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Constructing Solutions of a Single Conservation Law*

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1. INTRODUCTION

In this paper we will construct the solution of a single conservation law

$$\begin{aligned} u_t + f(u)_x = 0, \qquad t > 0, \qquad -\infty < x < \infty, \\ u(x, 0) = u_0(x), \qquad -\infty < x < \infty. \end{aligned} \tag{1}$$

We assume that $f(\cdot)$ is smooth and $f''(\cdot)$ vanishes at isolated points only. The initial data $u_0(\cdot)$ are bounded and piecewise monotone.

It is well known that, in general, (1) does not have global smooth solutions even if the initial data $u_0(\cdot)$ are smooth. Hence we consider a weak solution for (1): a weak solution of (1) is a bounded and measurable function u such that for any C^∞ function $g: R \times R \to R$ with compact support

$$\int_{R\times R^+} (ug_t + f(u)\, g_x)\, dx\, dt + \int_{-\infty}^{\infty} u_0(x)\, g(x, 0)\, dx = 0, \tag{2}$$

where $R^+ = \{t \in R: t \geqslant 0\}$.

In general, (1) does not have a unique weak solution. In order to single out a unique solution of (1), one requires that $u: R \times R^+ \to R$ satisfy an additional entropy condition. Across a discontinuity line $x = x(t)$, the solution u satisfies [5]

$$\sigma(u_-, u_+) \leqslant \sigma(u_-, u) \qquad \text{for all } u \text{ between } u_- \text{ and } u_+, \tag{E}$$

where $u_\pm = u(x(t) \pm 0, t)$ and $\sigma(u_1, u_2)$ is the shock speed defined as

$$\sigma(u_1, u_2) = [f(u_1) - f(u_2)]/(u_1 - u_2).$$

For other formulation of the entropy condition and the existence and uniqueness theorems, we refer to Krushkov [6] and Vol'pert [7].

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In the case $f(\cdot)$ is strictly convex (or concave), Lax [3] discovered an explicit solution for (1). Using an explicit representation of solutions similar to Lax's, Oleinik [4] studies the structure of solutions of (1) and shows that solutions are continuous except on the union of an at most countable set of Lipschitz continuous curves (shocks). Dafermos [2], using a different approach, can also establish the above results. In the case $f(\cdot)$ is uniformly strictly convex ($f''(\cdot) \geqslant \varepsilon > 0$), Lax [3] establishes that $u(\cdot, t)$ is in the class of functions of locally bounded variation in the sense of Tonelli and Cesari (space BV) for bounded and measurable initial data $u_0(\cdot)$. (See also Dafermos [2].)

When we remove the convexity condition on $f(\cdot)$, we know little about the structure of solutions for (1). There is no known explicit formula of solutions similar to Lax's in the strictly convex case. Ballou [1] can construct the solutions for piecewise constant initial data and "admissible" initial data (for the definition, see Ballou [1]). From these constructions we know the qualitative behavior of the solutions and the structure of the shock curves. It is the purpose of this paper to provide a different method to construct the solutions for a bounded and piecewise monotone initial data. Our method employs Lax's explicit formula very naturally and effectively. It seems very likely that our method can be modified and generalized to include more general initial data.

2. The Known Results

We collect the well-known results of Lax [3], Oleinik [4, 5] and Dafermos [2] into the following theorem. We omit the proof.

THEOREM 2.1. *Let $f(\cdot)$ be strictly convex (concave) in $[a, b]$ (a can be $-\infty$ and b can be ∞) and $u_0(\cdot)$ be measurable and bounded by a and b. Define*

$$F(x, t; u) = t(uf'(u) - f(u)) + \int_0^{x - f'(u)t} u_0(y)\, dy.$$

Then $G(x, t) = \min_{a \leqslant u \leqslant b} (\max) F(x, t; u)$ exists and

(i) *$G(\cdot, \cdot)$ is continuous on $R \times R^+$;*

(ii) *$\partial G(x, t)/\partial x \equiv u(x, t)$ exists on $R \times R^+$ except on Γ, which is the union of an at most countable set of Lipschitz continuous curves;*

(iii) *$\partial G(x, t)/\partial t = -f(u(x, t))$ for all $(x, t) \in R \times R^+ - \Gamma$;*

(iv) *the $u(\cdot, \cdot)$ in* (ii) *is the unique weak solution of* (1), *that is, $u(\cdot, \cdot)$ satisfies the entropy condition* (E);

(v) *$u(x \pm 0, t)$ exists for all $t > 0$;*

(vi) *for a fixed point (x_0, t_0), $t_0 > 0$, $u(x_0 - f'(u(x_0 \pm 0, t_0))(t_0 - t), t) = u(x_0 \pm 0, t_0)$ for all $0 < t < t_0$;*

(vii) *$u(\cdot, \cdot)$ is locally Lipschitz continuous on $R \times R^+ - \bar{\Gamma}$;*

(viii) *if $u_0(\cdot)$ is piecewise monotone, then $u(\cdot, t)$ is also piecewise monotone.*

3. The Case $f''(\cdot)$ Vanishes at Only One Point and Changes Sign There

Before we start our construction of solutions, we state and prove a theorem which will be used repeatedly.

Theorem 3.1. *Suppose that $G(x, t)$ is continuous in $R \times R^+$, $\partial G(x, t)/\partial x$ and $\partial G(x, t)/\partial t$ exist for all $R \times R^+ - \Gamma$ and $\partial G(x, t)/\partial t = -f(\partial G(x, t)/\partial x)$, where Γ is the union of an at most countable set of Lipschitz continuous curves. Then $\partial G(x, t)/\partial x$ is a weak solution of* (1), *that is, $\partial G(x, t)/\partial x$ satisfies* (2).

Proof. Let $g: R \times R \to R$ be a C^∞ function with compact support. Then

$$\iint_{R \times R^+} \left(\frac{\partial G(x, t)}{\partial x} g_t + f\left(\frac{\partial G(x, t)}{\partial x}\right) g_x\right) dx\, dt + \int_{-\infty}^{\infty} g(x, 0) \frac{\partial G(x, 0)}{\partial x}\, dx$$

$$= \iint_{R \times R^+} (G_x g_t - G_t g_x)\, dx\, dt - \int_{-\infty}^{\infty} g_x G(x, 0)\, dx$$

$$= \iint_{R \times R^+} (-G g_{tx} + G g_{xt})\, dx\, dt + \int_{-\infty}^{\infty} G(x, 0)\, g_x(x, 0)\, dx$$

$$- \int_{-\infty}^{\infty} G(x, 0)\, g_x(x, 0)\, dx = 0.$$

This completes the proof of this theorem.

