Model Reduction for Control Systems with Parameter Variations

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ABSTRACT: *The effects of model reduction on control systems with parameter variations are investigated. In order to reduce these effects, two modified Padé approximation methods are used so that the reduced models can approximate the frequency response of the original transfer function not only at* $s = 0$ *and* $s = \infty$ *but also at a selected point on the frequency response curve of the original transfer function. Examples are shown and compared with the methods given in the current literature.*

I. Introduction

The methods for model reduction such as continued-fraction expansion, time-moment matching and Padé approximations are equivalent under some conditions, and named as Padé type approximations (**1–9**). Since the approximations are taken at $s = 0$, these methods may yield poor frequency response. Some modified Padé approximations at two frequencies, i.e. $s = 0$ and $s = \infty$, have been suggested (**10**, **11**), but the frequency response at a specified frequency can not be matched by the reduced model. The method by use of continued-fraction expansion about a general point, $s = a$ $(a > 0)$, was proposed by Davidson and Lucas (**12**). However, the frequency response of the original transfer function at $s = 0$ and $s = \infty$ can not be approximated by the reduced model.

In the current literature, most of the methods for model reduction are based upon the assumption that the original system has constant parameters. However, it will be shown later in this paper that even if the reduced model is stable, the closed-loop system response characteristics may not be acceptable and the stability of the closed-loop system may not be preserved due to the effects of parameter variations. Therefore, a practical method for model reduction should consider the effects of parameter variations (**13–17**).

Recently, Hung and Han (**18**) proposed two methods for model reduction which extend modified Padé approximations to obtain "biased" reduced-order models (**19**). By use of these methods, a reduced model can be made to approximate the frequency response of the original transfer function not only at $s = 0$ and $s = \infty$ but also at a selected point on the frequency response curve of the original transfer function. This paper extends the application of Hung and Han's methods to control systems with parameter variations.

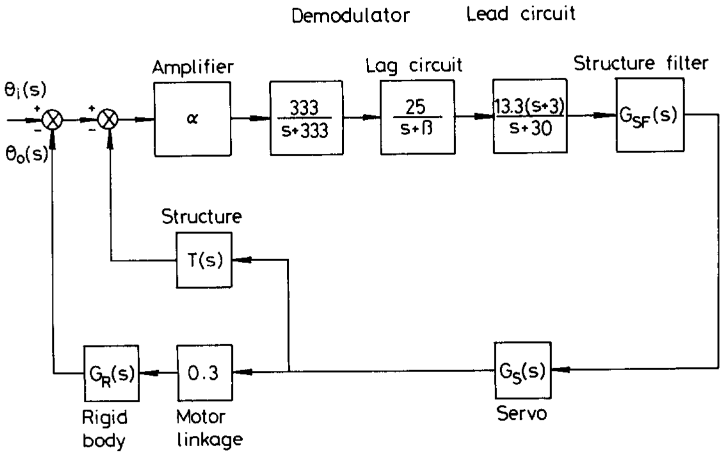

FIG. 1. Block diagram of a flexible missile control system.

II. Effects of Parameter Variations

In this section, three examples are presented to show the effects of parameter variations on the results of model reduction.

Example 1

Consider the system shown in Fig. 1. The transfer functions of the blocks are defined as (**20**)

$$G_R(s) = \frac{7.21}{(s+1.6)(s-1.48)} \tag{2.1}$$

$$G_S(s) = \frac{2750}{s^2+42.2s+2750} \tag{2.2}$$

$$G_{SF}(s) = \frac{(s^2+70s+4000)(s^2+22s+12800)}{(s^2+30s+5810)(s^2+30s+12800)} \tag{2.3}$$

and

$$T(s) = [0.686(s+53)(s-53)(s^2-152.2s+14500)(s^2+153.8s+14500)]/$$
$$[(s^2+s+605)(s^2+45.5s+2660)(s^2+2.51s+3900)(s^2+3.99s+22980)]. \tag{2.4}$$

The parameters α and β are defined as

$$15 \leqslant \alpha \leqslant 20 \tag{2.5}$$

and

$$40 \leqslant \beta \leqslant 100, \tag{2.6}$$

respectively. By use of the modified Padé approximation method and the Routh stability array method (**21**), one has the reduced models of $T(s)$ as follows:

(a) *Modified Padé approximation method.*

$$M_1(s) = M[3,5]_6^4(s)$$
$$= [0.686(s-111.3723)(s^2+321.1828s+38950.873)]/$$
$$[(s^2+1.2982s+401.8666)(s^2+152.1748s+28159.032)(s+107.7370)] \quad (2.7)$$

where $M[a,b]_j^i(s)$ represents that the reduced model has "a" zeros "b" poles, and is obtained by continued-fraction expansion of $T(s)$ about $s = 0$ and $s = \infty$ for "i" and "j" times, respectively.

