附錄 C　元素之節點內力及切線剛度之推導

在第二章中由虛功原理得知元素節點內力可以表示成：

$$\mathbf{f} = \int_V \mathbf{B}^T \boldsymbol{\sigma} dV \tag{C.1)(2.43}$$

$$\mathbf{f} = \{\mathbf{f}_A \quad \mathbf{f}_B \quad \mathbf{f}_\theta\} \tag{C.2)(2.37}$$

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon} \tag{C.3)(2.40}$$

$$\mathbf{D} = \begin{bmatrix} E & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & G \end{bmatrix} \tag{C.4)(2.42}$$

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^0 + \boldsymbol{\varepsilon}^L = \mathbf{HGq} + \frac{1}{2}\mathbf{AGq} \tag{C.5)(2.32}$$

$$\mathbf{B} = (\mathbf{B}_0 + \mathbf{B}_L) \text{，} \quad \mathbf{B}_0 = \mathbf{HG} \text{，} \quad \mathbf{B}_L = \mathbf{AG} \tag{C.6)(2.36}$$

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\bar{z} \\ 0 & 0 & 1 & \bar{y} \end{bmatrix} \tag{C.7)(2.18}$$

$$\mathbf{A} = \begin{bmatrix} u_{,x} & 0 & 0 & (\bar{y}^2 + \bar{z}^2)\theta_{,x} \\ u_{,y} & u_{,x} & 0 & 0 \\ u_{,z} & 0 & u_{,x} & 0 \end{bmatrix} \tag{C.8)(2.19}$$

$$\mathbf{G} = \begin{bmatrix} \mathbf{N}_{A,x}^t & z\mathbf{N}_{B,x}^t & 0 \\ \mathbf{N}_{A,y}^t & z\mathbf{N}_{B,y}^t & 0 \\ 0 & \mathbf{N}_B^t & 0 \\ 0 & 0 & \mathbf{N}_{\theta,x}^t \end{bmatrix} \tag{C.9)(2.33}$$

$$\bar{y} = y - y_C \tag{C.10)(2.13}$$

$$\bar{z} = z - z_C \tag{C.11)(2.14}$$

將(C.3)-(C.6)式代入(C.1)可得

$$\mathbf{f}=\int_V \mathbf{B}^T\boldsymbol{\sigma}dV=\int_V \mathbf{B}^T\mathbf{D}\boldsymbol{\varepsilon}dV=\int_V(\mathbf{B}_0+\mathbf{B}_L)^T\mathbf{D}(\boldsymbol{\varepsilon}^0+\boldsymbol{\varepsilon}^L)dV \tag{C.12}$$

本文中$\mathbf{f}_A$及$\mathbf{f}_B$都取到變形參數的二次項，$\mathbf{f}_\theta$則取到變形參數的三次項，因[41]中提到開口薄壁梁內力中，扭轉率的三次項不能忽略。

令

$$\mathbf{f}=\mathbf{f}^1+\mathbf{f}^2+\mathbf{f}^3 \tag{C.13}$$

其中$\mathbf{f}^i(i=1,2,3)$為節點內力中變形參數的i次項

將(C.2)、(C.3)、(C.5)、(C.6)、(C.8)及(C.13)式代入(C.12)式可得

$$\mathbf{f}^1=\int_V {\mathbf{B}_0}^T\mathbf{D}\boldsymbol{\varepsilon}^0dV=\begin{Bmatrix}\mathbf{f}_A^1 & \mathbf{f}_B^1 & \mathbf{f}_B^1\end{Bmatrix} \tag{C.14}$$

$$\mathbf{f}^2=\int_V({\mathbf{B}_L}^T\mathbf{D}\boldsymbol{\varepsilon}^0+{\mathbf{B}_0}^T\mathbf{D}\boldsymbol{\varepsilon}^L)dV=\begin{Bmatrix}\mathbf{f}_A^2 & \mathbf{f}_B^2 & \mathbf{f}_\theta^2\end{Bmatrix} \tag{C.15}$$

$$\mathbf{f}^3=\int_V {\mathbf{B}_L}^T\mathbf{D}\boldsymbol{\varepsilon}^LdV=\begin{Bmatrix}\mathbf{f}_A^3 & \mathbf{f}_B^3 & \mathbf{f}_\theta^3\end{Bmatrix} \tag{C.16}$$

