附錄 A　Green Strain 和工程應變之關係修正

文獻[23]中將 Green Strain $\varepsilon_{12},\varepsilon_{13}$和工程應變$\gamma_{12},\gamma_{13}$之關係誤寫為

$$\sin\gamma_{12}=\frac{2\varepsilon_{12}}{(1+e_{11})^{1/2}(1+e_{22})^{1/2}}$$

$$\sin\gamma_{13}=\frac{2\varepsilon_{13}}{(1+e_{11})^{1/2}(1+e_{33})^{1/2}}$$

其中e_{11},e_{22},e_{33}為工程應變。

正確的 Green Strain 和工程應變之關係應為[24]

$$\sin\gamma_{12}=\frac{2\varepsilon_{12}}{(1+2\varepsilon_{11})^{1/2}(1+2\varepsilon_{22})^{1/2}}$$

$$\sin\gamma_{13}=\frac{2\varepsilon_{13}}{(1+2\varepsilon_{11})^{1/2}(1+2\varepsilon_{33})^{1/2}}$$

其中ε_{11}, ε_{22}, ε_{33}為 Green strain。

附錄B 梁受不均勻彎矩作用時的主要平衡路徑推導

將一長度為L的簡支梁梁等分成N段，則有N個梁元素及$N+1$個節點，設κ_J、X_J、Y_J代表第$J\,(J=1,N+1)$個節點的曲率及在X、Y座標的座標值， ℓ_i及ϕ_i^e代表第$i\,(i=1,N)$個梁元素的弦長及水平座標軸和元素座標x_1軸的夾角，θ_{ij}代表第$i\,(i=1,N)$個梁元素第$j\,(j=1,2)$個節點的變形角

X_J、Y_J、ℓ_i及ϕ_i^e有如下的關係

$$X_J = X_1 + \sum_{k=1}^{J-1} \ell_k \cos\phi_k^e \tag{B.1}$$

$$Y_J = Y_1 + \sum_{k=1}^{J-1} \ell_k \sin\phi_k^e \tag{B.2}$$

其中 $J = 2,N+1$。

由簡支梁的邊界條件可知

$$X_1 = Y_1 = 0 \tag{B.3}$$

$$X_{N+1} = \overline{L} \tag{B.4}$$

$$Y_{N+1} = 0 \tag{B.5}$$

其中$\overline{L}$梁結構變形後的弦長。

若給一初值X_J、$Y_J\,(J=1,N+1)$及${}^0\phi_i^e\,(i=1,N)$，則在B端的反力F_z、節點J的彎矩M_J及節點J的曲率κ_J可表示成

$$F_z = \frac{M(1+\lambda)}{L} \tag{B.6}$$

$$M_J = M - F_z X_J \tag{B.7}$$

$$\kappa_J = \frac{M_J}{EI_y} \tag{B.8}$$

假設梁元素在其元素座標的變形曲線為一三次 Hermitian 曲線，則第 $i\,(i=1,N)$ 個梁元素的變形曲線可表示如下

$$w_i(\zeta) = D_1(-\zeta^2 + \frac{1}{4}) + D_2(\zeta^3 + \frac{1}{4}\zeta) + D_3(-\zeta) + D_4 \tag{B.9}$$

$$D_1 = \frac{(\bar{\kappa}_1 + \bar{\kappa}_2)\, s^2}{4}，D_2 = \frac{(\bar{\kappa}_1 - \bar{\kappa}_2)\, s^2}{6}，D_3 = \frac{(\bar{\kappa}_1 - \bar{\kappa}_2)\, s^2}{12}，D_4 = 0 \tag{B.10}$$

$$\bar{\kappa}_j = \kappa_{i+j-1} \tag{B.11}$$

$$s = \frac{L}{N} \tag{B.12}$$

$$\zeta = \frac{x}{s} - \frac{1}{2}，-\frac{1}{2} \le \zeta \le \frac{1}{2} \tag{B.13}$$

其中 $\bar{\kappa}_j\,(j=1,2)$ 為梁元素兩端節點的曲率，s 為梁元素變形前的長度。

由 $\theta_i = \dfrac{dw_i}{dx}$ 可得第 i 個元素兩端的變形角如下

$$\theta_{i1} = \frac{\bar{\kappa}_1 s}{3} + \frac{\bar{\kappa}_2 s}{6} \tag{B.14}$$

$$\theta_{i2} = -(\frac{\bar{\kappa}_1 s}{6} + \frac{\bar{\kappa}_2 s}{3}) \tag{B.15}$$

若不考慮梁的軸向應變，則梁元素的弦長可以表示成

$$\ell_i = s[1 - \frac{1}{15}(\theta_{i1}^2 - \frac{1}{2}\theta_{i1}\theta_{i2} + \theta_{i2}^2)] \tag{B.16}$$

由及相鄰元素在共同節點有相同的切線可得

$$\phi_i^e = \phi_{i-1}^e + \theta_{(i-1)2} - \theta_{i1} \tag{B.17}$$

其中 $i = 2, N$ 。

令

$$\bar{\phi}_1^e = {}^0\phi_1^e \tag{B.18}$$

$$\bar{\phi}_i^e = \bar{\phi}_{i-1}^e + \theta_{(i-1)2} - \theta_{i1} \tag{B.19}$$

$$\phi_1^e = \bar{\phi}_1^e + \Delta\phi \tag{B.20}$$

其中 $\Delta\phi$ 為一待定的改正量，則(B.17)式可以改寫成

$$\phi_i^e = \bar{\phi}_i^e + \Delta\phi \tag{B.21}$$

其中 $i = 1, N$ 。

由(B.21)式代入(B.1)及(B.2)式可得

$$X_J = \cos\Delta\phi \sum_{k=1}^{J-1} \ell_k \cos\bar{\phi}_k^e - \sin\Delta\phi \sum_{k=1}^{J-1} \ell_k \sin\bar{\phi}_k^e \tag{B.22}$$

$$Y_J = \cos\Delta\phi \sum_{k=1}^{J-1} \ell_k \sin\bar{\phi}_k^e + \sin\Delta\phi \sum_{k=1}^{J-1} \ell_k \cos\bar{\phi}_k^e \tag{B.23}$$

其中 $J = 2, N+1$ 。

由(B.5)式代入(B.23)式可得

$$Y_{N+1}=\cos\Delta\phi\sum_{k=1}^{N}\ell_k\sin\overline{\phi}_k^e+\sin\Delta\phi\sum_{k=1}^{N}\ell_k\cos\overline{\phi}_k^e=0 \tag{B.24}$$

由(B.24)式可得

$$\tan\Delta\phi=-\frac{\sum_{k=1}^{N}\ell_k\sin\overline{\phi}_k^e}{\sum_{k=1}^{N}\ell_k\cos\overline{\phi}_k^e} \tag{B.25}$$

由(B.25)式可得$\Delta\phi$，將$\Delta\phi$代入(B.21)、(B.22)、(B.23)式可以得到一組新的ϕ_i^e、X_J、$\overline{L}$、Y_J。若梁的變形不大，則(B.8)、(B.9)、(B.21)、(B.22)、(B.23)式可以當做主要平衡路徑的解。若梁的變形較大，則需將得到的X_J、$\overline{L}$、Y_J、ϕ_i^e當初值，重新迭代到$\overline{L}$收斂。

若梁僅受均勻彎矩作用，其主要平衡路徑為一圓弧，則節點J的曲率可表示成

$$\kappa_J=\frac{M}{EI_y} \tag{B.26}$$

令

$$\theta_0=\frac{ML}{EI_y}\ ,\quad \Delta\theta=\frac{\theta_0}{N} \tag{B.27}$$

則第i個元素兩端的變形角及ϕ_i^e如下

$$\bar{\theta}_{i1} = \frac{\Delta\theta}{2} \tag{B.28}$$

$$\bar{\theta}_{i2} = -\frac{\Delta\theta}{2} \tag{B.29}$$

$$\phi_i^e = \frac{\theta_0}{2} - (i - \frac{1}{2}) \cdot \Delta\theta \tag{B.30}$$

其中 $i = 1, 2, \cdots, N$

附錄 C 簡支梁受軸向拉力時之主要平衡路徑

如圖八所示之簡支梁所受的軸力為軸向拉力時，推導的方式和 3.2 節一樣，在此僅列出部分重要的方程式並做簡單的說明。與(3.2.16)式對應之主要平衡路徑之統御方程式需改成

$$\frac{EI_y}{\ell^2}\frac{d^4 w}{d\zeta^4} - F_1 \frac{d^2 w}{d\zeta^2} = 0 \tag{C.1}$$

其中 $F_1 = F_x \cos\phi^e - F_y \sin\phi^e$ ，$F_x = P$ ，F_1 為一正值。

(C.1)式的通解可表示為

$$w(\zeta) = \mathbf{N}^t(\zeta)\mathbf{q}^0 \tag{C.2}$$

$$\mathbf{N}^t(\zeta) = \{\sinh a\zeta \quad \cosh a\zeta \quad \zeta \quad 1\} \tag{C.3}$$

其中 $\mathbf{q}^0$ 的表示與(3.2.19)式相同，a 的表示為將(3.2.20)式中的 $-F_1$ 改為 F_1 即可，表示如下

$$a = \left(\frac{F_1 \ell^2}{EI_y}\right)^{1/2}$$

(C.2)中 $\mathbf{q}^0$ 和端點曲率間的關係可表示成

$$\mathbf{q}^0 = \mathbf{T}\kappa \tag{C.4}$$

$$\mathbf{T} = \frac{\ell^2}{2a^2}\begin{bmatrix} 1/\sinh\frac{a}{2} & -1/\sinh\frac{a}{2} \\ -1/\cosh\frac{a}{2} & -1/\cosh\frac{a}{2} \\ -2 & 2 \\ 1 & 1 \end{bmatrix} \tag{C.5}$$

其中 κ 和(3.2.26)式相同。

$\sin\phi^e$ 和(3.2.29)式相同，表示如下

$$\sin\phi^e = [\frac{EI_y}{\ell}(\kappa_1 - \kappa_2) - F_y \cos\phi^e] \Big/ F_x \tag{C.6}$$

與(3.2.37)式對應之方程式需改成

$$\kappa=\frac{-1}{2}\left[\frac{(1+\lambda)M}{EI_y}\frac{\sinh a\zeta}{\sinh a/2}-\frac{(1-\lambda)M}{EI_y}\frac{\cosh a\zeta}{\cosh a/2}\right] \tag{C.7}$$

與(3.2.42)式對應之方程式需改成

$$\begin{aligned}
k_{11}&=k_\phi-N'_{w,1}(\zeta_1)-\kappa_1T_{11}-\kappa_2T_{21}\quad ,\\
k_{12}&=-k_\phi-N'_{w,2}(\zeta_1)-\kappa_1T_{12}-\kappa_2T_{22}\quad ,\\
k_{21}&=-k_\phi+N'_{w,1}(\zeta_2)-\kappa_1T_{21}-\kappa_2T_{11}\quad ,\\
k_{22}&=k_\phi+N'_{w,2}(\zeta_2)-\kappa_1T_{22}-\kappa_2T_{12}\quad ,\\
k_\phi&=\frac{-EI_y}{(F_x\cos\phi^e-F_y\sin\phi^e)\,\ell}=\frac{-EI_y}{(P\cos\phi^e-F_y\sin\phi^e)\,\ell}\\
T_{11}&=\frac{\ell}{a}\frac{\partial a}{\partial\kappa_1}\left(\frac{-1}{\sinh^2 a}-\frac{\cosh a}{a\sinh a}+\frac{2}{a^2}\right)\\
T_{21}&=\frac{\ell}{a}\frac{\partial a}{\partial\kappa_1}\left(\frac{1}{a\sinh a}+\frac{\cosh a}{\sinh^2 a}-\frac{2}{a^2}\right)\\
T_{12}&=-T_{11}\\
T_{22}&=-T_{21}\\
\frac{\partial a}{\partial\kappa_1}&=\frac{-1}{2}(EI_y)^{1/2}\frac{F_x\sin\phi^e+F_y\cos\phi^e}{(F_x\cos\phi^e-F_y\sin\phi^e)^{3/2}}\\
\frac{\partial a}{\partial\kappa_2}&=-\frac{\partial a}{\partial\kappa_1}
\end{aligned} \tag{C.8}$$

附錄 D 懸臂梁之主要平衡路徑

圖五之懸臂梁的主要平衡路徑，可由一等效簡支梁的主要平衡路徑求得。

如圖十一(a)所示為一長度為 L 之懸臂梁之主要平衡路徑的示意圖，圖十一(b)所示為兩個長度為 L 的懸臂梁之主要平衡路徑的示意圖，圖十一(b)左右對稱，且右半部和圖十一(a)相同，圖十一(c)和圖十一(b)為等效結構，兩者有相同的變形，反力和內力。所以將一長度 2L 的簡支梁等分成 2N 個梁元素，利用 3.2 節的方法求得其主要平衡路徑，然後再取其右半部之節點，即可當作長度為 L，等分成 N 個元素的懸臂梁之主要平衡路徑。

