附錄 B　元素形狀函數之推導

圖 B-1 中 (x_j, y_j) 為元素節點 $j(j=1,2,3,4)$ 在 $\bar{x}_1$ 及 $\bar{x}_2$ 軸的座標值，$L = x_4 = x_3 - x_2$，$L_y^{12} = y_2$，$L_y^{34} = y_3 - y_4$，

$L_y(x) = L_y^{12} + \frac{x}{L}(L_y^{34} - L_y^{12})$，$L_y(\xi) = L_y^{12} + \frac{1+\xi}{2}(L_y^{34} - L_y^{12})$。

為了方便形狀函數的表示本文中令:

$$x = \frac{1+\xi}{2}L \tag{B.1}$$

$$\begin{aligned} y = y(\xi,\eta) &= \frac{1+\xi}{2}y_4 + \frac{1+\eta}{2}L_y(x) \\ &= \frac{1-\xi}{2}\frac{1+\eta}{2}L_y^{12} + \frac{1+\xi}{2}\frac{1+\eta}{2}L_y^{34} + \frac{1+\xi}{2}y_4 \end{aligned} \tag{B.2}$$

其中 ξ、η 為無因次座標，$(-1 \le \xi \le 1)$，$(-1 \le \eta \le 1)$，圖 B-2 為在無因次座標中，元素的節點及其座標值。L_y^{12} 為節點 1-2 的距離，L_y^{34} 為節點 3-4 的距離，L 為 L_y^{12} 邊至 L_y^{34} 邊的距離。