將(C.4)-(C.8)代入(C.14)-(C.16)可得

$$\mathbf{f}_A^1=E\int_V \mathbf{N}_{A,x}u_{,x}dV+G\int_V \mathbf{N}_{A,y}u_{,y}dV-G\int_V \bar{z}\mathbf{N}_{A,y}\theta_{,x}dV \tag{C.17}$$

$$\begin{aligned}\mathbf{f}_B^1=&E\int_V z\mathbf{N}_{B,x}u_{,x}dV+G\int_V \mathbf{N}_B u_{,z}dV+G\int_V \bar{y}\mathbf{N}_B\theta_{,x}dV\\&+G\int_V z\mathbf{N}_{B,y}u_{,y}dV-G\int_V z\bar{z}\mathbf{N}_{B,y}\theta_{,x}dV\end{aligned} \tag{C.18}$$

$$\begin{aligned}\mathbf{f}_\theta^1=&-G\int_V \bar{z}\mathbf{N}_{\theta,x}u_{,y}dV+G\int_V \bar{y}\mathbf{N}_{\theta,x}u_{,z}dV\\&+G\int_V \bar{y}^2\mathbf{N}_{\theta,x}\theta_{,x}dV+G\int_V \bar{z}^2\mathbf{N}_{\theta,x}\theta_{,x}dV\end{aligned} \tag{C.19}$$

$$\mathbf{f}_A^2 = \frac{E}{2}\int_V (\bar{y}^2 + \bar{z}^2)\mathbf{N}_{A,x}\theta_{,x}^2 dV + 2G\int_V \mathbf{N}_{A,y}u_{,y}u_{,x}dV$$
$$- G\int_V \bar{z}\mathbf{N}_{A,y}u_{,x}\theta_{,x}dV + G\int_V \mathbf{N}_{A,x}u_{,y}^2 dV + G\int_V \mathbf{N}_{A,x}u_{,z}^2 dV$$
$$+ \frac{3}{2}E\int_V \mathbf{N}_{A,x}u_{,x}^2 dV - G\int_V \bar{z}\mathbf{N}_{A,x}u_{,y}\theta_{,x}dV$$
$$+ G\int_V \bar{y}\mathbf{N}_{A,x}u_{,z}\theta_{,x}dV$$

(C.20)

$$\mathbf{f}_B^2 = \frac{E}{2}\int_V z(\bar{y}^2 + \bar{z}^2)\mathbf{N}_{B,x}\theta_{,x}^2 dV + 2G\int_V \mathbf{N}_B u_{,z}u_{,x}dV$$
$$+ G\int_V \bar{y}\mathbf{N}_B u_{,x}\theta_{,x}dV + 2G\int_V z\mathbf{N}_{B,y}u_{,y}u_x dV$$
$$- G\int_V z\bar{z}\mathbf{N}_{B,y}u_{,x}\theta_{,x}dV + G\int_V z\mathbf{N}_{B,x}u_{,y}^2 dV + G\int_V z\mathbf{N}_{B,x}u_{,z}^2 dV$$
$$+ \frac{3}{2}E\int_V z\mathbf{N}_{B,x}u_{,x}^2 dV - G\int_V z\bar{z}\mathbf{N}_{B,x}u_{,y}\theta_{,x}dV$$
$$+ G\int_V z\bar{y}\mathbf{N}_{B,x}u_{,z}\theta_{,x}dV$$

(C.21)

$$\mathbf{f}_\theta^2 = -G\int_V \bar{z}\mathbf{N}_{\theta,x}u_{,y}u_{,x}dV + G\int_V \bar{y}\mathbf{N}_{\theta,x}u_{,y}u_{,x}dV$$
$$+ E\int_V \bar{y}^2\mathbf{N}_{\theta,x}u_{,x}\theta_{,x}dV + E\int_V \bar{z}^2\mathbf{N}_{\theta,x}u_{,x}\theta_{,x}dV$$