附錄 E 擾動後的元素座標及節點旋轉參數

在本推導中所有的向量都是表示成擾動前之元素座標 x_i 的分量。令 $\mathbf{r}_1^0, \mathbf{r}_2^0$ 表示擾動前之元素節點 1 及 2 的位置向量，${}^0\mathbf{e}_{ij}^s$ 表示擾動前節點 j 之元素斷面座標軸 ${}^0x_{ij}^s$ 的單位向量，$\boldsymbol{\theta}_j^0$ 為擾動前節點 j 的旋轉參數向量，$\mathbf{r}_1^0, \mathbf{r}_2^0$ 及 $\boldsymbol{\theta}_j^0$ ，${}^0\mathbf{e}_{ij}^s (j=1,2 \quad i=1,2,3)$ 可表示成

$$\mathbf{r}_1^0 = \{0, 0, 0\} \tag{E.1}$$

$$\mathbf{r}_2^0 = \{\ell, 0, 0\} \tag{E.2}$$

$$\boldsymbol{\theta}_j^0 = \{0, \theta_{2j}^0, 0\} \tag{E.3}$$

$${}^0\mathbf{e}_{1j}^s = \{c_j, 0, -s_j\} \tag{E.4}$$

$${}^0\mathbf{e}_{2j}^s = \{0, 1, 0\} \tag{E.5}$$

$${}^0\mathbf{e}_{1j}^s = \{s_j, 0, c_j\} \tag{E.6}$$

其中

$$c_j = \cos\varphi_{2j}^0 \qquad s_j = \sin\varphi_{2j}^0 = \theta_{2j}^0$$

當元素節點 $j (j=1,2)$ 受到擾動位移

$$\mathbf{u}_j = \{u_j, v_j, w_j\} \tag{E.7}$$

作用時，其位置向量可表示成

$$\mathbf{r}_j = \mathbf{r}_j^0 + \mathbf{u}_j \tag{E.8}$$

擾動後之元素座標$\bar{x}_1$軸的單位向量可表示成

$$\bar{\mathbf{e}}_1 = (\mathbf{r}_2 - \mathbf{r}_1) / \|\mathbf{r}_2 - \mathbf{r}_1\| \tag{E.9}$$

若取到擾動量的一次項，則(E.9)式可表示成

$$\bar{\mathbf{e}}_1 = \{1\ ,\ \frac{\Delta v}{\ell}\ ,\ \frac{\Delta w}{\ell}\} \tag{E.10}$$

其中$\Delta v = v_2 - v_1$、$\Delta w = w_2 - w_1$，ℓ為擾動前元素的弦長。

當元素節點$j(\ j = 1,2\ )$之斷面座標受到擾動旋轉向量

$$\boldsymbol{\phi}_j = \{\ \phi_{1j}\ ,\phi_{2j}\ ,\phi_{3j}\ \} \tag{E.11}$$

作用後，若取到擾動量的一次項則其座標軸$x_{1j}^s\ ,x_{2j}^s$的單位向量可表示成

$$\begin{aligned}\mathbf{e}_{1j}^s &= {}^0\mathbf{e}_{1j}^s + \boldsymbol{\phi}_j \times {}^0\mathbf{e}_{1j}^s \\ &= \{c_j - s_j\phi_{2j}\ ,c_j\phi_{3j} + s_j\phi_{1j}\ ,-s_j - c_j\phi_{2j}\}\end{aligned} \tag{E.12}$$

$$\begin{aligned}\mathbf{e}_{2j}^s &= {}^0\mathbf{e}_{2j}^s + \boldsymbol{\phi}_j \times {}^0\mathbf{e}_{2j}^s \\ &= \{-\phi_{3j}\ ,1,\phi_{1j}\}\end{aligned} \tag{E.13}$$

本研究用文獻[14]中的方法決定擾動後的元素座標軸$\bar{x}_2$、$\bar{x}_3$的單位向量

及節點參數向量$\boldsymbol{\theta}_j = \{\theta_{1j}, \theta_{2j}, \theta_{3j}\}$，並說明如下

令旋轉向量$\boldsymbol{\theta}_{nj}$為(2.4.9)式中$\boldsymbol{\theta}_n$在節點j之節點值，由$\boldsymbol{\theta}_n$的定義可得

$$\sin\theta_{nj}\mathbf{n}_j = \bar{\mathbf{e}}_1 \times \mathbf{e}_{1j}^{s} \tag{E.14}$$

將(E.10)、(E.12)式代入(E.14)式且保留到擾動量的一次項可得

$$\mathbf{n}_j = \{-\frac{\Delta v}{\ell}s_j, s_j + c_j\phi_{2j} + \frac{\Delta w}{\ell}c_j, c_j\phi_{3j} + s_1\phi_{1j} - \frac{\Delta v}{\ell}c_j\}/s_j \tag{E.15}$$

擾動後元素座標軸$\bar{x}_2$、$\bar{x}_3$的單位向量$\bar{\mathbf{e}}_2$、$\bar{\mathbf{e}}_3$可由以下步驟決定：

將旋轉向量$-\boldsymbol{\theta}_{nj}$作用在$\mathbf{e}_{ij}^{s}$上，使其旋轉到$\mathbf{e}_{ij}^{s'}$，此$\mathbf{e}_{1j}^{s'}$和$\bar{\mathbf{e}}_1$重合，$\mathbf{e}_{2j}^{s'}$與$\mathbf{e}_{3j}^{s'}$和$\bar{\mathbf{e}}_1$垂直。由(2.3.1)式並保留到擾動量的一次項可得

$$\mathbf{e}_{2j}^{s'} = \{-\frac{\Delta v}{\ell}, 1 + s_j c_j(\frac{\Delta w}{\ell} + \phi_{2j}), \phi_{1j} - \frac{1}{2}s_j(\phi_{3j} - \frac{\Delta v}{\ell})\} \tag{E.16}$$

元素座標軸$\bar{x}_2$的單位向量可表示成

$$\bar{\mathbf{e}}_2 = \frac{\mathbf{e}_{22}^{s'} + \mathbf{e}_{21}^{s'}}{\left\|\mathbf{e}_{22}^{s'} + \mathbf{e}_{21}^{s'}\right\|} \tag{E.17}$$

若取到擾動量的一次項則$\bar{\mathbf{e}}_2$可表示成

$$\begin{aligned}\mathbf{e}_2 = \{&-\frac{\Delta v}{\ell}, 1 + \frac{1}{2}s_1c_1(\frac{\Delta w}{\ell} + \phi_{21}) + \frac{1}{2}s_2c_2(\frac{\Delta w}{\ell} + \phi_{22}),\\ &\frac{1}{2}(\phi_{11} + \phi_{12}) - \frac{1}{4}s_1(\phi_{31} - \frac{\Delta v}{\ell}) - \frac{1}{4}s_2(\phi_{32} - \frac{\Delta v}{\ell})\}\end{aligned} \tag{E.18}$$

$\bar{\mathbf{e}}_3$可表示成

$$
\begin{aligned}
\bar{\mathbf{e}}_3 &= \bar{\mathbf{e}}_1 \times \bar{\mathbf{e}}_2 \\
&= \{-\frac{\Delta w}{\ell}, -\frac{1}{2}(\phi_{11}+\phi_{12}) - \frac{1}{4}\frac{\Delta v}{\ell}(s_1+s_2) + \frac{1}{4}(s_1\phi_{31}+s_2\phi_{32}), \\
&\quad 1+\frac{1}{2}\frac{\Delta w}{\ell}(c_1 s_1 + c_2 s_2) + \frac{1}{2}(c_1 s_1 \phi_{21} + c_2 s_2 \phi_{22})\}
\end{aligned}
\tag{E.19}
$$

擾動後的旋轉參數θ_{1j}可表示成

$$
\sin\theta_{1j} = \bar{\mathbf{e}}_2 \times \mathbf{e}_{2j}^{s'} \cdot \bar{\mathbf{e}}_1 \tag{E.20}
$$

若取到擾動量的一次項則$\theta_{1j}\ j(j=1,2)$可表示成

$$
\begin{aligned}
\theta_{11} &= -\frac{1}{4\ell}(s_1-s_2)v_1 + (\frac{1}{2})\phi_{11} + (-\frac{1}{4}s_1)\phi_{31} \\
&\quad + \frac{1}{4\ell}(s_1-s_2)v_2 + (-\frac{1}{2})\phi_{12} + (\frac{1}{4}s_2)\phi_{32}
\end{aligned}
\tag{E.21}
$$

$$
\begin{aligned}
\theta_{12} &= \frac{1}{4\ell}(s_1-s_2)v_1 + (-\frac{1}{2})\phi_{11} + (\frac{1}{4}s_1)\phi_{31} \\
&\quad - \frac{1}{4\ell}(s_1-s_2)v_2 + (\frac{1}{2})\phi_{12} + (-\frac{1}{4}s_2)\phi_{32}
\end{aligned}
\tag{E.22}
$$

θ_{2j}和θ_{3j}可表示如下

$$
\theta_{ij} = \sin\theta_{nj}\mathbf{n}_j \cdot \mathbf{e}_i \tag{E.23}
$$

若保留到擾動量的一次項則θ_{ij} $(i=2,3)$、$(j=1,2)$可表示成

$$\theta_{21}=s_1+\frac{\Delta w}{\ell}(c_1+\frac{1}{2}c_1s_1^2+\frac{1}{2}c_2s_1s_2)$$
$$+\phi_{21}(c_1+\frac{1}{2}c_1s_1^2)+\frac{1}{2}c_2s_1s_2\phi_{22} \tag{E.24}$$

$$\theta_{22}=s_2+\frac{\Delta w}{\ell}(c_2+\frac{1}{2}c_2s_2^2+\frac{1}{2}c_1s_1s_2)$$
$$+\frac{1}{2}c_1s_1s_2\phi_{21}+\phi_{22}(c_2+\frac{1}{2}c_2s_2^2) \tag{E.25}$$

$$\theta_{31}=(\frac{c_1}{\ell}+\frac{s_1^2}{4\ell}+\frac{s_1s_2}{4\ell})v_1+(\frac{1}{2}s_1)\phi_{11}+(c_1+\frac{s_1^2}{4})\phi_{31}$$
$$-(\frac{c_1}{\ell}+\frac{s_1^2}{4\ell}+\frac{s_1s_2}{4\ell})v_2+(-\frac{1}{2}s_1)\phi_{12}+(\frac{s_1s_2}{4})\phi_{32} \tag{E.26}$$

$$\theta_{32}=(\frac{c_2}{\ell}+\frac{s_2^2}{4\ell}+\frac{s_1s_2}{4\ell})v_1+(-\frac{1}{2}s_2)\phi_{11}+(\frac{s_1s_2}{4})\phi_{31}$$
$$-(\frac{c_2}{\ell}+\frac{s_2^2}{4\ell}+\frac{s_1s_2}{4\ell})v_2+(\frac{1}{2}s_2)\phi_{12}+(c_2+\frac{s_2^2}{4})\phi_{32} \tag{E.27}$$

附錄 F 軸力為壓力時θ_2^0,ε_0及其微分的級數表示

由(2.4.4)式及其微分，及(3.3.26)式可得

$$
\begin{aligned}
&w_{,x}^0=\frac{dw}{dx}=\frac{1}{l}\frac{dw}{d\zeta} \quad w_{,xx}^0=\frac{1}{l^2}\frac{d^2w}{d\zeta^2}\\
&w_{,xxx}^0=\frac{1}{l^3}\frac{d^3w}{d\zeta^3} \quad w_{,xxxx}^0=\frac{1}{l^4}\frac{d^4w}{d\zeta^4}\\
&w_{,xxxxx}^0=\frac{1}{l^5}\frac{d^5w}{d\zeta^5} \quad w_{,xxxxxx}^0=\frac{1}{l^6}\frac{d^6w}{d\zeta^6}
\end{aligned}
\tag{F.1}
$$

由(3.2.17)式及其微分可得

$$
\begin{aligned}
&w(\zeta)=D_1\sin a\zeta+D_2\cos a\zeta+D_3\zeta+D_4\\
&w_{,\zeta}(\zeta)=aD_1\cos a\zeta-aD_2\sin a\zeta+D_3\\
&w_{,\zeta\zeta}(\zeta)=-a^2D_1\sin a\zeta-a^2D_2\cos a\zeta\\
&w_{,\zeta\zeta\zeta}(\zeta)=-a^3D_1\cos a\zeta+a^3D_2\sin a\zeta\\
&w_{,\zeta\zeta\zeta\zeta}(\zeta)=a^4D_1\sin a\zeta+a^4D_2\cos a\zeta=-a^2w_{,\zeta\zeta}(\zeta)\\
&w_{,\zeta\zeta\zeta\zeta\zeta}(\zeta)=a^5D_1\cos a\zeta-a^5D_2\sin a\zeta=-a^2w_{,\zeta\zeta\zeta}(\zeta)\\
&w_{,\zeta\zeta\zeta\zeta\zeta\zeta}(\zeta)=-a^6D_1\sin a\zeta-a^6D_2\cos a\zeta=-a^2w_{,\zeta\zeta\zeta\zeta}(\zeta)
\end{aligned}
\tag{F.2}
$$

將(F.2)式中$\sin a\zeta$及$\cos a\zeta$用泰勒級數展開取其前三項可得

$$
\begin{aligned}
&\sin a\zeta=a\zeta-\frac{(a\zeta)^3}{3!}+\frac{(a\zeta)^5}{5!}\\
&\cos a\zeta=1-\frac{(a\zeta)^2}{2!}+\frac{(a\zeta)^4}{4!}
\end{aligned}
\tag{F.3}
$$

由(F.1)~(F.3)式可得

$$w^0_{,x} = \frac{dw^0}{dx} = \frac{1}{l}\frac{dw^0}{d\zeta} = \frac{1}{l}(aD_1 \cos a\zeta - aD_2 \sin a\zeta + D_3)$$

$$= \frac{aD_1}{l}[1 - \frac{(a\zeta)^2}{2!} + \frac{(a\zeta)^4}{4!}] - \frac{aD_2}{l}[a\zeta - \frac{(a\zeta)^3}{3!} + \frac{(a\zeta)^5}{5!}] + \frac{D_3}{l}$$

$$= \frac{1}{l}(aD_1 + D_3 - D_2 a^2 \zeta - \frac{1}{2}D_1 a^3 \zeta^2 + \frac{1}{6}D_2 a^4 \zeta^3$$

$$+ \frac{1}{24}D_1 a^5 \zeta^4 - \frac{1}{120}D_2 a^6 \zeta^5)$$

(F.4)

$$w^0_{,xx} = \frac{1}{l^2}\frac{d^2 w}{d\zeta^2} = \frac{1}{l^2}(-a^2 D_1 \sin a\zeta - a^2 D_2 \cos a\zeta)$$

$$= \frac{-a^2 D_1}{l^2}[a\zeta - \frac{(a\zeta)^3}{3!} + \frac{(a\zeta)^5}{5!}] - \frac{a^2 D_2}{l^2}[1 - \frac{(a\zeta)^2}{2!} + \frac{(a\zeta)^4}{4!}]$$

$$= \frac{a^2}{l^2}(-D_2 - D_1 a\zeta + \frac{1}{2}D_2 a^2 \zeta^2 + \frac{1}{6}D_1 a^3 \zeta^3 - \frac{1}{24}D_2 a^4 \zeta^4 - \frac{1}{120}D_1 a^5 \zeta^5)$$

