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DEPENDENCE OF TREE COPY FUNCTIONS

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The graph copy function when restricted to the set of trees or of rooted trees is called the tree copy function or the rooted tree copy function, respectively. We show that both the set of tree copy functions and the set of rooted tree copy functions are strongly linearly independent. We also show that the set of all tree copy functions are algebraically independent but the set of all rooted tree copy functions are not. An algebraic base for the algebra generated by rooted tree copy functions is constructed in this paper.

1. Definition and introduction

For any graph H, the function c_H from the set of all graphs $\mathcal{G}$ into R is defined by setting $c_H(G) = |\{W \subseteq V(G) : |G|_W \cong H\}|$ for every $G \in \mathcal{G}$. In 1932, Whitney [4] proved that the functions c_H, H connected, are algebraically independent. From that time on, mathematicians have tried to get similar results by weakening the domain or by working with other graph functions. (See [1, 2, 3].) In this paper, we restrict our discussion to the sets of trees and rooted trees.

A tree is defined to be a connected graph without cycles and with at least one edge. $\mathcal{T}$ denotes the family of all trees. For a fixed tree T, c'_T is defined to by $c'_T = c_{T|\mathcal{T}}$, the restriction of c_T to the set of all trees. We say that $c'_{T_1}, c'_{T_2}, \ldots, c'_{T_n}$ are *linearly independent* if $\sum_{i=1}^n d_i c'_{T_i}(T) = 0$ for all $T \in \mathcal{T}$ implies $d_i = 0$ for every i; $c'_{T_1}, c'_{T_2}, \ldots, c'_{T_n}$ are *strongly linearly independent* if $\sum_{i=1}^n d_i c'_{T_i}(T) = d_0$ for all $T \in \mathcal{T}$ implies $d_i = 0$ for every i; and $c'_{T_1}, c'_{T_2}, \ldots, c'_{T_n}$ are algebraically *independent* if for any polynomial P in n variables such that $p(c'_{T_1}, c'_{T_2}, \ldots, c'_{T_n})(T) = 0$ for all $T \in \mathcal{T}$ implies $p \equiv 0$. If $B \subseteq \{c'_T \mid T \in \mathcal{T}\}$, we say B is *linearly* (*strongly linearly, algebraically*, respectively) *independent* if any finite elements in B are linearly (strongly linearly, algebraically, respectively) independent. If B is not linearly (strongly linearly, algebraically, respectively) independent, then we say that B is *linearly* (*strongly linearly, algebraically*, respectively) *dependent*. It is easy to see that algebraic independence implies strongly linear independence which in turn implies linear independence. But the converses need not be true.

2. Dependence of tree copy functions

Given a tree T, a vertex x is called a *brink point* if $|\{y \mid \deg_T(y) = 1, x \text{ is adjacent to } y\}| = \deg_T(x) - 1$. Let $\mathrm{Br}(T) = \{x \mid x \text{ is a brink point in } T\}$ and call $\mathrm{Br}(T)$ the *brink set* of T. Note that $\mathrm{Br}(T)=\emptyset$ if and only if $T\cong S_m$ for some m. We define the *brink degree* of T, $\deg \mathrm{Br}(T)$, to be $\max\{\deg_T(x)-1 \mid x\in \mathrm{Br}(T)\}$ if $\mathrm{Br}(T)\neq\emptyset$ and $\deg \mathrm{Br}(T)$ to be zero if $\mathrm{Br}(T)=\emptyset$. For those T with $\mathrm{Br}(T)\neq\emptyset$, we pick a fixed point $x\in \mathrm{Br}(T)$ with $\deg_T(x)=\deg \mathrm{Br}(T)$ and for those T with $\mathrm{Br}(T)=0$, we pick a fixed point $x\in V(T)$ such that $\deg_T(x)=m$ where $T\cong S_m$. We call this point a *cliff point* of T.