(C.22)

$$\mathbf{f}_\theta^3 = \frac{E}{2}\int_V (\bar{z}^2 + \bar{y}^2)^2\mathbf{N}_{\theta,x}\theta_{,x}^3 dV \qquad \text{(C.23)}$$

因(C.17)-(C.23)式中，z 方向的積分與 x 、y 方向的積分可以分開積，故可將(C.17)-(C.23)式中之體積分表示成

$$\int_V (\,)dV = \int_{-\frac{t}{2}}^{\frac{t}{2}}\int_A (\,)dAdz \qquad \text{(C.24)}$$

將(C.17)-(C.23)式中之厚度方向的積分積出可得

$$\mathbf{f}_A^1 = Et\int_A \mathbf{N}_{A,x}u_{A,x}dA + Gt\int_A \mathbf{N}_{A,y}u_{A,y}dA + Gtz_C\int_A \mathbf{N}_{A,y}\theta_{,x}dA$$

(C.25)(2.44)

$$
\mathbf{f}_B^1 = \frac{Et^3}{12}\int_A \mathbf{N}_{B,x} u_{B,x} dA + Gt\int_A \mathbf{N}_B u_B dA + Gt\int_A \bar{y}\mathbf{N}_B \theta_{,x} dA
- \frac{Gt^3}{12}\int_A \mathbf{N}_{B,y}\theta_{,x} dA + \frac{Gt^3}{12}\int_A \mathbf{N}_{B,y} u_{B,y} dA
$$

(C.26)(2.45)

$$
\mathbf{f}_\theta^1 = Gt\int_A \bar{y}\mathbf{N}_{\theta,x} u_B dA + (\frac{Gt^3}{12} + Gtz_C^2)\int_A \mathbf{N}_{\theta,x}\theta_{,x} dA
+ Gt\int_A \bar{y}^2\mathbf{N}_{\theta,x}\theta_{,x} dA + Gtz_C\int_A \mathbf{N}_{\theta,x} u_{A,y} dA
- \frac{Gt^3}{12}\int_A \mathbf{N}_{\theta,x} u_{B,y} dA
$$

(C.27)(2.46)

$$
\mathbf{f}_A^2 = Gt\int_A \mathbf{N}_{A,x} u_B^2 dA + Gt\int_A \bar{y}\mathbf{N}_{A,x} u_B \theta_{,x} dA
+ (\frac{Et^3}{24} + \frac{Etz_C^2}{2})\int_A \mathbf{N}_{A,x}\theta_{,x}^2 dA + \frac{Et}{2}\int_A \bar{y}^2\mathbf{N}_{A,x}\theta_{,x}^2 dA
+ Gtz_C\int_A \mathbf{N}_{A,x}\theta_{,x} u_{A,y} dA + Gt\int_A \mathbf{N}_{A,x} u_{A,y}^2 dA + Gtz_C\int_A \mathbf{N}_{A,y}\theta_{,x} u_{A,x} dA
+ 2Gt\int_A \mathbf{N}_{A,y} u_{A,y} u_{A,x} dA + \frac{3Et}{2}\int_A \mathbf{N}_{A,x} u_{A,x}^2 dA - \frac{Gt^3}{12}\int_A \mathbf{N}_{A,x}\theta_{,x} u_{B,y} dA
+ \frac{Gt^3}{12}\int_A \mathbf{N}_{A,x} u_{B,y}^2 dA - \frac{Gt^3}{12}\int_A \mathbf{N}_{A,y}\theta_{,x} u_{B,x} dA + \frac{Gt^3}{6}\int_A \mathbf{N}_{A,y} u_{B,x} u_{B,y} dA
+ \frac{Et^3}{8}\int_A \mathbf{N}_{A,x} u_{B,x}^2 dA
$$