(b) *Routh stability array method.*

$$T_1(s) = [-1.5899\times 10^{-2}(s+53)(s+150.2440)(s-157.0873)]/$$
$$[(s+161.1435)(s^2+0.7348s+611.5744)(s^2+2.0104s+3807.4562)]. \quad (2.8)$$

The Bode plots of $T(s)$, $M_1(s)$ and $T_1(s)$ are shown in Fig. 2. The unit-step responses of the original system and the systems with reduced models are shown in Fig. 3 (a and b). The maximum deviations (E_M), integral of absolute errors (I.A.E.) and integral of squared errors (I.S.E.) between the unit-step responses of the original system and the systems with reduced models are shown in Table I. Note that the closed-loop systems with the reduced models for $\alpha = 20$ and $\beta = 40$ are unstable, although the reduced models $M_1(s)$ and $T_1(s)$ are stable.

Example 2

Consider the system shown in Fig. 4. The transfer function and the gain are

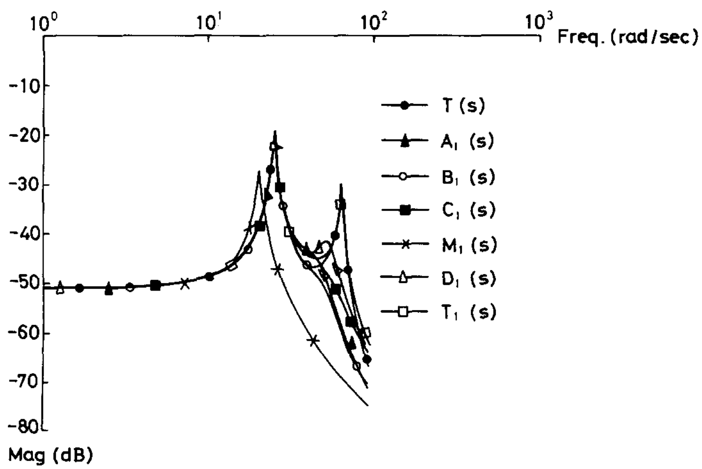

FIG. 2. Bode plots of $T(s)$ and its reduced models.

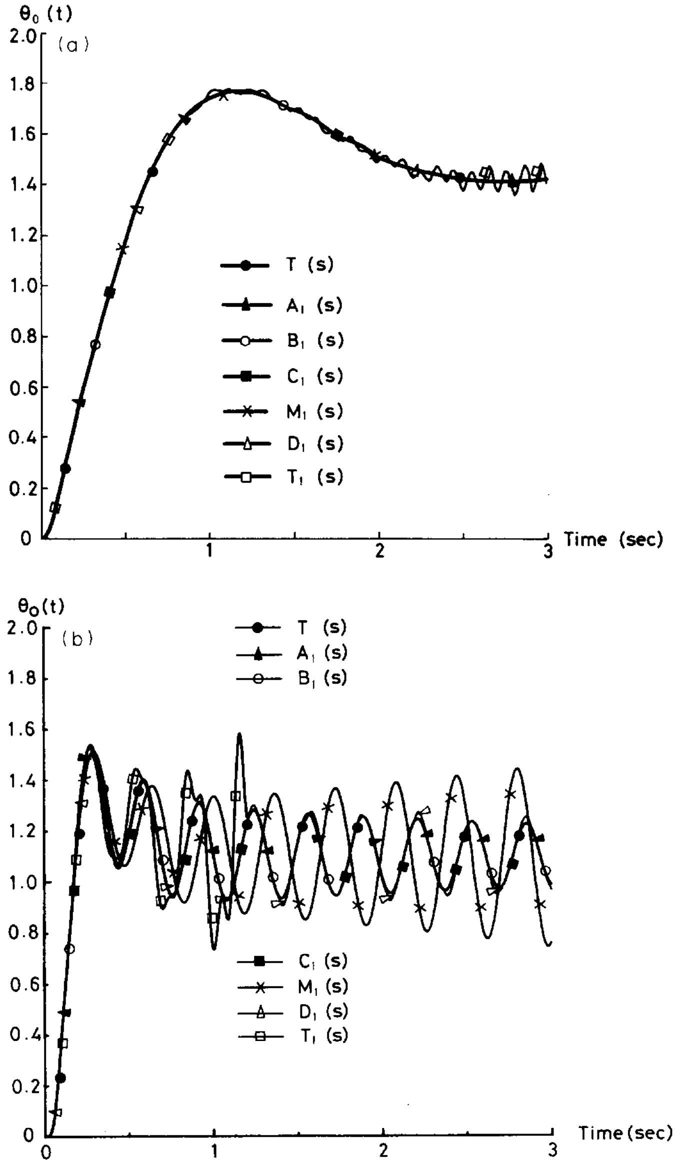

FIG. 3. (a) Closed-loop unit-step responses for $\alpha = 15$ and $\beta = 100$. (b) Closed-loop unit-step responses for $\alpha = 20$ and $\beta = 40$.