Lemma 2.1. *Let $B=\{T_1, T_2, \ldots, T_n\}$ be a finite set of distinct trees. Then there exists a tree T_s in B such that for any $\varepsilon>0$ and $K>0$, there exists a tree $T'=T'(\varepsilon, K)$ satisfying $c'_{T_s}(T')>K$ and $c'_{T_j}(T')<\varepsilon c'_{T_s}(T')$ for all $T_j\neq T_s$.*

Proof. Consider $C=\{T_i \mid \mathrm{Br}(T_i)=\emptyset\}$ a subset of B. If $C\neq\emptyset$, say $C=\{T_{i_1}, T_{i_2}, \ldots, T_{i_k}\}$, where $T_{i_j}\cong S_{m_{i_j}}$, then take T_s to be the tree in C with the largest m_s. Let $B(m)$ be S_m. We have $c'_{T_j}(B(m))=\binom{m}{m_j}$ if $T_j\in C$ and $c'_{T_j}(B(m))=0$ if $T_j\notin C$. By choosing m sufficiently large, we have $c'_{T_s}(B(m))>K$ and $c'_{T_j}(B(m))<\varepsilon c'_{T_s}(B(m))$ for $T_j\neq T_s$.

If $C=\emptyset$, take D to the subset of trees in B with the largest brink degree and take T_s to be a tree in D with the least number of edges. Construct $B(m)$ by adding $m-\deg \mathrm{Br}(T_s)$ pendant edges to the cliff point of T_s. For those tree T_j not in D, $c'_{T_j}(B(m))=P_j(m)$ where P_j is a polynomial with degree at most $\deg \mathrm{Br}(T_j)$ with $\deg \mathrm{Br}(T_j)<\deg \mathrm{Br}(T_s)$, and $c'_{T_s}(B(m))=P_s(m)$, where P_s is a polynomial with degree $\deg \mathrm{Br}(T_s)$. However, for those trees T_j in D and $T_j\neq T_s$, we have $c'_{T_j}(B(m))=0$. Therefore we can choose sufficiently large m such that $c'_{T_s}(B(m))>K$ and $c'_{T_j}(B(m))<\varepsilon c'_{T_s}(B(m))$ for $T_j\neq T_s$. □

Lemma 2.2. *The set of all tree copy functions is strongly linearly independent and hence it is linearly independent.*

Proof. If it is not strongly linearly independent, we can find $T_1, T_2, \ldots, T_n$ such that there exist $d_0, d_1, \ldots, d_n$ with $d_j\neq 0$ for $j\neq 0$ which satisfy $\sum_{j=1}^{n} d_j c'_{T_j}(T)=d_0$ for all $T\in\mathcal{T}$. By Lemma 2.1, we can find T_s such that for $0<\varepsilon<\min\{|d_s|/(|d_j|\, n) \mid j\neq s\}$ and $K>n\,|d_0/d_s|$, there exists a tree T' which satisfies $c'_{T_s}(T')>K$ and $c'_{T_j}(T')<\varepsilon c'_{T_s}(T')$ for $T_j\neq T_s$. Since $\sum_{j=1}^{n} d_j c'_{T_j}(T')=d_0$, we have $d_s c'_{T_s}(T')=d_0-\sum_{j\neq 0,s} d_j c'_{T_j}(T')$. Therefore

$$|c'_{T_s}(T')=\left|\frac{d_0}{d_s}-\sum_{j\neq 0,s}\frac{d_j}{d_s}c'_{T_j}(T')\right|<\left|\frac{d_0}{d_s}\right|+\sum_{j\neq 0,s}\left|\frac{d_j}{d_s}\right|c'_{T_j}(T')$$

$$<\frac{c'_{T_s}(T')}{n}+(n-1)\frac{c'_{T_s}(T')}{n}=c'_{T_s}(T').$$

We get a contradiction. Therefore the set of all tree copy functions is strongly linearly independent. □

Definition. A tree T is accessible from $T_1, T_2, \ldots, T_n$ if
(1) there exist subtrees $A_0, A_1, \ldots, A_n$ of T such that $T_i = A_i$ for $i \geq 1$,
(2) all the pendant of A_0 is identified with some node of A_i, $i \geq 1$ and
(3) the subtree of T generated by $A_0, A_1, \ldots, A_n$ is T itself.