(C.28)(2.47)

$$
\mathbf{f}_B^2 = -\frac{Et^3 z_C}{12}\int_A \mathbf{N}_{B,x}\theta_{,x}^2 dA - \frac{Gt^3}{12}\int_A \mathbf{N}_{B,x}\theta_{,x} u_{A,y} dA + 2Gt\int_A \mathbf{N}_B \theta_z u_{A,x} dA
+ Gt\int_A \bar{y}\mathbf{N}_B \theta_{,x} u_{A,x} dA - \frac{Gt^3}{12}\int_A \mathbf{N}_{B,y}\theta_{,x} u_{A,x} dA
$$

$$+\frac{Gt^3 z_C}{12}\int_A \mathbf{N}_{B,x}\theta_{,x}u_{B,y}dA+Gt\int_A \bar{y}\mathbf{N}_B\theta_{,x}u_{A,x}dA$$
$$-\frac{Gt^3}{12}\int_A \mathbf{N}_{B,y}\theta_{,x}u_{A,x}dA+\frac{Gt^3 z_C}{12}\int_A \mathbf{N}_{B,x}\theta_{,x}u_{B,y}dA$$
$$+\frac{Gt^3}{6}\int_A \mathbf{N}_{B,x}u_{A,y}u_{B,y}dA+\frac{Gt^3}{6}\int_A \mathbf{N}_{B,y}u_{A,x}u_{B,y}dA$$
$$+\frac{Gt^3 z_C}{12}\int_A \mathbf{N}_{B,y}\theta_{,x}u_{B,x}dA+\frac{Gt^3}{6}\int_A \mathbf{N}_{B,y}u_{A,y}u_{B,x}dA$$
$$+\frac{Et^3}{4}\int_A \mathbf{N}_{B,x}u_{A,x}u_{B,x}dA$$

(C.29)(2.48)

$$\mathbf{f}_\theta^2=Gt\int_A \bar{y}\mathbf{N}_{\theta,x}u_{A,x}u_B dA+(\frac{Et^3}{12}+Etz_C^2)\int_A \mathbf{N}_{\theta,x}u_{A,x}\theta_{,x}dA$$
$$+Et\int_A \bar{y}^2\mathbf{N}_{\theta,x}u_{A,x}\theta_{,x}dA+Gtz_C\int_A \mathbf{N}_{\theta,x}u_{A,y}u_{A,x}dA$$
$$-\frac{Gt^3}{12}\int_A \mathbf{N}_{\theta,x}u_{A,x}u_{B,y}dA-\frac{Et^3 z_C}{6}\int_A \mathbf{N}_{\theta,x}u_{B,x}\theta_{,x}dA$$
$$-\frac{Gt^3}{12}\int_A \mathbf{N}_{\theta,x}u_{A,y}u_{B,x}dA+\frac{Gt^3 z_C}{12}\int_A \mathbf{N}_{\theta,x}u_{B,x}u_{B,y}dA$$

(C.30)(2.49)

$$\mathbf{f}_\theta^3=(\frac{Et^5}{160}+\frac{Et^3 z_C^2}{4}+\frac{Etz_C^4}{2})\int_A \mathbf{N}_{\theta,x}\theta_{,x}^3 dA$$
$$+(\frac{Et^3}{12}+Etz_C^2)\int_A \bar{y}^2\mathbf{N}_{\theta,x}\theta_{,x}^3 dA+\frac{Et}{2}\int_A \bar{y}^4\mathbf{N}_{\theta,x}\theta_{,x}^3 dA$$

(C.31)(2.50)

將(C.1)式中的元素節點力對節點位移之微分可得元素之切線剛度矩陣

$$\mathbf{K}=\frac{\partial \mathbf{f}}{\partial \mathbf{q}}=\int_V \frac{\partial \mathbf{B}^T}{\partial \mathbf{q}}\boldsymbol{\sigma}dV+\int_V \mathbf{B}^T\frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{q}}dV \tag{C.32}$$

將(C.3)式微分可得

$$\frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{q}} = \mathbf{D}(\mathbf{B}^o + \mathbf{B}^L) \tag{C.33}$$