(F.5)

$$w^0_{,xxx} = \frac{1}{l^3}\frac{d^3 w}{d\zeta^3} = \frac{1}{l^3}(-a^3 D_1 \cos a\zeta + a^3 D_2 \sin a\zeta)$$

$$= \frac{-a^3 D_1}{l^3}[1 - \frac{(a\zeta)^2}{2!} + \frac{(a\zeta)^4}{4!}] + \frac{a^3 D_2}{l^3}[a\zeta - \frac{(a\zeta)^3}{3!} + \frac{(a\zeta)^5}{5!}]$$

$$= \frac{a^3}{l^3}(-D_1 + D_2 a\zeta + \frac{1}{2}D_1 a^2 \zeta^2 - \frac{1}{6}D_2 a^3 \zeta^3 - \frac{1}{24}D_1 a^4 \zeta^4 + \frac{1}{120}D_2 a^5 \zeta^5)$$

(F.6)

$$w^0_{,xxxx} = \frac{1}{l^4}\frac{d^4 w}{d\zeta^4} = \frac{-a^2}{l^4}w_{,\zeta\zeta}(\zeta)$$

$$= \frac{a^4}{l^4}(D_2 + D_1 a\zeta - \frac{1}{2}D_2 a^2 \zeta^2 - \frac{1}{6}D_1 a^3 \zeta^3$$

$$+ \frac{1}{24}D_2 a^4 \zeta^4 + \frac{1}{120}D_1 a^5 \zeta^5)$$

(F.7)

由(3.3.23)式及(F.1)~(F.3)式可得

$$\varepsilon_0 = \varepsilon_p - \frac{I_y w^0_{,x} w^0_{,xxx}}{A} = \varepsilon_p - \frac{I_y}{Al^4} w_{,\zeta} w_{,\zeta\zeta\zeta}$$
$$= \varepsilon_p - \frac{I_y}{Al^4}(aD_1 \cos a\zeta - aD_2 \sin a\zeta + D_3)(-a^3 D_1 \cos a\zeta + a^3 D_2 \sin a\zeta)$$
$$= \varepsilon_p - \frac{a^2 I_y}{Al^4}[-a^2(D_1^2 \frac{1+\cos 2a\zeta}{2} - D_1 D_2 \sin 2a\zeta + D_2^2 \frac{1-\cos 2a\zeta}{2})$$
$$+ D_3(-aD_1 \cos a\zeta + aD_2 \sin a\zeta)]$$
$$= \varepsilon_p + \frac{a^4 I_y}{Al^4}[D_1^2 - 2D_1 D_2 a\zeta + (D_2^2 - D_1^2)a^2\zeta^2 + \frac{4}{3}D_1 D_2 a^3 \zeta^3$$
$$+ \frac{1}{3}(D_1^2 - D_2^2)a^4\zeta^4 - \frac{4}{15}D_1 D_2 a^5 \zeta^5] - \frac{a^2 I_y}{Al^4} D_3(-aD_1 + D_2 a^2 \zeta$$
$$+ \frac{1}{2}D_1 a^3 \zeta^2 - \frac{1}{6}D_2 a^4 \zeta^3 - \frac{1}{24}D_1 a^5 \zeta^4 + \frac{1}{120}D_2 a^6 \zeta^5)$$

其中

$$\varepsilon_p = \frac{F_1}{EA}\text{ , } F_1 = F_x \cos\phi^e - F_y \sin\phi^e \text{ , } F_x = -P$$

(F.8)

$$\varepsilon_{0,x} = \frac{-I_y}{A}(w^0_{,xx} w^0_{,xxx} + w^0_{,x} w^0_{,xxxx})$$
$$= \frac{-I_y}{A}\{\frac{a^5}{l^5}[\frac{1}{2}(D_1^2 - D_2^2)\sin 2a\zeta + D_1 D_2 \cos 2a\zeta]$$
$$+ \frac{a^5}{l^5}[\frac{1}{2}(D_1^2 - D_2^2)\sin 2a\zeta + D_1 D_2 \cos 2a\zeta]$$
$$+ \frac{a^4}{l^5}(D_1 D_3 \sin a\zeta + D_2 D_3 \cos a\zeta)\}$$

$$= \frac{-I_y}{A}\{\frac{2a^5}{l^5}[D_1D_2 + (D_1^2 - D_2^2)a\zeta - 2D_1D_2a^2\zeta^2 - \frac{2}{3}(D_1^2 - D_2^2)a^3\zeta^3$$

$$+ \frac{2}{3}D_1D_2a^4\zeta^4 + \frac{2}{15}(D_1^2 - D_2^2)a^5\zeta^5] + \frac{a^4}{l^5}(D_2D_3 + D_1D_3a\zeta$$

$$- \frac{1}{2}D_2D_3a^2\zeta^2 - \frac{1}{6}D_1D_3a^3\zeta^3 + \frac{1}{24}D_2D_3a^4\zeta^4 + \frac{1}{120}D_1D_3a^5\zeta^5)\}$$

(F.9)

$$\varepsilon_{0,xx} = -\frac{I_y}{A}[(w^0_{,xxx})^2 + 2w^0_{,xx}w^0_{,xxxx} + w^0_{,x}w^0_{,xxxxx}]$$

$$= \frac{-I_y}{A}\{\frac{a^6}{l^6}[2(D_1^2 - D_2^2) - 8D_1D_2a\zeta - 4(D_1^2 - D_2^2)a^2\zeta^2 + \frac{16}{3}D_1D_2a^3\zeta^3$$

$$+ \frac{4}{3}(D_1^2 - D_2^2)a^4\zeta^4 - \frac{16}{15}D_1D_2a^5\zeta^5] + \frac{a^5}{l^6}(D_1D_3 - D_2D_3a\zeta$$

$$- \frac{1}{2}D_1D_3a^2\zeta^2 + \frac{1}{6}D_2D_3a^3\zeta^3 + \frac{1}{24}D_1D_3a^4\zeta^4 - \frac{1}{120}D_2D_3a^5\zeta^5)\}$$

(F.10)

$$\varepsilon_{0,xxx} = -\frac{I_y}{A}[4w^0_{,xxx}w^0_{,xxxx} + 3w^0_{,xx}w^0_{,xxxxx} + w^0_{,x}w^0_{,xxxxxx}]$$

$$= \frac{-I_y}{A}\{\frac{-8a^7}{l^7}[D_1D_2 + (D_1^2 - D_2^2)a\zeta - 2D_1D_2a^2\zeta^2$$

$$- \frac{2}{3}(D_1^2 - D_2^2)a^3\zeta^3 + \frac{2}{3}D_1D_2a^4\zeta^4 + \frac{2}{15}(D_1^2 - D_2^2)a^5\zeta^5]$$

$$- \frac{a^6}{L^7}(D_2D_3 + D_1D_3a\zeta - \frac{1}{2}D_2D_3a^2\zeta^2 - \frac{1}{6}D_1D_3a^3\zeta^3$$

$$+ \frac{1}{24}D_2D_3a^4\zeta^4 + \frac{1}{120}D_1D_3a^5\zeta^5)\}$$

(F.11)

附錄 G 軸力為壓力時**Q,R,S,T**的級數表示

將附錄 F 的結果代入(3.3.28)式，即可將**Q**,**R**,**S**,**T**表示成如(3.3.31)式的級數形式，其推導過程如下：

令

$$\mathbf{Q}=\begin{bmatrix} Q(1,1) & Q(1,2) \\ Q(2,1) & Q(2,2) \end{bmatrix}$$ 由附錄 F 及(3.3.28)式可得

$$Q(1,1)=0$$

$$\begin{aligned}Q(1,2)&=\frac{C\ell w^0_{,x}}{2}\\&=\frac{C}{2}(aD_1+D_3-D_2a^2\zeta-\frac{1}{2}D_1a^3\zeta^2+\frac{1}{6}D_2a^4\zeta^3\\&\quad+\frac{1}{24}D_1a^5\zeta^4-\frac{1}{120}D_2a^6\zeta^5)\end{aligned}$$

$$Q(2,1)=-Q(1,2)$$

$$\begin{aligned}Q(2,2)&=4EI_z\ell\varepsilon_{0,x}\\&=-2Q_2[D_1D_2+(D_1^2-D_2^2)a\zeta-2D_1D_2a^2\zeta^2-\frac{2}{3}(D_1^2-D_2^2)a^3\zeta^3\\&\quad+\frac{2}{3}D_1D_2a^4\zeta^4+\frac{2}{15}(D_1^2-D_2^2)a^5\zeta^5]-\frac{Q_2}{a}(D_2D_3+D_1D_3a\zeta\\&\quad-\frac{1}{2}D_2D_3a^2\zeta^2-\frac{1}{6}D_1D_3a^3\zeta^3+\frac{1}{24}D_2D_3a^4\zeta^4+\frac{1}{120}D_1D_3a^5\zeta^5)\end{aligned}$$

其中　$Q_2 = \dfrac{4EI_yI_z}{\ell^4 A}a^5$

收集$Q(i,j)$中ζ^i的同次項，即可將$\mathbf{Q}$表示成$\mathbf{Q} = \sum_{i=0}^{5}\zeta^i\mathbf{Q}_i$

$$\mathbf{Q}_0 = \begin{bmatrix} 0 & \dfrac{C}{2}(aD_1 + D_3) \\ -\dfrac{C}{2}(aD_1 + D_3) & -2Q_2D_1D_2 - \dfrac{Q_2}{a}D_2D_3 \end{bmatrix}$$

$$\mathbf{Q}_1 = \begin{bmatrix} 0 & -\dfrac{C}{2}D_2a^2 \\ \dfrac{C}{2}D_2a^2 & -2Q_2(D_1^2 - D_2^2)a - Q_2D_1D_3 \end{bmatrix}$$

$$\mathbf{Q}_2 = \begin{bmatrix} 0 & -\dfrac{C}{4}D_1a^3 \\ \dfrac{C}{4}D_1a^3 & 4Q_2D_1D_2a^2 + \dfrac{1}{2}Q_2D_2D_3a \end{bmatrix}$$

$$\mathbf{Q}_3 = \begin{bmatrix} 0 & \dfrac{C}{12}D_2a^4 \\ -\dfrac{C}{12}D_2a^4 & \dfrac{4}{3}Q_2(D_1^2 - D_2^2)a^3 + \dfrac{1}{6}Q_2D_1D_3a^2 \end{bmatrix}$$

$$\mathbf{Q}_4 = \begin{bmatrix} 0 & \dfrac{C}{48}D_1a^5 \\ -\dfrac{C}{48}D_1a^5 & \dfrac{-4}{3}Q_2D_1D_2a^4 - \dfrac{1}{24}Q_2D_2D_3a^3 \end{bmatrix}$$

$$\mathbf{Q}_5 = \begin{bmatrix} 0 & -\dfrac{C}{240}D_2a^6 \\ \dfrac{C}{240}D_2a^6 & \dfrac{-4}{15}Q_2(D_1^2 - D_2^2)a^5 - \dfrac{1}{120}Q_2D_1D_3a^4 \end{bmatrix}$$

(G.1)

同理，令

$$\mathbf{R}=\begin{bmatrix} R(1,1) & R(1,2) \\ R(2,1) & R(2,2) \end{bmatrix}$$

$$\begin{aligned}
R(1,1) &= \ell^2(E\varepsilon_0 I_P + C) \\
&= EI_p\varepsilon_p\ell^2 + R_2[D_1^2 - 2D_1D_2a\zeta + (D_2^2 - D_1^2)a^2\zeta^2 + \frac{4}{3}D_1D_2a^3\zeta^3 \\
&\quad + \frac{1}{3}(D_1^2 - D_2^2)a^4\zeta^4 - \frac{4}{15}D_1D_2a^5\zeta^5] - \frac{R_2}{a}(-D_1D_3 + D_2D_3a\zeta \\
&\quad + \frac{1}{2}D_1D_3a^2\zeta^2 - \frac{1}{6}D_2D_3a^3\zeta^3 - \frac{1}{24}D_1D_3a^4\zeta^4 + \frac{1}{120}D_2D_3a^5\zeta^5) \\
&\quad + \ell^2 C
\end{aligned}$$

$$\begin{aligned}
R(1,2) &= E\ell^2(I_y - I_z)w_{,xx}^0 \\
&= R_3(-D_2 - D_1a\zeta + \frac{1}{2}D_2a^2\zeta^2 + \frac{1}{6}D_1a^3\zeta^3 \\
&\quad - \frac{1}{24}D_2a^4\zeta^4 - \frac{1}{120}D_1a^5\zeta^5)
\end{aligned}$$

$$\begin{aligned}
R(2,1) &= -\ell^2[\frac{3}{2}C - E(I_y - I_z)]w_{,xx}^0 \\
&= -R_4(-D_2 - D_1a\zeta + \frac{1}{2}D_2a^2\zeta^2 + \frac{1}{6}D_1a^3\zeta^3 - \frac{1}{24}D_2a^4\zeta^4 \\
&\quad - \frac{1}{120}D_1a^5\zeta^5)
\end{aligned}$$

$$
\begin{aligned}
R(2,2) &= \ell^2(3EI_z\varepsilon_{0,xx}+F_1) \\
&= -3R_5\{[D_1^2-D_2^2-4D_1D_2a\zeta-2(D_1^2-D_2^2)a^2\zeta^2 \\
&\quad +\frac{8}{3}D_1D_2a^3\zeta^3+\frac{2}{3}(D_1^2-D_2^2)a^4\zeta^4-\frac{8}{15}D_1D_2a^5\zeta^5]+\frac{1}{a}(D_1D_3 \\
&\quad -D_2D_3a\zeta-\frac{1}{2}D_1D_3a^2\zeta^2+\frac{1}{6}D_2D_3a^3\zeta^3+\frac{1}{24}D_1D_3a^4\zeta^4 \\
&\quad -\frac{1}{120}D_2D_3a^5\zeta^5)\}+\ell^2F_1
\end{aligned}
$$