Without loss of generality, we assume that $f''(u) > 0$ for $u > 0$, $f''(0) = 0$, $f''(u) < 0$ for $u < 0$ and $f(0) = 0$. We define some notations. Let $\eta < 0$ be given. We define $\eta^* > 0$ to be the unique number which satisfies

$$f'(\eta^*) = [f(\eta^*) - f(\eta)]/(\eta^* - \eta).$$

Let $\eta > 0$ be given. We define $\eta_* < 0$ to be the unique number which satisfies

$$f'(\eta_*) = [f(\eta_*) - f(\eta)]/(\eta_* - \eta).$$

Here $\eta^* = \infty$ and $\eta_* = -\infty$ are possible. For details of these definitions, see Ballou [1].

We start our theorems from the simplest initial data. The method of construction will indicate how to understand why our method works. The results are also needed later.

THEOREM 3.2. *Let the initial data* $u_0(\cdot)$ *be* $u_0(x) = u_l$ *for* $x < 0$ *and* $u_0(x) = u_r$ *for* $x > 0$, u_l *and* u_r *are two constants. Define*

$$u_0^i(x) = \min(\max)\{u_0(x), 0\} \qquad \textit{for} \quad i = 1(2),$$

and

$$F_i(x, t; u) = t(uf'(u) - f(u)) + \int_0^{x - f'(u)t} u_0^i(y)\, dy, \qquad i = 1, 2.$$

Then $G_1(x, t) = \max_{u \leqslant 0} F_1(x, t; u)$ *and* $G_2(x, t) = \min_{u \geqslant 0} F_2(x, t; u)$ *exists and* $\partial G(x, t)/\partial x$ *is the unique weak solution of* (1) *for this initial data, where*

$$G(x, t) = \max\{G_1(x, t), G_2(x, t)\} \qquad \textit{if} \quad u_l < 0 < u_r,$$

and

$$G(x, t) = \min\{G_1(x, t), G_2(x, t)\} \qquad \textit{if} \quad u_l > 0 > u_r.$$

Proof. We consider only the case $u_l < 0 < u_r$. The other case $u_l > 0 > u_r$ can be similarly treated. We prove this theorem by direct constructions of $G_1(x, t)$ and $G_2(x, t)$. It is easy to see that $u_0^1(x) = u_l$ for $x < 0$ and $u_0^1(x) = 0$ for $x > 0$. Hence if $x \leqslant f'(0)t$, we have $\partial F_1(x, t; u)/\partial u = tf''(u)(u - u_l)$ for all $u \leqslant 0$. Thus $\partial F_1(x, t; u)/\partial u > 0$ for $u < u_l$ and $\partial F_1(x, t; u)/\partial u < 0$ for $0 > u > u_l$. We conclude that $G_1(x, t) = F_1(x, t; u_l) = -tf(u_l) + xu_l$ in this case. If $f'(0)t < x < f'(u_l)t$, then there exists u_1, $u_l \leqslant u_1 < 0$, such that $x = f'(u_1)t$. In this region of (x, t), we have $\partial F_1(x, t; u)/\partial u = tf''(u)(u - 0)$ for $u_1 < u \leqslant 0$ and $\partial F_1(x, t; u)/\partial u = tf''(u)(u - u_l)$ for $u < u_1$. Hence $u = 0$ and $u = u_l$ are the local maximum points. We obtain $G_1(x, t) = \max\{F_1(x, t; 0), F_1(x, t; u_l)\} = \max\{0, [-tf(u_l) + xu_l]\}$. If $f'(u_l)t < x$, we have $u_1 < u_l$ $(x = f'(u_1)t)$ and $u = 0$ is the only maximum point. Hence $G_1(x, t) = F_1(x, t; 0) = 0$. Combining these results we have

$$G_1(x, t) = \max\{0, -tf(u_l) + xu_l\}.$$

Now we calculate $G_2(x, t)$. It is easy to see that $u_0^2(x) = 0$ for $x < 0$ and $u_0^2(x) = u_r$ for $x > 0$. If $x \leqslant f'(0)t$, we have $\partial F_2(x, t; u)/\partial u = tf''(u)(u - 0)$ which is >0 for all $u > 0$. Hence $G_2(x, t) = F_2(x, t; 0) = 0$. If $f'(0)t < x \leqslant f'(u_r)t$, then there exists u_2, $0 < u_2 \leqslant u_r$, such that $x = f'(u_2)t$ or $u_2 = h_2(x/t)$, where h_2 is the inverse function of f' restricted in the region $u \geqslant 0$.

In this region of (x, t), we have $\partial F_2(x, t; u)/\partial u = tf''(u)(u - u_r)$ for $0 \leqslant u < u_2$ and $=tf''(u)(u-0)$ for $u_2 < u$. Hence $u = u_2$ is the unique minimum point. We have $G_2(x, t) = F_2(x, t; u_2) = -tf(h_2(x/t)) + xh_2(x/t)$ in this region. If $f'(u_r)t < x$, we have $u_2 = h_2(x/t) > u_r$ and $u = u_r$ is the minimum point. Hence $G_2(x, t) = F_2(x, t; u_r) = -tf(u_r) + xu_r$. Combining these results we have

$$
\begin{aligned}
G_2(x, t) &= 0 && \text{for} \quad x \leqslant f'(0)t, \\
&= -tf(h_2(x/t)) + xh_2(x/t) && \text{for} \quad f'(0)t < x \leqslant f'(u_r)t, \\
&= -tf(u_r) + xu_r && \text{for} \quad f'(u_r)t < x.
\end{aligned}
$$

It is easy to see that $G_1(\cdot, t)$ is monotone decreasing and $G_2(\cdot, t)$ is monotone increasing. Hence we can find a line $x = \gamma(t)$ such that $G_1(\gamma(t), t) = G_2(\gamma(t), t)$ for all $t \geqslant 0$. Thus we can see that $G(x, t) = G_1(x, t)$ for $x \leqslant \gamma(t)$ and $G(x, t) = G_2(x, t)$ for $x > \gamma(t)$. It is easy to check that $\partial G_i(x, t)/\partial t = -f(\partial G_i(x, t)/\partial x)$ for $i = 1, 2$. From Theorem 3.1, $\partial G(x, t)/\partial x \equiv u(x, t)$ is a weak solution for (1). The only thing we have to check is that $x = \gamma(t)$ satisfies the entropy condition (E). If $u_r < u_l^*$, we have $f'(h_2(x/t)) = x/t < [f(h_2(x/t)) - f(u_l)]/(h_2(x/t) - u_l)$ for $f'(0)t < x \leqslant f'(u_r)t$. Hence $-tf(u_l) + xu_l > -tf(h_2(x/t)) + xh_2(x/t)$. Hence we have $-tf(u_l) + \gamma(t)\, u_l = -tf(u_r) + \gamma(t)\, u_r$ or $x = \gamma(t) = [f(u_r) - f(u_l)]t/(u_r - u_l)$. This means that $d\gamma(t)/dt = \sigma(u_l, u_r) < \sigma(u_l, u)$ for all $u_l \leqslant u \leqslant u_r$. (E) is satisfied. If $u_r \geqslant u_l^*$, then we have $-tf(u_l) + \gamma(t)\, u_l = -tf(h_2(\gamma(t)/t)) + \gamma(t)\, h_2(\gamma(t)/t)$, or, $\gamma(t)/t = f'(h_2(\gamma(t)/t)) = [f(h_2(\gamma(t)/t)) - f(u_l)]/(h_2(\gamma(t)/t) - u_l)$. This means that $h_2(\gamma(t)/t) = u_l^*$, or $\gamma(t)/t = f'(u_l^*)$. Obviously, (E) is satisfied. This completes the proof of this Theorem.