將(C.6)-(C.9)式代入(C.32)式之$\dfrac{\partial \mathbf{B}^T}{\partial \mathbf{q}}\boldsymbol{\sigma}$中可將其改寫成

$$\frac{\partial \mathbf{B}^T}{\partial \mathbf{q}}\boldsymbol{\sigma} = \frac{\partial {\mathbf{B}_L}^T}{\partial \mathbf{q}}\boldsymbol{\sigma} = \mathbf{G}^T \frac{\partial \mathbf{A}^T \boldsymbol{\sigma}}{\partial \mathbf{q}} = \mathbf{G}^T \mathbf{S}\mathbf{G} = \mathbf{G}^T(\mathbf{S}^o + \mathbf{S}^L)\mathbf{G} \tag{C.34}$$

其中

$$\mathbf{A}^T\boldsymbol{\sigma} = \begin{Bmatrix} \sigma_{11}u_{,x} + \sigma_{12}u_{,y} + \sigma_{13}u_{,z} \\ \sigma_{12}u_{,x} \\ \sigma_{13}u_{,x} \\ (\bar{y}^2 + \bar{z}^2)\sigma_{11}\theta_{,x} \end{Bmatrix} = \mathbf{S}\begin{Bmatrix} u_{,x} \\ u_{,y} \\ u_{,z} \\ \theta_{,x} \end{Bmatrix} = \mathbf{S}\mathbf{G}\mathbf{q} \tag{C.35}$$

$$\begin{aligned}
\mathbf{S} = \mathbf{S}^o + \mathbf{S}^L &= \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & 0 \\ \sigma_{12} & 0 & 0 & 0 \\ \sigma_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & (\bar{y}^2 + \bar{z}^2)\sigma_{11} \end{bmatrix} \\
&= \begin{bmatrix} E\varepsilon_{11}^L & G\varepsilon_{12}^L & G\varepsilon_{13}^L & 0 \\ G\varepsilon_{12}^L & 0 & 0 & 0 \\ G\varepsilon_{13}^L & 0 & 0 & 0 \\ 0 & 0 & 0 & (\bar{y}^2 + \bar{z}^2)E\varepsilon_{11}^L \end{bmatrix} \\
&+ \begin{bmatrix} E\varepsilon_{11}^{NL} & G\varepsilon_{12}^{NL} & G\varepsilon_{13}^{NL} & 0 \\ G\varepsilon_{12}^{NL} & 0 & 0 & 0 \\ G\varepsilon_{13}^{NL} & 0 & 0 & 0 \\ 0 & 0 & 0 & (\bar{y}^2 + \bar{z}^2)E\varepsilon_{11}^{NL} \end{bmatrix}
\end{aligned} \tag{C.36}$$

將(C.33)、(C.34)式代入(C.32)式可得

$$\mathbf{K} = \int_V \mathbf{G}^T(\mathbf{S}^o + \mathbf{S}^L)\mathbf{G}\,dV + \int_V (\mathbf{B}^o + \mathbf{B}^L)^T \mathbf{D}(\mathbf{B}^o + \mathbf{B}^L)\,dV \tag{C.37}$$

將(C.6)-(C.8)、(C.35)、(C.36)代入(C.37)式並保留到變形參數的一次

項及扭轉率的二次項可得

$$\mathbf{K}=\begin{bmatrix}\mathbf{k}_{aa} & \mathbf{k}_{ab} & \mathbf{k}_{a\theta} \\ & \mathbf{k}_{bb} & \mathbf{k}_{b\theta} \\ sym. & & \mathbf{k}_{\theta\theta}\end{bmatrix} \qquad \text{(C.38)(2.54)}$$

