'(t_2))(t_2 - t_1)} \equiv F(t_2, t_1).$$

From Lemma 4.5, we have

$$z''(t) \leqslant 0.$$

Thus $F(t_2, t_1) > 0$ for all $t_2 > t_1$. Now

$$\lim_{t_2 \to t_1+} F(t_2, t_1) = \tfrac{1}{2}$$

uniformly in t_1. Hence we can choose $\delta > 0$, depending only on f and M, sufficiently small such that

$$F(t_2, t_1) \geqslant \tfrac{1}{4}$$

for all $(t_2 - t_1) \leqslant \delta$. This completes the proof of Lemma 4.6. Q.E.D.

LEMMA 4.7. *Let* $\{u_i\}_{i=1}^{\infty}$ *and* $\{v_i\}_{i=1}^{\infty}$ *be two sequences which satisfy*:

(i) u_i *and* v_i *have the same sign*, $i = 0, 1, 2, \ldots$,

(ii) $|u_0| \leqslant M$, $|v_0| \leqslant M$,

(iii) $u_{i+1} = (u_i)^*$ *or* $u_{i+1} = (u_i)_*$, $v_{i+1} = (v_i)^*$ *or* $v_{i+1} = (v_i)_*$ *for all* $i = 0, 1, 2, \ldots$. *Let*

$$\lambda_n = \frac{f'(u_{n-1}) - f'(v_{n-1})}{f'(u_n) - f'(v_n)}. \tag{4.13}$$

Then

$$\lim_{n \to \infty} \lambda_n = (1 + \delta)^{(k-1)} \tag{4.14}$$

where k is the k in assumption (iii) *of the function f and* $\delta > 0$ *is a constant depending only on k.*

Proof. From assumption (iii) of the function f, if ξ is sufficiently small, we can approximate the function f by $f'(0)\, u + Ku^k$ for $|u| \leqslant \xi$, where K is a positive constant and $k \geqslant 3$ is an odd integer. Now let $\eta < 0$ and $|\eta| \leqslant \xi$ be given. η^* satisfies

$$f'(\eta^*) = \frac{f(\eta^*) - f(\eta)}{\eta^* - \eta}. \tag{4.15}$$

Or approximately

$$k(\eta^*)^{k-1} \cong (\eta^*)^{k-1} + (\eta^*)^{k-2}\eta + \cdots + (\eta)^{k-1}. \tag{4.16}$$

Let $(\eta^*/\eta) = \rho$ and (4.16) becomes

$$(k-1)\rho^{k-1} - \rho^{k-2} - \cdots - 1 = 0. \tag{4.17}$$

Dividing the obvious factor $(\rho - 1)$ we have

$$G(\rho) \equiv (k-1)\rho^{k-2} + (k-2)\rho^{k-3} + \cdots + 2\rho + 1 = 0. \tag{4.18}$$

It is easy to see that $G(\rho) \leqslant G(-1) = -(k-1)/2 < 0$ for all $\rho \leqslant -1$ and $G(\rho) \geqslant G(-\frac{1}{2}) > 0$ for all $\rho \geqslant -\frac{1}{2}$. Hence the only real root ρ_0 of $G(\rho) = 0$ is between -1 and $-\frac{1}{2}$. Now for $n \geqslant N$ and N sufficiently large we have $|u_n| \leqslant \xi$ and $|v_n| \leqslant \xi$. Thus for $n \geqslant N$

$$\lambda_n = \frac{f'(u_{n-1}) - f'(v_{n-1})}{f'(u_n) - f'(v_n)} \cong \frac{(u_{n-1})^{k-1} - (v_{n-1})^{k-1}}{(\rho_0 u_{n-1})^{k-1} - (\rho_0 v_{n-1})^{k-1}}$$
$$= \left(\frac{1}{\rho_0}\right)^{k-1}.$$

Hence

$$\lim_{n \to \infty} \lambda_n = \left(\frac{1}{\rho_0}\right)^{k-1}. \tag{4.19}$$

This completes the proof. Q.E.D.

LEMMA 4.8. *Assume that $f'(u(\cdot, T))$ is increasing in the interval $[c, d]$. Let $C_b(c, T)$ and $C_b(d, T)$ be two generalized backward characteristics from $(c+0, T)$ and $(d-0, T)$. Assume that $(d-c)$ is sufficiently small so that $C_b(c, T)$ consists of $x_0(t), x_1(t), \ldots, x_n(t)$ and $C_b(d, T)$ consists of $y_0(t), y_1(t), \ldots, y_n(t)$, where $x_i(t)$ is defined between $T_i \leqslant t \leqslant T_{i+1}$ with speed $f'(u_i)$ and $y_i(t)$ is defined between $t_i \leqslant t \leqslant t_{i+1}$ with speed $f'(v_i)$, $i = 0, 1, 2, \ldots, n$. Here for convenience we let $T_0 = t_0 = 0$ and $T_{n+1} = t_{n+1} = T$. Furthermore* *assume that $(d-c)$ is sufficiently small such that there is a type-*II *shock $z_k(t)$ passing through points $(y_k(t_k), t_k)$ and $(x_k(T_k), T_k)$ and*

$$t_k \geqslant T_{k-1}, \qquad k=1,2,\ldots,n, \tag{4.20}$$

$$0 < T_k - t_k \leqslant \delta, \qquad k=1,2,\ldots,n, \tag{4.21}$$

where δ is the δ in Lemma 4.6. *Then there exists a constant C depending only on f and M, such that*

$$0 \leqslant f'(v_n) - f'(u_n) \leqslant \frac{C}{T^2} A(c, d; T), \tag{4.22}$$

where $A(c, d; T)$ is the area of region bounded by $C_b(c, T)$, $C_b(d, T)$, $t=0$ and $t=T$.

Proof. We extend $x_k(t)$ (below T_k) to intersect $y_k(t)$ at time t'_k. Let Δ_n be the area of triangle bounded by the following three lines:

$$\begin{aligned} &\{(x,t)\colon x = c + f'(u_n)(t-T),\ t \leqslant T\},\\ &\{(x,t)\colon x = d + f'(v_n)(t-T),\ t \leqslant T\},\\ &\{(x,t)\colon t = T\}, \end{aligned} \tag{4.23}$$

Δ_k, $k=1,2,\ldots,(n-1)$, be the area of triangle bounded by the three lines

$$\begin{aligned} &\{(x,t)\colon x = x_{k+1}(T_{k+1}) + f'(u_k)(t-T_{k+1}),\ t \leqslant T_{k+1}\},\\ &\{(x,t)\colon x = y_{k+1}(t_{k+1}) + f'(v_k)(t-t_{k+1}),\ t \leqslant t_{k+1}\},\\ &\{(x,t)\colon t = t_{k+1}\}, \end{aligned} \tag{4.24}$$

and Δ_0 be the area of region bounded by the lines

$$\begin{aligned} &\{(x,t)\colon x = x_1(T_1) + f'(u_0)(t-T_1),\ t \leqslant T_1\},\\ &\{(x,t)\colon x = y_1(t_1) + f'(v_0)(t-t_1),\ t \leqslant t_1\},\\ &\{(x,t)\colon t = t_1\},\\ &\{(x,t)\colon t = 0\}. \end{aligned} \tag{4.25}$$

It is easy to see that

$$\Delta_n = \tfrac{1}{2}(f'(v_n) - f'(u_n))(T - t'_n)^2, \tag{4.26}$$

$$\Delta_k = \tfrac{1}{2}(f'(v_k) - f'(u_k))(t_{k+1} - t'_k)^2, \qquad k=1,2,\ldots,(n-1) \tag{4.27}$$

and

$$\Delta_0 = \tfrac{1}{2}(f'(v_0) - f'(u_0))(t_1)^2 + (y_0(0) - x_0(0))\, t_1. \tag{4.28}$$

From the concavity of $z_k(\cdot)$ (Lemma 4.5),

$$A(c, d; T) \geqslant \Delta_0 + \Delta_1 + \cdots + \Delta_n. \tag{4.29}$$

Let $\tau_0 = t_1$, $\tau_k = t_{k+1} - t'_k$ for $k = 1, 2, \ldots, (n-1)$ and $\tau_n = (T - t'_n)$ and

$$\lambda_k = \frac{f'(v_{k-1}) - f'(u_{k-1})}{f'(v_k) - f'(u_k)}, \qquad k = 1, 2, \ldots, n. \tag{4.30}$$

From (4.21) and Lemma 4.6, we have

$$\tau_k \geqslant \tfrac{1}{4}(t_{k+1} - t_k), \qquad k = 1, 2, \ldots, (n-1) \tag{4.31}$$

and

$$\tau_n \geqslant \tfrac{1}{4}(T - t_n). \tag{4.32}$$

Now

$$\Delta_n + \Delta_{n-1} + \cdots + \Delta_0 \geqslant \tfrac{1}{2}(f'(v_n) - f'(u_n))\, \tau_n^2 + \cdots + \tfrac{1}{2}(f'(v_0) - f'(u_0))\, \tau_0^2. \tag{4.33}$$

Hence

$$(f'(v_n) - f'(u_n)) \leqslant \frac{2(\Delta_n + \Delta_{n-1} + \cdots + \Delta_0)}{\tau_n^2 + \lambda_n \tau_{n-1}^2 + \lambda_n \lambda_{n-1} \tau_{n-2}^2 + \cdots + \lambda_n \lambda_{n-1} \cdots \lambda_1 \tau_0^2}. \tag{4.34}$$

Let

$$B(\lambda_n, \ldots, \lambda_1; \tau_n, \ldots, \tau_0) = \tau_n^2 + \lambda_n \tau_{n-1}^2 + \lambda_n \lambda_{n-1} \tau_{n-2}^2 + \cdots + \lambda_n \lambda_{n-1} \cdots \lambda_1 \tau_0^2. \tag{4.35}$$

From (4.31), (4.32) we have

$$\tau_n + \tau_{n-1} + \cdots + \tau_0 \geqslant T/4. \tag{4.36}$$

Hence we have

$$B(\lambda_n, \ldots, \lambda_1; \tau_n, \ldots, \tau_0) \geqslant \left(\frac{T}{4}\right)^2 \left[1 + \frac{1}{\lambda_n} + \cdots + \frac{1}{\lambda_n \lambda_{n-1} \cdots \lambda_1}\right]^{-1}. \tag{4.37}$$

Now we take

$$C = 32 \max_{\substack{|u_0| \leqslant M \\ |v_0| \leqslant M}} \max_n \left[1 + \frac{1}{\lambda_n} + \cdots + \frac{1}{\lambda_n \lambda_{n-1} \cdots \lambda_0} \right], \tag{4.38}$$

where the $\max_n$ in (4.38) exists from Lemma 4.7.

Combining (4.29), (4.34), (4.37), and (4.38), we have

$$0 < (f'(v_n) - f'(u_n)) \leqslant \frac{C}{T^2} A(c, d; T). \tag{4.39}$$

This completes the proof of this lemma. Q.E.D.