defined as

$$G_2(s) = [1441.53(s+1.4706)(s+6.1350) \times (s+46.7248)]/[(s+1.8972)(s+49.3777)(s+52.5174) \times (s^2+0.5456s+1.1621)(s^2+7.7022s+108.0056)] \quad (2.9)$$

TABLE I

Parameters α	β	Errors	Models $M_1(s)$	$A_1(s)$	$B_1(s)$	$C_1(s)$	$D_1(s)$	$T_1(s)$
		E_M	$1.5185E-2$	$2.9371E-3$	$2.1794E-3$	$2.2348E-3$	$2.3766E-3$	†
15	100	I.A.E.	$1.4504E-2$	$1.5148E-3$	$1.2349E-3$	$1.2669E-3$	$1.7264E-3$	†
		I.S.E.	$1.1304E-3$	$1.6345E-6$	$1.1978E-6$	$1.2454E-6$	$1.7328E-6$	†
		E_M	†	$2.0390E-2$	$1.3898E-2$	$1.5170E-2$	$3.3728E-2$	†
20	40	I.A.E.	†	$8.4078E-3$	$6.2302E-3$	$7.5369E-3$	$3.4952E-2$	†
		I.S.E.	†	$7.4285E-5$	$4.3113E-5$	$4.8746E-5$	$6.3379E-4$	†

† Unstable, integration interval; 3 s.

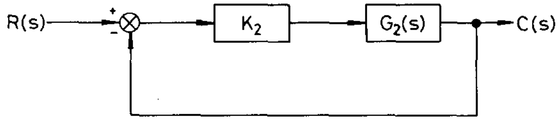

FIG. 4. Block diagram of a control system.

and

$$7.5 \leqslant k_2 \leqslant 12, \tag{2.10}$$

respectively. By use of the modified Padé approximation method, the reduced model of $G_2(s)$ is obtained as

$$\begin{aligned} M_2(s) &= M[1,5]_5^5(s) \\ &= [1441.53(s+1.8953)]/[s+45.3585) \\ &\quad \times (s^2+0.5688s+1.1664)(s^2+13.6776s+52.4722)]. \end{aligned} \tag{2.11}$$

The Nyquist plots of $G_2(s)$ and $M_2(s)$ are shown in Fig. 5. The unit-step responses of the original system and the system with reduced model $M_2(s)$ are shown in Fig. 6 (a and b) for $k_2 = 7.5$ and $k_2 = 12$, respectively. The maximum deviations, integral of absolute errors and integral of squared errors are shown in Table II. It can be seen that the closed-loop system with the reduced model for $k_2 = 12$ is unstable.

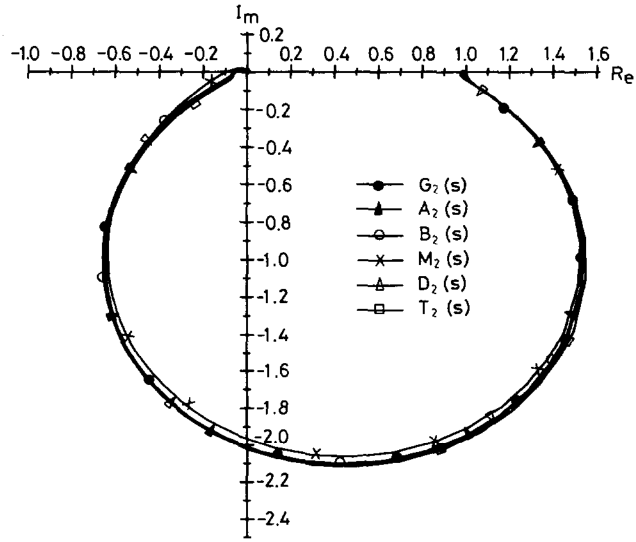

FIG. 5. Nyquist plots of $G_2(s)$ and its reduced models.