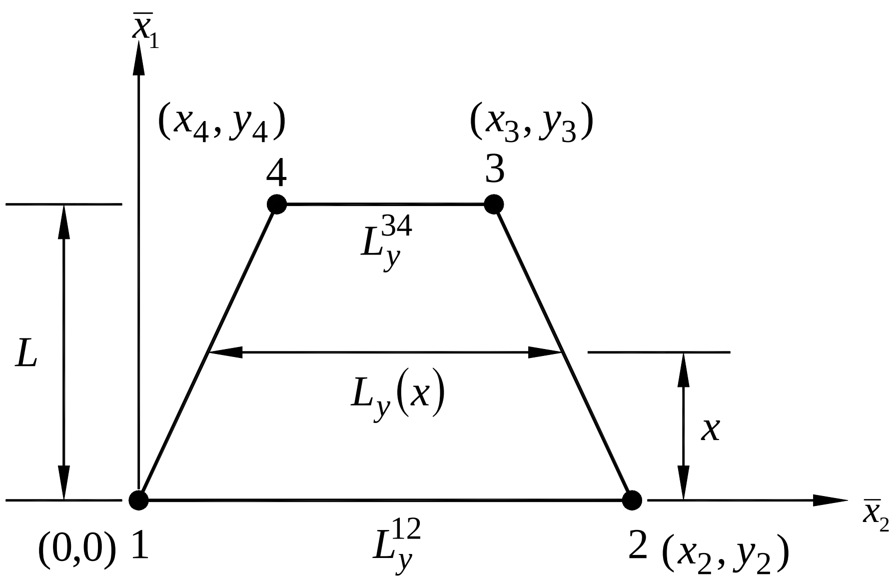

圖 B-1 元素的節點及其座標

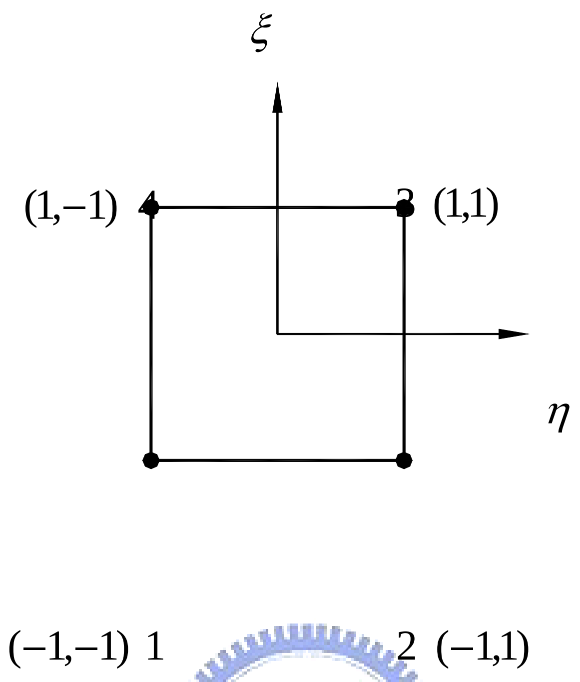

圖 B-2 元素在無因次座標中的節點及其座標值

基於本文對元素變形的假設，元素的位移函數$u(x,y,z)$可表示如下：

$$u(x,y,z)=u_A(x,y)+zu_B(x,y) \qquad \text{(B.3)(2.18)}$$

其中u_B為u在厚度方向的變化率，假設u_A和u_B可以表示成以下節點變形參數的函數

$$u_A(x,y)=\mathbf{N}_A^t\mathbf{q}_A \qquad \text{(B.4)(2.19)}$$

$$u_B(x,y)=\mathbf{N}_B^t\mathbf{q}_B \qquad \text{(B.5)(2.20)}$$

方程式(B.4)、(B.5)式中$\mathbf{N_A}$、$\mathbf{N_B}$為形狀函數，$\mathbf{q}_A$、$\mathbf{q}_B$為節點變形參數向量。本文假設(B.4)式中位移函數$u_A(\xi,\eta)$為ξ及η的三次多項式，其形狀函數$\mathbf{N}_A$及節點變形參數向量$\mathbf{q}_A$可表示如下

$$\mathbf{N}_A=\left\{N_{11}^A,N_{21}^A,N_{31}^A,N_{12}^A,N_{22}^A,N_{32}^A,N_{13}^A,N_{23}^A,N_{33}^A,N_{14}^A,N_{24}^A,N_{34}^A\right\}$$
$$=\{N_1^yN_1^x,N_1^yN_1^{x'},N_1^xN_1^{y'},N_2^yN_2^x,N_2^yN_2^{x'},N_2^xN_2^{y'},$$
$$N_3^yN_3^x,N_3^yN_3^{x'},N_3^xN_3^{y'},N_4^yN_4^x,N_4^yN_4^{x'},N_4^xN_4^{y'}\}$$

(B.6)

$$\mathbf{q}_A=\{u_1,u_{,x1},u_{,y1},u_2,u_{,x2},u_{,y2},u_3,u_{,x3},u_{,y3},u_4,u_{,x4},u_{,y4}\} \qquad \text{(B.7)}$$

其中

$$N_1^y=N_4^y=\frac{1}{4}(1-\eta)^2(2+\eta) \quad , \quad N_2^y=N_3^y=\frac{1}{4}(1+\eta)^2(2-\eta)$$

$$N_1^{y'}=N_4^{y'}=\frac{L_y(\xi)}{8}(1-\eta^2)(1-\eta)$$

$$N_2^{y'}=N_3^{y'}=\frac{L_y(\xi)}{8}(-1+\eta^2)(1+\eta)$$

$$N_1^x=N_2^x=\frac{1}{4}(1-\xi)^2(2+\xi) \quad , \quad N_3^x=N_4^x=\frac{1}{4}(1+\xi)^2(2-\xi)$$

$$N_1^{x'}=N_2^{x'}=\frac{L}{8}(1-\xi^2)(1-\xi) \quad , \quad N_3^{x'}=N_4^{x'}=\frac{L}{8}(-1+\xi^2)(1+\xi)$$

(B.8)

其中 u_j $(j=1,2,3,4)$ 為 u_A 在節點 j 之值，$u_{,xj}=(\frac{\partial u}{\partial x})_j$、$u_{,yj}=(\frac{\partial u}{\partial y})_j$ $(j=1,2,3,4)$ 為 $\frac{\partial u}{\partial x}$ 及 $\frac{\partial u}{\partial y}$ 在節點 j 之值。