Now we are in a position to prove Theorem 4.4.

Proof of Theorem 4.4. Assume that $f'(u(\cdot, T))$ is monotonically increasing in the interval $(c, d) \subset [a, b]$. If necessary we can divide (c, d) into small intervals. Hence we can assume that $C_b(c, T)$ and $C_b(d, T)$ are of the type in Lemma 4.8. Across a discontinuity $f'(u(\cdot, T))$ is decreasing. Hence from Lemma 4.8, we have

$$\begin{aligned} V^+(f'(u(\cdot, T)); [a, b]) &= \sum_{(c,d)} [f'(u(d-0, T)) - f'(u(c+0, T))] \\ &\leqslant \frac{C}{T^2} \left(\sum_{(c,d)} A(c, d; T) \right), \end{aligned} \tag{4.40}$$

where the sum $\sum$ in (4.40) is summing over all disjoint small open intervals (c, d) such that $f'(u(\cdot, T))$ is increasing in (c, d). This sum is at most a countable sum.

As in the proof of Theorem 3.5, we have

$$\sum_{(c,d)} A(c, d; T) \leqslant T(b-a) + VT^2, \tag{4.41}$$

where $V = \max\{|f'(u)|: |u| \leqslant M\}$. Combining (4.40) and (4.41) we finally have

$$V^+(f'(u(\cdot, T)); [a, b]) \leqslant \frac{C_1(b-a)}{T} + C_2, \tag{4.42}$$

where $C_1 = C$ and $C_2 = CV$. This completes the proof of Theorem 4.4. Q.E.D.

Now we can state our main results.

THEOREM 4.9. *Assume that* $u_0(x)$ *is a measurable function satisfying* $|u_0(x)| \leqslant M$. *Let* $u(x, t)$ *be the generalized solution of* (1.1) *with these initial data. Then*

(i) *for all* $-\infty < a < b < \infty$ *and* $T > 0$,

$$V(f'(u(\cdot, T)), [a, b]) \leqslant \frac{C_1(b-a)}{T} + C_2, \tag{4.43}$$

where C_1 *and* C_2 *are two constants depending only on* f *and* M, *in particular for all* $t > 0$ $f'(u(\cdot, t)) \in \mathrm{BV}$.

(ii) $u(x \pm 0, t)$ *exists for all* $x \in (-\infty, \infty)$ *and all* $t > 0$.

(iii) *The set* $\{(x, t): u(\cdot, t)$ *is discontinuous at* $x\}$ *is a countable set for every* $t > 0$.

In order to prove these main results, we need more lemmas.

LEMMA 4.10. *If* $u(x, t)$ *is a generalized solution of* (1.1) *satisfying* $|u(x, t)| \leqslant M$, *then there exists a constant* C *depending only on* f *and* M, *such that for all* $-\infty < a < b < \infty$, $t_1 > t_2 > 0$,

$\int_a^b |f'(u(x, t_2)) - f'(u(x, t_1))|\, dx \leqslant C\, |t_2 - t_1| \cdot$ *total variation of* $f'(u(\cdot, t_1)$ *between* $(a - V(t_2 - t_1))$ *and* $(b + V(t_2 - t_1))$. (4.44)

Proof. This is an easy consequence of the proof of Theorem 4.4. We omit the details.

LEMMA 4.11. *Let* $u(x, t)$ *be a generalized solution of* (1.1) *satisfying* $|u(x, t)| \leqslant M$ *with piecewise monotone initial data* $u_0(x)$. *If for a given* $t > 0$, (1) $u(\cdot, t)$ *is continuous on the interval* $[c, d]$, (2) $f'(u(\cdot, t))$ *is monotonically increasing on* $[c, d]$, *and* (3) $u(d-0, t) = \bar{u}$, $u(c+0, t) = \bar{v}$ *are fixed numbers, then there exists a constant* β *depending only on* f, M, $\bar{u}$, *and* $\bar{v}$, *such that*

$$(d - c) \geqslant \beta t. \tag{4.45}$$

Proof. If the backward characteristics drawing from $(c+0, t)$ and $(d-0, t)$ with speed $f'(\bar{v})$ and $f'(\bar{u})$, respectively, terminate at $t = 0$, then

$$d - f'(\bar{u})\, t \geqslant c - f'(\bar{v})\, t. \tag{4.46}$$

Hence we can choose $\beta = f'(\bar{u}) - f'(\bar{v})$. So, in general, there are type-II shocks crossing the two generalized backward characteristics $C_b(c, t)$ and $C_b(d, t)$. Assume that the number of type-II shocks crossing these two characteristics is n. From Theorem 4.3(iv) and $|u(x, t)| \leqslant M$, we know that n is bounded by N which is a constant depending only on f, M, $\bar{u}$, and $\bar{v}$. Using the notations of Theorem 4.3(iv), we have

$$d = d_0 + f'(u_0)\, t_1 + f'(u_1)(t_2 - t_1) + \cdots + f'(u_n)(t - t_n), \qquad (4.47)$$

$$c = c_0 + f'(v_0)\, T_1 + f'(v_1)(T_2 - T_1) + \cdots + f'(v_n)(t - T_n), \qquad (4.48)$$

where $u_n = \bar{u}$ and $v_n = \bar{v}$. Since $\bar{u}$ and $\bar{v}$ are fixed, u_k and v_k, $k = 0, 1, 2,..., n$, are also fixed. The variable t_1 can vary from 0 to a time $t_1^u < t$. T_1 is a continuous function of t_1. t_2 can vary from t_1^* to t_2^u, t_1^* is a continuous function of t_1. Continuing this processes, we know that T_k is a continuous function of $t_1, t_2,..., t_k$ for $k = 1, 2,..., n$ and

$$t_{k-1}^* \leqslant t_k \leqslant t_k^u, \qquad k = 1, 2,..., n, \qquad (4.49)$$

where $t_0^* = 0$ and t_{k-1}^* is a continuous function of $t_1,..., t_{k-1}$ for $k \geqslant 2$. Actually, t_k^u is the number corresponding to

$$T_k = T_{k+1} = \cdots = T_n = t \qquad (4.50)$$

and t_{k-1}^* is the number corresponding to $T_k = T_{k-1}$. Hence from (4.47), (4.48), and the fact that $d_0 - c_0 \geqslant 0$ we have

$$(d - c) \geqslant [f'(u_0)\, t_1 + \cdots + f'(u_n)(t - t_n)] - [f'(v_0)\, T_1 + \cdots + f'(v_n)(t - T_n)] \equiv H(t_1, t_2,..., t_n). \qquad (4.51)$$

But from the above consideration we know that H is a positive continuous function on a compact set. Hence H has a positive minimum. Now Eq. (1.1) is invariant under the dilation $x \to \alpha x$, $t \to \alpha t$ for $\alpha > 0$. Thus we have a factor t in (4.45). We can choose $\beta t = \min_{n \leqslant N} \min H(t_1,..., t_n)$. Of course, there may be some type-II shocks which cross $C_b(c, t)$ but not $C_b(d, t)$. These cases are similar to the case we just discussed. We omit the details. This completes the proof. Q.E.D.

LEMMA 4.12. *Let $u(x, t)$ be a generalized solution of* (1.1) *satisfying $|u(x, t)| \leqslant M$ with piecewise monotone initial data $u_0(x)$. If for a given $t > 0$, $d > c$, $u(c \pm 0, t) = u_1$ and $u(d \pm 0, t) = u_2$ are two different fixed numbers such that u_1 at left cannot be connected to u_2 at right to form a shock satisfying* (E), *then there exists a constant γ depending only on f, M, u_1, and u_2, such that*

$$(d - c) \geqslant \gamma t. \qquad (4.52)$$

Proof. Without loss of generality, we assume that $u_1 < 0$ and $u_2 > 0$.

Since $u_0(x)$ is piecewise monotone, $f'(u(\cdot, t))$ is also piecewise monotone. Because u_1 cannot be connected to u_2 to form a shock satisfying the entropy condition (E), there must be increasing variation of $f'(u(\cdot, t))$ in the interval (c, d). From the entropy condition, this increasing variation of $f'(u(\cdot, t))$ in (c, d) is at least

$$[f'(v)-f'(u_1)]+[f'(u_2)-f'(v^*)],$$

where $U_2 \leqslant v \leqslant u_1$ and $U_2^* = u_2$. Using Lemma 4.11, we can obtain (4.52). This completes the proof. Q.E.D.

Now we are in a position to prove Theorem 4.9.

Proof of Theorem 4.9. Assume that $u_0(x)$ is bounded by M. We can find a sequence of piecewise monotone functions $\{u_0^n\}$ which are also bounded by M, such that for all $-\infty < a < b < \infty$,

$$\lim_{n \to \infty} \int_a^b u_0^n(x)\, dx = \int_a^b u_0(x)\, dx. \tag{4.53}$$

Now for the given $u_0^n(x)$ we can construct a generalized solution $u^n(x, t)$ of (1.1) with initial data $u_0^n(x)$. $u^n(x, t)$ is also bounded by M. Furthermore from Theorem 4.4, the total variation of $f'(u^n(x, t))$ on $[a, b]$ is bounded for $t > 0$, that is,

$$V(f'(u^n(\cdot, t)); [a, b]) \leqslant \frac{\tilde{C}_1(b-a)}{t} + \tilde{C}_2, \tag{4.54}$$

where $\tilde{C}_1$ and $\tilde{C}_2$ are constants depending only on f and M. Using Helly's theorem and Lemma 4.10, we can find a subsequence of $\{u^n(x, t)\}$, also denoted by $\{u^n(x, t)\}$, such that $f'(u^n(x, t))$ converges almost everywhere on $\{(x, t): 1/N \leqslant t \leqslant N, -N \leqslant x \leqslant N\}$ as $n \to \infty$. Using diagonal process, we can find another subsequence of $\{u^n(x, t)\}$, also denoted by $\{u^n(x, t)\}$, such that $f'(u^n(x, t))$ converges almost everywhere in $t > 0$ to a bounded measurable function $g(x, t)$. From (4.54) it is easy to see that

$$V(g(\cdot, t); [a, b]) \leqslant \frac{\tilde{C}_1(b-a)}{t} + \tilde{C}_2 \tag{4.55}$$

for all $-\infty < a < b < \infty$ and $t > 0$. Thus for a given $t > 0$, $g(\cdot, t)$ is continuous except on a countable set. We claim that $u^n(x, t)$ also converges for those x at which $g(\cdot, t)$ is continuous. Suppose that $g(\cdot, t)$ is continuous at x_0. If $g(x_0, t) = f'(0)$, then $f'(u^n(x_0, t)) \to f'(0)$ as $n \to \infty$. Hence from assumption (iii) of f,

$$Kk(u^n(x_0, t))^{k-1} \to 0, \qquad \text{as} \quad n \to \infty,$$

or

$$u^n(x_0, t) \to 0, \qquad \text{as} \quad n \to \infty.$$

Now assume that $g(x_0, t) \neq f'(0)$. Let $u_1 < 0$ and $u_2 > 0$ be such that

$$f'(u_1) = f'(u_2) = g(x_0, t). \tag{4.56}$$

Choose $\varepsilon < \frac{1}{2}\min\{-u_1, u_2\}$ sufficiently small such that, if

$$|u^n(c \pm 0, t) - u_1| < \varepsilon \qquad \text{and} \qquad |u^n(d \pm 0, t) - u_2| < \varepsilon,$$

we have

$$|d - c| \geqslant \eta t > 0, \tag{4.57}$$

where η is a number depending only on f, M, and ε. Equation (4.57) is possible from Lemmas 4.11 and 4.12. Now since $g(\cdot, t)$ is continuous at x_0, we can find a number ξ, $0 < \xi < \frac{1}{2}\eta t$, such that

$$|g(x, t) - g(x_0, t)| < \delta \tag{4.58}$$

for all $|x - x_0| < \xi$, where

$$\delta = \min\{\sup_{|u - u_1| \leqslant \varepsilon/2} |f'(u) - f'(u_1)|, \sup_{|u - u_2| \leqslant \varepsilon/2} |f'(u) - f'(u_2)|\}. \tag{4.59}$$