Now we consider a more general initial data.

LEMMA 3.3. *Let $u_0(\cdot)$ be piecewise monotone and $u_0(x) \leqslant 0$ for $x < 0$ and $u_0(x) \geqslant 0$ for $x > 0$. Define $u^i(x)$, $F_i(x, t; u)$ and $G_i(x, t)$, $i = 1, 2$, as in Theorem* 3.2. *Then $\partial G(x, t)/\partial x \equiv u(x, t)$ is a weak solution of* (1) *which satisfies* (E) *except possibly on the curve γ, where $G(x, t) = \max\{G_1(x, t), G_2(x, t)\}$ and $G_1(\gamma(t), t) = G_2(\gamma(t), t)$ for all $t \geqslant 0$.*

Proof. From Theorem 2.1, we know that $\partial G_i(x, t)/\partial t = -f(\partial G_i(x. t)/\partial x)$, $i = 1, 2$. Hence $\partial G(x, t)/\partial t = -f(\partial G(x, t)/\partial x)$. Thus from Theorem 3.1, we know that $\partial G(x, t)/\partial x \equiv u(x, t)$ is a weak solution for (1). Now $G(x, t) = G_1(x, t)$ for $x \leqslant \gamma(t)$ and $G(x, t) = G_2(x, t)$ for $x \geqslant \gamma(t)$, since $G_1(\cdot, t)$ is monotone decreasing and $G_2(\cdot, t)$ is monotone increasing. But the shocks in $\partial G_i(x, t)/\partial x$, $i = 1, 2$, all satisfy (E). Hence the Lemma is proved.

LEMMA 3.4. *Consider the curve γ in Lemma* 3.3. *If γ does not satisfy the entropy condition, then there exist t_1 and t_2, $0 \leqslant t_1 < t_2$, such that γ satisfies* *the entropy condition for $t < t_1$ and does not satisfy the entropy condition for $t_1 < t < t_2$. Furthermore, if we let $u_\gamma^+(t) = u(\gamma(t)+0, t)$ and $u_\gamma^-(t) = u(\gamma(t)-0, t)$, we have*

(i) $u_\gamma^+(t_1+0) = [u_\gamma^-(t_1+0)]^*$,

(ii) *$u_\gamma^-(t)$ is Lipschitz continuous and satisfies $d(u_\gamma^-(t))/dt \geqslant 0$ for $t \in (t_1, t_2)$.*

Proof. It $t_1 = 0$, then in the vicinity of the origin, the initial data look like Riemann data (recall the piecewise monotone assumption of $u_0(\cdot)$). From Theorem 3.2, we know that (i) holds. If $t_1 > 0$ and $u_\gamma^+(t_1+0) > [u_\gamma^-(t_1+0)]^*$, then there exist $\varepsilon > 0$ and $\delta > 0$, such that $u_\gamma^-(t) \geqslant u_\gamma^-(t_1+0) - \delta$ and $u_\gamma^+(t) \geqslant u_\gamma^+(t_1+0) - \delta$ for all $t \in [t_1-\varepsilon, t_1]$ with $u_\gamma^+(t_1+0) - \delta > [u_\gamma^-(t_1+0) - \delta]^*$. This means that γ does not satisfy the entropy condition for $t \in [t_1-\varepsilon, t_1]$. This is a contradiction. If, on the other hand, $u_\gamma^+(t_1+0) < [u_\gamma^-(t_1+0)]^*$, then there exist $\varepsilon > 0$ and $\delta > 0$ such that $u_\gamma^-(t) \leqslant u_\gamma^-(t_1+0) + \delta$ and $u_\gamma^+(t) \leqslant u_\gamma^+(t_1+0) + \delta$ for all $t \in [t_1, t_1+\varepsilon]$ with $u_\gamma^+(t_1+0) + \delta < [u_\gamma^-(t_1+0) + \delta]^*$. This means that γ satisfies the entropy condition for $t \in [t_1, t_1+\varepsilon]$. This also leads to a contradiction. Hence (i) is proved.

Now, since $\gamma(t) - f'(u_\gamma^-(t))t \leqslant 0 \leqslant \gamma(t) - f'(u_\gamma^+(t))t$ for all $t > t_1$, we have $f'(u_\gamma^-(t)) > f'(u_\gamma^+(t))$ for all $t > t_1$. Hence, the only possibility for γ to violate the entropy condition is $f'(u_\gamma^-(t)) > f'(u_\gamma^+(t)) > d\gamma(t)/dt$ for all $t \in (t_1, t_2)$. From the piecewise monotone assumption of $u_0(\cdot)$ and Theorem 2.1(viii), we can choose $(t_2 - t_1)$ so small that $u_\gamma^-(t)$ is monotone for $t \in (t_1, t_2)$. Assume $d(u_\gamma^-(t))/dt \leqslant 0$ for $t \in (t_1, t_2)$ and $\{t\colon d(u_\gamma^-(t))/dt < 0,\ t \in (t_1, t_1+\delta)\}$ has positive measure for all $\delta > 0$. Let $A = (\gamma(t_1), t_1)$ and $B = (\gamma(t_1) + f'(u_\gamma^+(t_1+0))(t-t_1), t)$. For $(t-t_1)$ sufficiently small, we have

$$\begin{aligned} G_2(B) &= G_2(A) + \int_{\overline{AB}} \left(\frac{\partial G_2}{\partial x}\,dx + \frac{\partial G_2}{\partial t}\,dt \right) \\ &= G_2(A) + \int_{t_1}^{t} [u_\gamma^+(t_1+0)\,f'(u_\gamma^+(t_1+0)) - f(u_\gamma^+(t_1+0))]\,dt' \\ &= G_2(A) + (t-t_1)[u_\gamma^+(t_1+0)\,f'(u_\gamma^+(t_1+0)) - f(u_\gamma^+(t_1+0))], \\ G_1(B) &= G_1(A) + \int_{\overline{AB}} \left(\frac{\partial G_1}{\partial x}\,dx + \frac{\partial G_1}{\partial t}\,dt \right) \\ &= G_1(A) + \int_{t_1}^{t} \{u_1^-(t')\,f'(u_\gamma^+(t_1+0)) - f(u_1^-(t'))\}\,dt', \end{aligned}$$

where

$$u_1^-(t') = \frac{\partial G_1}{\partial x}(\gamma(t_1) + f'(u_\gamma^+(t_1+0))(t'-t_1) - 0, t').$$