其中

$$R_2=\frac{EI_pI_y}{\ell^2A}a^4 \quad , \quad R_3=E(I_y-I_z)a^2$$

$$R_4=[\frac{3}{2}C-E(I_y-I_z)]a^2 \quad , \quad R_5=EI_z\frac{2I_y}{\ell^4A}a^6, \ \varepsilon_p=\frac{F_1}{EA},$$

$$F_1=F_x\cos\phi^e-F_y\sin\phi^e, \ F_x=-P$$

收集 $R(i,j)$ 中 ζ^i 的同次項，即可將 $\mathbf{R}$ 表示成 $\mathbf{R}=\sum_{i=0}^{5}\zeta^i\mathbf{R}_i$

$$\mathbf{R}_0=\begin{bmatrix} EI_p\varepsilon_p\ell^2+R_2D_1^2+\frac{R_2}{a}D_1D_3+l^2C & -R_3D_2 \\ R_4D_2 & -3R_5(D_1^2-D_2^2)-\frac{3R_5}{2a}D_1D_3+\ell^2F_1 \end{bmatrix}$$

$$\mathbf{R}_1=\begin{bmatrix} -2R_2D_1D_2a-R_2D_2D_3 & -R_3D_1a \\ R_4D_1a & 12R_5D_1D_2a+3R_5D_2D_3 \end{bmatrix}$$

$$\mathbf{R}_2=\begin{bmatrix} -R_2(D_1^2-D_2^2)a^2-\frac{R_2}{2}D_1D_3a & \frac{1}{2}R_3D_2a^2 \\ -\frac{1}{2}R_4D_2a^2 & 6R_5(D_1^2-D_2^2)a^2+\frac{3}{2}R_5D_1D_3a \end{bmatrix}$$

$$\mathbf{R}_3 = \begin{bmatrix} \frac{4R_2}{3} D_1 D_2 a^3 + \frac{R_2}{6} D_2 D_3 a^2 & \frac{1}{6} R_3 D_1 a^3 \\ -\frac{1}{6} R_4 D_1 a^3 & -8R_5 D_1 D_2 a^3 - \frac{R_5}{2} D_2 D_3 a^2 \end{bmatrix}$$

$$\mathbf{R}_4 = \begin{bmatrix} \frac{R_2}{3} (D_1^2 - D_2^2) a^4 + \frac{R_2}{24} D_1 D_3 a^3 & -\frac{1}{24} R_3 D_2 a^4 \\ \frac{1}{24} R_4 D_2 a^4 & -2R_5 (D_1^2 - D_2^2) a^4 - \frac{R_5}{8} D_1 D_3 a^3 \end{bmatrix}$$

$$\mathbf{R}_5 = \begin{bmatrix} \frac{-4R_2}{15} D_1 D_2 a^5 - \frac{R_2}{120} D_2 D_3 a^4 & -\frac{1}{120} R_3 D_1 a^5 \\ \frac{1}{120} R_4 D_1 a^5 & \frac{8}{5} R_5 D_1 D_2 a^5 - \frac{R_5}{40} D_2 D_3 a^4 \end{bmatrix}$$

(G.2)

令

$$\mathbf{S} = \begin{bmatrix} S(1,1) & S(1,2) \\ S(2,1) & S(2,2) \end{bmatrix}$$

$$\begin{aligned} S(1,1) &= \ell^3 (E\varepsilon_{0,x} I_P - 2C\varepsilon_{0,x}) \\ &= S_1[D_1 D_2 + (D_1^2 - D_2^2) a\zeta - 2D_1 D_2 a^2 \zeta^2 - \frac{2}{3}(D_1^2 - D_2^2) a^3 \zeta^3 \\ &+ \frac{2}{3} D_1 D_2 a^4 \zeta^4 + \frac{2}{15}(D_1^2 - D_2^2) a^5 \zeta^5] + \frac{S_1}{2a}(D_2 D_3 + D_1 D_3 a\zeta \\ &- \frac{1}{2} D_2 D_3 a^2 \zeta^2 - \frac{1}{6} D_1 D_3 a^3 \zeta^3 + \frac{1}{24} D_2 D_3 a^4 \zeta^4 + \frac{1}{120} D_1 D_3 a^5 \zeta^5) \end{aligned}$$

$$S(1,2) = -\frac{1}{2}\ell^3 C w^0_{,xxx}$$

$$= \frac{-Ca^3}{2}(-D_1 + D_2 a\zeta + \frac{1}{2}D_1 a^2\zeta^2 - \frac{1}{6}D_2 a^3\zeta^3$$

$$-\frac{1}{24}D_1 a^4\zeta^4 + \frac{1}{120}D_2 a^5\zeta^5)$$

$$S(2,1) = -\ell^3[C - 2E(I_y - I_z)]w^0_{,xxx}$$

$$= -S_2(-D_1 + D_2 a\zeta + \frac{1}{2}D_1 a^2\zeta^2 - \frac{1}{6}D_2 a^3\zeta^3$$

$$-\frac{1}{24}D_1 a^4\zeta^4 + \frac{1}{120}D_2 a^5\zeta^5)$$

$$S(2,2) = \ell^3 EI_z \varepsilon_{0,xxx}$$

$$= 8S_3[D_1D_2 + (D_1^2 - D_2^2)a\zeta - 2D_1D_2 a^2\zeta^2 - \frac{2}{3}(D_1^2 - D_2^2)a^3\zeta^3$$

$$+\frac{2}{3}D_1D_2 a^4\zeta^4 + \frac{2}{15}(D_1^2 - D_2^2)a^5\zeta^5] + \frac{S_3}{a}(D_2D_3 + D_1D_3 a\zeta$$

$$-\frac{1}{2}D_2D_3 a^2\zeta^2 - \frac{1}{6}D_1D_3 a^3\zeta^3 + \frac{1}{24}D_2D_3 a^4\zeta^4 + \frac{1}{120}D_1D_3 a^5\zeta^5)$$

其中

$$S_1 = \frac{-2(EI_P - 2C)I_y a^5}{A\ell^2}, \quad S_2 = [C - 2E(I_y - I_z)]a^3, \quad S_3 = \frac{EI_y I_z}{A\ell^4}a^7$$

收集 $S(i,j)$ 中 ζ^i 的同次項，即可將 $\mathbf{S}$ 表示成 $\mathbf{S} = \sum_{i=0}^{5}\zeta^i \mathbf{S}_i$

$$\mathbf{S}_0 = \begin{bmatrix} S_1D_1D_2 + \frac{S_1}{2a}D_2D_3 & \frac{1}{2}CD_1a^3 \\ S_2D_1 & 8S_3D_1D_2 + \frac{S_3}{a}D_2D_3 \end{bmatrix}$$

$$\mathbf{S}_1 = \begin{bmatrix} S_1(D_1^2 - D_2^2)a + \frac{1}{2}S_1D_1D_3 & \frac{-1}{2}CD_2a^4 \\ -S_2D_2a & 8S_3(D_1^2 - D_2^2)a + S_3D_1D_3 \end{bmatrix}$$

$$\mathbf{S}_2 = \begin{bmatrix} -2S_1D_1D_2a^2 - \frac{S_1}{4}D_2D_3a & \frac{-1}{4}CD_1a^5 \\ -\frac{1}{2}S_2D_1a^2 & -16S_3D_1D_2a^2 - \frac{1}{2}S_3D_2D_3a \end{bmatrix}$$

$$\mathbf{S}_3 = \begin{bmatrix} -\frac{2}{3}S_1(D_1^2 - D_2^2)a^3 - \frac{1}{12}S_1D_1D_3a^2 & \frac{1}{12}CD_2a^6 \\ \frac{1}{6}S_2D_2a^3 & -\frac{16}{3}S_3(D_1^2 - D_2^2)a^3 - \frac{1}{6}S_3D_1D_3a^2 \end{bmatrix}$$

$$\mathbf{S}_4 = \begin{bmatrix} \frac{2}{3}S_1D_1D_2a^4 + \frac{S_1}{48}D_2D_3a^3 & \frac{1}{48}CD_1a^7 \\ \frac{1}{24}S_2D_1a^4 & \frac{16}{3}S_3D_1D_2a^4 + \frac{S_3}{24}D_2D_3a^3 \end{bmatrix}$$

$$\mathbf{S}_5 = \begin{bmatrix} \frac{2}{15}S_1(D_1^2 - D_2^2)a^5 + \frac{S_1}{240}D_1D_3a^4 & -\frac{1}{240}CD_2a^8 \\ -\frac{1}{120}S_2D_2a^5 & \frac{16}{15}S_3(D_1^2 - D_2^2)a^5 + \frac{S_3}{120}D_1D_3a^4 \end{bmatrix} \qquad \text{(G.3)}$$

令

$$\mathbf{T} = \begin{bmatrix} T(1,1) & T(1,2) \\ T(2,1) & T(2,2) \end{bmatrix}$$

$$T(1,1) = T(1,2) = T(2,2) = 0$$

$$T(2,1) = \ell^4 E(I_y - I_z) w^0_{,xxxx}$$

$$= T_1(D_2 + D_1 a\zeta - \frac{1}{2} D_2 a^2 \zeta^2 - \frac{1}{6} D_1 a^3 \zeta^3 + \frac{1}{24} D_2 a^4 \zeta^4 + \frac{1}{120} D_1 a^5 \zeta^5)$$

其中 $T_1 = E(I_y - I_z) a^4$

收集 $T(i,j)$ 中 ζ^i 的同次項，即可將 $\mathbf{T}$ 表示成 $\mathbf{T} = \sum_{i=0}^{5} \zeta^i \mathbf{T}_i$

$$\mathbf{T}_0 = \begin{bmatrix} 0 & 0 \\ T_1 D_2 & 0 \end{bmatrix} \qquad \mathbf{T}_1 = \begin{bmatrix} 0 & 0 \\ T_1 D_1 a & 0 \end{bmatrix}$$

$$\mathbf{T}_2 = \begin{bmatrix} 0 & 0 \\ -\frac{1}{2} T_1 D_2 a^2 & 0 \end{bmatrix} \qquad \mathbf{T}_3 = \begin{bmatrix} 0 & 0 \\ -\frac{1}{6} T_1 D_1 a^3 & 0 \end{bmatrix}$$

$$\mathbf{T}_4 = \begin{bmatrix} 0 & 0 \\ \frac{1}{24} T_1 D_2 a^4 & 0 \end{bmatrix} \qquad \mathbf{T}_5 = \begin{bmatrix} 0 & 0 \\ \frac{1}{120} T_1 D_1 a^5 & 0 \end{bmatrix} \tag{G.4}$$

附錄 H 軸力為拉力時$\theta_2^0, \varepsilon_0$及其微分的級數表示

由(2.4.4)式及其微分，及(3.3.26)式可得

$$w_{,x}^0 = \frac{dw}{dx} = \frac{1}{\ell}\frac{dw}{d\zeta} \qquad w_{,xx}^0 = \frac{1}{\ell^2}\frac{d^2 w}{d\zeta^2}$$
$$w_{,xxx}^0 = \frac{1}{\ell^3}\frac{d^3 w}{d\zeta^3} \qquad w_{,xxxx}^0 = \frac{1}{\ell^4}\frac{d^4 w}{d\zeta^4} \tag{H.1}$$

由(3.2.17)式及其微分可得

$$
\begin{aligned}
w(\zeta) &= D_1 \sinh a\zeta + D_2 \cosh a\zeta + D_3\zeta + D_4 \\
w_{,\zeta}(\zeta) &= aD_1 \cosh a\zeta + aD_2 \sinh a\zeta + D_3 \\
w_{,\zeta\zeta}(\zeta) &= a^2 D_1 \sinh a\zeta + a^2 D_2 \cosh a\zeta \\
w_{,\zeta\zeta\zeta}(\zeta) &= a^3 D_1 \cosh a\zeta + a^3 D_2 \sinh a\zeta \\
w_{,\zeta\zeta\zeta\zeta}(\zeta) &= a^4 D_1 \sinh a\zeta + a^4 D_2 \cosh a\zeta = a^2 w_{,\zeta\zeta}(\zeta) \\
w_{,\zeta\zeta\zeta\zeta\zeta}(\zeta) &= a^5 D_1 \cosh a\zeta + a^5 D_2 \sinh a\zeta = a^2 w_{,\zeta\zeta\zeta}(\zeta) \\
w_{,\zeta\zeta\zeta\zeta\zeta\zeta}(\zeta) &= a^6 D_1 \sinh a\zeta + a^6 D_2 \cosh a\zeta \\
&= a^2 w_{,\zeta\zeta\zeta\zeta}(\zeta) = a^4 w_{,\zeta\zeta}(\zeta)
\end{aligned} \tag{H.2}
$$

將(H.2)式中$\sinh a\zeta$及$\cosh a\zeta$用泰勒級數展開取其前三項可得

$$\sinh a\zeta = a\zeta + \frac{(a\zeta)^3}{3!} + \frac{(a\zeta)^5}{5!}$$
$$\cosh a\zeta = 1 + \frac{(a\zeta)^2}{2!} + \frac{(a\zeta)^4}{4!} \tag{H.3}$$