In the interval $(x_0 - \xi, x_0 + \xi)$, $f'(u^n(x, t))$ converges everywhere. Hence there exists a measurable set $E \subset (x_0 - \xi, x_0 + \xi)$ with measure $m(E) < \xi$ such that $f'(u^n(x, t))$ converges uniformly for all $x \in (x_0 - \xi, x_0 + \xi) - E$. Thus there exists a number N sufficiently large such that

$$|f'(u^n(x, t)) - g(x, t)| < \delta \tag{4.60}$$

for all $n \geqslant N$ and all $x \in (x_0 - \xi, x_0 + \xi) - E$. Now assume that $u^n(x_0, t)$ does not converge. Since $f(u^n(x_0, t))$ converges to $g(x_0, t)$, there exists a subsequence of $u^n(x_0, t)$, denoted by $u^{n_i}(x_0, t)$, such that

$$u^{n_i}(x_0, t) \to u_1 \qquad \text{for} \quad i \text{ odd} \tag{4.61}$$

and

$$u^{n_i}(x_0, t) \to u_2 \qquad \text{for} \quad i \text{ even}. \tag{4.62}$$

Using (4.57), (4.58), (4.59), (4.60), (4.61), and (4.62) we have for $n_i \geqslant N$, $x \in (x_0 - \xi, x_0 + \xi) - E$,

$$|u^{n_i}(x, t) - u_1| < \varepsilon \qquad \text{for} \quad i \text{ odd} \tag{4.63}$$

and

$$|u^{n_i}(x,t)-u_2|<\varepsilon \qquad \text{for} \quad i \text{ even}. \tag{4.64}$$

Hence for $n_i, n_j \geqslant N$, i odd, j even,

$$\begin{aligned}\int_{(x_0-\xi,x_0+\xi)} &|u^{n_i}(x,t)-u^{n_j}(x,t)|\,dx \\ &\geqslant \int_{(x_0-\xi,x_0+\xi)-E} |u^{n_i}(x,t)-u^{n_j}(x,t)|\,dx \\ &\geqslant 2\xi\varepsilon. \end{aligned}\tag{4.65}$$

But from Theorem 2.2 and (4.53) we have

$$\begin{aligned}\int_{x_0-\xi}^{x_0+\xi} |u^{n_i}(x,t)-u^{n_j}(x,t)|\,dx &\leqslant \int_{x_0-\xi-Vt}^{x_0+\xi+Vt} |u_0^{n_i}(x)-u_0^{n_j}(x)|\,dx \\ &\to 0 \qquad \text{as} \quad i,j\to\infty. \end{aligned}\tag{4.66}$$

Thus we have a contradiction. This proves that $u^n(x_0, t)$ converges.

Hence we prove that $u^n(x,t)$ converges almost everywhere to a function $\tilde{u}(x,t)$ and $g(x,t)=f'(\tilde{u}(x,t))$. It is easy to see that $\tilde{u}(x,t)$ is a generalized solution of (1.1) with initial value $u_0(\cdot)$. From Theorem 2.3, $u(x,t)$ and $\tilde{u}(x,t)$ are equal almost everywhere. Hence we prove (i). Parts (ii) and (iii) are easy consequences of the above proof. This completes the proof.

Q.E.D.

From Theorem 4.9, we have the following compactness results. The strictly convex counterpart was obtained by Lax in [12].

THEOREM 4.13. *Let $\{u_0^n\}$ be a sequence of bounded measurable functions with uniform bound M, which converges to u_0 in the sense*

$$\lim_{n\to\infty}\int_a^b u_0^n(x)\,dx=\int_a^b u_0(x)\,dx \tag{4.67}$$

for all $-\infty<a<b<\infty$. Let $u^n(x,t)$ be the generalized solution of (1.1) *with initial data $u_0^n(x)$ and let $u(x,t)$ be the generalized solution of* (1.1) *corresponding to the initial data $u_0(x)$. Then the sequence u^n converges almost everywhere to u.*

Proof. From the results and the proof of Theorem 4.9, we know that Lemmas 4.11 and 4.12 are still valid for general bounded and measurable initial data. Again, using the same arguments as in the proof of

Theorem 4.9, we can find a subsequence $\{u^{n_i}\}$ of $\{u^n\}$ such that u^{n_i} converges to u almost everywhere. If the whole sequence $\{u^n\}$ does not converge to u almost everywhere, then we can find an $\varepsilon > 0$, a set E with positive measure in $(-\infty, \infty) \times (0, \infty)$, and a subsequence $\{u^{n_j}\}$ of $\{u^n\}$, such that

$$|u^{n_j}(x, t) - u(x, t)| \geqslant \varepsilon$$

for all $(x, t) \in E$. But from the subsequence $\{u^{n_j}\}$, using the same arguments as above, we can find another subsequence $\{u^{n'_j}\}$ of $\{u^{n_j}\}$ such that $u^{n'_j}$ converges to u almost everywhere. This is a contradiction. This completes the proof of this theorem. Q.E.D.

5. The Case When $f''(u)$ Vanishes and Changes Sign at Exactly Two Points

5.1. *Construction of Solutions for Piecewise Monotone Initial Data*

We assume that $f''(u)$ has exactly two zero points a_1 and a_2. For definiteness, we assume that

$$\begin{aligned}
\text{(A):}\quad & f''(u) < 0 \qquad \text{for} \quad u > a_2 \quad \text{and} \quad u < a_1,\\
& f''(u) > 0 \qquad \text{for} \quad a_1 < u < a_2,\\
& f(a_1 + \eta) \cong f(a_1) + f'(a_1)\,\eta + K_1 \eta^{k_1}, \qquad |\eta| \text{ small},\\
& f(a_2 + \eta) \cong f(a_2) + f'(a_2)\,\eta - K_2 \eta^{k_2}, \qquad |\eta| \text{ small},
\end{aligned}$$

where K_1 and K_2 are two positive constants and k_1 and k_2 are two odd integers which are greater than three.

In order to consider the most complete case, we also assume that

$$\text{(B):}\quad \lim_{u \to \infty} f(u) = \lim_{u \to -\infty} f(u) = -\infty.$$

We need some properties of f.

Lemma 5.1. *Assume that f satisfies assumptions* (A) *and* (B). *Then there exist b_1 and b_2 such that*

$$f'(b_1) = \frac{f(b_1) - f(b_2)}{b_1 - b_2} = f'(b_2) \tag{5.1}$$

where $b_1 < a_1$ and $a_2 < b_2$. Furthermore, b_1 and b_2 are uniquely determined.

Proof. Let

$$S=\left\{c\leqslant a_1\colon \exists d\geqslant a_2, \text{ such that } f'(c)<\frac{f(c)-f(d)}{c-d}<f'(d)\right\}.$$

Since f is convex on a_1, a_2, it is easy to see that

$$f'(a_1)<\frac{f(a_1)-f(a_2)}{a_1-a_2}<f'(a_2). \tag{5.2}$$

Hence $a_1\in S$. Now let $b_1=\inf S$. From assumption (B), we know that b_1 is finite. Let $\{c_n\}_{n=1}^{\infty}$ be a sequence of S such that

$$\lim_{n\to\infty} c_n=b_1. \tag{5.3}$$

Let $\{d_n\}_{n=1}^{\infty}$, $d_n\geqslant a_2$ for $n=1,2,\ldots$, be a sequence corresponding to the sequence $\{c_n\}_{n=1}^{\infty}$ satisfying

$$f'(c_n)<\frac{f(c_n)-f(d_n)}{c_n-d_n}<f'(d_n), \qquad n=1,2,\ldots. \tag{5.4}$$

From assumption (B), $\{d_n\}_{n=1}$ is a bounded sequence. Hence, by passing to a subsequence if necessary, we can assume that

$$\lim_{n\to\infty} d_n=b_2. \tag{5.5}$$

We shall prove that

$$f'(b_1)=\frac{f(b_1)-f(b_2)}{b_1-b_2}=f'(b_2). \tag{5.6}$$

Assume that

$$f'(b_1)<\frac{f(b_1)-f(b_2)}{b_1-b_2}. \tag{5.7}$$

Then from the assumption (A), there exists a sufficiently small $\varepsilon>0$, such that

$$\begin{aligned} f'(b_1)<f'(b_1-\varepsilon)&<\frac{f(b_1-\varepsilon)-f(b_2)}{b_1-\varepsilon-b_3}\\ &<\frac{f(b_1)-f(b_2)}{b_1-b_2}\leqslant f'(b_2). \end{aligned} \tag{5.8}$$

This contradicts the definition $b_1 = \inf S$. Hence

$$f'(b_1) = \frac{f(b_1) - f(b_2)}{b_1 - b_2}. \tag{5.9}$$

Now assume that

$$\frac{f(b_1) - f(b_2)}{b_1 - b_2} < f'(b_2). \tag{5.10}$$

Then there exists a sufficiently small $\varepsilon < 0$, such that

$$\begin{aligned} f'(b_1) &\leqslant \frac{f(b_1) - f(b_2)}{b_1 - b_2} < \frac{f(b_1) - f(b_2 + \varepsilon)}{b_1 - (b_2 + \varepsilon)} \\ &< f'(b_2 + \varepsilon) < f'(b_2). \end{aligned} \tag{5.11}$$

But in this case there still exists a sufficiently small $\delta > 0$, such that

$$\begin{aligned} f'(b_1) < f'(b_1 - \delta) &< \frac{f(b_1 - \delta) - f(b_2 + \varepsilon)}{(b_1 - \delta) - (b_2 + \varepsilon)} \\ &< \frac{f(b_1) - f(b_2 + \varepsilon)}{b_1 - (b_2 + \varepsilon)} < f'(b_2 + \varepsilon). \end{aligned} \tag{5.12}$$

This means that $(b_1 - \delta) \in S$, which is a contradiction to $b_1 = \inf S$. Thus

$$\frac{f(b_1) - f(b_2)}{b_1 - b_2} = f'(b_2). \tag{5.13}$$

It is not difficult to see that b_1 and b_2 are uniquely determined. This completes the proof of this lemma. Q.E.D.