Hence

$$G_1(B) - G_2(B) = \int_{t_1}^{t} \{[u_1^-(t') - u_\gamma^-(t_1+0)]\, f'(u_\gamma^+(t_1+0)) - [f(u_1^-(t')) - f(u_\gamma^-(t_1+0))]\}\, dt' + \int_{t_1}^{t} \{[u_\gamma^-(t_1+0) - u_\gamma^+(t_1+0)]\, f'(u_\gamma^+(t_1+0)) - [f(u_\gamma^-(t_1+0) - f(u_\gamma^+(t_1+0))]\}\, dt' > 0,$$

since the second term is zero and the first term is positive due to $u_1^-(t') - u_\gamma^-(t_1+0) \leqslant 0$ or all sufficiently small $(t'-t_1) > 0$. But $G_1(\gamma(t), t) = G_2(\gamma(t), t)$, thus we have $\gamma(t) > \gamma(t_1) + f'(u_\gamma^+(t_1+0))(t-t_1)$ for all sufficiently small $(t-t_1) > 0$. This fact contradicts the inequality $f'(u_\gamma^+(t)) > d\gamma(t)/dt$ for all $t \in (t_1, t_2)$ because this inequality means that $\gamma(t+\varepsilon) < \gamma(t) + f'(u_\gamma^+(t))\varepsilon$ for sufficiently small $\varepsilon > 0$ and hence $\gamma(t) \leqslant \gamma(t_1) + f'(u_\gamma^+(t_1+0))(t-t_1)$ for sufficiently small $(t-t_1) > 0$. This completes the proof of this lemma.

THEOREM 3.5. *Take the same assumption as in Lemma* 3.3. *Then we can construct two curves γ_1 and γ_2 with $\gamma_1(t) \leqslant \gamma_2(t)$ for all $t \geqslant 0$ and one function $\tilde{G}_2(x,t)$ which is defined in $\gamma_1(t) \leqslant x \leqslant \gamma_2(t)$, such that $\partial G(x,t)/\partial x \equiv u(x,t)$ is the unique weak solution of* (1), *where $G(x,t) = G_1(x,t)$ for $x < \gamma_1(t)$, $G(x,t) = \tilde{G}_2(x,t)$ for $\gamma_1(t) \leqslant x \leqslant \gamma_2(t)$ and $G(x,t) = G_2(x,t)$ for $\gamma_2(t) < x$.*

Proof. From Lemmas 3.3 and 3.4, if γ satisfies the entropy condition, then we can choose $\gamma_1 = \gamma = \gamma_2$. We are done. So assume that γ does not satisfy the entropy condition. Take t_1 as the t_1 in Lemma 3.3 and choose $\gamma_1(t) = \gamma_2(t) = \gamma(t)$ for $0 \leqslant t \leqslant t_1$. For $t \geqslant t_1$, we take γ_2 to be the generalized characteristic of $u_2(x,t) \equiv \partial G_2(x,t)/\partial x$ passing through the point $(\gamma(t_1), t_1)$ (see Dafermos [2] for the definition of generalized characteristic). From Lemma 3.4 and Theorem 2.1(viii), we can find a maximum interval $(\gamma(t_1) - \delta_{\max}, \gamma(t_1))$ such that $u_1(\cdot, t_1)$ is monotone decreasing in this interval. Let the generalized characteristics of $u_1(x,t)$ passing through the points $(\gamma(t_1) - \delta_{\max}, t_1)$ and $(\gamma(t_1), t_1)$ be Γ and Γ', respectively. From Therem 2.1 we know that $u_1(x,t)$ is Lipschitz continuous in the region between Γ and Γ'. Hence the vector field $(f'(u_1^*(x,t), 1))$ is also Lipschitz continuous in this region. Thus there exists a unique integral curve of this vector field passing the point $(\gamma(t_1), t_1)$. This integral curve will intersect Γ at $t = t_2$. We let $\gamma_1(t)$ be this integral curve in the range $t_1 \leqslant t \leqslant t_2$. It is easy to see that $d\gamma_1(t)/dt = f'(u_1^*(\gamma_1(t), t))$ is Lipschitz continuous and $d^2\gamma_1(t)/dt^2 \leqslant 0$ for $t_1 < t < t_2$. Now at every point $(\gamma_1(t), t)$, $t_1 \leqslant t < t_2$, we can draw a line with slope $dx/dt = f'(u_1^*(\gamma_1(t), t))$ in the increasing time direction. These lines will cover a fan-like region H without intersection with each other. Along these lines we assign a function $H_2(x, t)$, such that $H_2(x, t) = G_1(\gamma_1(\bar{t}), \bar{t}) + (t - \bar{t})[u_1^*(\gamma_1(\bar{t}), \bar{t}) f'(u_1^*(\gamma_1(\bar{t}), \bar{t})) - f(u_1^*(\gamma_1(\bar{t}), \bar{t}))]$ if (x, t) is on the line passing through the point $(\gamma_1(\bar{t}), \bar{t})$ with slope $dx/dt = f'(u_1^*(\gamma_1(\bar{t}), \bar{t}))$, or if $x = \gamma_1(\bar{t}) + (t - \bar{t}) \times f'(u_1^*(\gamma_1(\bar{t}), \bar{t}))$. We can calculate $\partial H_2(x, t)/\partial x$ and $\partial H_2(x, t)/\partial t$ as follows.

$$\begin{aligned} \partial H_2(x, t)/\partial x &= [u_1(\gamma_1(\bar{t}), \bar{t}) f'(u_1^*(\gamma_1(\bar{t}), \bar{t})) - f(u_1(\gamma_1(\bar{t}), \bar{t}))](\partial \bar{t}/\partial x) \\ &\quad - [u_1^*(\gamma_1(\bar{t}), \bar{t}) f'(u_1^*(\gamma_1(\bar{t}), \bar{t})) - f(u_1^*(\gamma_1(\bar{t}), \bar{t}))](\partial \bar{t}/\partial x) \\ &\quad + (t - \bar{t})\, u_1^*(\gamma_1(\bar{t}), \bar{t}) f''(u_1^*(\gamma_1(\bar{t}), \bar{t}))(\partial u_1^*(\gamma_1(\bar{t}), \bar{t})/\partial x) \\ &= (t - \bar{t})\, u_1^*(\gamma_1(\bar{t}), \bar{t}) f''(u_1^*(\gamma_1(\bar{t}), \bar{t}))(\partial u_1^*(\gamma_1(\bar{t}), \bar{t})/\partial x). \end{aligned}$$