由(H.1)~(H.3)式可得

$$w^0_{,x} = \frac{1}{\ell}\frac{dw}{d\zeta} = \frac{1}{\ell}(aD_1 \cosh a\zeta + aD_2 \sinh a\zeta + D_3)$$

$$= \frac{1}{\ell}\{aD_1[1 + \frac{(a\zeta)^2}{2!} + \frac{(a\zeta)^4}{4!}] + aD_2[a\zeta + \frac{(a\zeta)^3}{3!} + \frac{(a\zeta)^5}{5!}] + D_3\}$$

$$= \frac{1}{\ell}(aD_1 + D_3 + D_2 a^2 \zeta + \frac{1}{2}D_1 a^3 \zeta^2 + \frac{1}{6}D_2 a^4 \zeta^3 + \frac{1}{24}D_1 a^5 \zeta^4$$

$$+ \frac{1}{120}D_2 a^6 \zeta^5)$$

(H.4)

$$w^0_{,xx} = \frac{1}{\ell^2}\frac{d^2 w}{d\zeta^2} = \frac{a^2}{\ell^2}(D_1 \sinh a\zeta + D_2 \cosh a\zeta)$$

$$= \frac{a^2}{\ell^2}\{D_1[a\zeta + \frac{(a\zeta)^3}{3!} + \frac{(a\zeta)^5}{5!}] + D_2[1 + \frac{(a\zeta)^2}{2!} + \frac{(a\zeta)^4}{4!}]\}$$

$$= \frac{a^2}{\ell^2}(D_2 + D_1 a\zeta + \frac{1}{2}D_2 a^2 \zeta^2 + \frac{1}{6}D_1 a^3 \zeta^3 + \frac{1}{24}D_2 a^4 \zeta^4$$

$$+ \frac{1}{120}D_1 a^5 \zeta^5)$$

(H.5)

$$w^0_{,xxx} = \frac{1}{\ell^3}\frac{d^3 w}{d\zeta^3} = \frac{1}{\ell^3}(a^3 D_1 \cosh a\zeta + a^3 D_2 \sinh a\zeta)$$

$$= \frac{a^3}{\ell^3}\{D_1[1 + \frac{(a\zeta)^2}{2!} + \frac{(a\zeta)^4}{4!}] + D_2[a\zeta + \frac{(a\zeta)^3}{3!} + \frac{(a\zeta)^5}{5!}]\}$$

$$= \frac{a^3}{\ell^3}(D_1 + D_2 a\zeta + \frac{1}{2}D_1 a^2 \zeta^2 + \frac{1}{6}D_2 a^3 \zeta^3 + \frac{1}{24}D_1 a^4 \zeta^4$$

$$+ \frac{1}{120}D_2 a^5 \zeta^5)$$

(H.6)

$$w^0_{,xxxx} = \frac{1}{\ell^4}\frac{d^4 w}{d\zeta^4} = \frac{1}{\ell^4} a^2 w_{,\zeta\zeta}(\zeta)$$

$$= \frac{a^4}{\ell^4}(D_2 + D_1 a\zeta + \frac{1}{2}D_2 a^2\zeta^2 + \frac{1}{6}D_1 a^3\zeta^3 + \frac{1}{24}D_2 a^4\zeta^4 + \frac{1}{120}D_1 a^5\zeta^5) \tag{H.7}$$

由(3.3.23)式及(H.1)~(H.3)式可得

$$\varepsilon_0 = \varepsilon_p - \frac{I_y w^0_{,x} w^0_{,xxx}}{A}$$

$$= \varepsilon_p - \frac{I_y}{\ell^4 A} w_{,\zeta} w_{,\zeta\zeta\zeta}$$

$$= \varepsilon_p - \frac{I_y}{\ell^4 A}[a^4(D_1^2\frac{1+\cosh 2a\zeta}{2} + D_1D_2\sinh 2a\zeta + D_2^2\frac{\cosh 2a\zeta - 1}{2}) + a^3(D_1D_3\cosh a\zeta + D_2D_3\sinh a\zeta)]$$

$$= \varepsilon_p - \frac{I_y a^4}{\ell^4 A}[D_1^2 + 2D_1D_2 a\zeta + (D_1^2 + D_2^2)a^2\zeta^2 + \frac{4}{3}D_1D_2 a^3\zeta^3 + \frac{1}{3}(D_1^2 + D_2^2)a^4\zeta^4 + \frac{4}{15}D_1D_2 a^5\zeta^5] - \frac{I_y a^3}{\ell^4 A}(D_1D_3 + D_2D_3 a\zeta + 2D_1D_3 a^2\zeta^2 + \frac{1}{6}D_2D_3 a^3\zeta^3 + \frac{2}{3}D_1D_3 a^4\zeta^4 + \frac{1}{120}D_2D_3 a^5\zeta^5)$$

其中

$$\varepsilon_p = \frac{F_1}{EA}\text{，} F_1 = F_x\cos\phi^e - F_y\sin\phi^e\text{，} F_x = P \tag{H.8}$$

$$
\begin{aligned}
\varepsilon_{0,x} &= \frac{-I_y}{A}(w^0_{,xx}w^0_{,xxx} + w^0_{,x}w^0_{,xxxx}) \\
&= \frac{-I_y}{A}\{\frac{a^5}{\ell^5}(D_1^2 + D_2^2)\frac{\sinh 2a\zeta}{2} + \frac{a^5}{\ell^5}D_1D_2[\frac{\cosh 2a\zeta - 1}{2} + \frac{1 + \cosh 2a\zeta}{2}] \\
&\quad + \frac{a^5}{\ell^5}(D_1^2 + D_2^2)\frac{\sinh 2a\zeta}{2} + \frac{a^5}{\ell^5}D_1D_2(\frac{1 + \cosh 2a\zeta}{2} + \frac{\cosh 2a\zeta - 1}{2}) \\
&\quad + \frac{a^4}{\ell^5}(D_1D_3\sinh a\zeta + D_2D_3\cosh a\zeta)\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{-2I_y a^5}{A\ell^5}[D_1D_2 + (D_1^2 + D_2^2)a\zeta + 2D_1D_2a^2\zeta^2 + \frac{2}{3}(D_1^2 + D_2^2)a^3\zeta^3 \\
&\quad + \frac{2}{3}D_1D_2a^4\zeta^4 + \frac{2}{15}(D_1^2 + D_2^2)a^5\zeta^5] - \frac{I_y a^4}{A\ell^5}(D_2D_3 + D_1D_3a\zeta \\
&\quad + \frac{1}{2}D_2D_3a^2\zeta^2 + \frac{1}{6}D_1D_3a^3\zeta^3 + \frac{1}{24}D_2D_3a^4\zeta^4 + \frac{1}{120}D_1D_3a^5\zeta^5)
\end{aligned} \tag{H.9}
$$

$$
\varepsilon_{0,xx} = -\frac{I_y}{A}[(w^0_{,xxx})^2 + 2w^0_{,xx}w^0_{,xxxx} + w^0_{,x}w^0_{,xxxxx}] \tag{H.10}
$$

其中

$$
\begin{aligned}
(w^0_{,xxx})^2 &= (\frac{1}{\ell^3}w_{,\zeta\zeta\zeta})^2 = \frac{1}{\ell^6}w^2_{,\zeta\zeta\zeta} \\
&= \frac{1}{\ell^6}(a^3D_1\cosh a\zeta + a^3D_2\sinh a\zeta)^2 \\
&= \frac{1}{\ell^6}(a^6D_1^2\cosh^2 a\zeta + 2a^6D_1D_2\cosh a\zeta\sinh a\zeta + a^6D_2^2\sinh^2 a\zeta) \\
&= \frac{a^6}{\ell^6}(D_1^2\frac{1 + \cosh 2a\zeta}{2} + D_1D_2\sinh 2a\zeta + D_2^2\frac{\cosh 2a\zeta - 1}{2})
\end{aligned}
$$

$$=\frac{a^6}{\ell^6}[\frac{1}{2}(D_1^2-D_2^2)+\frac{1}{2}(D_1^2+D_2^2)\cosh 2a\zeta+D_1D_2\sinh 2a\zeta]$$

$$=\frac{a^6}{\ell^6}\{\frac{1}{2}(D_1^2-D_2^2)+\frac{1}{2}(D_1^2+D_2^2)[1+\frac{(2a\zeta)^2}{2!}+\frac{(2a\zeta)^4}{4!}]$$

$$+D_1D_2[2a\zeta+\frac{(2a\zeta)^3}{3!}+\frac{(2a\zeta)^5}{5!}]\}$$

$$=\frac{a^6}{\ell^6}[D_1^2+2D_1D_2a\zeta+(D_1^2+D_2^2)a^2\zeta^2+\frac{4}{3}D_1D_2a^3\zeta^3$$

$$+\frac{1}{3}(D_1^2+D_2^2)a^4\zeta^4+\frac{4}{15}D_1D_2a^5\zeta^5]$$

(H.11a)

$$w^0_{,xx}w^0_{,xxxx}$$

$$=\frac{1}{\ell^6}w_{,\zeta\zeta}w_{,\zeta\zeta\zeta\zeta}$$

$$=\frac{1}{\ell^6}w_{,\zeta\zeta}a^2w_{,\zeta\zeta}=\frac{a^2}{\ell^6}w^2_{,\zeta\zeta}$$

$$=\frac{a^2}{\ell^6}(a^2D_1\sinh a\zeta+a^2D_2\cosh a\zeta)^2$$

$$=\frac{a^6}{\ell^6}(D_1^2\frac{\cosh 2a\zeta-1}{2}+D_1D_2\sinh 2a\zeta+D_2^2\frac{1+\cosh 2a\zeta}{2})$$

$$=\frac{a^6}{\ell^6}[D_2^2+2D_1D_2a\zeta+(D_1^2+D_2^2)a^2\zeta^2+\frac{4}{3}D_1D_2a^3\zeta^3$$

$$+\frac{1}{3}(D_1^2+D_2^2)a^4\zeta^4+\frac{4}{15}D_1D_2a^5\zeta^5]$$

(H.11b)

$$
\begin{aligned}
&w^0_{,x}w^0_{,xxxxx} \\
&=\frac{1}{\ell^6}w_{,\zeta}w_{,\zeta\zeta\zeta\zeta\zeta} \\
&=\frac{a^2}{\ell^6}w_{,\zeta}w_{,\zeta\zeta\zeta} \\
&=\frac{a^6}{\ell^6}[D_1^2+2D_1D_2a\zeta+(D_1^2+D_2^2)a^2\zeta^2+\frac{4}{3}D_1D_2a^3\zeta^3+\frac{1}{3}(D_1^2+D_2^2)a^4\zeta^4 \\
&\quad+\frac{4}{15}D_1D_2a^5\zeta^5]+\frac{a^5}{\ell^6}(D_1D_3+D_2D_3a\zeta+2D_1D_3a^2\zeta^2+\frac{1}{6}D_2D_3a^3\zeta^3 \\
&\quad+\frac{2}{3}D_1D_3a^4\zeta^4+\frac{1}{120}D_2D_3a^5\zeta^5)
\end{aligned}
$$

(H.11c)

由上面(H.11a)、(H.11b)、(H.11c)三式代入$\varepsilon_{0,xx}$可得

$$
\begin{aligned}
\varepsilon_{0,xx}&=\frac{-I_ya^6}{A\ell^6}[2(D_1^2+D_2^2)+8D_1D_2a\zeta+4(D_1^2+D_2^2)a^2\zeta^2+\frac{16}{3}D_1D_2a^3\zeta^3 \\
&\quad+\frac{4}{3}(D_1^2+D_2^2)a^4\zeta^4+\frac{16}{15}D_1D_2a^5\zeta^5]-\frac{I_ya^5}{A\ell^6}(D_1D_3+D_2D_3a\zeta+2D_1D_3a^2\zeta^2 \\
&\quad+\frac{1}{6}D_2D_3a^3\zeta^3+\frac{2}{3}D_1D_3a^4\zeta^4+\frac{1}{120}D_2D_3a^5\zeta^5)
\end{aligned}
$$

(H.12)

$$
\varepsilon_{0,xxx}=\frac{-I_y}{A}(4w^0_{,xxx}w^0_{,xxxx}+3w^0_{,xx}w^0_{,xxxxx}+w^0_{,x}w^0_{,xxxxxx})
$$

(H.13)