LEMMA 5.2. *For $b_1 \leqslant u < a_1$, there exists unique u^*, $u^* > a_1$, such that*

$$f'(u) = \frac{f(u) - f(u^*)}{u - u^*} \leqslant f'(u^*). \tag{5.14}$$

Proof. The line passing through $(u, f(u))$ with slope $f'(u)$ must intersect the graph of f at exactly two points $(u^*, f(u^*))$ and $(\bar{u}^*, f(\bar{u}^*))$, where $a_1 < u^* < b_2$ and $\bar{u}^* > b_2$. It is easy to see that

$$\frac{f(u) - f(u^*)}{u - u^*} \leqslant f'(u^*) \qquad \text{and} \qquad \frac{f(u) - f(\bar{u}^*)}{u - \bar{u}^*} > f'(\bar{u}^*).$$

This completes the proof of this lemma. Q.E.D.

LEMMA 5.3. *For $a_2 < u \leqslant b_2$, there exists unique u_*, $u_* < a_2$, such that*

$$f'(u_*) \leqslant \frac{f(u) - f(u_*)}{u - u_*} = f'(u). \tag{5.15}$$

Proof. The proof is similar to that of Lemma 5.2. We omit it. Q.E.D.

LEMMA 5.4. *For $a_1 < u < a_2$, there exist unique u_* and u^* such that $u_* < a_1$ and $u^* > a_2$, and*

$$f'(u_*) > \frac{f(u_*) - f(u)}{u_* - u} = f'(u) = \frac{f(u) - f(u^*)}{u - u^*} > f'(u^*). \tag{5.16}$$

Proof. The proof is also similar to that of Lemma 5.2. We omit it. Q.E.D.

We need some definitions.

DEFINITION 5.5. Let $x(\cdot): (a, b) \to R$ be a shock of a generalized solution $u(x, t)$ of (1.1), that is, $u(x(t)+0, t) \neq u(x(t)-0, t)$ for all $t \in (a, b)$ and $x(\cdot)$ is Lipschitz continuous on (a, b). We call this shock

(i) a type-I shock, if

$$f'(u(x(t)-0, t)) > x'(t) > f'(u(x(t)+0, t))$$

for almost all $t \in (a, b)$.

(ii) a type-II-L shock, if

$$f'(u(x(t)-0, t)) > x'(t) = f'(u(x(t)+0, t))$$

for almost all $t \in (a, b)$ and $f'(u(x(t)-0, t))$ is a monotone decreasing function of t.

(iii) a type-II-R shock, if

$$f'(u(x(t)-0, t)) = x'(t) > f'(u(x(t)+0, t))$$

for almost all $t \in (a, b)$ and $f'(u(x(t)+0, t))$ is a monotone increasing function of t.

(iv) a type-III-L shock, if

$$f'(u(x(t)-0, t)) > x'(t) = f'(u(x(t)+0, t))$$

for almost all $t \in (a, b)$ and $f'(u(x(t)-0, t))$ is a monotone increasing function of t.

(v) a type-III-R shock, if

$$f'(u(x(t)-0,\,t))=x'(t)>f'(u(x(t)+0,\,t))$$

for almost all $t\in(a,b)$ and $f'(u(x(t)+0,t))$ is a monotone decreasing function of t.

(vi) a type-IV shock, if

$$f'(u(x(t)+0,\,t))=x'(t)=f'(u(x(t)-0,\,t))$$

for all $t\in(a,b)$.

We give some examples of these shocks.

EXAMPLE 5.6. Let $u_0(x)=u_l$ for $x<0$ and $u_0(x)=u_r$ for $x>0$, where $b_1<u_r<u_l<b_2$. Then the function u defined by

$$\begin{aligned} u(x,t)&=u_l \qquad \text{if} \quad x<\sigma t,\\ &=u_r \qquad \text{if} \quad x>\sigma t, \end{aligned} \tag{5.17}$$

is a generalized solution of (1.1) with these initial data, where $\sigma=(f(u_l)-f(u_r))/(u_l-u_r)$ and $x(t)=\sigma t$ is a type-I shock.

Proof. It is easy to see that u is a weak solution. From the definition of b_1 and b_2, we easily obtain that

$$\frac{f(u)-f(u_r)}{u-u_r}\leqslant\frac{f(u_l)-f(u_r)}{u_l-u_r}\leqslant\frac{f(u_l)-f(u)}{u_l-u} \tag{5.18}$$

for all $u\in[u_r,u_l]$. Hence u satisfies the entropy condition E and u is a generalized solution of (1.1). Taking the limits $u\to u_l$ and $u\to u_r$ in (5.18), we obtain

$$f'(u_r)\leqslant\frac{f(u_l)-f(u_r)}{u_l-u_r}\leqslant f'(u_l). \tag{5.19}$$

Since $b_1<u_r<u_l<b_2$, we have

$$f'(u_r)<\frac{f(u_l)-f(u_r)}{u_l-u_r}<f'(u_l).$$

Thus $x(t)=\sigma t$ is a type-I shock. Q.E.D.

For type-II-L shock (type-II-R shock is similar), there is already an example in Section 4. So we give a type-III-L shock in the following example.

EXAMPLE 5.7. Let $u_0(\cdot)$ be monotonically increasing for $x<0$, $a_2 \leqslant u_0(x) \leqslant b_2$ for $x<0$, and $u_0(x)=u_r$ for $x>0$, where $b_1 \leqslant u_r \leqslant a_1$ and $(u_r)^* = u_0(0-0)$ (recall the definition for $(u_r)^*$ in Lemma 5.2). We shall briefly describe the construction process of the generalized solution of (1.1) with these initial data.

Let $u_1(x,t)$ be the generalized solution of (1.1) with the initial data $u_0'(x)$,

$$\begin{aligned} u_0'(x) &= u_0(x) && \text{if} \quad x<0, \\ &= a_2 && \text{if} \quad x>0. \end{aligned}$$

Since $u_0'(x) \in [a_2, b_2]$, $u_1(x,t)$ can be obtained from the method in Section 3. We can divide the $x-t$ half plane $t>0$ into three regions D_1, D_2, and D_3,

$$\begin{aligned} D_1 &= \{(x,t)\colon x<\gamma(t), 0<t\}, \\ D_2 &= \{(x,t)\colon \gamma(t)<x<f'(a_2)\, t, 0<t\}, \\ D_3 &= \{(x,t)\colon f'(a_2)\, t \leqslant x, 0<t\}, \end{aligned}$$

where $\gamma(t)$ is a type-I shock. $u_1(\cdot,t)$ is monotonically increasing in $(-\infty, \gamma(t))$, $u_1(x,t)=h_3(x/t)$ for $(x,t)\in D_2$, and $u_1(x,t)=a_2$ for $(x,t)\in D_3$, where h_3 is the inverse function of f' restricted in $[a_2, \infty)$. Now consider $u_2(x,t)$ which is defined in D_1 by

$$(u_2(x,t))^* = u_1(x,t), \qquad (x,t)\in D_1.$$

Let $x=x(t)$ be a Lipschitz continuous curve satisfying

$$\begin{aligned} &\frac{dx(t)}{dt} = f'(u_2(x(t),t)) \qquad \text{for almost all } t\in(0,\infty), \\ &x(0)=0. \end{aligned}$$

It is easy to see that $x(t)$ is a concave Lipschitz continuous curve in D_1 and $dx(t)/dt$ is monotonically decreasing. From every point of $(x(t),t)$, we draw all lines $L_t(v)$ with speed v between $x'(t+0)$ and $x'(t-0)$ in the positive time direction. It is easy to see that all these lines $L_t(v)$ cover a fan-like region H without intersection with each other, where

$$H = \{(x',t')\colon x(t')<x'<f'(u_r)\, t', 0<t'\}.$$

Let $u(x, t)$ be defined by

$$\begin{aligned} u(x', t') &= u_1(x', t') && \text{if } x' < x(t'), \\ &= h_1(v) && \text{if } (x', t') \in L_t(v) \subset H, \\ &= u_r && \text{if } f'(u_r)\, t' < x', \end{aligned} \tag{5.20}$$

where h_1 is the inverse function of f' restricted in $(-\infty, a_1]$. Now the function u defined in (5.20) is a generalized solution of (1.1) and $x = x(t)$ is a type-III-L shock.

EXAMPLE 5.8. Let $u_0(\cdot)$ be a monotonically decreasing function in $(-\infty, \infty)$ with $u_0(0+0) = b_1$ and $u_0(0-0) = b_2$. For every $y < 0$, we draw all lines $L_y(v)$ from $(y, 0)$ with speed v between $f'(u_0(y-0))$ and $f'(u_0(y+0))$. Along the line $L_y(v)$ we assign $u = h_3(v)$. For every $y > 0$, we draw all lines $L_y(v)$ from $(y, 0)$ with speed between $f'(u_0(y-0))$ and $f'(u_0(y+0))$. Along this line $L_y(v)$ we assign $u = h_1(v)$. Here h_1 and h_3 are inverse functions of f' restricted in regions $(-\infty, a_1]$ and (a_2, ∞), respectively. It is easy to see that these lines $L_y(v)$ cover the upper $x-t$ plane except the line $x = \sigma t$, where $\sigma = (f(b_2) - f(b_1))/(b_2 - b_1)$. The function u defined by the above arguments is continuous except on line $x = \sigma t$. It is easy to see that this u is a generalized solution for (1.1) and $x = \sigma t$ is a type-IV shock.

For piecewise monotone and bounded initial data $u_0(\cdot)$, we shall briefly describe the method of construction of the generalized solution for (1.1). For details we refer to Cheng [5]. For convenience we let $a_0 = -\infty$ and $a_3 = \infty$.