But from $x = \gamma_1(\bar{t}) + (t - \bar{t}) f'(u_1^*(\gamma_1(\bar{t}), \bar{t}))$, we have

$$\begin{aligned} 1 &= [d\gamma_1(\bar{t})/d\bar{t} - f'(u_1^*(\gamma_1(\bar{t}), \bar{t}))](\partial \bar{t}/\partial x) \\ &\quad + (t - \bar{t}) f''(u_1^*(\gamma_1(\bar{t}), \bar{t}))(\partial u_1^*(\gamma_1(\bar{t}), \bar{t})/\partial x). \end{aligned}$$

Hence we obtain $\partial H_2(x, t)/\partial x = u_1^*(\gamma_1(\bar{t}), \bar{t})$. Similarly we can obtain $\partial H_2(x, t)/\partial t = -f(u_1^*(\gamma_1(\bar{t}), \bar{t}))$. Thus $H_2(\cdot, t)$ is an increasing function in H.

Along γ_2, we can find a time $t_3 > t_1$, such that $\gamma_2(t)$ is a genuine characteristic of $u_2(x, t)$ (see Dafermos [2]) for $t_1 < t < t_3$ and $\gamma_2(t)$ is a shock curve for $t > t_3$. At every point $(\gamma_2(t'), t)$, $t_3 < t'$, we draw a half line $C(t'; t)$ with slope $dx/dt = f'(u_2(\gamma_2(t') + 0, t'))$ in the positive time direction. (It is easy to see that $\gamma_2(t)$ meets the shock at t_3 from the left side of the shock.) Along $C(t'; t)$, we let

$$\begin{aligned} G_2'(C(t'; t), t) &= G_2(\gamma_2(t'), t') + (t - t')[u_2(\gamma_2(t') + 0, t') \\ &\quad \times f'(u_2(\gamma_2(t') + 0, t')) - f(u_2(\gamma_2(t') + 0, t'))]. \end{aligned}$$

These lines $C(t'; t)$ cover a region $H' \subset H$ and generally intersect with each other. Let $(x, t) \in H'$ be fixed. Define $T(x, t) = \{t' : x = C(t'; t)\}$. Let

$$\begin{aligned} G_2^C(x, t) &= \min_{t' \in T(x,t)} G_2'(C(t'; t), t) \\ &= G_2'(C(t''; t), t). \end{aligned}$$

Performing a calculation similar to the calculation of $\partial H_2(x, t)/\partial x$ and $\partial H_2(x, t)/\partial t$, we can obtain

$$\begin{aligned} \partial G_2^C(x, t)/\partial x &= u_2(\gamma_2(t'') + 0, t''), \\ \partial G_2^C(x, t)/\partial t &= -f(u_2(\gamma_2(t'') + 0, t'')). \end{aligned}$$

It is easy to see that the boundary of region H is

$$\begin{aligned}&\{(x,t): x=\gamma_2(t)\}\\ &\quad\cup\{(x,t): x=\gamma_1(t), t_1\leqslant t\leqslant t_2\}\\ &\quad\cup\{(x,t): x=\gamma_1(t_2)+f'(u_1^*(\gamma_1(t_2-0), t_2-0))(t-t_2), t\geqslant t_2\}.\end{aligned}$$

For $(x, t)\in H$, we define

$$\begin{aligned}\tilde{G}_2'(x,t)&=H_2(x,t) && \text{if} \quad (x,t)\in H-H',\\ &=\min\{H_2(x,t), G_2^c(x,t)\} && \text{if} \quad (x,t)\in H'.\end{aligned}$$

It is easy to see that $\tilde{G}_2'(x,t)$ is continuous and $\tilde{G}_2'(\cdot,t)$ is monotone increasing.

For $t\geqslant t_2$, $G_1(x,t)=\tilde{G}_2'(x,t)$ determines a curve $\gamma_1'(t)$ with $\gamma_1'(t_2)=\gamma_1(t_2)$. We define

$$\begin{aligned}G'(x,t)&=G_1(x,t) && \text{for} \quad x\leqslant\gamma_1(t), && \\ &=G_2(x,t) && \text{for} \quad \gamma_1(t)<x, && 0\leqslant t\leqslant t_1,\\ G'(x,t)&=G_1(x,t) && \text{for} \quad x\leqslant\gamma_1(t), &&\\ &=G_2^c(x,t) && \text{for} \quad \gamma_1(t)\leqslant x\leqslant\gamma_2(t), && t_1\leqslant t\leqslant t_2,\\ &=G_2(x,t) && \text{for} \quad \gamma_2(t)<x, &&\\ G'(x,t)&=G_1(x,t) && \text{for} \quad x\leqslant\gamma_1'(t), &&\\ &=G_2^c(x,t) && \text{for} \quad \gamma_1'(t)<x\leqslant\gamma_2(t), && t_2<t,\\ &=G_2(x,t) && \text{for} \quad \gamma_2(t)<x. &&\end{aligned}$$

From the construction of $G'(x,t)$ and Theorem 3.1, we know that $\partial G'(x,t)/\partial x\equiv u'(x,t)$ is a weak solution of (1) satisfying the entropy condition except possibly on $\gamma_1'(t)$ for $t>t_2$. But from the construction of $\gamma_1(t)$, $t_1<t\leqslant t_2$, we know that $u_1(\cdot,t_2)$ is monotone increasing in some interval $(\gamma_1(t_2)-\delta, \gamma_1(t_2))$. Hence there exists $t_4>t_2$ such that $\gamma_1'(t)$, $t_2<t<t_4$, satisfies the entropy condition. Now we can repeat our processes to remove the portion of $\gamma_1'(t)$ which does not satisfy the entropy condition. It is precisely in this way that we construct the curve γ_1. Note that from the piecewise monotone assumption of $u_0(\cdot)$, we need only repeat this process a finite time. This complete the proof of this theorem.

Now we are in a position to construct solutions for more general initial data. Assume that $u_0(\cdot)$ is bounded and piecewise monotone. Then there exist $y_1<y_2<\cdots<y_N$, such that $u_0(x)\in(-\infty,0]$ for $x\in(y_i,y_{i+1})$ if i is even and $u_0(x)\in[0,\infty)$ for $x\in(y_i,y_{i+1})$ if i is odd. We already set $y_0=-\infty$

and $y_{N+1} = \infty$ for convenience in the above expression. Of course it is equally possible that $u_0(x) \in [0, \infty)$ for $x \in (y_i, y_{i+1})$ if i is even. But we consider only the first case. The later case can be similarly treated.