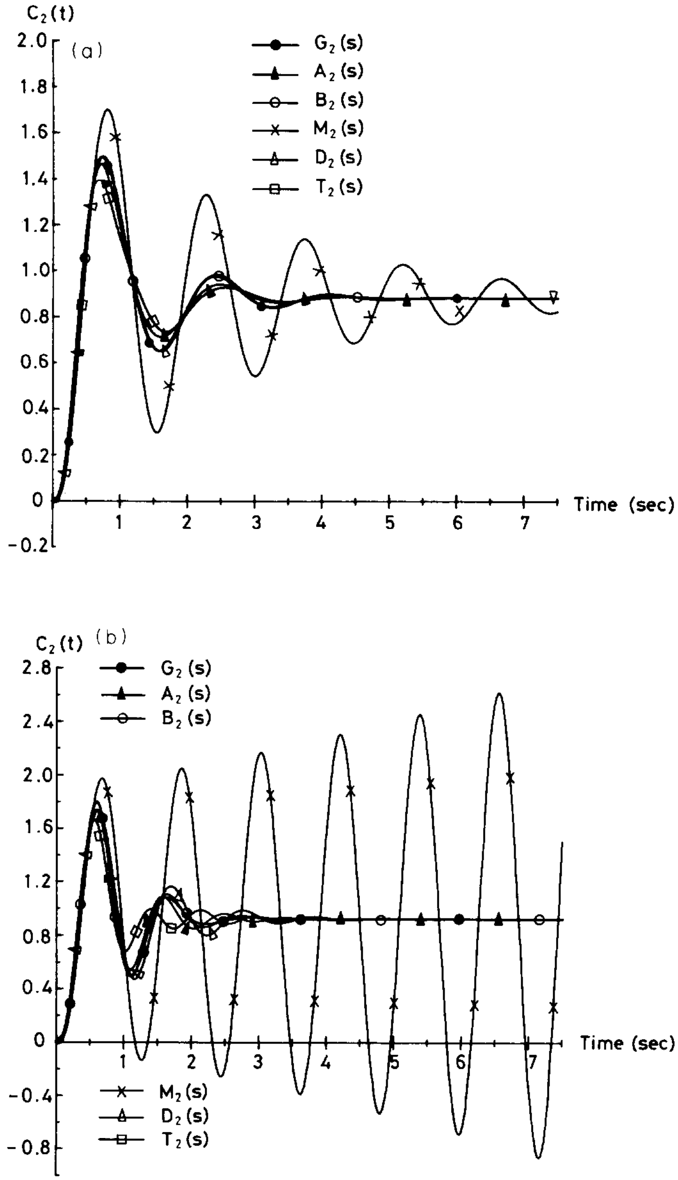

FIG. 6. (a) Closed-loop unit-step responses for $k_2 = 7.5$. (b) Closed-loop unit-step responses for $k_2 = 12$.

TABLE II

Parameter k_2	Errors	Models $M_2(s)$	$A_2(s)$	$B_2(s)$	$D_2(s)$	$T_2(s)$
	E_M	$3.6947E-1$	$6.3364E-1$	$1.4326E-2$	$4.4852E-2$	$1.3213E-1$
7.5	I.A.E.	$1.0135E+0$	$1.0362E-1$	$2.5909E-2$	$3.4664E-2$	$1.9254E-1$
	I.S.E.	$2.1329E-1$	$3.3249E-2$	$1.9872E-4$	$7.1119E-4$	$1.3196E-2$
	E_M	†	$8.1413E-1$	$2.5873E-2$	$1.1917E-1$	$2.9989E-1$
12	I.A.E.	†	$1.3098E-1$	$4.4793E-2$	$1.7343E-1$	$3.2550E-1$
	I.S.E.	†	$6.4556E-2$	$7.1451E-4$	$1.1254E-2$	$5.5259E-2$

† Unstable, integration interval; 7.5 s.

Example 3

Consider the system shown in Fig. 7. The transfer functions of $G_3(s)$ and $H_3(s)$ are defined as

$$G_3(s) = [(s+0.1263)(s+0.6883)(s^2+1.4210s+1.0832) \times (s^2-1.6962s+66.2273)(s^2+99.4606s + 8294.2128)]/[13(s+0.1263)(s+1.0617) \times (s+1.2812)(s+1.8152)(s+3.3039)(s+8.6994) \times (s+20.1418)(s^2+1.3140s+22.0430)] \tag{2.12}$$

and

$$H_3(s) = \frac{s+z_3}{s(s+p_3)} \tag{2.13}$$

where the parameters are defined as

$$k_3 = 0.5 \tag{2.14}$$

$$6 \leqslant z_3 \leqslant 10 \tag{2.15}$$

$$4 \leqslant p_3 \leqslant 6. \tag{2.16}$$

By use of the methods of modified Padé approximation and continued-fraction (3), one has the reduced models of $G_3(s)$ as follows.