Assume that $u_0(\cdot)$ is bounded and piecewise monotone. Furthermore

$$u_0(x) \in (-\infty, a_1] \qquad \text{for} \quad x < 0$$

and

$$u_0(x) \in [a_2, \infty) \qquad \text{for} \quad x > 0. \tag{5.21}$$

First we use the construction method of Section 4 to construct the generalized solution of (1.1) with the initial data

$$\begin{aligned} \tilde{u}_0(x) &= u_0(x) && \text{if } x < 0, \\ &= a_2 && \text{if } x > 0. \end{aligned} \tag{5.22}$$

We obtain a $\tilde{G}(x, t)$ which is Lipschitz continuous and $\partial \tilde{G}(x, t)/\partial x \equiv \tilde{u}(x, t)$ is a generalized solution with initial data (5.22). It is easy to see that $\tilde{u}$ may contain type-II-L shocks and type-I shocks only. Now consider the initial data

$$u_0^R(x) = a_2 \quad \text{if} \quad x < 0, \qquad = u_0(x) \quad \text{if} \quad x > 0. \tag{5.23}$$

We can obtain a Lipschitz continuous $G^R(x, t)$ such that $\partial G^R(x, t)/\partial x \equiv u^R(x, t)$ is a generalized solution of (1.1) with initial data (5.23). Now let

$$\tilde{G}(x, t) = G^R(x, t) \tag{5.24}$$

to determine a curve $x = \gamma(t)$, $\gamma(0) = 0$. Let $G(x, t)$ be

$$\bar{G}(x, t) = \tilde{G}(x, t) \quad \text{if} \quad x \leqslant \gamma(t), \qquad = G^R(x, t) \quad \text{if} \quad x > \gamma(t).$$

Then $\partial \bar{G}(x, t)/\partial x \equiv \bar{u}(x, t)$ is a weak solution of (1.1) in the sense of (1.2). $\bar{u}(x, t)$ is not a generalized solution of (1.1) in general. This is because $x = \gamma(t)$ need not satisfy the entropy condition (E). We can introduce type-II-R shocks to modify $\gamma(t)$. Finally, we can obtain a $G(x, t)$ such that $\partial G(x, t)/\partial x \equiv u(x, t)$ is the generalized solution of (1.1) with the correct initial data $u_0(\cdot)$ of (5.21). We note that the generalized solution $u(x, t)$ may contain only type-I, type-II-R, and type-II-L shocks.

Now consider the following initial data (bounded and piecewise monotone):

$$u_0(x) \in [a_2, \infty) \quad \text{if} \quad x < 0, \qquad u_0(x) \in (-\infty, a_1] \quad \text{if} \quad x > 0. \tag{5.25}$$

First, as in the above case, we use the construction method of Section 4 to construct the generalized solution of (1.1) with the initial data

$$\tilde{u}_0(x) = u_0(x) \quad \text{if} \quad x < 0, \qquad = a_1 \quad \text{if} \quad x > 0. \tag{5.26}$$

We obtain a Lipschitz continuous function $\tilde{G}(x, t)$ such that $\partial G(x, t)/\partial x \equiv \tilde{u}(x, t)$ is a generalized solution of (1.1) with initial data (5.26). Now consider the initial data

$$u_0^R(x) = a_1 \quad \text{if} \quad x < 0, \qquad = u_0(x) \quad \text{if} \quad x > 0. \tag{5.27}$$

Using the method in Section 3, we can obtain a $G^R(x, t)$ such that

$\partial G^R(x, t)/\partial x \equiv u^R(x, t)$ is a generalized solution of (1.1) with initial data (5.27). Now we let

$$\tilde{G}(x, t) = G^R(x, t)$$

to determine a curve $x = \gamma(t)$ and to modify it if it does not satisfy the entropy condition (E). Finally, we obtain a $G(x, t)$ such that $\partial G(x, t)/\partial x \equiv u(x, t)$ is a generalized solution of (1.1) with initial data (5.25). We have to stress here that $u(x, t)$ may contain type-III shocks now.

For more general bounded and piecewise monotone initial data, we refer to Cheng [5] for details of construction of generalized solution.

5.2. *Properties of Solutions for Piecewise Monotone Initial Data*

From the construction of generalized solutions for (1.1) we have

THEOREM 5.9. *Let $f(\cdot)$ satisfy the assumptions* (A) *and* (B) *and $u_0(\cdot)$ be piecewise monotone. Let $u(x, t)$ be the generalized solution of* (1.1) *with these initial data. Then*

(i) *Regions* $\{(x, t): u(x, t) \in (-\infty, a_1]\}$, $\{(x, t): u(x, t) \in [a_1, a_2]\}$, *and* $\{(x, t): u(x, t) \in [a_2, \infty)\}$ *are separated by Lipschitz continuous curves which are genuine characteristics, type-*I *shocks, type-*II-*L shocks, type-*II-*R shocks, type-*III-*L shocks, type-*III-*R shocks, or type-*IV *shocks.*

(ii) *$u(\cdot, t)$ is piecewise monotone, especially $u(x \pm 0, t) = u_\pm$ exists for all $t > 0$, for $x \in (-\infty, \infty)$, and*

$$f'(u_-) \geqslant \frac{f(u_+) - f(u_-)}{u_+ - u_-} \geqslant f'(u_+),$$

$$\frac{f(u) - f(u_-)}{u - u_-} \geqslant \frac{f(u_+) - f(u_-)}{u_+ - u_-}$$

for all u's between u and u_+.

(iii) *$u(x, t)$ is continuous except on the union of an at most countable set of Lipschitz continuous curves (shocks).*

Proof. These are easy consequences of the construction method. We omit the details. Q.E.D.

THEOREM 5.10. *Let $u(x, t)$ be the solution in Theorem* 5.9. *Then*

(i) *If $x(\cdot): (a, b) \to R$ is a type-*II-*R shock, then $x'(\cdot)$ is Lipschitz continuous, $x''(t) \geqslant 0$ for almost all $t \in (a, b)$, and*

$$u(x(t) + 0, t) = (u(x(t) - 0, t))_*$$

or

$$u(x(t)+0,\, t)=(u(x(t)-0,\, t))^*.$$

(ii) *If* $x(\cdot)\colon (a, b)\to R$ *is a type*-II-L *shock, then* $x'(\cdot)$ *is Lipschitz continuous,* $x''(t)\leqslant 0$ *for almost all* $t\in(a, b)$*, and*

$$u(x(t)-0,\, t)=(u(x(t)+0,\, t))_*$$

or

$$u(x(t)-0,\, t)=(u(x(t)+0,\, t))^*.$$

(iii) *If* $x(\cdot)\colon (a, b)\to R$ *is a type*-III-L *shock, then* $x(\cdot)$ *is Lipschitz continuous,* $x'(t)$ *is a decreasing function of* t*. Furthermore, for fixed* $t_0\in(a, b)$,

$$x'(t_0+0)=\lim_{t\to t_0+} f'(u(x(t)+0,\, t)),$$

$$x'(t_0-0)=\lim_{t\to t_0-} f'(u(x(t)+0,\, t)),$$

$$b_2\geqslant u(x(t_0)-0,\, t_0)>a_2,$$

$$a_1>u(x(t_0)+0,\, t_0)\geqslant b_1,$$

$$\lim_{t\to t_0+} u(x(t)-0,\, t)=(\lim_{t\to t_0+} u(x(t)+0,\, t))^*,$$

and

$$\lim_{t\to t_0-} u(x(t)-0,\, t)=(\lim_{t\to t_0-} u(x(t)+0,\, t))^*.$$

(iv) *If* $x(\cdot)\colon (a, b)\to R$ *is a type*-III-R *shock, then* $x(\cdot)$ *is Lipschitz continuous,* $x'(t)$ *is an increasing function of* t*. Furthermore, for fixed* $t_0\in(a, b)$,

$$x'(t_0+0)=\lim_{t\to t_0+} f'(u(x(t)-0,\, t)),$$

$$x'(t_0-0)=\lim_{t\to t_0-} f'(u(x(t)-0,\, t)),$$

$$b_1\leqslant u(x(t_0)+0,\, t_0)\leqslant a_1,$$

$$a_2\leqslant u(x(t_0)-0,\, t_0)\leqslant b_2,$$

$$\lim_{t\to t_0+} u(x(t)+0,\, t)=(\lim_{t\to t_0+} u(x(t)-0,\, t))_*,$$

and

$$\lim_{t \to t_0-} u(x(t)+0, t) = (\lim_{t \to t_0-} u(x(t)-0, t))_*.$$

Proof. (i) From the definition of type-II-R shock, we have

$$f'(u(x(t)-0, t)) = x'(t) > f'(u(x(t)+0, t)) \tag{5.28}$$

for almost all $t \in (a, b)$ and $f'(u(x(t)+0, t))$ is monotone increasing. From conditions (R–H) and (E) of Section 1, we have

$$x'(t) = \frac{f(u(x(t)+0, t)) - f(u(x(t)-0, t))}{u(x(t)+0, t) - u(x(t)-0, t)}, \tag{5.29}$$

$$\frac{f(u(x(t)+0, t)) - f(u(x(t)-0, t))}{u(x(t)+0, t) - u(x(t)-0, t)} \leqslant \frac{f(u) - f(u(x(t)-0, t))}{u - u(x(t)-0, t)} \tag{5.30}$$

for all u's between $u(x(t)+0, t)$ and $u(x(t)-0, t)$. From (5.28) and (5.29), we have

$$u(x(t)+0, t) = (u(x(t)-0, t))_* \tag{5.31}$$

if $u(x(t)+0, t) \in [a_1, a_2]$ and

$$u(x(t)+0, t) = (u(x(t)-0, t))^* \tag{5.32}$$

if $u(x(t)+0, t) \in [a_2, \infty)$. Furthermore, since $f'(u(x(t)+0, t))$ is monotone increasing, $u(x(t)+0, t)$ is Lipschitz continuous. From (5.31) and (5.32), we can regard $u(x(t)-0, t)$ as a differentiable function of $u(x(t)+0, t)$. Hence from (5.29), $x'(t)$ is a Lipschitz continuous function and

$$x''(t) = \frac{d}{d(u(x(t)+0, t))}\left[\frac{f(u(x(t)+0, t)) - f(u(x(t)-0, t))}{u(x(t)+0, t) - u(x(t)-0, t)}\right] \cdot \frac{du(x(t)+0, t)}{dt}.$$

For either case (5.31) or (5.32), it is easy to see that $x''(t) \geqslant 0$ and $x''(t) = 0$ only if $du(x(t)+0, t)/dt = 0$. This proves (i). Case (ii) can be similarly proved.

Cases (iii) and (iv) can also be similarly proved. The only difference is that $u(x(t)-0, t)$ can be discontinuous in case (iii) and $u(x(t)+0, t)$ can be discontinuous in case (iv). See also Example 5.7 to get a feeling of type-III shocks. This completes the proof. Q.E.D.

THEOREM 5.11. *Let $u(x, t)$ be the generalized solution in Theorem 5.9. If $f'(u(\cdot, t_0))$ is monotonically decreasing or is a constant in the interval (a, b), then*

$$L_1 = \{(x, t): x = a + f'(u(a+0, t_0))(t - t_0), 0 < t < t_0\}$$

and

$$L_2 = \{(x, t): x = b + f'(u(b-0, t_0))(t - t_0), 0 < t < t_0\}$$

are two genuine characteristics, that is,

$$u(x+0, t) = u(x-0, t) = u(a+0, t_0) \qquad \text{for all } (x, t) \in L_1$$

and

$$u(x+0, t) = u(x-0, t) = u(b-0, t_0) \qquad \text{for all } (x, t) \in L_2.$$

Proof. From our construction of solution $u(x, t)$, each type-II or type-III shock generates a fan with monotonically increasing $f'(u(\cdot, t))$. Hence the backward genuine characteristics from $(a+0, t_0)$ and $(b-0, t_0)$ cannot terminate at a type-II or type-III shock. Obviously they also cannot terminate to a type-I or type-IV shock (entropy condition (E)). Hence they must extend to $t = 0$. This completes the proof. Q.E.D.