其中

$$
\begin{aligned}
&w^0_{,xxx}w^0_{,xxxx} \\
&=\frac{1}{\ell^7}w_{,\zeta\zeta\zeta}w_{,\zeta\zeta\zeta\zeta} \\
&=\frac{a^2}{\ell^7}w_{,\zeta\zeta\zeta}w_{,\zeta\zeta}
\end{aligned}
$$

$$=\frac{a^2}{\ell^7}a^5[D_1D_2+(D_1^2+D_2^2)a\zeta+2D_1D_2a^2\zeta^2+\frac{2}{3}(D_1^2+D_2^2)a^3\zeta^3$$
$$+\frac{2}{3}D_1D_2a^4\zeta^4+\frac{2}{15}(D_1^2+D_2^2)a^5\zeta^5]$$
$$=\frac{a^7}{\ell^7}[D_1D_2+(D_1^2+D_2^2)a\zeta+2D_1D_2a^2\zeta^2+\frac{2}{3}(D_1^2+D_2^2)a^3\zeta^3$$
$$+\frac{2}{3}D_1D_2a^4\zeta^4+\frac{2}{15}(D_1^2+D_2^2)a^5\zeta^5] \qquad \text{(H.14a)}$$

$$w^0_{,xx}w^0_{,xxxxx}$$
$$=\frac{1}{\ell^7}w_{,\zeta\zeta}w_{,\zeta\zeta\zeta\zeta\zeta}$$
$$=\frac{a^2}{\ell^7}w_{,\zeta\zeta}w_{,\zeta\zeta\zeta}$$
$$=\frac{a^2}{\ell^7}a^5[D_1D_2+(D_1^2+D_2^2)a\zeta+2D_1D_2a^2\zeta^2+\frac{2}{3}(D_1^2+D_2^2)a^3\zeta^3$$
$$+\frac{2}{3}D_1D_2a^4\zeta^4+\frac{2}{15}(D_1^2+D_2^2)a^5\zeta^5]$$
$$=\frac{a^7}{\ell^7}[D_1D_2+(D_1^2+D_2^2)a\zeta+2D_1D_2a^2\zeta^2+\frac{2}{3}(D_1^2+D_2^2)a^3\zeta^3$$
$$+\frac{2}{3}D_1D_2a^4\zeta^4+\frac{2}{15}(D_1^2+D_2^2)a^5\zeta^5] \qquad \text{(H.14b)}$$

$$w^0_{,x}w^0_{,xxxxxx}$$
$$=\frac{1}{\ell^7}w_{,\zeta}w_{,\zeta\zeta\zeta\zeta\zeta\zeta}$$
$$=\frac{a^2}{\ell^7}w_{,\zeta}w_{,\zeta\zeta\zeta\zeta}=\frac{a^4}{\ell^7}w_{,\zeta}w_{,\zeta\zeta}$$

$$
=\frac{a^7}{\ell^7}[D_1D_2+(D_1^2+D_2^2)a\zeta+2D_1D_2a^2\zeta^2+\frac{2}{3}(D_1^2+D_2^2)a^3\zeta^3
+\frac{2}{3}D_1D_2a^4\zeta^4+\frac{2}{15}(D_1^2+D_2^2)a^5\zeta^5]+\frac{a^6}{\ell^7}(D_2D_3+D_1D_3a\zeta
+\frac{1}{2}D_2D_3a^2\zeta^2+\frac{1}{6}D_1D_3a^3\zeta^3+\frac{1}{24}D_2D_3a^4\zeta^4+\frac{1}{120}D_1D_3a^5\zeta^5)
\tag{H.14c}
$$

由上面(H.14a)、(H.14b) 、(H.14c)三式代入$\varepsilon_{0,xxx}$可得

$$
\varepsilon_{0,xxx}=\frac{-8I_ya^7}{A\ell^7}[D_1D_2+(D_1^2+D_2^2)a\zeta+2D_1D_2a^2\zeta^2+\frac{2}{3}(D_1^2+D_2^2)a^3\zeta^3
+\frac{2}{3}D_1D_2a^4\zeta^4+\frac{2}{15}(D_1^2+D_2^2)a^5\zeta^5]-\frac{I_ya^6}{A\ell^7}(D_2D_3+D_1D_3a\zeta
+\frac{1}{2}D_2D_3a^2\zeta^2+\frac{1}{6}D_1D_3a^3\zeta^3+\frac{1}{24}D_2D_3a^4\zeta^4+\frac{1}{120}D_1D_3a^5\zeta^5)
\tag{H.15}
$$

附錄I 軸力為拉力時**Q,R,S,T**的級數表示

將附錄H的結果代入(3.3.28)式，即可將**Q,R,S,T**表示成如(3.3.31)式的級數形式，其推導過程如下：

令

$$\mathbf{Q}=\begin{bmatrix} Q(1,1) & Q(1,2) \\ Q(2,1) & Q(2,2) \end{bmatrix}$$ 由附錄H及(3.3.28)式可得

$$Q(1,1)=0$$

$$Q(1,2)=\frac{C\ell w_{,x}^{0}}{2}$$
$$=\frac{C}{2}(aD_1+D_3+D_2a^2\zeta+\frac{1}{2}D_1a^3\zeta^2+\frac{1}{6}D_2a^4\zeta^3$$
$$+\frac{1}{24}D_1a^5\zeta^4+\frac{1}{120}D_2a^6\zeta^5)$$

$$Q(2,1)=-Q(1,2)$$

$$Q(2,2)=4EI_z\ell\varepsilon_{0,x}$$
$$=-2Q_2[D_1D_2+(D_1^2+D_2^2)a\zeta+2D_1D_2a^2\zeta^2+\frac{2}{3}(D_1^2+D_2^2)a^3\zeta^3$$
$$+\frac{2}{3}D_1D_2a^4\zeta^4+\frac{2}{15}(D_1^2+D_2^2)a^5\zeta^5]-\frac{Q_2}{a}(D_2D_3+D_1D_3a\zeta$$
$$+\frac{1}{2}D_2D_3a^2\zeta^2+\frac{1}{6}D_1D_3a^3\zeta^3+\frac{1}{24}D_2D_3a^4\zeta^4+\frac{1}{120}D_1D_3a^5\zeta^5)$$

其中 $Q_2=\dfrac{4EI_yI_z}{\ell^4A}a^5$

收集 $Q(i,j)$ 中 ζ^i 的同次項，即可將 $\mathbf{Q}$ 表示成 $\mathbf{Q}=\sum_{i=0}^{5}\zeta^i\mathbf{Q}_i$

$$\mathbf{Q}_0 = \begin{bmatrix} 0 & \frac{C}{2}(aD_1 + D_3) \\ -\frac{C}{2}(aD_1 + D_3) & -2Q_2 D_1 D_2 - \frac{Q_2}{a} D_2 D_3 \end{bmatrix}$$

$$\mathbf{Q}_1 = \begin{bmatrix} 0 & \frac{C}{2} D_2 a^2 \\ -\frac{C}{2} D_2 a^2 & -Q_2[2(D_1^2 + D_2^2)a + D_1 D_3] \end{bmatrix}$$

$$\mathbf{Q}_2 = \begin{bmatrix} 0 & \frac{C}{4} D_1 a^3 \\ -\frac{C}{4} D_1 a^3 & -Q_2 a(4D_1 D_2 a + \frac{1}{2} D_2 D_3) \end{bmatrix}$$

$$\mathbf{Q}_3 = \begin{bmatrix} 0 & \frac{C}{12} D_2 a^4 \\ -\frac{C}{12} D_2 a^4 & -Q_2 a^2[\frac{4}{3}(D_1^2 + D_2^2)a + \frac{1}{6} D_1 D_3] \end{bmatrix}$$

$$\mathbf{Q}_4 = \begin{bmatrix} 0 & \frac{C}{48} D_1 a^5 \\ -\frac{C}{48} D_1 a^5 & \frac{-Q_2 a^3}{3}(4D_1 D_2 a + \frac{1}{8} D_2 D_3) \end{bmatrix}$$

$$\mathbf{Q}_5 = \begin{bmatrix} 0 & \frac{C}{240} D_2 a^6 \\ -\frac{C}{240} D_2 a^6 & \frac{-Q_2 a^4}{15}[4(D_1^2 + D_2^2)a + \frac{1}{8} D_1 D_3] \end{bmatrix}$$

(I.1)

同理，令

$$\mathbf{R}=\begin{bmatrix} R(1,1) & R(1,2) \\ R(2,1) & R(2,2) \end{bmatrix}$$

$$R(1,1)=\ell^2(E\varepsilon_0 I_P + C)$$
$$=\ell^2 EI_p\varepsilon_p - R_2[D_1^2 + 2D_1D_2a\zeta + (D_2^2 + D_1^2)a^2\zeta^2 + \frac{4}{3}D_1D_2a^3\zeta^3$$
$$+\frac{1}{3}(D_2^2 + D_1^2)a^4\zeta^4 + \frac{4}{15}D_1D_2a^5\zeta^5] - \frac{R_2}{a}(D_1D_3 + D_2D_3a\zeta$$
$$+2D_1D_3a^2\zeta^2 + \frac{1}{6}D_2D_3a^3\zeta^3 + \frac{2}{3}D_1D_3a^4\zeta^4 + \frac{1}{120}D_2D_3a^5\zeta^5)$$
$$+\ell^2 C$$

$$R(1,2)=E\ell^2(I_y - I_z)w^0_{,xx}$$
$$=R_3D_2 + R_3D_1a\zeta + \frac{1}{2}R_3D_2a^2\zeta^2 + \frac{1}{6}R_3D_1a^3\zeta^3$$
$$+\frac{1}{24}R_3D_2a^4\zeta^4 + \frac{1}{120}R_3D_1a^5\zeta^5$$

$$R(2,1)=-\ell^2[\frac{3}{2}C - E(I_y - I_z)]w^0_{,xx}$$
$$=-R_4(D_2 + D_1a\zeta + \frac{1}{2}D_2a^2\zeta^2 + \frac{1}{6}D_1a^3\zeta^3 + \frac{1}{24}D_2a^4\zeta^4$$
$$+\frac{1}{120}D_1a^5\zeta^5)$$

$$R(2,2)=\ell^2(3EI_z\varepsilon_{0,xx} + F_1)$$
$$=-3R_5[D_1^2 + D_2^2 + 4D_1D_2a\zeta + 2(D_1^2 + D_2^2)a^2\zeta^2$$
$$+\frac{8}{3}D_1D_2a^3\zeta^3 + \frac{2}{3}(D_1^2 + D_2^2)a^4\zeta^4 + \frac{8}{15}D_1D_2a^5\zeta^5]$$
$$-\frac{3R_5}{2a}(D_1D_3 + D_2D_3a\zeta + 2D_1D_3a^2\zeta^2 + \frac{1}{6}D_2D_3a^3\zeta^3$$
$$+\frac{2}{3}D_1D_3a^4\zeta^4 + \frac{1}{120}D_2D_3a^5\zeta^5) + \ell^2 F_1$$

其中

$$R_2=\frac{EI_pI_y}{\ell^2 A}a^4\ ,\ R_3=E(I_y-I_z)a^2,\ R_4=[\frac{3}{2}C-E(I_y-I_z)]a^2,$$

$$R_5=EI_z\frac{2I_y}{\ell^4 A}a^6,\ \varepsilon_p=\frac{F_1}{EA},\ F_1=F_x\cos\phi^e-F_y\sin\phi^e,\ F_x=P$$

收集 $R(i,j)$ 中 ζ^i 的同次項，即可將 $\mathbf{R}$ 表示成 $\mathbf{R}=\sum_{i=0}^{5}\zeta^i\mathbf{R}_i$

$$\mathbf{R}_0=\begin{bmatrix}\ell^2C+\ell^2EI_p\varepsilon_p-R_2(D_1^2+\frac{D_1D_3}{a}) & R_3D_2\\ -R_4D_2 & -3R_5[(D_1^2+D_2^2)+\frac{D_1D_3}{2a}]+\ell^2F_1\end{bmatrix}$$

$$\mathbf{R}_1=\begin{bmatrix}-R_2(2D_1D_2a+D_2D_3) & R_3D_1a\\ -R_4D_1a & -3R_5(4D_1D_2a+\frac{D_2D_3}{2})\end{bmatrix}$$

$$\mathbf{R}_2=\begin{bmatrix}-R_2a[(D_1^2+D_2^2)a+2D_1D_3] & \frac{1}{2}R_3D_2a^2\\ -\frac{1}{2}R_4D_2a^2 & -R_5a[6(D_1^2+D_2^2)a+3D_1D_3]\end{bmatrix}$$

$$\mathbf{R}_3=\begin{bmatrix}-\frac{1}{3}R_2a^2(4D_1D_2a+\frac{1}{2}D_2D_3) & \frac{1}{6}R_3D_1a^3\\ -\frac{1}{6}R_4D_1a^3 & -\frac{1}{4}R_5a^2(32D_1D_2a+D_2D_3)\end{bmatrix}$$

$$\mathbf{R}_4=\begin{bmatrix}-\frac{R_2}{3}a^3[(D_1^2+D_2^2)a+2D_1D_3] & \frac{1}{24}R_3D_2a^4\\ -\frac{1}{24}R_4D_2a^4 & -R_5a^3[2(D_1^2+D_2^2)a+D_1D_3]\end{bmatrix}$$

$$\mathbf{R}_5 = \begin{bmatrix} \dfrac{-R_2 a^4}{15}(4D_1D_2a + \dfrac{1}{8}D_2D_3) & \dfrac{1}{120}R_3D_1a^5 \\ -\dfrac{1}{120}R_4D_1a^5 & -\dfrac{R_5}{5}a^4(8D_1D_2a + \dfrac{1}{16}D_2D_3) \end{bmatrix}$$

(I.2)

令

$$\mathbf{S} = \begin{bmatrix} S(1,1) & S(1,2) \\ S(2,1) & S(2,2) \end{bmatrix}$$

$$\begin{aligned} S(1,1) &= \ell^3(EI_P - 2C)\varepsilon_{0,x} \\ &= S_1[D_1D_2 + (D_1^2 + D_2^2)a\zeta + 2D_1D_2a^2\zeta^2 + \frac{2}{3}(D_1^2 + D_2^2)a^3\zeta^3 \\ &\quad + \frac{2}{3}D_1D_2a^4\zeta^4 + \frac{2}{15}(D_1^2 + D_2^2)a^5\zeta^5] + \frac{S_1}{2a}(D_2D_3 + D_1D_3a\zeta \\ &\quad + \frac{1}{2}D_2D_3a^2\zeta^2 + \frac{1}{6}D_1D_3a^3\zeta^3 + \frac{1}{24}D_2D_3a^4\zeta^4 + \frac{1}{120}D_1D_3a^5\zeta^5) \end{aligned}$$

$$\begin{aligned} S(1,2) &= -\frac{1}{2}\ell^3 C w^0_{,xxx} \\ &= -\frac{Ca^3}{2}(D_1 + D_2a\zeta + \frac{1}{2}D_1a^2\zeta^2 + \frac{1}{6}D_2a^3\zeta^3 + \frac{1}{24}D_1a^4\zeta^4 \\ &\quad + \frac{1}{120}D_2a^5\zeta^5) \end{aligned}$$

$$\begin{aligned} S(2,1) &= -\ell^3[C - 2E(I_y - I_z)]w^0_{,xxx} \\ &= -S_2(D_1 + D_2a\zeta + \frac{1}{2}D_1a^2\zeta^2 + \frac{1}{6}D_2a^3\zeta^3 + \frac{1}{24}D_1a^4\zeta^4 \\ &\quad + \frac{1}{120}D_2a^5\zeta^5) \end{aligned}$$

$$
\begin{aligned}
S(2,2) &= \ell^3 EI_z \varepsilon_{0,xxx} \\
&= -8S_3[D_1D_2 + (D_1^2 + D_2^2)a\zeta + 2D_1D_2a^2\zeta^2 + \frac{2}{3}(D_1^2 + D_2^2)a^3\zeta^3 \\
&\quad + \frac{2}{3}D_1D_2a^4\zeta^4 + \frac{2}{15}(D_1^2 + D_2^2)a^5\zeta^5] - \frac{S_3}{a}(D_2D_3 + D_1D_3a\zeta \\
&\quad + \frac{1}{2}D_2D_3a^2\zeta^2 + \frac{1}{6}D_1D_3a^3\zeta^3 + \frac{1}{24}D_2D_3a^4\zeta^4 + \frac{1}{120}D_1D_3a^5\zeta^5)
\end{aligned}
$$