將(B.8)式代入(B.6)式可將形狀函數 N_{ij}^A $(i=1,2,3$ ，$j=1,2,3,4)$ 表示成

$$N_{11}^A=\frac{1}{4}(1-\eta)^2(2+\eta)\times\frac{1}{4}(1-\xi)^2(2+\xi)$$

$$N_{21}^A=\frac{1}{4}(1-\eta)^2(2+\eta)\times\frac{L}{8}(1-\xi^2)(1-\xi)$$

$$N_{31}^A=\frac{1}{4}(1-\xi)^2(2+\xi)\times\frac{L_y(\xi)}{8}(1-\eta^2)(1-\eta)$$

$$N_{12}^{A}=\frac{1}{4}(1+\eta)^{2}(2-\eta)\times\frac{1}{4}(1-\xi)^{2}(2+\xi)$$

$$N_{22}^{A}=\frac{1}{4}(1+\eta)^{2}(2-\eta)\times\frac{L}{8}(1-\xi^{2})(1-\xi)$$

$$N_{32}^{A}=\frac{1}{4}(1-\xi)^{2}(2+\xi)\times\frac{L_{y}(\xi)}{8}(-1+\eta^{2})(1+\eta)$$

$$N_{13}^{A}=\frac{1}{4}(1+\eta)^{2}(2-\eta)\times\frac{1}{4}(1+\xi)^{2}(2-\xi)$$

$$N_{23}^{A}=\frac{1}{4}(1+\eta)^{2}(2-\eta)\times\frac{L}{8}(-1+\xi^{2})(1+\xi)$$

$$N_{33}^{A}=\frac{1}{4}(1+\xi)^{2}(2-\xi)\times\frac{L_{y}(\xi)}{8}(-1+\eta^{2})(1+\eta)$$

$$N_{14}^{A}=\frac{1}{4}(1-\eta)^{2}(2+\eta)\times\frac{1}{4}(1+\xi)^{2}(2-\xi)$$

$$N_{24}^{A}=\frac{1}{4}(1-\eta)^{2}(2+\eta)\times\frac{L}{8}(-1+\xi^{2})(1+\xi)$$

$$N_{34}^{A}=\frac{1}{4}(1+\xi)^{2}(2-\xi)\times\frac{L_{y}(\xi)}{8}(1-\eta^{2})(1-\eta)$$

(B.9)

本文假設$u_B(\xi,\eta)$為ξ和η的一次多項式，故(B.5)式中之形狀函數$\mathbf{N}_B$及變形參數向量$\mathbf{q}_B$可表示如下

$$\mathbf{N}_B=\{N_1^B,N_2^B,N_3^B,N_4^B\} \qquad \text{(B.10)(2.27)}$$

$$\mathbf{q}_B=\{u_{,z1},u_{,z2},u_{,z3},u_{,z4}\} \qquad \text{(B.11)(2.28)}$$

其中$u_{,zj}=(\frac{\partial u}{\partial z})_j$為$\frac{\partial u}{\partial z}$在節點$j(j=1,2,3,4)$之值。

$$N_i^B=\frac{1}{4}(1+\xi_i\xi)(1+\eta_i\eta) \quad (i=1,2,3,4)$$

即

$$N_1^B=\frac{1}{4}(1-\xi)(1-\eta)$$

$$N_2^B = \frac{1}{4}(1-\xi)(1+\eta)$$

$$N_3^B = \frac{1}{4}(1+\xi)(1+\eta)$$

$$N_4^B = \frac{1}{4}(1+\xi)(1-\eta)$$

(B.12)

本文假設元素斷面的扭轉角$\theta(x)$為x的三次 Hermitian 氏多項式，因此$\theta(x)$可表示成

$$\theta(x) = \mathbf{N}_\theta^t \mathbf{q}_\theta \qquad \text{(B.13)(2.29)}$$

$$\mathbf{N}_\theta = \{N_1^\theta, N_2^\theta, N_3^\theta, N_4^\theta\} \qquad \text{(B.14)(2.30)}$$

$$\mathbf{q}_\theta = \{\theta_5, \theta,_{x5}, \theta_6, \theta,_{x6}\} \qquad \text{(B.15)(2.31)}$$

