Identification of Nonlinear Systems from Steady-state Frequency Response

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ABSTRACT: *Methods are proposed for identifying the* NL, LN *and* LNL *models. The nonlinearity is assumed to be independent of frequency. For each model, the linear part is first identified based on extracting the amplitude and/or phase responses due to linear systems from the overall response. After the linear system has been identified, the nonlinearity is identified graphically. An advantage of the proposed methods is that multiple-valued nonlinearity can also be considered. An application of identifying a unity feedback nonlinear system is also discussed.*

I. Introduction

Several methods exist for identifying a nonlinear system. These methods are developed in either the time domain (**1–4**), or the frequency domain (**5–9**) based on the Volterra expansion. However, the Volterra series is not suited for representing a system containing a multiple-valued nonlinearity (**10**). To identify a system containing the multiple-valued nonlinearity, a dither signal is added to the system to smooth the jump phenomena in the nonlinearity so that methods for single-valued nonlinearities can be applied. However, it is still not an effective method (**10**).

In this paper, three nonlinear models, as shown in Fig. 1, are considered: (1) the *NL* model which consists of a nonlinearity cascaded in front of a linear system; (2) the *LN* model of a nonlinearity cascaded after a linear system; and (3) the *LNL* model of a nonlinearity sandwiched between two linear systems. The nonlinearity may be single-valued or multiple-valued. In all cases, the only assumption imposed on a nonlinearity is that its input–output characteristic be independent of frequency. The problem is to identify the transfer function(s) of the linear systems and the nonlinearity from the open-loop frequency response data. A simple application for identifying a unity feedback system is also discussed. However, the closed-loop application is restricted to the situation where the high-order harmonics generated by the nonlinearity are severely attenuated by the linear system so that the overall feedback system behaves like a quasi-linear system.

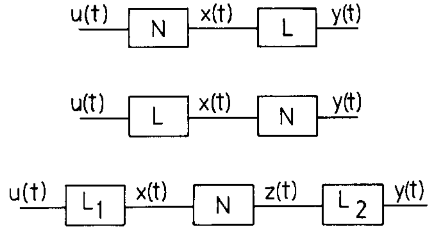

FIG. 1. Nonlinear models.

II. Identification of the NL *Model*

Consider the NL model as shown in Fig. 1. Since it is clear that the nonlinearity and the linear system can only be identified to within a constant gain factor, we arbitrarily normalize the dc gain, the gain at zero frequency, of the linear system to unity and ascribe it to the nonlinearity. Thus, without loss of generality, we let the linear system L be described by

$$L(s) = \frac{K(1+b_1 s)\dots(1+b_m s)}{(1+a_1 s)(1+a_2 s)\dots(1+a_n s)} \tag{1}$$

where the orders n and m are known *a priori.* To a sinusoidal input

$$u(t) = A \cos \omega_i t, \tag{2}$$

the corresponding signals (steady state) $x(t)$ and $y(t)$ are, respectively,

$$x(t) = N_0(A) + N_1(A)\cos[\omega_i t + \phi_1(A)] + N_2(A)\cos[2\omega_i t + \phi_2(A)] + \cdots \tag{3}$$

and

$$y(t) = N_0(A)\cdot|L(\mathrm{j}0)| + N_1(A)\cdot|L(\mathrm{j}\omega_i)|\cdot\cos[\omega_i t + \phi_1(A) + \angle L(\mathrm{j}\omega_i)]$$
$$+ N_2(A)\cdot|L(\mathrm{j}2\omega_i)|\cdot\cos[2\omega_i t + \phi_2(A) + \angle L(\mathrm{j}2\omega_i)] + \cdots \tag{4}$$

where $N_l(A)$ and $\phi_l(A)$ are the amplitude and phase shift, respectively, of the lth harmonic component generated by the nonlinearity N and are functions of the input amplitude A. Practically, we can approximate $y(t)$ by a finite Fourier series (**11**) as

$$\hat{y}(t) = \frac{\alpha_0}{2} + \sum_{l=1}^{\bar{k}} \alpha_l \cos(l\omega_i t + \theta_l) \tag{5}$$

where

$$\theta_l \triangleq \phi_l(A) + \angle L(\mathrm{j}l\omega_i).$$

The procedure for identifying an NL model consists of two steps. The first is to identify the linear part L from the frequency response. Then based on the identified L, the signal x(t) is computed so that the non-linearity N can be identified from its input–output response. Two methods which can be used to identify the linear part $L(s)$ are as follows:

Method I. Based on amplitude frequency response alone

With the assumption that N_l and ϕ_l are independent of the frequency ω_i, it is clear that the fundamental amplitude of the signal $x(t)$, i.e. $N_1(A)$ in (3), remains constant for all sinusoidal inputs of arbitrary frequencies as long as their amplitudes remain constant. However, the fundamental amplitude of the output signal $y(t)$ is modified by a factor $N_1(A)\cdot|L(\mathrm{j}\omega_i)|$. Consider a set of k fundamental amplitude responses of the NL model at frequencies ω_i, $i = 1, 2, \ldots, k$:

$$\left\{\frac{N_1(A)\cdot|L(\mathrm{j}\omega_1)|}{A}, \frac{N_1(A)\cdot|L(\mathrm{j}\omega_2)|}{A}, \ldots, \frac{N_1(A)\cdot|L(\mathrm{j}\omega_k)|}{A}\right\}. \tag{6}$$

It is clear that the data set given by (6) corresponds to the amplitude response of an artificial system L' which is obtained from L by a constant amplification factor $N_1(A)$. Therefore, if L, hence L', is of minimum phase, the system L' can be identified from the data set (6) by Jong and Shanmugam's method (**12**). By normalizing the dc gain to unity, we identify the linear system $L(s)$ as

$$L(s) = \frac{(1+\hat{b}_1 s)\ldots(1+\hat{b}_m s)}{(1+\hat{a}_1 s)(1+\hat{a}_2 s)\ldots(1+\hat{a}_n s)}. \tag{7}$$

Method II. Based on both amplitude and phase responses

If $L(s)$ is not minimum phase, the NL model can be identified as follows. For convenience, let the fundamental response of the NL model at frequency ω_i, i.e., $\frac{N_1(A)\angle[\phi_1(A)]}{A} L(\mathrm{j}\omega_i)$, be described by

$$\frac{N_1(A)\angle[\phi_1(A)]}{A} L(\mathrm{j}\omega_i) \equiv R_i + \mathrm{j}I_i \tag{8}$$

where

$$R_i = \frac{N_1(A)}{A}\cdot|L(\mathrm{j}\omega_i)|\cdot\cos[\phi_1(A) + \angle L(\mathrm{j}\omega_i)], \tag{9}$$

$$I_i = \frac{N_1(A)}{A}\cdot|L(\mathrm{j}\omega_i)|\cdot\sin[\phi_1(A) + \angle L(\mathrm{j}\omega_i)]. \tag{10}$$

With an input of the same amplitude A but different frequency $\omega_i' = r\omega_i$, where $r \neq 1$, the response of this NL model is

$$\frac{N_1(A)\angle[\phi_1(A)]}{A} L(\mathrm{j}\omega_i') \equiv R_i' + \mathrm{j}I_i'. \tag{11}$$

Dividing (8) by (11), we obtain

$$\frac{L(\mathrm{j}\omega_i)}{L(\mathrm{j}\omega_i')} = \frac{R_i + \mathrm{j}I_i}{R_i' + \mathrm{j}I_i'} = R_i'' + \mathrm{j}I_i''. \tag{12}$$

Now $R_i'' + \mathrm{j}I_i''$ corresponds to the frequency response of the following fictitious linear system:

$$L'(s) = \frac{(1+b_1 s)\ldots(1+b_m s)(1+ra_1 s)\ldots(1+ra_n s)}{(1+a_1 s)\ldots(1+a_n s)(1+rb_1 s)\ldots(1+rb_m s)} \equiv L(s)\cdot\frac{1}{L(rs)} \tag{13}$$

at frequency ω_i. If the frequency responses R_i+jI_i of the NL model are obtained at input frequencies ω_i, $i = 1,\ldots,k$, and the frequency responses $R_i'+jI_i'$ are obtained at input frequencies $\omega_i' = r\omega_i$, $i = 1,\ldots,k$, the frequency responses $R_i''+jI_i''$ of the fictitious transfer function $L'(s)$ can be computed. Thus $\hat{L}'(s)$ can be identified by means of the complex curve fitting method of Sanathanan and Koerner (**13**).