其中

$$
S_1 = \frac{-2(EI_P - 2C)I_y a^5}{A\ell^2},\quad S_2 = [C - 2E(I_y - I_z)]a^3,\quad S_3 = \frac{EI_yI_z}{A\ell^4}a^7
$$

收集 $S(i,j)$ 中 ζ^i 的同次項，即可將 $\mathbf{S}$ 表示成 $\mathbf{S} = \sum_{i=0}^{5}\zeta^i\mathbf{S}_i$

$$
\mathbf{S}_0 = \begin{bmatrix} S_1(D_1D_2 + \dfrac{D_2D_3}{2a}) & -\dfrac{1}{2}CD_1a^3 \\ -S_2D_1 & -S_3(8D_1D_2 + \dfrac{D_2D_3}{a}) \end{bmatrix}
$$

$$
\mathbf{S}_1 = \begin{bmatrix} S_1[(D_1^2 + D_2^2)a + \dfrac{1}{2}D_1D_3] & \dfrac{-1}{2}CD_2a^4 \\ -S_2D_2a & -S_3[8(D_1^2 + D_2^2)a + D_1D_3] \end{bmatrix}
$$

$$
\mathbf{S}_2 = \begin{bmatrix} \dfrac{aS_1}{4}(8D_1D_2a + D_2D_3) & \dfrac{-1}{4}CD_1a^5 \\ -\dfrac{1}{2}S_2D_1a^2 & \dfrac{-aS_3}{2}(32D_1D_2a + D_2D_3) \end{bmatrix}
$$

$$
\mathbf{S}_3 = \begin{bmatrix} \dfrac{S_1a^2}{12}[8(D_1^2 + D_2^2)a + D_1D_3] & -\dfrac{1}{12}CD_2a^6 \\ -\dfrac{1}{6}S_2D_2a^3 & -\dfrac{1}{6}S_3a^2[32(D_1^2 + D_2^2)a + D_1D_3] \end{bmatrix}
$$

$$
\mathbf{S}_4 = \begin{bmatrix} \dfrac{S_1a^3}{48}(32D_1D_2a + D_2D_3) & -\dfrac{1}{48}CD_1a^7 \\ -\dfrac{1}{24}S_2D_1a^4 & -\dfrac{S_3a^3}{24}(128D_1D_2a + D_2D_3) \end{bmatrix}
$$

$$
\mathbf{S}_5 = \begin{bmatrix} \frac{S_1 a^4}{240}[32(D_1^2 + D_2^2)a + D_1 D_3] & -\frac{1}{240} C D_2 a^8 \\ -\frac{1}{120} S_2 D_2 a^5 & -\frac{a^4 S_3}{120}[128(D_1^2 + D_2^2)a + D_1 D_3] \end{bmatrix}
$$

(I.3)

令

$$
\mathbf{T} = \begin{bmatrix} T(1,1) & T(1,2) \\ T(2,1) & T(2,2) \end{bmatrix}
$$

$$
T(1,1) = T(1,2) = T(2,2) = 0
$$

$$
\begin{aligned}
T(2,1) &= \ell^4 E(I_y - I_z) w^0_{,xxxx} \\
&= T_1 (D_2 + D_1 a\zeta + \frac{1}{2} D_2 a^2 \zeta^2 + \frac{1}{6} D_1 a^3 \zeta^3 \\
&\quad + \frac{1}{24} D_2 a^4 \zeta^4 + \frac{1}{120} D_1 a^5 \zeta^5)
\end{aligned}
$$

其中 $T_1 = E(I_y - I_z) a^4$

收集$T(i,j)$中ζ^i的同次項，即可將$\mathbf{T}$表示成$\mathbf{T} = \sum_{i=0}^{5} \zeta^i \mathbf{T}_i$

$$
\mathbf{T}_0 = \begin{bmatrix} 0 & 0 \\ T_1 D_2 & 0 \end{bmatrix} \qquad \mathbf{T}_1 = \begin{bmatrix} 0 & 0 \\ T_1 D_1 a & 0 \end{bmatrix}
$$

$$
\mathbf{T}_2 = \begin{bmatrix} 0 & 0 \\ \frac{1}{2} T_1 D_2 a^2 & 0 \end{bmatrix} \qquad \mathbf{T}_3 = \begin{bmatrix} 0 & 0 \\ \frac{1}{6} T_1 D_1 a^3 & 0 \end{bmatrix}
$$

$$
\mathbf{T}_4 = \begin{bmatrix} 0 & 0 \\ \frac{1}{24} T_1 D_2 a^4 & 0 \end{bmatrix} \qquad \mathbf{T}_5 = \begin{bmatrix} 0 & 0 \\ \frac{1}{120} T_1 D_1 a^5 & 0 \end{bmatrix}
$$

(I.4)

附錄J 梁元素的節點內力

本附錄中的推導是在擾動後的元素座標$\bar{x}_i$中推導，但為了式子的簡明，本附錄在推導的過程中將變數$(\bar{\ })$用$(\ )$代替。

所以由(3.3.6)及(2.6.28)式並取到擾動量的一次項可得

$$A_\varepsilon = A_\varepsilon^\theta + M_2^\theta \theta_2^0 \tag{J.1}$$

$$M_1 = M_1^\theta - M_2^\theta \theta_3 + M_3^\theta \theta_2^0 \tag{J.2}$$

$$M_2 = (1+\varepsilon_0)\, M_2^\theta \tag{J.3}$$

$$M_3 = (1+\varepsilon_0)\, M_3^\theta - \frac{1}{2} M_1^\theta \theta_2^0 \tag{J.4}$$

其中A_ε^θ、A_ε、$M_i^\theta\,(\,i=1,2,3\,)$為廣義力矩，在(2.6.3)及(2.6.28)式中已有定義。

令

$$\xi = 1 - 2\varepsilon_0 \tag{J.5a}$$

$$\eta = 1 + \varepsilon_0 \tag{J.5b}$$

由(2.6.26)式，廣義力矩A_ε^θ可表示成

$$A_\varepsilon^\theta = -EI_y w_{,x}^0 w_{,xx}^0 \tag{J.6}$$

由(2.6.24)式，雙力矩可表示成

$$B^\theta = C_1 \theta_{1,xx} \tag{J.7}$$

由(2.6.19)式，梁截面在x_2方向的合力可以表示成

$$F_2 = -M_{3,x}^\theta + F_1 v_{,x} - \frac{1}{2} C w_{,xx}^0 \theta_{1,x} - EI_z \varepsilon_{0,x} v_{,xx} \tag{J.8}$$

因F_2僅取到擾動量的一次項，所以(J.8)式中之F_1式採用擾動前的F_1。

由(2.4.5)、(3.3.6)、(2.6.21)式及(2.6.24)式的微分式，可將M_1^θ表示成

$$M_1^{\theta} = EI_p \varepsilon_0 \theta_{1,x} + C(\xi\theta_{1,x} + \frac{1}{2}w_{,x}^0 v_{,xx} - \frac{1}{2}w_{,xx}^0 v_{,x}) - C_1\theta_{1,xxx} \qquad \text{(J.9)}$$

由(2.4.5)、(3.3.6)及(2.6.22)式，可將M_2^{θ}表示成

$$M_2^{\theta} = E(-\xi I_y w_{,xx}^0 + I_y \varepsilon_{0,x} w_{,x}^0) \qquad \text{(J.10)}$$

由(2.4.5)、(3.3.6)及(2.6.23)式，可將M_3^{θ}表示成

$$M_3^{\theta} = E[\xi I_z v_{,xx} - (I_y - I_z)w_{,xx}^0\theta_1 - I_z\varepsilon_{0,x}v_{,x}] + \frac{1}{2}Cw_{,x}^0\theta_{1,x} \qquad \text{(J.11)}$$

將(J.11)式微分可得

$$M_{3,x}^{\theta} = E[-2\varepsilon_{0,x}I_z v_{,xx} + \xi I_z v_{,xxx} - (I_y - I_z)(w_{,xxx}^0\theta_1 + w_{,xx}^0\theta_{1,x}) - (I_z\varepsilon_{0,xx}v_{,x} + I_z\varepsilon_{0,x}v_{,xx})] + \frac{1}{2}Cw_{,xx}^0\theta_{1,x} + \frac{1}{2}Cw_{,x}^0\theta_{1,xx} \qquad \text{(J.12)}$$

將(J.6)、(J.10)式代入(J.1)式，並忽略旋轉參數的三次項可得

$$A_{\varepsilon} = 0$$

將(J.9)~(J.11)式代入(J.2)~(J.4)式可得

$$M_1 = EI_p\varepsilon_0\theta_{1,x} + C(\xi\theta_{1,x} + \frac{1}{2}w_{,x}^0 v_{,xx} - \frac{1}{2}w_{,xx}^0 v_{,x}) - C_1\theta_{1,xxx} - \frac{1}{\eta}M_2 v_{,x} - w_{,x}^0 E[\xi I_z v_{,xx} - (I_y - I_z)w_{,xx}^0\theta_1 - I_z\varepsilon_{0,x}v_{,x}] - \frac{1}{2}C(w_{,x}^0)^2\theta_{1,x} \qquad \text{(J.13)}$$

$$M_3 = E[\eta\xi I_z v_{,xx} - (I_y - I_z)\eta w_{,xx}^0\theta_1 - I_z\eta\varepsilon_{0,x}v_{,x}] + \frac{1}{2}\eta C w_{,x}^0\theta_{1,x} + \frac{1}{2}w_{,x}^0[EI_p\varepsilon_0\theta_{1,x} + C(\xi\theta_{1,x} + \frac{1}{2}w_{,x}^0 v_{,xx} - \frac{1}{2}w_{,xx}^0 v_{,x}) - C_1\theta_{1,xxx}] \qquad \text{(J.14)}$$

將(J.12)式代入(J.8)式

$$F_2 = -E[-2\varepsilon_{0,x} I_z v_{,xx} + \xi I_z v_{,xxx} - (I_y - I_z)(w^0_{,xxx}\theta_1 + w^0_{,xx}\theta_{1,x}) - I_z\varepsilon_{0,xx} v_{,x}] - Cw^0_{,xx}\theta_{1,x} - \frac{1}{2}Cw^0_{,x}\theta_{1,xx} + F_1 v_{,x} \tag{J.15}$$

將(3.3.52)及(3.3.53)式代入(J.13)、(J.14)、(J.15)及(J.7)式中，並取 $\zeta = \zeta_j$ $(j = 1, 2)$ 其中 $\zeta_1 = -0.5, \zeta_2 = 0.5$，可得在元素座標上梁元素在節點 $j(j = 1, 2)$ 的節點內力。

$$\begin{aligned} F_{2j} = \{&-E[(1-2\varepsilon_0) I_z \mathbf{N}^t_{v,xxx}(\zeta_j) - \varepsilon_{0,x} I_z \mathbf{N}^t_{v,xx}(\zeta_j) - (I_y - I_z)(w^0_{,xx}\mathrm{N}^t_{1,x}(\zeta_j) \\ &+ w^0_{,xxx}\mathbf{N}^t_1(\zeta_j)) - I_z(\varepsilon_{0,xx}\mathbf{N}^t_{v,x}(\zeta_j) + \varepsilon_{0,x}\mathbf{N}^t_{v,xx}(\zeta_j))] - \frac{1}{2}Cw^0_{,x}\mathbf{N}^t_{1,xx}(\zeta_j) \\ &- Cw^0_{,xx}\mathbf{N}^t_{1,x}(\zeta_j) + F_1\mathbf{N}^t_{v,x}(\zeta_j)\}\,\mathbf{q} = \mathbf{N}^t_{F2}(\zeta_j)\mathbf{q} \end{aligned} \tag{J.16}$$

$$\begin{aligned} M_{1j} = \{&E\varepsilon_0 I_p \mathbf{N}^t_{1,x} + C[\xi\mathbf{N}^t_{1,x}(\zeta_j) - \frac{1}{2}w^0_{,xx}\mathbf{N}^t_{v,x}(\zeta_j) + \frac{1}{2}w^0_{,x}\mathbf{N}^t_{v,xx}(\zeta_j)] \\ &- C_1\mathbf{N}^t_{1,xxx}(\zeta_j) - \frac{1}{\eta}M_2\mathbf{N}^t_{v,x}(\zeta_j) - w^0_{,x}E[\xi I_z\mathbf{N}^t_{v,xx}(\zeta_j) \\ &- (I_y - I_z)w^0_{,xx}\mathbf{N}^t_1(\zeta_j) - I_z\varepsilon_{0,x}\mathbf{N}^t_{v,x}(\zeta_j)] - \frac{1}{2}C(w^0_{,x})^2\mathbf{N}^t_{1,x}\}\,\mathbf{q} \\ = {}&\mathbf{N}^t_{M1}(\zeta_j)\mathbf{q} \end{aligned} \tag{J.17}$$

$$\begin{aligned} M_{3j} = \{&\frac{1}{2}w^0_{,x}[E\varepsilon_0 I_p\mathbf{N}^t_{1,x}(\zeta_j) + C\xi\mathbf{N}^t_{1,x}(\zeta_j) - \frac{1}{2}Cw^0_{,xx}\mathbf{N}^t_{v,x}(\zeta_j) \\ &+ \frac{1}{2}Cw^0_{,x}\mathbf{N}^t_{v,xx}(\zeta_j) - C_1\mathbf{N}^t_{1,xxx}(\zeta_j)] + E[\xi\eta I_z\mathbf{N}^t_{v,xx}(\zeta_j) \\ &- (I_y - I_z)\eta w^0_{,xx}\mathbf{N}^t_1(\zeta_j) - I_z\varepsilon_{0,x}\eta\mathbf{N}^t_{v,x}(\zeta_j)] + \frac{1}{2}C\eta w^0_{,x}\mathbf{N}^t_{1,x}(\zeta_j)\}\,\mathbf{q} \\ = {}&\mathbf{N}^t_{M3}(\zeta_j)\mathbf{q} \end{aligned} \tag{J.18}$$

$$B_j^{\theta} = C_1 \mathbf{N}_{1,xx}^t(\zeta_j)\mathbf{q} = \mathbf{N}_B^t(\zeta_j)\mathbf{q} \tag{J.19}$$