其中$\theta_j\,(j=5,6)$， $\theta_{xj} = (\frac{\partial\theta}{\partial x})_j$為斷面在節點$j$的扭轉角及扭轉率。

$$N_1^\theta = \frac{1}{4}(1-\xi)^2(2+\xi)$$

$$N_2^\theta = \frac{L}{8}(1-\xi^2)(1-\xi)$$

$$N_3^\theta = \frac{1}{4}(1+\xi)^2(2-\xi)$$

$$N_4^\theta = \frac{L}{8}(-1+\xi^2)(1+\xi)$$

(B.16)

在(2.33)式中需要計算形狀函數$\mathbf{N}_A, \mathbf{N}_B$及$\mathbf{N}_\theta$對$x$及$y$的微分即$\mathbf{N}_{A,x}, \mathbf{N}_{A,y}, \mathbf{N}_{B,x}, \mathbf{N}_{B,y}$及$\mathbf{N}_{\theta,x}$但因$\mathbf{N}_A, \mathbf{N}_B$, $\mathbf{N}_\theta$為ξ及η的函數，故需先求出$\mathbf{N}_A, \mathbf{N}_B$及$\mathbf{N}_\theta$對$\xi$及$\eta$的微分。令$N$代表形狀函數$N_{ij}^A$、$N_j^B$及$N_j^\theta$ $(i=1,2,3)$，$(j=1,2,3)$，利用 chain rule 將形狀函數$N(\xi,\eta)$對ξ、η的偏微分表示如下

$$\frac{\partial N}{\partial \xi}=\frac{\partial N}{\partial x}\frac{\partial x}{\partial \xi}+\frac{\partial N}{\partial y}\frac{\partial y}{\partial \xi} \tag{B.17}$$

$$\frac{\partial N}{\partial \eta}=\frac{\partial N}{\partial x}\frac{\partial x}{\partial \eta}+\frac{\partial N}{\partial y}\frac{\partial y}{\partial \eta} \tag{B.18}$$

將(B.17) 、(B.18)式表示成矩陣形式：

$$\begin{Bmatrix}\frac{\partial N}{\partial \xi}\\ \frac{\partial N}{\partial \eta}\end{Bmatrix}=\begin{bmatrix}\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi}\\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta}\end{bmatrix}\begin{Bmatrix}\frac{\partial N}{\partial x}\\ \frac{\partial N}{\partial y}\end{Bmatrix}=\mathbf{J}\begin{Bmatrix}\frac{\partial N}{\partial x}\\ \frac{\partial N}{\partial y}\end{Bmatrix} \tag{B.19}$$

由(B.19)式可得

$$\begin{Bmatrix}\frac{\partial N}{\partial x}\\ \frac{\partial N}{\partial y}\end{Bmatrix}=\begin{bmatrix}\frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x}\\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y}\end{bmatrix}\begin{Bmatrix}\frac{\partial N}{\partial \xi}\\ \frac{\partial N}{\partial \eta}\end{Bmatrix}=\mathbf{J}^{-1}\begin{Bmatrix}\frac{\partial N}{\partial \xi}\\ \frac{\partial N}{\partial \eta}\end{Bmatrix} \tag{B.20}$$

其中 **J** 為二維的賈可比矩陣(Jacobian Matrix)，$\mathbf{J}^{-1}$ 為其反矩陣。

將式(B.1)、(B.2)代入(B.19)可得賈可比矩陣及其行列式值如下

$$\mathbf{J}=\begin{bmatrix}\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi}\\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta}\end{bmatrix}=\begin{bmatrix}\frac{L}{2} & (\frac{2y_4-(1+\eta)L_y^{12}+(1+\eta)L_y^{34}}{4})\\ 0 & \frac{(1-\xi)L_y^{12}+(1+\xi)L_y^{34}}{4}\end{bmatrix} \tag{B.21}$$

$$\|\mathbf{J}\|=\frac{L}{8}((1-\xi)L_y^{12}+(1+\xi)L_y^{34}) \tag{B.22}$$

其中$\|\mathbf{J}\|$為 **J** 的行列式值，在(2.37)、(2.54)式中計算之元素內力及剛度矩陣的積分時，需要用到$\|\mathbf{J}\|$，即

$$\int_A(\,)dxdy=\int_{-1}^{1}\int_{-1}^{1}(\,)\|\mathbf{J}\|\,d\xi d\eta \tag{B.23}$$