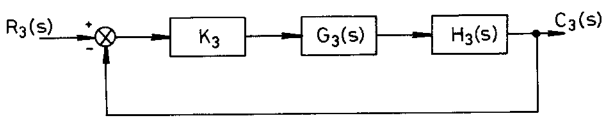

FIG. 7. Block diagram of a control system.

(a) *Modified Padé approximation method.*

$$M_3(s) = M[4,5]_3^7(s)$$
$$= [7.6923 \times 10^{-2}(s^2+0.06171s+0.002628)$$
$$\times (s^2+9.9840s+799.1396)]/[(s+37.5501)$$
$$\times (s^2+0.06171s+0.002628)(s^2+0.5886s+16.3711)]. \qquad (2.17)$$

(b) *Continued-fraction method.*

$$C_3(s) = C[4,5]_0^{10}(s)$$
$$= [1.6447(s+0.1263)(s+1.2792)(s^2+2.0818s+1.1653)]/$$
$$[(s+0.1263)(s^2+0.8198s+4.3035)(s^2+1.3826s+0.5697)]. \qquad (2.18)$$

The Nyquist plots of $G_3(s)$, $M_3(s)$ and $C_3(s)$ are shown in Fig. 8. For various values of z_3, p_3 and k_3, the unit-step responses of the original system and the system with the reduced models are shown in Fig. 9 (a and b). The other data are shown in Table III. It can be seen that the closed-loop systems with the reduced models are unstable when $k_3 = 0.5$, $z_3 = 10$ and $p_3 = 4$.

From the results of the above presented examples, it is clear that, due to the effects of parameter variations, the closed-loop system response characteristics and the stability characteristics of control systems with reduced models may not be acceptable, even if the reduced models are very accurate.

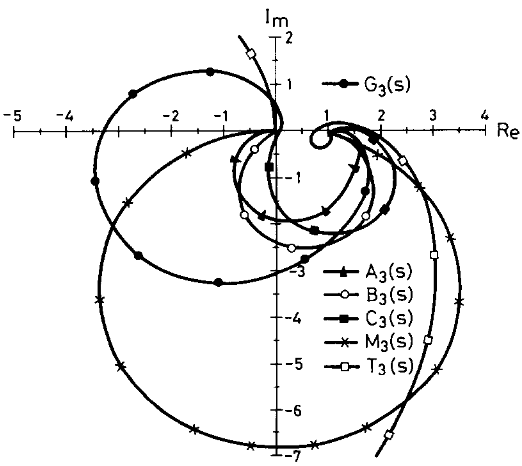

FIG. 8. Nyquist plots of $G_3(s)$ and its reduced models.