THEOREM 5.12. *Let $u(x, t)$ be the generalized solution in Theorem 5.9. Let (x_0, t_0) be a point with $t_0 > 0$. If $u(x_0 - 0, t_0) = b_2$ and $u(x_0 + 0, t_0) = b_1$, then $u(x-0, t) = b_2$ and $u(x+0, t) = b_1$ for all $(x, t) \in L$, where*

$$L = \{(x, t): x = x_0 + f'(b_1)(t - t_0), 0 < t < t_0\}.$$

Proof. It is obvious that the backward genuine characteristics from $(x_0 - 0, t_0)$ and $(x_0 + 0, t_0)$ are the line L. This line L is a type-IV shock which cannot terminate to any type-II or type-III shock. This completes the proof. Q.E.D.

THEOREM 5.13. *Let $u(x, t)$ be the generalized solution in Theorem 5.9. If $f'(u(\cdot, t_0))$ is increasing in the interval (x_1, x_2), then $f'(u(\cdot, t))$ is continuous in (x_1, x_2), where $t_0 > 0$. Furthermore, if there is no type-IV shock passing through the line segment $\{(x, t_0): x_1 < x < x_2\}$, then $u(\cdot, t_0)$ is also continuous in (x_1, x_2) and*

$$\{u(x, t_0): x_1 < x < x_2\} \subset (-\infty, a_1] \text{ or } [a_1, a_2] \text{ or } [a_2, \infty). \tag{5.33}$$

Proof. Assume that $f'(u(\cdot, t_0)$ has a jump discontinuity at $x_0 \in (x_1, x_2)$, then since $f'(u(\cdot, t_0))$ is increasing in (x_1, x_2), we have $f'(u(x_0 - 0, t_0)) <$

$f'(u(x_0+0, t_0))$, which contradicts Theorem 5.9(ii). Hence $f'(u(\cdot, t_0))$ is continuous in (x_1, x_2). Now assume that $u(\cdot, t_0)$ is discontinuous at $x_0 \in (x_1, x_2)$, then $f'(u(x_0-0, t_0)) = f'(u(x_0+0, t_0))$. But in this case, $f'(u(x_0-0, t_0)) = (f(u(x_0+0, t_0)) - f(u(x_0-0, t_0))/(u(x_0+0, t_0) - u(x_0-0, t_0))$. Hence from Lemma 5.1, entropy condition (E), and the assumptions (A) and (B) of f, we have $u(x_0-0, t_0) = b_2$ and $u(x_0+0, t_0) = b_1$. From Theorem 5.12, there is a type-IV shock passing through (x_0, t_0). Thus if there is no type-IV shock passing through the segment (x_1, x_2), then $u(\cdot, t_0)$ is also continuous on (x_1, x_2). Finally, (5.33) is an easy consequence of the continuity of $u(\cdot, t_0)$ and the monotonicity of $f'(u(\cdot, t_0))$. This completes the proof. Q.E.D.

THEOREM 5.14. *Let $u(x, t)$ be the generalized solution in Theorem 5.9. Let (x_0, T) be a point in $R \times R^+$ with $T > 0$. Then there exists a backward generalized characteristic from (x_0, T) which is the union of a finite number of genuine characteristics, that is, there exist a nonnegative integer n, times $t_0, t_1, \ldots, t_n$, $0 \leqslant t_0 \leqslant t_1 < \cdots < t_n \leqslant T$, $u_0, u_1, \ldots, u_n$, and $x_0(t), x_1(t), \ldots, x_n(t)$, such that*

(a) $x_n(t) = x_0 + f'(u_n)(t-T)$ *for* $t_n \leqslant t \leqslant T$,
$x_k(t) = x_{k+1}(t_{k+1}) + f'(u_k)(t - t_{k+1})$ *for* $t_k \leqslant t \leqslant t_{k+1}$,
$k = 0, 1, \ldots, (n-1)$.

(b) $u(x_k(t) \pm 0, t) = u_k$ *for* $t_k < t < t_{k+1}$, $k = 0, 1, \ldots, n$.

(c) $u_n = u(x_0+0, T)$ *or* $u_n = u(x_0-0, T)$ *and* $u_k = (u_{k+1})^*$ *or* $u_k = (u_{k+1})_*$, $k = 0, 1, 2, \ldots, (n-1)$.

(d) *There is a type*-II-*R of type*-II-*L shock $z_k(t)$ passing through $(x_k(t_k), t_k)$ for each $k = 1, 2, \ldots, n$, with shock speed*

$$z_k'(t_k) = f'(u_k).$$

(e) *If $u_0 \in (a_1, a_2)$, then $t_0 = 0$.*

(f) *If $u_0 \in (-\infty, a_1)$, then t_0 can be greater than zero. In that case, either there is a type*-III-*L shock passing through $(x_0(t_0), t_0)$ or $(x_0(t_0), t_0)$ is an interaction point of two shocks.*

(g) *If $u_0 \in (a_2, \infty)$, then t_0 can be greater than zero. In that case, either there is a type*-III-*R shock passing through $(x_0(t_0), t_0)$ or $(x_0(t_0), t_0)$ is an interaction point of two shocks.*

Proof. If $u(x_0+0, t) = u(x_0-0, T)$, we draw a genuine backward characteristic from (x_0, T) with speed $f'(u(x_0+0, t))$. This backward characteristic can be extended to $t = 0$ or must terminate to a type-II or a type-III shock or terminate at points of shock interactions. If it extends to $t = 0$, then we are done. If it terminates at a type-III shock or at points of shock interactions, then we are also done. If it terminates to a type-II shock, we can again draw another backward genuine characteristic from the point of termination. Continuing this process, we can find all these genuine characteristics. Now if $u_0 \in (a_1, a_2)$, $t_0 > 0$, then there must be a type-II shock passing through $(x_0(t_0), t_0)$. We can draw still another backward genuine characteristic. Hence $t_0 = 0$. The only mechanism to generate new genuine characteristics in the positive time direction besides type-II and type-III shocks is the interactions of different types of shocks. Thus (f) and (g) are proved. If $u(x_0+0, t) \neq u(x_0-0, t)$, then we may draw two backward genuine characteristics from (x_0, T) with speed $f'(u(x_0+0, T))$ and $f'(u(x_0-0, T))$. The conclusion is the same. This completes the proof. Q.E.D.

We call this generalized backward characteristic $C_b(x_0, T)$. In general, $C_b(x_0, T)$ is not uniquely determined by (x_0, T). $C_b(x_0, T)$ and $C_b(y_0, T)$ cannot intersect each other except at end point, if $x_0 \neq y_0$.

5.3. *Estimates of Total Variation of $f'(u(\cdot, T))$*

Now assume that $u_0(\cdot)$ is piecewise monotone and $|u_0(x)| \leqslant M$ for almost all $x \in (-\infty, \infty)$. Let $u(x, t)$ be the generalized solution of (1.1) corresponding to these initial data. We still have the fundamental theorem.

THEOREM 5.15. *There exist constants C_1 and C_2 depending only on f and M, such that for all $-\infty < a < b < \infty$ and $T > 0$,*

$$V^+(f'(u(\cdot, T)); [a, b]) \leqslant \frac{C_1(b-a)}{T} + C_2. \tag{5.34}$$

To prove this theorem we need some lemmas. We wish to point out that Lemma 4.6 is still valid for type-II-L and type-II-R shocks. Lemma 4.7 is also valid except that δ depends on k_1 and k_2 of assumption (A). Lemma 4.8 has to be modified.

LEMMA 5.16. *Assume that $f'(u(\cdot, T))$ is increasing in the interval $[c, d]$. Let $C_b(c, T)$ and $C_b(d, T)$ be two generalized backward characteristics from $(c+0, T)$ and $(d-0, T)$, respectively. Assume that $(d-c)$ is sufficiently small so that $C_b(c, T)$ consists of $x_0(t), \ldots, x_n(t)$ and $C_b(d, T)$ consists of $y_0(t), y_1(t), \ldots, y_n(t)$, where $x_i(t)$ is defined between $T_i \leqslant t \leqslant T_{i+1}$ with speed $f'(u_i)$ and $y_i(t)$ is defined between $t_i \leqslant t \leqslant t_{i+1}$ with speed $f'(v_i)$, $i = 0, 1, 2, \ldots, n$. Assume that $t_0 = T_0 = 0$. Furthermore assume that $(d-c)$ is suf-*

*ficiently small such that there is a type-*II-*L shock or type-*II-*R shock* $z_R(t)$ *passing through points* $(y_k(t_k), t_k)$ *and* $(x_k(T_k), T_k)$ *and*

$$|t_k - T_k| \leqslant \delta, \qquad k = 1, 2, \ldots, n,$$

$$T_k \geqslant t_{k-1}, \qquad t_k \geqslant T_{k-1}, \qquad k = 1, 2, \ldots, n,$$

where δ *is the constant in Lemma* 4.6 *which is still valid in this* f. *Then there exists a constant* C *depending only on* f *and* M, *such that*

$$0 \leqslant f'(v_n) - f'(u_n) \leqslant \frac{C}{T^2} A(c, d; T), \tag{5.35}$$

where $A(c, d; T)$ *is the area of region bounded by* $C_b(c, T)$ *and* $C_b(d, T)$, $t = 0$ *and* $t = T$.

Proof. The proof is similar to the proof of Lemma 4.7 or the proof of the following Lemma, Lemma 5.17. Since we will give a complete proof for Lemma 5.17, we just omit the proof of this one. Q.E.D.

LEMMA 5.17. *The same assumptions as in Lemma* 5.16 *except that* $t_0 > 0$ *and* $T_0 > 0$. *Assume that* $(d - c)$ *is sufficiently small such that there is a type-*III-*L shock or a type-*III-*R shock passing through* $(x_0(T_0), T_0)$ *and* $(y_0(t_0), t_0)$. *We extend* $C_b(c, T)$ *and* $C_b(y, T)$ *to* $t = 0$ *by lines*

$$x_{-1}(t) = x_0(T_0) + f'(u_{-1})(t - T_0), \qquad 0 < t < T_0,$$

$$y_{-1}(t) = y_0(t_0) + f'(v_{-1})(t - t_0), \qquad 0 < t < t_0,$$

where

$$u_{-1} = (u_0)^* \qquad or \qquad u_{-1} = (u_0)_*$$

$$v_{-1} = (v_0)^* \qquad or \qquad v_{-1} = (v_0)_*.$$

Then the conclusion of Lemma 5.16 *in* (5.35) *is still valid except now* $A(c, d; T)$ *is the area of the region bounded by* $C_b(c, T)$, $C_b(d, T)$, $x_{-1}(t)$, $y_{-1}(t)$, $t = 0$ *and* $t = T$.