The first step is to use Theorem 3.5 to construct the unique weak solution $u_{12}(x, t) \equiv \partial G_{12}(x, t)/\partial x$ for the initial data $u_0^{12}(x)$, where $u_0^{12}(x) = u_0(x)$ for $x < y_2$ and $u_0^{12}(x) = 0$ for $y_2 < x$. Now we let $u_0^3(x) = u_0(x)$ for $y_2 < x < y_3$ and $u_0^3(x) = 0$ otherwise. Define $F_3(x, t; u) = t(uf'(u) - f(u)) + \int_{y_2}^{x-f'(u)t} u_0^3(y)\, dy$. Then $G_3(x, t) = \max_{u \leqslant 0} F_3(x, t; u)$ exists from Theorem 2.1. Now set $G_{12}(x, t) - \int_{y_1}^{y_2} u_0(y)\, dy = G_3(x, t)$ to determine a Lipschitz continuous curve $x = \gamma_2(t)$ with $\gamma_2(0) = y_2$. It is easy to see that this curve γ_2 exists and is unique. Choose $G'_{123}(x, t) = G_{12}(x, t) - \int_{y_1}^{y_2} u_0(\xi)\, d\xi$ for $x < \gamma_2(t)$ and $G'_{123}(x, t) = G_3(x, t)$ for $\gamma_2(t) < x$. Then from the construction we know that $\partial G'_{123}(x, t)/\partial t = -f(\partial G'_{123}(x, t)/\partial x)$ and hence $\partial G'_{123}(x, t)/\partial x$ is a weak solution of (1) with initial data $u_0^{123}(\cdot)$, where $u_0^{123}(x) = u_0(x)$ for $x < y_3$ and $u_0^{123}(x) = 0$ for $y_3 < x$. $\partial G'_{123}(x, t)/\partial x$ satisfies (E) except possibly on γ_2. Using the same processes as in the proofs of Lemmas 3.3, 3.4 and Theorem 3.5, we can construct a $G_{123}(x, t)$ with $G_{123}(y_2, 0) = 0$ such that $\partial G_{123}(x, t)/\partial x \equiv u_{123}(x, t)$ is the unique weak solution of (1) with initial data $u_0^{123}(\cdot)$. Repeating this process, we finally can construct $G(x, t) \equiv G_{123\cdots(N+1)}(x, t)$ such that $\partial G(x, t)/\partial x \equiv u(x, t)$ is the unique weak solution of (1) with the correct initial data.

4. The Case $f''(u)$ Vanishes at Finite Points and Changes Sign There

Without loss of generality, we assume that $f''(u)$ vanishes at $a_1 < a_2 < \cdots < a_M$ and $f''(u) < 0$ for $u \in (a_i, a_{i+1})$ if i is even, and $f''(u) > 0$ for $u \in (a_i, a_{i+1})$ if i is odd, where for convenience we set $a_0 = -\infty$ and $a_{M+1} = \infty$.

We consider the simplest initial data $u_0(\cdot)$ first.

Theorem 4.1. *Let* $u_0(x) = u_l$ *for* $x < 0$ *and* $u_0(x) = u_r$ *for* $x > 0$, *where* $u_l \in (a_0, a_1]$ *and* $u_r \in [a_M, a_{M+1})$ *are two constants. Define*

$$u_0^i(x) = \min\{a_{i+1}, \max\{a_i, u_0(x)\}\}, \qquad i = 0, 1, \ldots, M,$$

$$F_i(x, t; u) = t(uf'(u) - f(u)) + \int_0^{x-f'(u)t} u_0^i(\xi)\, d\xi, \qquad i = 0, 1, \ldots, M,$$

and

$$G_i(x, t) = \max_{u \in [a_i, a_{i+1}]} (\min) F_i(x, t; u) \qquad \textit{if } i \textit{ is even (odd).}$$

Let $G(x, t) = \max\{G_0(x, t), G_1(x, t),..., G_M(x, t)\}$. Then $\partial G(x, t)/\partial x \equiv u(x, t)$ is the unique weak solution of (1) *for this Riemann initial data $u_0(\cdot)$.*

Proof. Let $\bar{a}_0 = u_l$, $\bar{a}_i = a_i$ for $i = 1, 2,..., M$ and $\bar{a}_{M+1} = u_r$. Then we have

$$G_i(x, t) = \max\{(-tf(\bar{a}_i) + x\bar{a}_i), (-tf(\bar{a}_{i+1}) + x\bar{a}_{i+1})\} \qquad \text{if } i \text{ is even,}$$

and

$$\begin{aligned} G_i(x, t) &= -tf(\bar{a}_i) + x\bar{a}_i && \text{for } \ x \leqslant f'(\bar{a}_i)t, \\ &= -tf(h_i(x/t)) + xh_i(x/t) && \text{for } \ f'(\bar{a}_i)t < x \leqslant f'(\bar{a}_{i+1})t, \\ &= -tf(\bar{a}_{i+1}) + x\bar{a}_{i+1} && \text{for } \ f'(\bar{a}_{i+1})t < x, \end{aligned}$$

if i is odd, where $h_i(\cdot)$ is the inverse function of $f'(\cdot)$ restricted in $[a_i, a_{i+1}]$.

From the construction, we can easily verify that $\partial G(x, t)/\partial t = -f(\partial G(x, t)/\partial x)$. Hence $\partial G(x, t)/\partial x$ is a weak solution of (1). Furthermore, $G(x, 0) = xu_l$ for $x < 0$ and $G(x, 0) = xu_r$ for $x > 0$. Hence $\partial G(x, 0)/\partial x = u_0(x)$. It remains to show that the shocks of $u(x, t)$ satisfy (E). Let $\gamma_1,..., \gamma_k$ be the shocks of $u(x, t)$ with $\gamma_1(t) < \gamma_2(t) < \cdots < \gamma_k(t)$ for $t > 0$. It is easy to see that $G(x, t) = G_0(x, t)$ for $x < \gamma_1(t)$ and $G(x, t) = G_M(x, t)$ for $x > \gamma_k(t)$. Of course $k = 1$ is possible. Furthermore, since $G_i(x, t)/t$ depends only on (x/t) for every i, $\gamma_j(t)/t$ must be independent of t for every $j = 1, 2,..., k$ and $u(\gamma_j(t) \pm 0, t)$ are also independent of t. It is also easy to see that $d\gamma_j(t)/dt = \gamma_j(t)/t = [f(u(\gamma_j(t) + 0, t)) - f(u(\gamma_j(t) - 0, t))]/[u(\gamma_j(t) + 0, t) - u(\gamma_j(t) - 0, t)]$. Let $u(\gamma_1(t) + 0, t) = u_l^*$. We shall show that (1) $[f(u) - f(u_l)]/(u - u_l) \geqslant d\gamma_1(t)/dt$ for all $u \in [u_l, u_r]$ and (2) $f'(u_l^*) = d\gamma_1(t)/dt$ if $u_l^* \neq u_r$. Suppose that there is a $u \in [u_l, u_r]$ such that $[f(u) - f(u_l)]/(u - u_l) < d\gamma_1(t)/dt$. Then from mean value theorem we can find a $\bar{u} \in [u_l, u_r]$ such that $[f(\bar{u}) - f(u_l)]/(\bar{u} - u_l) < d\gamma_1(t)/dt$, $f'(\bar{u}) = d\gamma_1(t)/dt$ and $f''(\bar{u}) > 0$. But then on $x = \gamma_1(t)$ we have (assume $\bar{u} \in (a_i, a_{i+1})$)