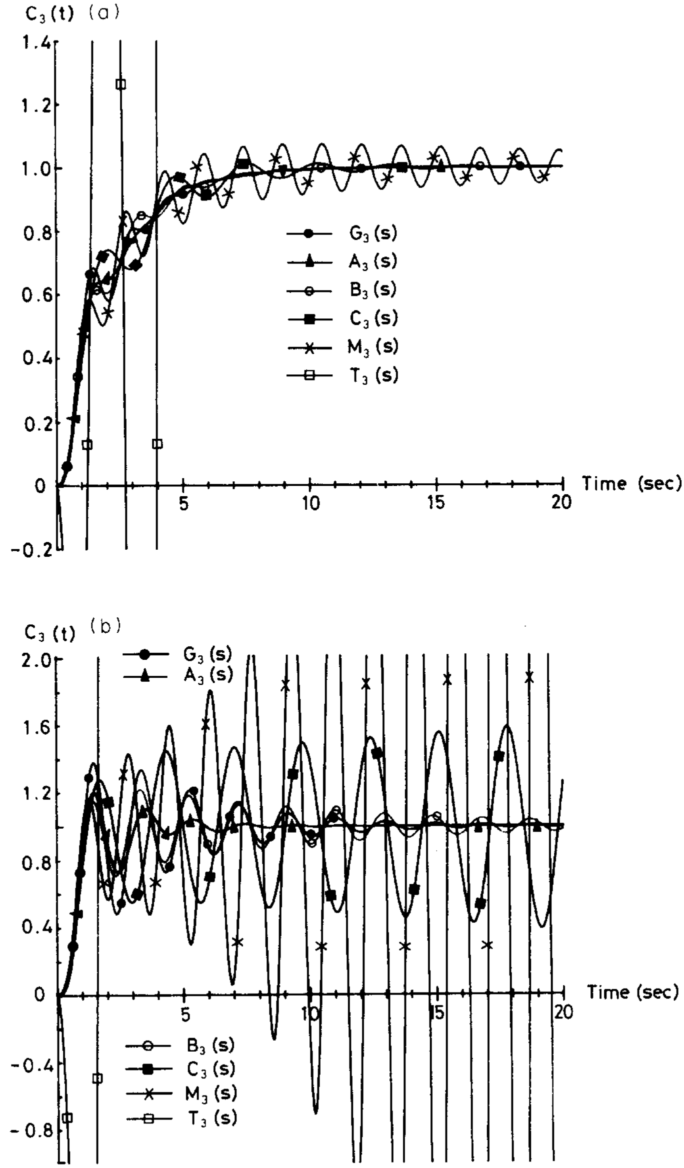

FIG. 9. (a) Closed-loop unit-step responses for $p_3 = 6$ and $z_3 = 6$. (b) Closed-loop unit-step responses for $p_3 = 4$ and $z_3 = 10$.

TABLE III

Parameters			Models				
z_3	p_3	Errors	$M_3(s)$	$A_3(s)$	$B_3(s)$	$C_3(s)$	$T_3(s)$
		E_M	1.3945$E-1$	7.3832$E-2$	8.2724$E-2$	1.6117$E-1$	†
6	6	I.A.E.	9.9302$E-1$	1.4150$E-1$	1.7397$E-1$	4.6494$E-1$	†
		I.S.E.	6.5610$E-2$	5.3165$E-3$	6.2723$E-3$	3.2237$E-2$	†
		E_M	†	2.6279$E-1$	2.3361$E-1$	†	†
10	4	I.A.E.	†	1.1183$E+0$	8.5478$E-1$	†	†
		I.S.E.	†	1.5140$E-1$	7.9249$E-2$	†	†

† Unstable, integration interval; 20 s.

III. A Review of Hung and Han's Methods (18)

Assume that the original transfer function and the reduced model are

$$G(s) = \frac{A_{21}+A_{22}s+A_{23}s^2+\cdots+A_{2n}s^{n-1}}{A_{11}+A_{12}s+A_{13}s^2+\cdots+A_{1,n+1}s^n} \tag{3.1}$$

and

$$(\omega)R[r-1,r]_j^i(s) = \frac{d_0+d_1s+d_2s^2+\cdots+d_{r-1}s^{r-1}}{c_0+c_1s+c_2s^2+\cdots+c_rs^r}, \tag{3.2}$$

respectively. In Eq. (3.2), r and $r-1$ represent the numbers of poles and zeros of $R(s)$, respectively; i and j are the numbers of times of the continued-fraction expansion of $G(s)$ about $s = 0$ and $s = \infty$, respectively and ω is the frequency at which the frequency response of $G(s)$ is matched by $R(s)$.

(i) *Method A*

Step 1. Expand $G(s)$ about $s = 0$ for i (even number) times, i.e.

$$G(s) = \cfrac{1}{h_1+\cfrac{1}{\cfrac{h_2}{s}+\cfrac{1}{\ddots+\cfrac{1}{\cfrac{h_i}{s}+\cfrac{H_N(s)}{H_D(s)}}}}} \tag{3.3}$$

where

$$H_N(s) = A_{i+2,1}+A_{i+2,2}s+\cdots+A_{i+2,n-i/2}s^{n-1-i/2} \tag{3.4}$$

and

$$H_D(s) = A_{i+1,1}+A_{i+1,2}s+\cdots+A_{i+1,n+1-i/2}s^{n-i/2}. \tag{3.5}$$

Step 2. Reverse the sequence of the polynomials in Eqs (3.4) and (3.5) and continue to expand (3.3) about $s = \infty$ for j (odd number) times yield

$$\frac{H_N(s)}{H_D(s)} = \cfrac{1}{H_1 s + \cfrac{1}{H_2 + \cfrac{1}{H_3 s + \cfrac{1}{\ddots + \cfrac{1}{H_j s + \cfrac{M_N(s)}{M_D(s)}}}}}} \tag{3.6}$$

where

$$M_N(s) = A_{i+j+2,n-(i+j-1)/2} s^{n-(i+j+1)/2} + \cdots + A_{i+j+2,1} \tag{3.7}$$

and

$$M_D(s) = A_{i+j+1,n-(i+j-1)/2} s^{n-(i+j+1)/2} + \cdots + A_{i+j+1,1}. \tag{3.8}$$

Step 3. Let

$$P(s) = s^2 + \omega^2 \tag{3.9}$$

and let $M_N(s)$ and $M_D(s)$ in (3.7) and (3.8) be divided by $P(s)$ in decreasing order of s, one has

$$M_N(s) = P(s)Q_N(s) + R_N(s) \tag{3.10}$$

and

$$M_D(s) = P(s)Q_D(s) + R_D(s) \tag{3.11}$$

where $Q_N(s)$, $Q_D(s)$ are the quotients and $R_N(s)$, $R_D(s)$ are the remainders, i.e.