其中 A 為斷面面積。

由(B.21)式，(B.20)式之$\mathbf{J}^{-1}$可以表示成

$$\mathbf{J}^{-1}=\begin{bmatrix} J_{11}^{I} & J_{12}^{I} \\ J_{21}^{I} & J_{22}^{I} \end{bmatrix}=\begin{bmatrix} \dfrac{2}{L} & \dfrac{2}{L}\left(\dfrac{2y_4-(1+\eta)L_y^{12}+(1+\eta)L_y^{34}}{(-1+\xi)L_y^{12}-(1+\xi)L_y^{34}}\right) \\ 0 & \dfrac{4}{(1-\xi)L_y^{12}+(1+\xi)L_y^{34}} \end{bmatrix} \tag{B.24}$$

(B.6)式中$\mathbf{N}_A$對ξ及η的偏微分可以表示成

$$\left\{\mathbf{N}_{A,\xi} \quad \mathbf{N}_{A,\eta}\right\}=\left\{\begin{matrix} N_1^{y}N_{1,\xi}^{x} & N_{1,\eta}^{y}N_1^{x} \\ N_1^{y}N_{1,\xi}^{x'} & N_{1,\eta}^{y}N_1^{x'} \\ N_1^{x}N_{1,L_y}^{y'}L_{y,\xi}+N_{1,\xi}^{x}N_1^{y'} & N_1^{x}N_{1,\eta}^{y'} \\ N_2^{y}N_{2,\xi}^{x} & N_{2,\eta}^{y}N_2^{x} \\ N_2^{y}N_{2,\xi}^{x'} & N_{2,\eta}^{y}N_2^{x'} \\ N_2^{x}N_{2,L_y}^{y'}L_{y,\xi}+N_{2,\xi}^{x}N_2^{y'} & N_2^{x}N_{2,\eta}^{y'} \\ N_3^{y}N_{3,\xi}^{x} & N_{3,\eta}^{y}N_3^{x} \\ N_3^{y}N_{3,\xi}^{x'} & N_{3,\eta}^{y}N_3^{x'} \\ N_3^{x}N_{3,L_y}^{y'}L_{y,\xi}+N_{3,\xi}^{x}N_3^{y'} & N_3^{x}N_{3,\eta}^{y'} \\ N_4^{y}N_{4,\xi}^{x} & N_{4,\eta}^{y}N_4^{x} \\ N_4^{y}N_{4,\xi}^{x'} & N_{4,\eta}^{y}N_4^{x'} \\ N_4^{x}N_{4,L_y}^{y'}L_{y,\xi}+N_{4,\xi}^{x}N_4^{y'} & N_4^{x}N_{4,\eta}^{y'} \end{matrix}\right\} \tag{B.25}$$

其中

$$\begin{Bmatrix} N_{1,\xi}^{x} \\ N_{3,\xi}^{x} \\ N_{1,\xi}^{x'} \\ N_{3,\xi}^{x'} \end{Bmatrix}=\begin{Bmatrix} N_{2,\xi}^{x} \\ N_{4,\xi}^{x} \\ N_{2,\xi}^{x'} \\ N_{4,\xi}^{x'} \end{Bmatrix}=\begin{Bmatrix} \dfrac{3}{4}(-1+\xi^2) \\ -\dfrac{3}{4}(-1+\xi^2) \\ \dfrac{L}{8}(-1+\xi)(1+3\xi) \\ \dfrac{L}{8}(1+\xi)(-1+3\xi) \end{Bmatrix} \tag{B.26}$$