Let $A(T_1, T_2, \ldots, T_n)$ denote the set of all trees accessible from $T_1, T_2, \ldots, T_n$; and for $T \in A(T_1, T_2, \ldots, T_n)$, we let $\psi_{T_1,T_2,\ldots,T_n}(T)$ be the number of all possible ways to select $A_0, A_1, \ldots, A_n$ such that
(1) $T_i \cong A_i$ for $i \geq 1$;
(2) all pendant of A_0 is identified with some node of A_i where $i \geq 1$;
(3) the subtree of T generated by $A_0, A_1, \ldots, A_n$ is T itself; and
(4) A_0 is minimum with the properties (1), (2) and (3).

Lemma 2.3.

$$c'_{T_1}c'_{T_2} \cdots c'_{T_n} = \sum_{T \in A(T_1,T_2,\ldots,T_n)} \psi_{T_1,T_2,\ldots,T_n}(T)c'_T.$$

Proof. For any $B \in \mathcal{T}$, we have $c'_{T_1}c'_{T_2} \ldots c'_{T_n}(B) = |\{(A_1, A_2, \ldots, A_n) \mid A_i$ is a subtree of B and $A_i \simeq T_i$ for all $i\}|$.

For each $(A_1, A_2, \ldots, A_n)$ with A_i a subtree of B and $A_i \cong T_i$ for all i, let T be a minimum subtree of B containing A_i for all i. Then $T \in A(T_1, T_2, \ldots, T_n)$. By summing over all $T \in A(T_1, T_2, \ldots, T_n)$, we have

$$c'_{T_1}c'_{T_2} \cdots c'_{T_n}(B) = \sum_{T \in A(T_1,T_2,\ldots,T_n)} \psi_{T_1,T_2,\ldots,T_n}(T)c'_T(B). \quad \square$$

Example 1. Let $T_1 = T_2 \cong P_1$. Then $A(T_1, T_2) = A(2P_1) = \{P_i \mid i \geq 1\}$, where P_i is the path graph of length i. Therefore we have $c'^2_{P_1} = c'_{P_1} + 2\sum_{i=2}^{\infty} c'_{P_i}$.

Theorem 2.1. *$\{c'_T \mid T \in \mathcal{T}\}$ is algebraically independent.*

Proof. If this theorem is not true, there is a polynomial which is zero for all trees but not identically zero. Let P be such a polynomial containing q variables $c'_{T_1}, c'_{T_2}, \ldots, c'_{T_q}$ where T_i are arranged in descending order on the number of edges. Then P can be written as

$$P(c'_{T_1}, c'_{T_2}, \ldots, c'_{T_q}) = \sum_{i=1}^{n} a_i c'^{\alpha_{i,1}}_{T_1} c'^{\alpha_{i,2}}_{T_2} \cdots c'^{\alpha_{i,q}}_{T_q},$$

where $(\sum_{j=1}^{q} \alpha_{i,j}, \alpha_{i,1}, \alpha_{i,2}, \ldots, \alpha_{i,q})$ is greater than $(\sum_{j=1}^{q} \alpha_{k,j}, \alpha_{k,1}, \alpha_{k,2}, \ldots, \alpha_{k,q})$ lexicographically if and only if $i < k$. By Lemma 2.3 P can be written as

$$\begin{aligned} &P(c'_{T_1}, c'_{T_2}, \ldots, c'_{T_q}) \\ &= \sum_{i=1}^{n} \sum_{T \in A(\alpha_{i,1}T_1, \alpha_{i,2}T_2, \ldots, \alpha_{i,q}T_q)} a_i \psi_{\alpha_{i,1}T_1, \alpha_{i,2}T_2, \ldots, \alpha_{i,q}T_q}(T) c'_T \\ &= \sum_{T \in A} a_T c'_T, \end{aligned} \qquad (*)$$

where $T \in A$ if and only if

$$\sum_{i=1}^{n} \sum_{T \in A(\alpha_{i,1}T_1, \alpha_{i,2}T_2, \ldots, \alpha_{i,q}T_q)} a_i \psi_{\alpha_{i,1}T_1, \alpha_{i,2}T_2, \ldots, \alpha_{i,q}T_q}(T)$$

is not zero.