由(J.16)~(J.19)式，並用 $\overline{F}_{2j}, \overline{M}_{1j}, \overline{M}_{3j}$ 取代 F_{2j}, M_{1j}, M_{3j} 可得

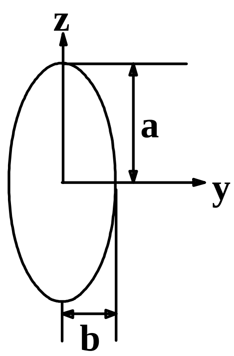
$$\overline{\mathbf{F}}_e = \mathbf{N}_F \mathbf{q} \tag{J.20}$$

其中 $\overline{\mathbf{F}}_e$ 在(3.3.70)式中已定義

$$\begin{aligned} \mathbf{N}_F = \{ & -\mathbf{N}_{F2}^t(\zeta_1) \quad -\mathbf{N}_{M1}^t(\zeta_1) \quad -\mathbf{N}_{M3}^t(\zeta_1) \quad -\mathbf{N}_B^t(\zeta_1) \\ & \mathbf{N}_{F2}^t(\zeta_2) \quad \mathbf{N}_{M1}^t(\zeta_2) \quad \mathbf{N}_{M3}^t(\zeta_2) \quad \mathbf{N}_B^t(\zeta_2) \} \end{aligned} \tag{J.21}$$

附錄 K　橢圓及 W 型鋼之斷面常數

A. 橢圓斷面



翹曲函數：

$$\omega = \frac{a^2 - b^2}{a^2 + b^2} xy$$

$$I_y = \frac{1}{4}\pi a^3 b \qquad (K.1)$$

$$I_z = \frac{1}{4}\pi a b^3 \qquad (K.2)$$

$$J = \frac{\pi a^3 b^3}{a^2 + b^2} \qquad (K.3)$$

$$I_\omega = \frac{\pi a^3 b^3}{24}\left(\frac{a^2 - b^2}{a^2 + b^2}\right)^2 \qquad (K.4)$$

$$\alpha_{\omega yz} = \frac{\pi a^3 b^3}{24}\left(\frac{a^2 - b^2}{a^2 + b^2}\right) \qquad (K.5)$$

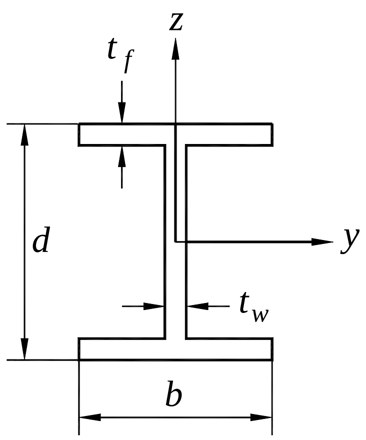
本研究中分析的橢圓斷面有兩種，一為 a:b=10:1 ，另一為 a:b=30:1 的比例。橢圓斷面之材料常數亦有兩種，一為 $E = 10000\, N/cm^2$ ，$G = 5000\, N/cm^2$ ；另一為 $E = 2\times10^7\ N/cm^2$ ，$G = 7.6923\times10^6\ N/cm^2$ 。利用上述所整理(K.1)~(K.5)式，將兩種橢圓的斷面尺寸及常數列表如下

a : b	10 : 1	30 : 1
a(cm)	2.5	3.0
b(cm)	0.25	0.1
$A(cm^2)$	1.9635	0.942478
$I_y(cm^4)$	3.06796	2.120575
$I_z(10^{-2}cm^4)$	3.06796	0.23562
$J(10^{-1}cm^4)$	1.21503	0.09414
$I_\omega(10^{-2}cm^6)$	3.07048	0.35186
$\alpha_{\omega yz}(10^{-2}cm^6)$	3.13251	0.35265
$\alpha(10^{-2})$	1.0	0.11111
$\beta(10^{-2})$	1.98019	0.22197
γ	0.505416	0.747525

註：

$$\alpha = \frac{I_z}{I_y}, \beta = \frac{GJ}{EI_y}, \gamma = \frac{EI_\omega}{GJ}$$

B. W 型鋼斷面



翹曲函數：

top flange

$$-\omega = y(z-d+t_f) \quad for \begin{cases} -0.5b \le y \le 0.5b \\ 0.5(d-2t_f) \le z \le 0.5d \end{cases}$$

web

$$\omega = yz \quad for \begin{cases} -0.5t_w \le y \le 0.5t_w \\ -0.5(d-2t_f) \le z \le 0.5(d-2t_f) \end{cases}$$

bottom flange

$$\omega = -y(z+d-t_f) \quad for \begin{cases} -0.5b \le y \le 0.5b \\ -0.5d \le z \le -0.5(d-2t_f) \end{cases}$$

$$\alpha_{\omega yz} = \frac{b^3h^2t_f}{24} - \frac{b^3{t_f}^3}{72} + \frac{(h-t_f)^3{t_w}^3}{144} \tag{K.6}$$

其中 $h = d - t_f$

本研究中所分析的 W 型鋼斷面有四種，斷面的常數除了$\alpha_{\omega yz}$(見(K.6)式)，其餘皆是參考 AISC 所出版 Manual of Steel Construction [29]，四種型鋼的斷面尺寸及常數列表如下

型鋼斷面	W14x159	W14x90	W10x100	W10x60	W10x30
d(in)	14.98	14.02	11.1	10.22	10.47
b(in)	15.565	14.52	10.34	10.08	5.81
t_f(in)	1.19	0.71	1.12	0.68	0.51
t_w(in)	0.745	0.44	0.68	0.37	0.3
$A(in^2)$	46.7	26.5	29.4	17.6	8.84
$I_y(10^3 in^4)$	1.9	0.999	0.623	0.341	0.170
$I_z(10^3 in^4)$	0.748	0.362	0.207	0.116	0.0167
$J(in^4)$	19.8	4.06	10.9	2.48	0.62
$I_\omega(10^4 in^6)$	3.56	1.6	0.515	0.264	0.0414
$\alpha_{\omega yz}(10^4 in^6)$	3.5462	1.6027	0.5115	0.2636	0.0413
α	0.3937	0.3624	0.3323	0.3402	0.0982
$\beta(10^{-2})$	0.4025	0.1570	0.6757	0.2809	0.1409
$\gamma(10^4)$	0.4655	1.0204	0.1223	0.2756	0.1729

註：

$$\alpha = \frac{I_z}{I_y}, \beta = \frac{GJ}{EI_y}, \gamma = \frac{EI_\omega}{GJ}$$

$E = 29000ksi$ $G = 11200ksi$

附錄 L　簡支梁和懸臂梁的線性挫屈軸力及彎矩

如圖四所示之簡支梁在軸向壓力 P 作用時，造成側向挫屈的軸力可表式成[30]

$$P_{cr} = \frac{\pi^2 EI_{\min}}{L^2} \tag{L.1}$$

其中 $I_{\min}$ 為 I_y 和 I_z 中較小者

如圖五所示之懸臂梁在軸向壓力 P 作用時，造成側向挫屈的軸力可表示成[30]

$$P_{cr} = \frac{\pi^2 EI_{\min}}{4L^2} \tag{L.2}$$

其中 $I_{\min}$ 為 I_y 和 I_z 中較小者

固端梁在軸向壓力 P 作用時，造成側向挫屈的軸力可表式成[30]

$$P_{cr} = \frac{4\pi^2 EI_{\min}}{L^2} \tag{L.3}$$

其中 $I_{\min}$ 為 I_y 和 I_z 中較小者

如(圖 L.1)所示之簡支梁在軸向壓力 P 作用時，造成扭轉挫屈的軸力可表示成[9]

當 $\phi(0) = \phi(L) = 0 , \phi'(0) = \phi'(L) = 0$

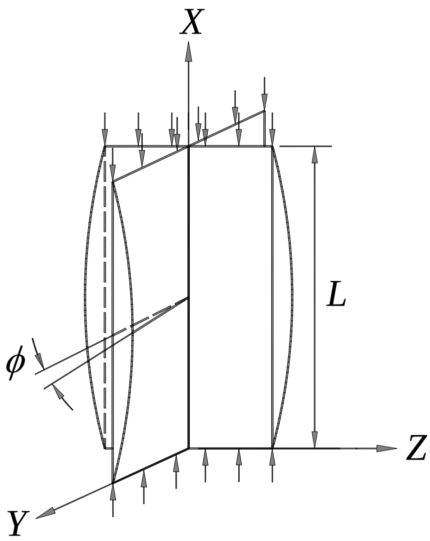
$$P_{\phi} = \frac{A}{I_0}(C + C_1 \frac{4\pi^2}{L^2}) \tag{L.4}$$

當 $\phi(0) = \phi(L) = 0 , \phi''(0) = \phi''(L) = 0$

$$P_{\phi} = \frac{A}{I_0}(C + C_1 \frac{\pi^2}{L^2}) \tag{L.5}$$

其中 $I_0 = I_y + I_z$ ， $C = GJ$ ， $C_1 = EI_{\omega}$

在本文中邊界條件為 BC1 及 BC3 時之P_{ϕ}值為(L.4)式；為 BC2 及 BC4 時之P_{ϕ}值為(L.5)式



(圖 L.1)

如(圖 L.2)所示之簡支梁受到軸力 P 及不均勻彎矩作用時，其線性挫屈彎矩$M_{cr}^{(P_r,\lambda)}$可表示成[31]

$$\frac{M_{cr}^{(P_r,\lambda)}}{M_{cr}^{(0,\lambda)}}=[(1-\frac{P}{P_{ycr}})(1-\frac{P}{P_{zcr}})(1-\frac{P}{P_{\phi}})]^{1/2} \tag{L.6}$$

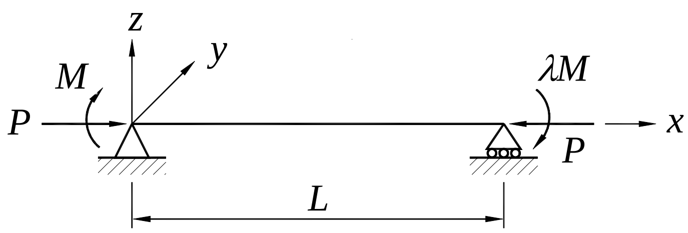
$$P_r=\frac{P}{P_{cr}} \tag{L.7}$$

$$M_{cr}^{(0,\lambda)}=C_b\frac{\pi EI_y}{k_bL}\sqrt{\alpha\beta}\ \sqrt{1+K^2} \tag{L.8}$$

$$C_b=1.75+1.05\lambda+0.3\lambda^2\leq 2.56 \tag{L.9}$$

其中

$$\alpha=\frac{I_z}{I_y}\ ,\ \beta=\frac{GJ}{EI_y}\ ,\ \gamma=\frac{EI_\omega}{GJ}\ ,\ K=\sqrt{\gamma\pi^2/(k_tL)^2}\quad -1\le\lambda\le1$$



(圖 L.2)

上式中 $P_{cr}=Min(P_{ycr},P_{zcr})$ 為梁僅受軸力時的側向挫屈軸力，P_{ycr} 及 P_{zcr} 分別為梁在 X_1X_3 及 X_1X_2 平面的側向挫屈軸力((L.1)式)。k_b,k_t 二常數分別代表不同邊界條件時之有效長度和實際長度的比值，其值如下表：

k_b	k_t	邊界條件
0.883	0.492	$M_3^G(0)=M_3^G(L)=0$ ， $\beta(0)=\beta(L)=0$
1.000	1.000	$M_3^G(0)=M_3^G(L)=0$ ， $B(0)=B(L)=0$
0.492	0.492	$\phi_3^G(0)=\phi_3^G(L)=0$ ， $\beta(0)=\beta(L)=0$
0.434	1.000	$\phi_3^G(0)=\phi_3^G(L)=0$ ， $B(0)=B(L)=0$

當懸臂梁之固定端的截面為抑制翹曲時，若其自由端受到 QT 或 ST 型的彎矩作用時，其線性挫屈彎矩的古典解為[31]

QT 型彎矩

$$M_{cr}=\frac{\pi EI_y}{2l}\sqrt{\alpha\beta}\ \sqrt{1+\gamma\pi^2/4l^2} \qquad \text{(L.10)}$$

ST 型彎矩

$$M_{cr} = \frac{\pi E I_y}{l} \sqrt{\alpha\beta} \sqrt{1 + \gamma\pi^2 / 4l^2} \quad \text{(L.11)}$$

其中

$$\alpha = \frac{I_z}{I_y} \text{，} \beta = \frac{GJ}{EI_y} \text{，} \gamma = \frac{EI_\omega}{GJ} \quad \text{(L.12)}$$