Proof. From the properties of type-III shock, Theorem 5.10, it is easy to see that $f'(u_{-1}) > f'(v_{-1})$. It is to be noted that $x_{-1}(t)$ and $y_{-1}(t)$ need not be true backward genuine characteristics from points $(x_0(T_0), T_0)$ and $(y_0(t_0), t_0)$. But $A(c, d; T)$ and $A(c', d'; T)$ do not have any overlapping if (c, d) and (c', d') do not overlap. Now if $z_k(t)$ is a type-II-L shock, then $T_k > t_k$. We extend $x_k(t)$ backward to intersect $y_k(t)$ at time t_k'. If $z_k(t)$ is a type-II-R shock, then $t_k > T_k$. We extend $y_k(t)$ backward to intersect $x_k(t)$ at time T'_k. It is easy to see that

$$\Delta_n + \Delta_{n-1} + \cdots + \Delta_0 + \Delta_{-1} \leqslant A(c, d; T), \tag{5.36}$$

where Δ_n is the area of triangle bounded by the three lines

$$\{(x, t): x = c + f'(u_n)(t - T), t \leqslant T\},$$
$$\{(x, t): x = d + f'(v_n)(t - T), t \leqslant T\},$$
$$\{(x, t):\ t = T, c \leqslant x \leqslant d\},$$

Δ_k, $k = 1, \ldots, (n-1)$, is the area of triangle bounded by the three lines

$$\{(x, t): x = x_{k+1}(T_{k+1}) + f'(u_k)(t - T_{k+1}), t \leqslant T_{k+1}\},$$
$$\{(x, t): x = y_{k+1}(t_{k+1}) + f'(v_k)(t - t_{k+1}), t \leqslant t_{k+1}\},$$
$$\{(x, t):\ t = \min\{t_{k+1}, T_{k+1}\}\},$$

Δ_0 is the area of region bounded by the lines

$$\{(x, t): x = x_1(T_1) + f'(u_0)(t - T_1), t \leqslant T_1\},$$
$$\{(x, t): x = y_1(t_1) + f'(v_0)(t - t_1), t \leqslant t_1\},$$
$$\{(x, t):\ t = \min\{t_1, T_1\}\},$$
$$\{(x, t):\ t = \max\{t_0, T_0\}\},$$

and Δ_{-1} is the area of triangle bounded by the lines

$$\{(x, t): x = s_0 + f'(u_{-1})(t - \tau_0), t \leqslant \tau_0\},$$
$$\{(x, t): x = s_0 + f'(v_{-1})(t - \tau_0), t \leqslant \tau_0\},$$
$$\{(x, t):\ t = 0\}.$$

Here

$$\tau_0 = \max\{T_0, t_0\}$$

and

$$s_0 = x_0(\tau_0) \qquad \text{or} \qquad y_0(\tau_0).$$

Thus we have

$$\Delta_n = \tfrac{1}{2}(f'(v_n) - f'(u_n))(T - t'_n)^2 \tag{5.37}$$

or

$$\Delta_n = \tfrac{1}{2}(f'(v_n) - f'(u_n))(T - T'_n)^2, \tag{5.37)'}$$

$$\Delta_k = \tfrac{1}{2}(f'(v_k) - f'(u_k))[\min\{T_{k+1}, t_{k+1}\} - t'_k]^2 \tag{5.38}$$

or

$$\Delta_k = \tfrac{1}{2}(f'(v_k) - f'(u_k))[\min\{T_{k+1}, t_{k+1}\} - T'_k]^2, \qquad k = 1, 2, ..., (n-1), \tag{5.38)'}$$

$$\Delta_0 \geqslant \tfrac{1}{2}(f'(v_0) - f(u_0))[\min\{T_1, t_1\} - \tau_0]^2 \tag{5.39}$$

and

$$\Delta_{-1} = \tfrac{1}{2}(f'(u_{-1}) - f(v_{-1}))\,\tau_0^2. \tag{5.40}$$

Let

$$[\min\{T_{k+1}, t_{k+1}\} - t'_k] \qquad \text{or} \qquad [\min\{T_{k+1}, t_{k+1}\} - T'_k] = \tau_{k+1}, \qquad k = 2, ..., (n-1),$$

$$\tau_{n+1} = (T - t'_n) \qquad \text{or} \qquad (T - T'_n)$$

and

$$\tau_1 = \min\{T_1, t_1\} - \tau_0.$$

Let

$$\lambda_n = \frac{f'(v_{n-1}) - f'(u_{n-1})}{f'(u_n) - f'(u_n)},$$

$$\lambda_{n-1} = \frac{f'(v_{n-2}) - f'(u_{n-2})}{f'(v_{n-1}) - f'(u_{n-1})}, ...$$

$$\lambda_1 = \frac{f(v_0) - f'(u_0)}{f'(u_1) - f'(u_1)}$$

and

$$\lambda_0 = \frac{f'(u_{-1}) - f'(v_{-1})}{f'(v_0) - f'(u_0)}. \tag{5.41}$$

Then from the assumption $|t_k - T_k| \leqslant \delta$ and Lemma 4.6, we have

$$\tau_0 + \tau_1 + \cdots + \tau_{n+1} \geqslant \tfrac{1}{4}T.$$

Furthermore, from Lemma 4.7, we have

$$\lim_{n\to\infty} \lambda_n = (1+\delta_1)^{k_1-1} \qquad \text{or} \qquad (1+\delta_2)^{k_2-1}. \tag{5.42}$$

From (5.36)–(5.41), we have

$$0 \leqslant f'(v_n) - f'(u_n) \leqslant \frac{2A(c, d; T)}{B(\lambda_0, \lambda_1, ..., \lambda_n; \tau_0, \tau_1, ..., \tau_{n+1})} \tag{5.43}$$

where

$$\begin{aligned} &B(\lambda_0, \lambda_1, ..., \lambda_n; \tau_0, \tau_1, ..., \tau_{n+1}) \\ &\quad = \tau_{n+1}^2 + \lambda_n \tau_n^2 + \cdots + \lambda_n \lambda_{n-1} \cdots \lambda_0 \tau_0^2 \\ &\quad \geqslant \left(\frac{T}{4}\right)^2 \left[1 + \frac{1}{\lambda_n} + \cdots + \frac{1}{\lambda_n \lambda_{n-1} \cdots \lambda_0}\right]^{-1}. \end{aligned} \tag{5.44}$$

From (5.42), the sum in (5.44) is bounded for all n. Now take

$$C = 32 \max_{\substack{|u_0| \leqslant M \\ |v_0| \leqslant M}} \max_n \left[1 + \frac{1}{\lambda_n} + \cdots + \frac{1}{\lambda_n \lambda_{n-1} \cdots \lambda_0}\right]. \tag{5.45}$$

We have from (5.43), (5.44), and (5.45)

$$0 \leqslant f'(v_n) - f'(u_n) \leqslant \frac{C}{T^2} A(c, d; T).$$

This completes the proof of this lemma. Q.E.D.

Remark. If $t_0 = T_0 > 0$, $x_0(T_0) = y_0(t_0)$, and there is no type-III shock passing through $(x_0(T_0), T_0)$, from the construction method, we know that $(x_0(T_0), T_0)$ is a point of interaction of two shocks. In this case, if u_{-1} and v_{-1} are both belonging to $[a_2, \infty)$ or $(-\infty, a_1]$ (actually, $[a_2, b_2]$ or $[b_1, a_1]$), then we can still extend $C_b(c, T)$ and $C_b(d, T)$ by $x_{-1}(t)$ and $y_{-1}(t)$. It is easy to see that the above lemma still holds. The point $(x_0(T_0), T_0)$ can be regarded as a limiting type-III shock.

LEMMA 5.18. *Let $u(c \pm 0, T) = u_1$, $u(d \pm 0, T) = u_2$, $d > c$, and $C_b(c, T)$ and $C_b(d, T)$ are two backward genuine characteristics, that is,*

$$C_b(c, T) = \{(x, t): x = c + f'(u_1)(t - T), 0 < t < T\},$$
$$C_b(d, T) = \{(x, t): x = d + f'(u_2)(t - T), 0 < t < T\}.$$

Assume that $u_1 \in [b_1, a_1]$ and $u_2 \in [a_2, b_2]$; then there exists a constant β depending only on f and M, such that

$$(d-c) \geqslant \beta T.$$

Proof. From the assumptions, it is easy to see that

$$c_0 \equiv c - f'(u_1)\, T \leqslant d - f'(u_2)\, T \equiv d_0. \tag{5.46}$$

Let

$$\begin{aligned} \tilde{u}(x, t) &= u_1 && \text{if} \quad x \leqslant c_0 + f'(u_1)\, t, \\ &= u(x, t) && \text{if} \quad c_0 + f'(u_1)\, t < x < d_0 + f'(u_2)\, t, \\ &= u_2 && \text{if} \quad d_0 + f'(u_2)\, t < x. \end{aligned}$$

Then it is easy to see that $\tilde{u}(x, t)$ is a generalized solution of (1.1) in the strip π_T. Now assume that $0 \in (a_1, a_2)$. Consider the generalized solution $u_m(x, t)$ corresponding to the initial data $u_m(x, 0)$,

$$\begin{aligned} u_m(x, 0) &= u_1 && \text{if} \quad x < c_0, \\ &= -M \equiv m && \text{if} \quad c_0 < x < d_0, \\ &= u_2 && \text{if} \quad d_0 < x. \end{aligned}$$

This solution $u_m(x, t)$ can be easily constructed. We just write down explicitly $u_m(x, t)$. For details of construction, see Cheng [5] and Ballou [1].