The cardinality of A can be zero, a positive integer or infinite. First, we claim that $|A| \neq 0$. Consider the first term of P, $a_1 c_{T_1}'^{\alpha_{1,1}} c_{T_2}'^{\alpha_{1,2}} \cdots c_q'^{\alpha_{1,q}}$. Let $k = \max\{d(T_i) \mid i = 1, 2, \ldots, q\}$, where $d(T)$ is the length of the longest path in T. Form a tree B as follows: For each of the m given P_k where $m = \sum_{j=1}^{q} \alpha_{1,j}$, pick up an endpoint and identity all these m points to get a star like graph, $P_{m,k}$. We identify each cliff point of trees in $\alpha_{1,1}T_1, \alpha_{1,2}T_2, \ldots, \alpha_{1,q}T_q$ with one and only one pendant vertex of $P_{m,k}$. B is the resultant graph. It is easy to check that B is an element of $A(\alpha_{1,1}T_1, \alpha_{1,2}T_2, \ldots, \alpha_{1,q}T_q)$ but not of $A(\alpha_{i,1}T_1, \alpha_{i,2}T_2, \ldots, \alpha_{i,q}T_q)$ where $i \geq 2$. Therefore the term $a_B c_B'$ cannot be cancelled out in (*). Hence $|A| \neq 0$. Second, we assume that $|A|$ is a positive integer. Then we have $0 = P(c_{T_1}', c_{T_2}', \ldots, c_{T_q}') = \sum_{\text{finite}} a_T c_T'$ which contradicts Lemma 2.2. Finally, assume that $|A|$ is infinite. Choose B in A with the least number of edges. We have $\sum_{T \in A} a_T c_T'(B) = a_B c_B'(B) \neq 0$, a contradiction. From the discussions above, we conclude that $\{c_T' \mid T \in \mathcal{T}\}$ is algebraically independent. □

Remark. We must exclude K_1 as a tree for otherwise we have $c_{K_1}' - c_{K_2}' = 1$.

3. Dependence of rooted tree copy functions

A tree with one point, its root, distinguished from the other points is called a rooted tree. Let T_1 be a rooted tree with root x_1, and T_2 be a rooted tree with root y_1. $c_{T_1}''(T_2)$ is the number of rooted subtrees in T_2 which is isomorphic to T_1. Let T_1, T_2 be the rooted trees in Fig. 1, with roots x_1, y_1, respectively. Then $c_{T_1}''(T_2) = 1$. Note that the subtree generated by y_2, y_3 is not a rooted subtree of T_2.

The term of linearly (strongly linearly, algebraically) dependence for rooted tree copy function is similarly defined to that for tree copy function. In this section we are going to discuss the dependence of rooted tree copy functions. Our result

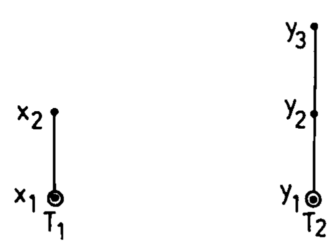


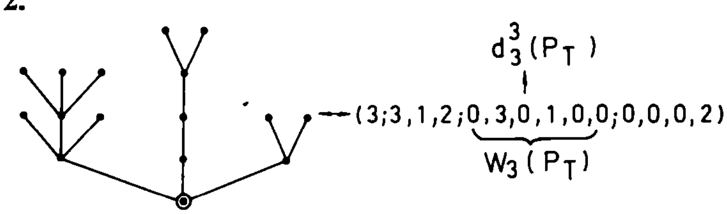
Fig. 1.

is that they are strongly linearly independent but not algebraically independent but not algebraically independent. One trivial example is

$$2C''_{Y} = C''^{2}_{\text{I}} - C''_{\text{I}}$$

Since every rooted tree T is planar, we can map it on the plane in such a way that the root x of T is always at bottom and those vertices with equal distance to the root x are on the same level. The height of a rooted tree is the maximum distance from the root x to any particular point. Therefore for every rooted tree T, we can associate with it at least one pictured-tree on paper, say P_T. We may represent P_T by a unique *sentence*, the first *word* of which reflects the number of branches P_T has, and the $(i+1)$th word reflects the number of sub-branches for that of the ith step, recorded from left to right. We denote the ith word by $W_i(P_T)$; the jth digit in $W_i(P_T)$ by $d_i^j(P_T)$. The sentence representing P_T is called a representation of T.