其中

$$\begin{aligned}\mathbf{k}_{aa} &= Gt\int_A \mathbf{N}_{A,y}\mathbf{N}^t_{A,y}dA + Et\int_A \mathbf{N}_{A,x}\mathbf{N}^t_{A,x}dA \\ &+ Gtz_C\int_A \mathbf{N}_{A,x}\mathbf{N}^t_{A,y}\theta_{,x}dA + Gtz_C\int_A \mathbf{N}_{A,y}\mathbf{N}^t_{A,x}\theta_{,x}dA \\ &+ 2Gt\int_A \mathbf{N}_{A,x}\mathbf{N}^t_{A,y}u_{A,y}dA + 2Gt\int_A \mathbf{N}_{A,y}\mathbf{N}^t_{A,x}u_{A,y}dA \\ &+ 2Gt\int_A \mathbf{N}_{A,y}\mathbf{N}^t_{A,y}u_{A,x}dA + 3Et\int_A \mathbf{N}_{A,x}\mathbf{N}^t_{A,x}u_{A,x}dA\end{aligned}$$

(C.39)(2.55)

$$\begin{aligned}\mathbf{k}_{ab} &= 2Gt\int_A \mathbf{N}_{A,x}\mathbf{N}^t_B u_B dA + Gt\int_A \bar{y}\mathbf{N}_{A,x}\mathbf{N}^t_B\theta_{,x}dA \\ &-\frac{Gt^3}{12}\int_A \mathbf{N}_{A,x}\mathbf{N}^t_{B,y}\theta_{,x}dA - \frac{Gt^3}{12}\int_A \mathbf{N}_{A,y}\mathbf{N}^t_{B,x}\theta_{,x}dA \\ &+\frac{Gt^3}{6}\int_A \mathbf{N}_{A,x}\mathbf{N}^t_{B,y}u_{B,y}dA + \frac{Gt^3}{6}\int_A \mathbf{N}_{A,y}\mathbf{N}^t_{B,x}u_{B,y}dA \\ &+\frac{Gt^3}{6}\int_A \mathbf{N}_{A,y}\mathbf{N}^t_{B,y}u_{B,x}dA + \frac{Et^3}{4}\int_A \mathbf{N}_{A,x}\mathbf{N}^t_{B,x}u_{B,x}dA\end{aligned}$$

(C.40)(2.56)

$$\begin{aligned}\mathbf{k}_{a\theta} &= Gtz_C\int_A \mathbf{N}_{A,y}\mathbf{N}^t_{\theta,x}dA + Gt\int_A \bar{y}\mathbf{N}_{A,x}\mathbf{N}^t_{\theta,x}u_B dA \\ &+ \left(\frac{Et^3}{12} + Etz_C^2\right)\int_A \mathbf{N}_{A,x}\mathbf{N}^t_{\theta,x}\theta_{,x}dA + Et\int_A \bar{y}^2\mathbf{N}_{A,x}\mathbf{N}^t_{\theta,x}\theta_{,x}dA \\ &+ Gtz_C\int_A \mathbf{N}_{A,x}\mathbf{N}^t_{\theta,x}u_{A,y}dA + Gtz_C\int_A \mathbf{N}_{A,y}\mathbf{N}^t_{\theta,x}u_{A,x}dA \\ &-\frac{Gt^3}{12}\int_A \mathbf{N}_{A,x}\mathbf{N}^t_{\theta,x}u_{B,y}dA - \frac{Gt^3}{12}\int_A \mathbf{N}_{A,y}\mathbf{N}^t_{\theta,x}u_{B,x}dA\end{aligned}$$

(C.41)(2.57)

$$\mathbf{k}_{bb} = Gt\int_A \mathbf{N}_B \mathbf{N}_B^t dA + \frac{Gt^3}{12}\int_A \mathbf{N}_{B,y}\mathbf{N}_{B,y}^t dA + \frac{Et^3}{12}\int_A \mathbf{N}_{B,x}\mathbf{N}_{B,x}^t dA$$
$$+\frac{Gt^3 z_C}{12}\int_A \mathbf{N}_{B,x}\mathbf{N}_{B,y}^t \theta_{,x} dA + \frac{Gt^3 z_C}{12}\int_A \mathbf{N}_{B,y}\mathbf{N}_{B,x}^t \theta_{,x} dA$$
$$+\frac{Gt^3}{6}\int_A \mathbf{N}_{B,x}\mathbf{N}_{B,y}^t u_{A,y} dA + \frac{Gt^3}{6}\int_A \mathbf{N}_{B,y}\mathbf{N}_{B,x}^t u_{A,y} dA$$
$$+2Gt\int_A \mathbf{N}_B \mathbf{N}_B^t u_{A,x} dA + \frac{Gt^3}{6}\int_A \mathbf{N}_{B,y}\mathbf{N}_{B,y}^t u_{A,x} dA$$
$$+\frac{Et^3}{4}\int_A \mathbf{N}_{B,x}\mathbf{N}_{B,x}^t u_{A,x} dA$$