$$\begin{aligned} G_i(\gamma_1(t), t) - G_0(\gamma_1(t), t) &= [-f(\bar{u}) + \gamma_1(t)\bar{u}] - [-tf(u_l) + \gamma_1(t)u_l] \\ &= -t(\bar{u} - u_l)\left[\frac{f(\bar{u}) - f(u_l)}{\bar{u} - u_l} - \frac{\gamma_1(t)}{t}\right] \\ &> 0. \end{aligned}$$

This contradicts to $G_0(\gamma_1(t), t) = \max\{G_i(\gamma(t), t),\ i = 0,..., M\}$. Hence (1) is proved and γ_1 satisfies (E). If $u_l^* \neq u_r$, then $u_l^* = u(\gamma_1(t) + 0, t) = h_m(\gamma_1(t)/t)$ for some odd m. Hence $\gamma_1(t)/t = f'(u_l^*)$. This proves (2). Similar arguments can be applied to prove that $\gamma_2,..., \gamma_k$ satisfy (E), $f'(u(\gamma_j(t) \pm 0, t)) = d\gamma_j(t)/dt$

for $j = 2, 3, \ldots, (k-1)$ and $f'(u(\gamma_k(t)-0, t)) = d\gamma_k(t)/dt$ if $u(\gamma_k(t)-0, t) \neq u_l$ or $k \neq 1$. This completes the proof of the theorem.

Now we consider a slightly more general initial data.

THEOREM 4.2. *Let $u_0(x) \in (a_0, a_1]$ for $x < 0$ and $u_0(x) \in [a_M, a_{M+1})$ for $x > 0$. Define $u_0^i(x)$, $F_i(x, t; u)$, $G_i(x, t)$ and $G(x, t)$ as in Theorem 4.1. Then $\partial G(x, t)/\partial x \equiv u(x, t)$ is a weak solution of (1) which satisfies the entropy condition (E) except possibly on two curves γ_1 and γ_2, where $\gamma_1(t) \leqslant \gamma_2(t)$ are defined by $G(x, t) = G_0(x, t)$ for $x < \gamma_1(t)$, $G(x, t) = G_M(x, t)$ for $x > \gamma_2(t)$ and $G(x, t) \neq G_0(x, t)$, $G(x, t) \neq G_M(x, t)$ for $\gamma_1(t) < x < \gamma_2(t)$.*

Proof. The proof is similar to that of Lemma 3.3. The only thing we have to add is that the shocks between γ_1 and γ_2 satisfy (E). But this is an easy consequence of Theorem 4.1. We omit the detail.

THEOREM 4.3. *Consider the two curves γ_1 and γ_2 in Theorem 4.2. There exist four curves Γ_1^-, Γ_1^+, Γ_2^- and Γ_2^+ and two functions $H_1(x, t)$ and $H_2(x, t)$ such that if we redefine the $G(x, t)$ in Theorem 4.2 as*

$$\begin{aligned} G^N(x, t) &= H_1(x, t) && \text{for} \quad \Gamma_1^-(t) \leqslant x \leqslant \Gamma_1^+(t), \\ &= H_2(x, t) && \text{for} \quad \Gamma_2^-(t) \leqslant x \leqslant \Gamma_2^+(t), \\ &= G(x, t) && \text{otherwise}, \end{aligned}$$

then $\partial G^N(x, t)/\partial x$ is the unique weak solution of (1) for this initial data.

Proof. We assume first that M is an even integer. Let $\eta < a_1$ be given. Define $\eta^* > a_1$ to be

$$\eta^* = \sup\{u : \sigma(\eta, v) \geqslant \sigma(\eta, u) \text{ for all } v \in (\eta, a_M)\}.$$

Let $\eta > a_M$ be given. Define $\eta_* < a_M$ to be

$$\eta_* = \inf\{u : \sigma(\eta, v) \leqslant \sigma(\eta, u) \text{ for all } v \in (a_1, \eta)\}.$$

Although $\eta^*(\eta)$ $(\eta_*(\eta))$ is not a continuous function of η in general, $f'(\eta^*(\eta))$ is a continuous function of η.

Now if $\gamma_1(t)$ satisfies (E) for $0 < t < t_1$ and violates (E) for $t_1 < t < t_2$, then following the same arguments as in the proof of Lemma 3.4 we can prove that $u(\gamma_1(t_1)+0, t) = [u(\gamma_1(t_1)-0, t_1)]^*$ if $t_1 > 0$ (if $t_1 = 0$, then we take $t_1 \to 0+$ along $\gamma_1(t)$) and $du(\gamma_1(t)-0, t)/dt \geqslant 0$ for $t \in (t_1, t_2)$. We can follow the same procedure as in Theorem 3.5 to construct $\Gamma_1^-(t)$, $\Gamma_1^+(t)$ and $H_1(x, t)$. (These curves and function correspond to γ_1, γ_2 and $\tilde{G}_2(x, t)$ in Theorem 3.5.) The only difference is that $\partial H_1(x, t)/\partial x$ may have discontinuity in the fan-like region. These correspond to the fact that $\eta^*(\eta)$ is not a continuous function of η in general. But these discontinuities satisfy (E) (double side contact discontinuity). There is one thing we have to mention. Γ_1^+ is a genuine characteristic except the portion which coincides with Γ_1^-. Similar method (use $\eta_*(\eta)$) can be used to construct Γ_2^- and Γ_2^+ and $H_2(x, t)$. We omit the detail. This completes the proof of the theorem for M as an even integer.