Example 2.



It is easy to see that $\sum_j d_i^j =$ the number of digits in W_{i+1} and $\sum_{i,j} d_i^j =$ the number of edges in T. Since every rooted tree T may have many P_T's associated with it, the representation of T is not uniquely defined. To avoid this complexity, we may define the *principal representation*, PR(T), of T to be the largest representations for T. Then PR(T) is the representation for the most left-tilted pictured-tree P_T which T may form. In the following discussions, the representation of T refers to the principal representation of T; and the order refers to the lexicographical order. We have the following lemma whose proof is similar to that of the strongly linear independence for tree copy functions.

Lemma 3.1. *Let $B_0 = \{T_1, T_2, \ldots, T_n\}$ be a family of distinct rooted trees. Then there exists a rooted tree in B_0, say T_s, such that for any $\varepsilon > 0$ and $K > 0$, there exists a rooted tree T which satisfies $c''_{T_s}(T) > K$ and $c''_{T_i}(T) < \varepsilon c_{T_s}(T)$ for every $T_i \neq T_s$.*

Proof. We need an algorithm to find T_s and to construct T. We define a sequence of sets by $B_i = \{T \in B_{i-1} \mid W_i(T)$ is the least with respect to the lexicographic order$\}$. After finite steps, we have an α such that B_α contains only one element, say T_k. In fact, T_k is the least element in B_0. Let α be the first index such that $B_\alpha = \{T_k\}$. If the height of T_k is greater than α, take T_s to be T_k. Let d_i^j be the first nonzero digit of the last word in PR(T_s). Construct $T(m)$ to be the rooted tree whose representation differs from that of T_s by changing d_i^j to m. If the height of

T_k equals α, consider $A=\{T\in B_{\alpha-1} \mid \text{height of } T \text{ is } \alpha\}$. Since $A\neq\emptyset$, we can choose T_s to be the largest element in A. Let $i(1)<i(2)<\cdots<i(p)$ be the indices such that $d_\alpha^{i(j)}\neq 0$. Construct $T(m)$ to be the rooted tree whose representation differs from that of T_s by a replacement of $d_\alpha^{i(j)}$ by $m(p-i+1)$, where $m(1)=m$, $m(i+1)=2^{m(i)}$. Now we can show that $T(m)$ and T_s satisfy our requirement for large m. For the first case, that is, the height of T_k is greater than α, we have $c''_{T_i}((T(m))=0$ for all $i\neq s$. Therefore a sufficiently large m may be chosen to do the job. For the second case, that is, the height of T_k equals α, we have $c''_{T_i}(T(m))=0$ for those rooted trees not in A; and

$$c''_{T_s}(T(m))=(a_s+\mathrm{o}(1))\prod_{j=1}^{P}\binom{m(p-j+1)}{d_\alpha^{i(j)}(T_s)},$$

where a_s is a positive constant. For those rooted trees T_i in A with $T_i\neq T_s$, $c''_{T_i}(T(m))$ is at most

$$(a_i+\mathrm{o}(1))\prod_{j=1}^{P}\binom{m(p-j+1)}{d_\alpha^{i(j)}(T_i)},$$

where a_i is a positive constant. Since $W_\alpha(T_i)<W_\alpha(T_s)$, we can choose a sufficiently large m such that $c''_{T_s}(T(m))>K$ and $c''_{T_i}(T(m))<\varepsilon c_{T_s}(T(m))$. □

Theorem 3.1. *$\{c''_T \mid T$ is a rooted tree$\}$ is strongly linearly independent.*

Proof. Similar to the proof of Lemma 2.2. □

As we mentioned above, $\{c''_T \mid T \text{ is a rooted tree}\}$ is algebraically dependent. We would like to find an algebraic base for it. For this purpose, we consider the set of stem trees.