$$R_N(s) = q_1 s + q_0 \tag{3.12}$$

and

$$R_D(s) = v_1 s + v_0. \tag{3.13}$$

Step 4. Replace $M_N(s)/M_D(s)$ in (3.6) by $R_N(s)/R_D(s)$ defined by (3.12) and (3.13) and invert the continued fraction, the reduced model given in (3.2) is obtained. The order of the denominator of the reduced model is

$$r = (i+j+3)/2. \tag{3.14}$$

From (3.9)–(3.11), it can be seen that

$$M_N(j\omega) = R_N(j\omega) \tag{3.15}$$

and

$$M_D(j\omega) = R_D(j\omega). \tag{3.16}$$

Therefore, one has

$$G(j\omega) = R(j\omega). \tag{3.17}$$

Equation (3.17) indicates that the frequency response of $G(s)$ at $s = j\omega$ is exactly matched by $R(s)$. Note that if i is odd and j is even, the above equations may have some minor differences, but the procedure is the same.

(ii) *Method B*

Step 1. Same as *Step* 1 of Method A.

Step 2. Reverse the polynomial sequences in (3.4) and (3.5) and continue to expand (3.3) about $s = \infty$ for j (even number) times yield

$$\frac{H_N(s)}{H_D(s)} = \cfrac{1}{E_1 s + \cfrac{1}{E_2 + \cfrac{1}{\ddots + \cfrac{1}{E_j + \cfrac{F_N(s)}{F_D(s)}}}}} \tag{3.18}$$

where

$$F_N(s) = A_{i+j+2,n-(i+j)/2} s^{n-1-(i+j)/2} + \cdots + A_{i+j+2,2} s + A_{i+j+2,1} \tag{3.19}$$

and

$$F_D(s) = A_{i+j+1,n+1-(i+j)/2} s^{n-(i+j)/2} + \cdots + A_{i+j+1,2} s + A_{i+j+1,1}. \tag{3.20}$$

Step 3. Let

$$\frac{T_N(s)}{T_D(s)} = \frac{Y_0}{X_1 s + X_0} \tag{3.21}$$

and with

$$\frac{F_N(s)}{F_D(s)} = \frac{T_N(s)}{T_D(s)}, \quad \text{for } s = j\omega. \tag{3.22}$$

The unknowns X_1 and X_0 can be obtained as

$$X_1 = \frac{Y_0}{\omega} \operatorname{Im}\left[\frac{F_D(s)}{F_N(s)}\right]\Bigg|_{s=j\omega} \tag{3.23}$$

and

$$X_0 = Y_0 \operatorname{Re}\left[\frac{F_D(s)}{F_N(s)}\right]\Bigg|_{s=j\omega} \tag{3.24}$$

where Im[·] and Re[·] denote the imaginary part and the real part of $F_D(s)/F_N(s)$ for $s = j\omega$, respectively, and Y_0 can be chosen as any non-zero real number.

Step 4. Replace $F_N(s)/F_D(s)$ in Eq. (3.18) by (3.21) and invert the continued fraction, the reduced model defined in (3.2) is obtained. The order of the denominator of the reduced model is

$$r = (i+j+2)/2. \tag{3.25}$$

The above equations may have some minor differences if both i and j are odd numbers.

IV. Applications to Systems with Parameter Variations

The main purpose of this section is to extend Hung and Han's methods to control systems with parameter variations. For ease in presentation, the examples considered in Section II are reconsidered here.

Example 4

Consider the system in Example 1. By use of Hung and Han's methods, and let the frequency responses of the reduced models be matched with that of $T(s)$ at $\omega_1 = 21$ rad s^{-1}, then the following models are obtained:

(a) *Method A.*

$$\begin{aligned} A_1(s) &= (\omega_1)R[3,5]_1^6(s) \\ &= [-4.0825\times 10^{-1}(s+43.6677)(s-49.6197) \\ &\quad \times (s-428.8551)]/[(s+107.3613)(s^2+18.6332s \\ &\quad +2067.7344)(s^2+0.9456s+608.3490)]. \end{aligned} \tag{4.1}$$

(b) *Method B.*

$$\begin{aligned} B_1(s) &= (\omega_1)R[4,5]_0^8(s) \\ &= [-2.6594\times 10^{-2}(s^2+78.0282s+1555.1648)(s^2-125.0310s \\ &\quad +3962.3308)]/[(s+38.0753)(s^2+0.9930s+605.2786) \\ &\quad \times (s^2+25.7674s+2531.3237)]. \end{aligned} \tag{4.2}$$

For comparison purpose the models obtained by use of the continued-fraction method, and the stability-equation and Padé approximation method (**22**), are as follows.