Now we assume that M is an odd integer. Let $\eta < a_1$ be given. Define $\eta^* = \eta^*(\eta)$ to be

$$\eta^* = \sup\{u: \sigma(\eta, v) \geqslant \sigma(\eta, u) \text{ for all } v \varepsilon (\eta, \infty)\}.$$

Note that $\eta^* = \infty$ is possible. The constructions of Γ_1^-, Γ_1^+ and $H_1(x, t)$ are the same as in the case of even M. But the constructions of Γ_2^-, Γ_2^+ and $H_2(x, t)$ are a little bit different. First of all, if γ_2 satisfies (E) for $0 < t < t_1$ and violates (E) for $t_1 < t < t_2$, then $u(\gamma_2(t_1) - 0, t_1) = [u(\gamma_2(t_1) + 0, t_1)]_*$ and $d(u(\gamma_2(t) + 0, t))/dt \leqslant 0$. These facts can be proved as in Lemma 3.4. We now find the maximum interval $(\gamma_2(t_1), \gamma_2(t_1) + \delta_{\max})$ such that $d(u(x, t_1))/dx \leqslant 0$ for all $x \in (\gamma_2(t_1), \gamma_2(t_1) + \delta_{\max})$. Let C_1 and C_2 be the generalized characteristics of $\partial G_M(x, t)/\partial x$ passing through points $(\gamma_2(t_1), t_1)$ and $((\gamma_2(t_1) + \delta_{\max}), t_1)$, respectively. Between C_1 and C_2, we form the vector field $(f'([u(x, t)]_*), 1)$. We can find a continuous curve Γ_2^+ such that $d\Gamma_2^+(t)/dt = f'([u(\Gamma_2^+(t), t)]_*)$. It should be noted that $d\Gamma_2^+(t)/dt$ may be discontinuous. But $d\Gamma_2^+(t)/dt$ is monotone increasing. Γ_2^+ will meet C_2 and terminate there. Now at every point of Γ_2^+, say, $(\Gamma_2^+(t'), t')$, if $d\Gamma_2^+(t)/dt$ is continuous at $(\Gamma_2^+(t'), t')$, we draw a line with speed $d\Gamma_2^+(t')/dt'$ and assign a function $H_2(x, t)$ on this line such that

$$H_2(x, t) = G_M(\Gamma_2^+(t'), t') + (t - t')[u_2^+(t') f'(u_2^+(t')) - f(u_2^+(t'))],$$

where $u_2^+(t) = u(\Gamma_2^+(t) + 0, t)$ and $x = \Gamma_2^+(t') + (t - t')\, d\Gamma_2^+(t')/dt'$. On the other hand, if $d\Gamma_2^+(t)/dt$ is discontinuous at $(\Gamma_2^+(t'), t')$, then we draw a line with every speed between $d\Gamma_2^+(t' - 0)/dt$ and $d\Gamma_2^+(t' + 0)/dt$ and assign a function $H_2(x, t)$ similarly. Then $H_2(x, t) = G_M(x, t)$ will determine a curve which starts from the intersection point of Γ_2^+ and C_2. An initial portion of this curve satisfies condition (E). Continuing this process, we can construct Γ_2^-, Γ_2^+ and $H_2(x, t)$. They are almost the same as in Lemma 3.3, 3.4 and Theorem 3.5. We omit the detail. This completes the proof.

Now we are in a position to describe how to construct solutions for more general initial data. Assume that $u_0(\cdot)$ is bounded and piecewise monotone. Then there exist $y_1 < y_2 < \cdots < y_N$, such that $\{u_0(x): x \in (y_i, y_{i+1})\} \subseteq [a_{j(i)}, a_{j(i)+1}]$, where $j(i)$ is a function of i and we take $y_0 = -\infty$, $y_{N+1} = \infty$ for convenience.

The first step is to use Theorems 4.2 and 4.3 to construct the unique weak solution $u_{01}(x, t) \equiv \partial G_{01}(x, t)/\partial x$ for the initial data $u_0^{01}(x)$, where $u_0^{01}(x) =$

$u_0(x)$ for $x < y_2$ and $u_0^{01}(x) = a_{j(1)}$ if $j(2) < j(1)$, $u_0^{01}(x) = a_{j(1)+1}$ if $j(1) < j(2)$ for $x > y_2$. Now we let $u_0^2(x) = u_0(x)$ for $y_2 < x < y_3$, $u_0^2(x) = a_{j(2)+1}$ if $j(2) < j(1)$ and $u_0^2(x) = a_{j(2)}$ if $j(1) < j(2)$ for $x < y_2$, $u_0^2(x) = a_{j(2)}$ if $j(3) < j(2)$ and $u_0^2(x) = a_{j(2)+1}$ if $j(2) < j(3)$ for $y_3 < x$. Define

$$F_2(x, t; u) = t(uf'(u) - f(u)) + \int_{y_2}^{x - f'(u)t} u_0^2(y)\, dy.$$

Then if $j(2)$ is even, we have

$$G_2(x, t) = \max_{u \in [a_{j(2)}, a_{j(2)+1}]} F_2(x, t; u),$$

and if $j(2)$ is odd, we have

$$G_2(x, t) = \min_{u \in [a_{j(2)}, a_{j(2)+1}]} F_2(x, t; u).$$

Now define $u_R^2(x) = u_0^{01}(y_2 + 0)$ for $x < y_2$ and $u_R^2(x) = u_0^2(y_2 - 0)$ for $x > y_2$. Using Theorem 4.1 to construct the unique weak solution for this initial data $u_R^2(\cdot)$, we obtain $G_2^R(x, t)$. Now let $G_{01}(x, t) - \int_{y_1}^{y_2} u_0(y)\, dy = G_2^R(x, t)$ to determine a curve which starts at $(y_1, 0)$. Using the same method of Theorem 4.3 to remove the portions of this curve which violate (E), we can obtain $G_{01}^R(x, t)$ such that $\partial G_{01}^R(x, t)/\partial x$ is the unique weak solution of (1) for the initial data which are $u_0(x)$ for $x < y_2$ and $u_R^2(x)$ for $x > y_2$. Now let $G_{01}^R(x, t) = G_2(x, t)$ to determine another curve with end point at $(y_2, 0)$. Again we can use the method of Theorem 4.3 to remove the portions of this curve which violates (E). Finally we obtain $G_{012}(x, t)$ such that $\partial G_{012}(x, t) \partial x$ is the unique weak solution of (1) for the initial data $u_0^{012}(x)$, where $u_0^{012}(x) = u_0(x)$ for $x < y_3$ and $u_0^{012}(x) = u_0^2(y_3 + 0)$ for $y_3 < x$. Continuing this process we can construct the unique weak solution for bounded and piecewise monotone initial data $u_0(\cdot)$.

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