A rooted tree T with root x is called a *stem tree* if $\deg_T(x)=1$. Given an arbitrary rooted tree T, we can decompose it into branches of stem trees $B_1, B_2, \ldots, B_n$. Then we write $T=B_1\oplus B_2\oplus\cdots\oplus B_n$. We let iT denote $T\oplus T\oplus\cdots\oplus T$ (i times); and $\mathrm{ST}=\{T \mid T \text{ is a stem tree}\}$. (See Fig. 2).

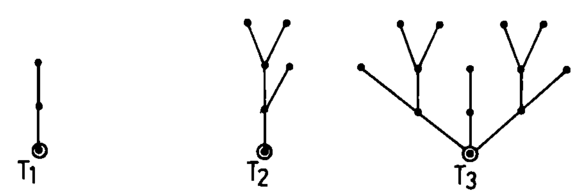


Fig. 2. T_1 and T_2 are both stem trees but T_3 is not. Moreover $T_3=T_1\oplus 2T_2$.

Lemma 3.2. *$\{c_T \mid T \in ST\}$ is algebraically independent.*

Proof. We follow Whitney's idea. If the assertion is not true, there is a non-zero polynomial which is zero for all graphs. Let P be such a polynomial containing the least possible number, say $q(>0)$, of variables. Let us call the variables $c''_{T_1}, c''_{T_2}, \ldots, c''_{T_q}$ where T_1 is one of the stem trees with the least number of vertices among $\{T_i\}_{i=1}^{q}$. Arrange the polynomial in descending power of c''_T and assume that it is of degree α in this variable:

$$P(C''_{T_1}, C''_{T_2}, \ldots, C''_{T_q}) = P_0(c''_{T_2}, \ldots, c''_{T_q})c''^{\alpha}_{T_1} + P_1(c''_{T_2}, \ldots, c''_{T_q})c''^{\alpha-1}_{T_1} + \cdots + P_\alpha(c''_{T_2}, \ldots, c''_{T_q}) \quad (P_0 \neq 0).$$

Take any rooted tree B_0 and form $B_1, B_2, \ldots, B_\alpha$ by letting $B_i = B_0 \oplus iT_1$ for $i = 1, 2, \ldots, \alpha$. We get $c''_{T_1}(B_i) = c''_{T_1}(B_{i-1}+1$ and $c''_{T_j}(B_i) = c''_{T_j}(B_{i-1})$ for $i = 1, 2, \ldots, \alpha, 2 \leq j \leq q$. P vanishes for these $\alpha+1$ distinct values of c''_{T_1} and the coefficients $P_0, P_1, \ldots, P_\alpha$ are constants for these rooted trees. They vanish, in particular, for the rooted tree B_0. Since B_0 is arbitrary, we get $P_0(c''_{T_2}, \ldots, c''_{T_q}) = 0$ for all rooted trees, which contradict the choice of P. Therefore $\{c''_T \mid T \in \mathrm{ST}\}$ is algebraically independent. □

Our goal is to prove that $\{c''_T \mid T \in \mathrm{ST}\}$ is actually an algebraic base for the algebra generated by $\{c''_T \mid T$ is a rooted tree$\}$. All we need is to prove that for any given rooted tree T, c''_T can be expressed algebraically by elements of $\{c''_T \mid T \in \mathrm{ST}\}$.

Let $T_1, T_2, \ldots, T_n$ be stem trees. We say a stem tree T is constructible by $T_1, T_2, \ldots, T_n$ if there exist rooted subtrees $A_1, A_2, \ldots, A_n$ of T such that $T_i \cong A_i$ for every i and the rooted subtree of T generated by the union of $A_1, A_2, \ldots, A_n$ is T itself. Let $I(T_1, T_2, \ldots, T_n)$ denote the set of all constructible stem trees of $T_1, T_2, \ldots, T_n$; and for $T \in I(T_1, T_2, \ldots, T_n)$, let $\Psi_{T_1, T_2, \ldots, T_n}(T)$ be the number of all possible ways to select rooted subtrees $A_1, A_2, \ldots, A_n$ of T such that $T_i \cong A_i$ for every i and their union generates T.