(C.42)(2.58)

$$\mathbf{k}_{b\theta} = Gt\int_A \bar{y}\mathbf{N}_B \mathbf{N}_{\theta,x}^t dA - \frac{Gt^3}{12}\int_A \mathbf{N}_{B,y}\mathbf{N}_{\theta,x}^t dA$$
$$-\frac{Et^3 z_C}{6}\int_A \mathbf{N}_{B,x}\mathbf{N}_{\theta,x}^t \theta_{,x} dA - \frac{Gt^3}{12}\int_A \mathbf{N}_{B,x}\mathbf{N}_{\theta,x}^t u_{A,y} dA$$
$$Gt\int_A \bar{y}\mathbf{N}_B \mathbf{N}_{\theta,x}^t u_{A,x} dA - \frac{Et^3 z_C}{6}\int_A \mathbf{N}_{B,x}\mathbf{N}_{\theta,x}^t \theta_{,x} dA$$
$$-\frac{Gt^3}{12}\int_A \mathbf{N}_{B,x}\mathbf{N}_{\theta,x}^t u_{A,y} dA + Gt\int_A \bar{y}\mathbf{N}_B \mathbf{N}_{\theta,x}^t u_{A,x} dA$$
$$-\frac{Gt^3}{12}\int_A \mathbf{N}_{B,y}\mathbf{N}_{\theta,x}^t u_{A,x} dA + \frac{Gt^3 z_C}{12}\int_A \mathbf{N}_{B,x}\mathbf{N}_{\theta,x}^t u_{B,y} dA$$
$$+\frac{Gt^3 z_C}{12}\int_A \mathbf{N}_{B,y}\mathbf{N}_{\theta,x}^t u_{B,x} dA$$

(C.43)(2.59)

$$
\begin{aligned}
\mathbf{k}_{\theta\theta} &= \left(\frac{Gt^3}{12} + Gtz_C^2\right)\int_A \mathbf{N}_{\theta,x}\mathbf{N}_{\theta,x}^t dA + Gt\int_A \bar{y}^2 \mathbf{N}_{\theta,x}\mathbf{N}_{\theta,x}^t dA \\
&+ \left(\frac{Et^3}{12} + Etz_C^2\right)\int_A \mathbf{N}_{\theta,x}\mathbf{N}_{\theta,x}^t u_{A,x} dA \\
&+ Et\int_A \bar{y}^2 \mathbf{N}_{\theta,x}\mathbf{N}_{\theta,x}^t u_{A,x} dA - \frac{Et^3 z_C}{6}\int_A \mathbf{N}_{\theta,x}\mathbf{N}_{\theta,x}^t u_{B,x} dA \\
&+ \left(\frac{3Et^5}{160} + \frac{3Et^3 z_C^2}{4} + \frac{3Etz_C^4}{2}\right)\int_A \mathbf{N}_{\theta,x}\mathbf{N}_{\theta,x}^t \theta_{,x}^2 dA \\
&+ \left(\frac{Et^3}{4} + 3Etz_C^2\right)\int_A \bar{y}^2 \mathbf{N}_{\theta,x}\mathbf{N}_{\theta,x}^t \theta_{,x}^2 dA \\
&+ \frac{3Et}{2}\int_A \bar{y}^4 \mathbf{N}_{\theta,x}\mathbf{N}_{\theta,x}^t \theta_{,x}^2 dA
\end{aligned}
$$

(C.44)(2.60)