(c) *Continued-fraction method.*

$$\begin{aligned} C_1(s) &= C[4,5]_0^{10}(s) \\ &= [-2.44\times 10^{-2}(s-61.3836)(s-67.5164)(s^2+81.5000s \\ &\quad +1748.6381)]/[(s+41.3000)(s^2+0.9700s+605.9021) \\ &\quad \times (s^2+25.4000s+2447.8033)]. \end{aligned} \tag{4.3}$$

(d) *Stability-equation and Padé approximation method.*

$$D_1(s) = [-2.3873 \times 10^{-2}(s+38.1380)(s+130.4869)(s^2-101.2398s +2997.6391)]/[(s+72.5638)(s^2+0.91s+602.1438) \times (s^2+12.4262s+2902.0190)]. \quad (4.4)$$

Same as in Example 1, the results are also shown in Table I and in Figs 2 and 3. It can be seen that the system response with the model $B_1(s)$ of Hung and Han's method B is the best one.

Example 5

Consider the system in Example 2. Assume that the frequency responses of the reduced models are matched with that of $G_2(s)$ at $\omega_{p2} = 7.7155$ rad s^{-1}, which is the phase crossover frequency. The following models are obtained:

(a) *Method A.*

$$A_2(s) = (\omega_{p2})R[2,5]_2^5(s) = [2.4478 \times 10^1(s+1.2126)(s+4.6310)]/[(s+1.3640) \times (s^2+0.5328s+1.1532)(s^2+7.8318s+88.8049)]. \quad (4.5)$$

(b) *Method B.*

$$B_2(s) = (\omega_{p2})R[2,5]_2^6(s) = [1.9304 \times 10^1(s+1.4880)(s+7.6536)]/[(s+1.9625) \times (s^2+0.5456s+1.1608)(s^2+6.5298s+97.8933)]. \quad (4.6)$$

(c) *Stability-equation method* (**23**).

$$D_2(s) = [3.9424 \times 10^{-1}(s+1.4706)(s+6.1350)(s+46.7248)]/ \times [(s+1.8563)(s^2+0.5454s+1.1610)(s^2+7.8478s+78.3993)]. \quad (4.7)$$

(d) *Routh stability array method.*

$$T_2(s) = [4.2382 \times 10^{-1}(s+1.4706)(s+6.1350)(s+46.7248)]/ [(s+1.7321)(s^2+0.5328s+1.1735)(s^2+5.9616s+89.3227)]. \quad (4.8)$$

The results are also shown in Table II and in Figs 5 and 6, which indicate that the reduced model $B_2(s)$ of Hung and Han's method B can give the best result.

Example 6

Consider the system in Example 3. If the frequency responses of the reduced models are matched with that of $G_3(s)$ at $\omega_3 = 1.5$ rad s^{-1} and $\omega_4 = 3.3$ rad s^{-1}. The following models are obtained.

(a) *Method A.*

$$A_3(s) = (\omega_3)R[4,5]_4^3(s) = [7.6923 \times 10^{-2}(s^2+1.0958s+2.3388)(s^2+102.8190s+8201.5786)]/ [(s+38.4017)(s^2+2.3560s+14.7175)(s^2+0.9006s+2.6109)]. \quad (4.9)$$

(b) *Method B.*

$$B_3(s) = (\omega_4)R[4,5]_0^8(s)$$
$$= [8.4719 \times 10^{-1}(s+0.1139)(s+13.5001)(s^2+0.1786s+0.1008)]/$$
$$[(s+0.1139)(s^2+0.1794s+0.1008)(s^2+1.3364s+11.4298)]. \quad (4.10)$$

(c) *Routh stability array method.*

$$T_3(s) = [-23.1355(s+0.09605)(s-1.1446)(s^2+0.3195s+0.2851)]/$$
$$[(s+0.1207)(s^2+1.0592s+0.5177)(s^2+1.7500s+11.6037)]. \quad (4.11)$$

The results are also shown in Table III and in Figs 8 and 9. It can be seen that the reduced models $A_3(s)$ and $B_3(s)$ of Hung and Han's methods are the best ones.

V. Conclusions

The effects of parameter variations on the characteristics of closed-loop systems with reduced models have been illustrated. The applications of Hung and Han's method to the systems with parameter variations have been presented. In a comparison of the results obtained by other current methods the advantages of Hung and Han's methods are quite evident.

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