For simplicity, we work on 2-branch rooted trees.

Lemma 3.3. *Let $T = T_1 \oplus T_2$, where T_1, T_2 are stem trees. Then c''_T can be algebraically expressed in terms of c''_{T_1}, c''_{T_2} and those $c''_{T''}$ with $T'' \in I(T_1, T_2)$. To be more precise,*

$$kc''_T = c''_{T_1}c''_{T_2} - \sum_{T'' \in I(T_1, T_2)} \Psi_{T_1, T_2}(T'')c''_{T''},$$

where $k = 1$ if $T_1 \neq T_2$ and $k = 2$ otherwise.

Proof. For any n-branch rooted tree T' with branches $B_1, B_2, \ldots, B_n$, we have $kc''_T(T') = |\{(A, B) \mid A, B$ are rooted subtrees of T' such that $A \cong T_1$ and $B \cong T_2$; A, B are not in the same branch$\}|$ and $c''_{T_1}c''_{T_2}(T') = |\{(A, B) \mid A, B$ are rooted subtrees of T' such that $A \cong T_1, B \cong T_2\}|$. Thus

$$\begin{aligned}(c''_{T_1}c''_{T_2} - kc''_T)(T') &= |\{(A, B) \mid A, B \text{ are in the same branch of } T' \text{ and } A \cong T_1, B \cong T_2\}| \\ &= \sum_{i=1}^{n} |\{(A, B) \mid A \cong T_1, B \cong T_2,\ A, B \text{ both in } B_i\}|.\end{aligned}$$

For $T'' \in I(T_1, T_2)$, we get

$$\sum_{T'' \in I(T_1,T_2)} \Psi_{T_1,T_2}(T'')c''_{T''}(T') = \sum_{i=1}^{n} \sum_{T'' \in I(T_1,T_2)} \Psi_{T_1,T_2}(T'')c''_{T''}(B_i).$$

For each B_i, we define an equivalence relation '$\sim_i$' on the set $\{(A, B) \mid A \cong T_1, B \cong T_2,\ A, B \text{ in } B_i\}$: $(A, B) \sim_i (A', B')$ if the union of A and B is equal to the union of A' and B'. Consider the equivalent class of $\sim_i$, we get

$$(c''_{T_1}c''_{T_2} - kc''_T)(B_i) = \sum_{T'' \in I(T_1,T_2)} \Psi_{T_1,T_2}(T'')c''_{T''}(B_i).$$

Summing over all i, we have

$$(c''_{T_1}c''_{T_2} - kc''_T)(T') = \sum_{T'' \in I(T_1,T_2)} \Psi_{T_1,T_2}(T'')c''_{T''}(T'). \quad \square$$

Corollary. *Let $T = T_1 \oplus T_2 \oplus \cdots \oplus T_n$, where each T_i is a stem tree. Then*

$$kc''_T = c''_{T_1}c''_{T_2} \cdots c''_{T_n} + \sum_{\text{finite}} c''_{T'_1}c''_{T'_2} \cdots c''_{T'_q}$$

where T'_i is a stem tree and k depend on the repetition type of $T_1, T_2, \ldots, T_n$.

Combining Lemma 3.2 and the above corollary, we have the following theorem.

Theorem 3.2. $\{c''_T \mid T \in \mathrm{ST}\}$ *is a base for the algebra generated by* $\{c''_T \mid T$ *is a rooted tree*$\}$.

References

[1] P. Erdös, L. Lovász and J. Spencer, Strong independence of graph copy functions, in: Bondy and Murty, eds., Graph Theory and Related Topics (Academic Press, New York, 1979) 165–172.
[2] R.L. Graham and Szemerdi, On subgraph number independence in trees, J. Combin. Theory Ser. B 24 (1978) 213–222.
[3] L.-H. Hsu, Monotone multiplicative graph functions and dependence of tree copy functions, Ph.D. Dissertation (1981).
[4] H. Whitney, The coloring of a graph, Ann. of Math. 33 (1932) 688–718.