$$\begin{Bmatrix} N_{1,\eta}^{y} \\ N_{2,\eta}^{y} \\ N_{1,\eta}^{y'} \\ N_{2,\eta}^{y'} \end{Bmatrix} = \begin{Bmatrix} N_{4,\eta}^{y} \\ N_{3,\eta}^{y} \\ N_{4,\eta}^{y'} \\ N_{3,\eta}^{y'} \end{Bmatrix} = \begin{Bmatrix} \frac{3}{4}(-1+\eta^2) \\ -\frac{3}{4}(-1+\eta^2) \\ \frac{L_y(\xi)}{8}(-1+\eta)(1+3\eta) \\ \frac{L_y(\xi)}{8}(1+\eta)(-1+3\eta) \end{Bmatrix} \tag{B.27}$$

$$L_{y,\xi} = \frac{\partial L_y(\xi)}{\partial \xi} = \frac{1}{2}(L_y^{34} - L_y^{12}) \tag{B.28}$$

(B.12)式中$\mathbf{N}_B$對ξ及η的偏微分可以表示成

$$\mathbf{N}_{B,\xi} = \begin{Bmatrix} N_{1,\xi}^{B} \\ N_{2,\xi}^{B} \\ N_{3,\xi}^{B} \\ N_{4,\xi}^{B} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{4}(-1+\eta) \\ \frac{1}{4}(-1-\eta) \\ \frac{1}{4}(1+\eta) \\ \frac{1}{4}(1-\eta) \end{Bmatrix} \tag{B.29}$$

$$\mathbf{N}_{B,\eta} = \begin{Bmatrix} N_{1,\eta}^{B} \\ N_{2,\eta}^{B} \\ N_{3,\eta}^{B} \\ N_{4,\eta}^{B} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{4}(-1+\xi) \\ \frac{1}{4}(1-\xi) \\ \frac{1}{4}(1+\xi) \\ \frac{1}{4}(-1-\xi) \end{Bmatrix} \tag{B.30}$$

(B.16)式中$\mathbf{N}_\theta$對ξ的偏微分可以表示成

$$\mathbf{N}_{\theta,\xi} = \begin{Bmatrix} N_{1,\xi}^{\theta} \\ N_{2,\xi}^{\theta} \\ N_{3,\xi}^{\theta} \\ N_{4,\xi}^{\theta} \end{Bmatrix} = \begin{Bmatrix} \frac{3}{4}(-1+\xi^2) \\ \frac{L}{8}(-1+\xi)(1+3\xi) \\ -\frac{3}{4}(-1+\xi^2) \\ \frac{L}{8}(1+\xi)(-1+3\xi) \end{Bmatrix} \tag{B.31}$$

由(B.20)及(B.24)式可將$\mathbf{N}_A$及$\mathbf{N}_B$對x及y的偏微分表示成

$$\begin{Bmatrix} \dfrac{\partial \mathbf{N}_A}{\partial x} \\ \dfrac{\partial \mathbf{N}_A}{\partial y} \end{Bmatrix} = \begin{bmatrix} J_{11}^{I} & J_{12}^{I} \\ J_{21}^{I} & J_{22}^{I} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial \mathbf{N}_A}{\partial \xi} \\ \dfrac{\partial \mathbf{N}_A}{\partial \eta} \end{Bmatrix} \tag{B.32}$$

$$\begin{Bmatrix} \dfrac{\partial \mathbf{N}_B}{\partial x} \\ \dfrac{\partial \mathbf{N}_B}{\partial y} \end{Bmatrix} = \begin{bmatrix} J_{11}^{I} & J_{12}^{I} \\ J_{21}^{I} & J_{22}^{I} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial \mathbf{N}_B}{\partial \xi} \\ \dfrac{\partial \mathbf{N}_B}{\partial \eta} \end{Bmatrix} \tag{B.33}$$

由(B.26)及(B.1)式可得$\mathbf{N}_{\theta,x}$表示成

$$\mathbf{N}_{\theta,x} = \mathbf{N},_{\xi} \frac{\partial \xi}{\partial x} = \frac{2}{L} \mathbf{N},_{\xi} \tag{B.34}$$