We have for $0 < t \leqslant t_1$

$$\begin{aligned} u_m(x, t) &= u_1 && \text{if} \quad x \leqslant x_1(t), \\ &= h_1((x - c_0)/t) && \text{if} \quad x_1(t) < x < x_2(t), \\ &= m && \text{if} \quad x_2(t) \leqslant x < x_3(t), \\ &= h_2((x - d_0)/t) && \text{if} \quad x_3(t) < x < x_4(t), \\ &= u_2 && \text{if} \quad x_4(t) < x, \end{aligned}$$

where

$$\begin{aligned} x_1(t) &= c_0 + f'(u_1)\, t, \\ x_2(t) &= c_0 + f'(m)\, t, \\ x_3(t) &= d_0 + \frac{f(\tilde{m}) - f(m)}{\tilde{m} - m}\, t \qquad ((\tilde{m})_* = m), \\ x_4(t) &= d_0 + \frac{f(\tilde{u}_2) - f(u_2)}{\tilde{u}_2 - u_2}\, t \qquad ((\tilde{u}_2)^* = u_2), \end{aligned}$$

and t_1 is the time when $x_2(t)$ and $x_3(t)$ meet, that is,

$$c_0 + f'(m)\, t_1 = d_0 + \frac{f(\tilde{m}) - f(m)}{\tilde{m} - m}\, t_1.$$

For $t_1 \leqslant t < t_2$,

$$\begin{aligned} u_m(x, t) &= u_1 && \text{if } x \leqslant x_1(t), \\ &= h_1((x - c_0)/t) && \text{if } x_1(t) < x < x_5(t), \\ &= F(x, t) && \text{if } x_5(t) < x < x_3(t), \\ &= h_2((x - d_0)/t) && \text{if } x_3(t) < x < x_4(t), \\ &= u_2 && \text{if } x_4(t) < x, \end{aligned}$$

where $x_5(t)$ is a type-II-L shock satisfying

$$\frac{dx_5(t)}{dt} = f'\left(\tilde{h}_1\left(\frac{x_5(t) - c_0}{t}\right)\right), \qquad t_1 \leqslant t,$$

$$x_5(t_1) = x_2(t_1) = x_3(t_1)$$

and

$$\left(\tilde{h}_1\left(\frac{x - c_0}{t}\right)\right)_* = h_1\left(\frac{x - c_0}{t}\right),$$

$$x_5(t_2) = x_1(t_2).$$

For $t_2 \leqslant t$,

$$\begin{aligned} u_m(x, t) &= u_1 && \text{if } x < x_6(t), \\ &= F(x, t) && \text{if } x_6(t) < x < x_3(t), \\ &= h_2((x - d_0)/t) && \text{if } x_3(t) < x < x_4(t), \\ &= u_2 && \text{if } x_4(t) < x, \end{aligned}$$

where

$$x_6(t) = x_1(t_2) + f'(\tilde{u}_1)(t - t_2), \qquad (\tilde{u}_1)_* = u_1.$$

Now comparing the initial data of the two generalized solutions $u_m(x, t)$ and $\tilde{u}(x, t)$, we have

$$u_m(x, 0) \leqslant \tilde{u}(x, 0)$$

for all $x \in (-\infty, \infty)$. Hence from the ordering principle, Theorem 2.4, we have for all $(x, t) \in \pi_T$

$$u_m(x, t) \leqslant \tilde{u}(x, t). \tag{5.47}$$

Hence from the explicit solution $u_m(x, t)$, and (5.47), we have

$$t_2 > T. \tag{5.48}$$

Now from our construction of solution $u_m(x, t)$, or from the invariance property of (1.1) under similarity transformation $x \to \alpha x$, $t \to \alpha t$, there is a constant β depending only on f and M, such that

$$\frac{d_0 - c_0}{t_2} = \beta > 0. \tag{5.49}$$

From our assumptions of f, we also have

$$f'(u_1) \leqslant f'(b_1) = f'(b_2) \leqslant f'(u_2). \tag{5.50}$$

Combining (5.46), (5.48), (5.49), and (5.50), we obtain

$$\begin{aligned}(d - c) &= (d_0 - c_0) + (f'(u_2) - f'(u_1))\, T \\ &= \beta t_2 + (f'(u_2) - f'(u_1))\, T \\ &\geqslant \beta T.\end{aligned}$$

This completes the proof of Lemma 5.18. Q.E.D.

LEMMA 5.19. *Let (c, t) be an interaction point of shocks $\{u_l, u_m\}$ and $\{u_m, u_r\}$ and (d, T) be also a interaction point of shocks $\{\bar{u}_l, \bar{u}_m\}$ and $\{\bar{u}_m, \bar{u}_r\}$, where $c < d$ and $t > 0$. Assume that*

(i) $u_l, \bar{u}_l \in (\tilde{b}_2, a_2]$,
(ii) $u_m, \bar{u}_m \in (a_2, b_2]$,
(iii) $u_r, \bar{u}_r \in [b_1, a_1)$,

or

(i)′ $u_l, \bar{u}_l \in (a_2, b_2]$,
(ii)′ $u_m, \bar{u}_m \in (b_2, a_1]$,
(iii)′ $u_r, \bar{u}_r \in (a_1, \tilde{b}_1]$,

where $(\tilde{b}_1)_* = b_1$ and $(\tilde{b}_2)^* = b_2$. Then there exists a constant β depending only on f and M, such that

$$(d - c) \geqslant \beta T.$$

Remark. We say that shock $\{u_l, u_m\}$ interacts shock $\{u_m, u_r\}$ at (c, T). We mean that there are two shocks $x(t)$, $y(t)$ satisfying

(i) $x(T)=y(T)=c$,

(ii) $x(t)<y(t)$ for $t<T$,

(iii) $\lim_{t\to T-0} u(x(t)-0, t)=u_l$,

$$\lim_{t\to T-0} u(x(t)+0, t)=\lim_{t\to T-0} u(y(t)-0, t)=u_m,$$

$$\lim_{t\to T-0} u(y(t)+0, t)=u_r.$$

Proof of Lemma 5.19. We only consider the first case. The second case can be similarly treated. From our construction, we can draw a genuine characteristic L_d from d with speed $f'(\bar{u}_m)$, that is,

$$L_d=\{(x, t): x=d+f'(\bar{u}_m)(t-T), 0<t<T\} \tag{5.51}$$

is a generalized backward characteristic from (d, T). Let

$$A=\{x_0: u(x, T)\in(-\infty, a_1] \text{ for all } x\in(c, x_0)\}. \tag{5.52}$$

Set $\sup A=e$. From our construction method, we can draw a genuine characteristic L_e from (e, T) with speed $f'(u(e-0, T))$,

$$L_e=\{(x, t): x=e+f'(u(e-0, T))(t-T), 0<t<T\}.$$

If $u(e-0, T)\in[b_1, a_1)$, then from Lemma 5.18, we have

$$(d-c)\geqslant(d-e)\geqslant\beta T. \tag{5.53}$$

If $u(e-0, T)\notin[b_1, a_1)$, then $u(e-0, T)<b_1$. From our construction method, we can find a point (e', T), $c<e'<e$, such that $u(e'+0, T)=b_1$ or $u(e'-0, T)=b_1$. It is easy to see that we can draw a genuine backward characteristic $L_{e'}$ from (e', T) with speed $f'(b_1)$. That is,

$$L_{e'}=\{(x, t): x=e'+f(b_1)(t-T), 0<t<T\}. \tag{5.54}$$

Again, using Lemma 5.18, we have

$$(d-c)\geqslant(d-e')\geqslant\beta T.$$

This completes the proof. Q.E.D.

Now we are in a position to prove Theorem 5.15.

Proof of Theorem 5.15. Let $-\infty < a < b < \infty$ be fixed. Assume that $f'(u(\cdot, T))$ is monotonically increasing in the interval $(c, d) \subset [a, b]$. If necessary we can divide (c, d) into small intervals. Hence we can assume that $C_b(c, T)$ and $C_b(d, T)$ are of the type in Lemma 5.16 and Lemma 5.17. There are several cases that we have to consider.

Case A. $t_0 = T_0 = 0$. Lemma 5.16 can be applied.

Case B. $t_0 > 0$, $T_0 > 0$, $T_0 \neq t_0$. There must be a type-III shock passing through $(x_0(T_0), T_0)$ and $(y_0(t_0), t_0)$. Lemma 5.17 can be applied.

Case C. $t_0 = T_0 > 0$. In this case, $(x_0(T_0), T_0) = (y_0(t_0), t_0)$ is an interaction point of two shocks $\{u_l, u_m\}$ and $\{u_m, u_r\}$. But either $u_l, u_m \in [a_2, b_2]$ or $u_m, u_r \in [b_1, a_1]$. As remarked after the proof of Lemma 5.17, Lemma 5.17 can be applied.

Case D. $t_0 = T_0 > 0$. In this case, $(x_0(T_0), T_0) = (y_0(t_0), t_0)$ is an interaction point of two shocks $\{u_l, u_m\}$ and $\{u_m, u_r\}$. But $u_l \in (a_1, a_2)$ or $u_r \in (a_1, a_2)$.

Let

$$\begin{aligned} \mathrm{I} = \{&(c, d) \subset (a, b): f'(u(\cdot T)) \text{ is increasing in } (c, d), (d-c) \\ &\text{is sufficiently small, and } (c, d)\text{'s are disjoint}\}, \\ \mathrm{II} = \{&(c, d) \in \mathrm{I}: (c, d) \in \text{Case A, Case B, or Case C}, \\ &(c, d) \in \text{Case D with } t_0 = T_0 < T/2\}, \\ \mathrm{III} = \{&(c, d) \in \mathrm{I}: (c, d) \in \text{Case D with } t_0 = T_0 \geqslant T/2\}. \end{aligned}$$

Now across a discontinuity of $f'(u(\cdot, T))$, $f'(u(\cdot, T))$ is decreasing. Hence we have

$$\begin{aligned} &V^+(f'(u(\cdot, T)); [a, b]) \\ &\quad = \sum_{(c,d) \in \mathrm{I}} [f'(u(d-0, T)) - f'(u(c+0, T))] \\ &\quad = \left(\sum_{(c,d) \in \mathrm{II}} + \sum_{(c,d) \in \mathrm{III}} \right) [f'(u(d-0, T)) - f'(u(c+0, T))]. \end{aligned} \tag{5.55}$$

Now we can apply Lemmas 5.16 and 5.17 to the sum $(c, d) \in \mathrm{II}$ but from $t = T/2$ to $t = T$. This gives us

$$\begin{aligned} &\sum_{(c,d) \in \mathrm{II}} [f'(u(d-0, T)) - f'(u(c+0, t))] \\ &\qquad \leqslant \frac{C}{T^2} \left[\frac{T}{2}(b-a) + V\left(\frac{T}{2}\right)^2 \right]. \end{aligned} \tag{5.56}$$

For the sum $(c, d) \in \text{III}$, we first note that we can combine those (c, d)'s such that they have the same termination point. Hence we can assume that for (c, d)'s in III, they have different termination points. From Lemmas 5.18 and 5.19, we know that the total number of (c, d)'s in III is bounded by

$$\left\{\left[(b-a)+V\cdot\left(\frac{T}{2}\right)\cdot 2\right]\Big/\beta\cdot\left(\frac{T}{2}\right)\right\}\cdot 2$$
$$=\frac{4[(b-a)+VT]}{\beta T}. \tag{5.57}$$

It is also easy to see that for $(c, d) \in \text{III}$, $[f'(u(d-0, t))-f'(u(c+0, T))]$ is bounded by

$$\max\{[f'(b_1)-f'(\bar{b}_2)], [f'(\bar{b}_1)-f'(b_2)]\} \tag{5.58}$$

where $\bar{b}_2 \in [b_1, a_1]$, $\bar{b}_1 \in [a_2, b_2]$, $((\bar{b}_2)^*)^* = b_2$, and $((\bar{b}_1)_*)_* = b_1$.

Combining (5.55), (5.56), (5.57), and (5.58) we finally have (5.34). This completes the proof of Theorem 5.15. Q.E.D.

Finally, we point out that Theorems 4.9 and 4.13 with the f satisfying assumptions (A) and (B) are still valid. The proofs are similar to those of Theorems 4.9 and 4.13. We omit all the statements and details of proofs.

6. Discussion

For general totally nonlinear f, the solutions can be much more complicated. But from our analysis, we can see that Theorems 4.9 and 4.13 are still valid.

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