If there is no pole-zero cancellation in $L'(s)$, then the presence of the factor $(1+rx's)/(1+xs)$ in $\hat{L}'(s)$ when $x' \approx x$ will indicate the presence of the factor $1+xs$ in the denominator of $\hat{L}(s)$, and the factor $(1+ys)/(1+ry's)$ in $\hat{L}'(s)$ when $y' \approx y$ will indicate the factor $(1+ys)$ in the numerator of $\hat{L}(s)$. Thus, $\hat{L}(s)$ can be found completely. An ambiguity may occur, however, if r happens to be the number such that $ra_i = a_j$, so that one or more factors $(1+ra_is)$ and $(1+a_js)$ are cancelled identically. In this case, the order of L' (equal in denominator and numerator) must be determined so that erroneous pairs are not included. As in the most procedures, order determination can be accomplished by a residual test. Assume the fictitious system L' is of order J and has been identified as follows:

$$\hat{L}_J'(s) = \frac{A_0+A_1s+\cdots+A_Js^J}{1+B_1s+\cdots+B_Js^J}. \tag{14}$$

Define the associated cost function by

$$e_J = \sum_{i=1}^{k} |\hat{L}_J'(j\omega_i)-(R_i''+jI_i'')|^2. \tag{15}$$

Then, starting with $J = m+n$, the order of L' can be estimated using the following procedure:

Step 1. Compute e_J and e_{J-1} from (15).

Step 2. If the increase of e_{J-1} from e_J is insignificant and if $J > 1$, decrease J by 1 and go to Step 1. Otherwise, the order of $L'(s)$ is estimated as J.

If the estimated order of L' is J, then $m+n-J$ pole-zero pairs are cancelled in (13). We should therefore multiply both the denominator and the numerator by identical product factors $(1+z_1s)(1+z_2s)\ldots(1+z_{m+n-J}s)$ to restore the required form of $L'(s)$ as given in (13). Then $\hat{L}(s)$ can be found directly by observation. Note that the above procedure ceases at the test for $J = 1$ since it is the smallest possible order of L' to result from the assumption $r \neq 1$.

The two methods described above are used to determine the linear system $L(s)$. Identification of the nonlinearity N remains to be described. Since the output $y(t)$ can be approximated by a finite Fourier series as (5), the signal $x(t)$ can be estimated from the superposition property of the linear system $L(s)$ as

$$\hat{x}(t) = \frac{\alpha_0}{2} + \sum_{l=1}^{\bar{k}} \frac{\alpha_l}{|\hat{L}(jl\omega_i)|} \cos [l\omega_i t+\theta_l - \angle \hat{L}(jl\omega_i)]. \tag{16}$$

Then, by comparing $u(t)$ and $\hat{x}(t)$, the nonlinearity N can be identified graphically. Recall that the identified nonlinearity $\hat{N}$ absorbs the dc gain of the entire cascaded system.

III. Identification of the LN *Model*

Consider the *LN* model as shown in Fig. 1. The linear system $L(s)$ is as described by (1). To a sinusoidal input $u(t) = A \cos \omega_i t$, the corresponding steady state signals $x(t)$ and $y(t)$ are, respectively,

$$x(t) = A \cdot |L(j\omega_i)| \cdot \cos [\omega_i t + \angle L(j\omega_i)] \tag{17}$$

and

$$y(t) = N_0(A') + N_1(A') \cos [\omega_i t + \angle L(j\omega_i) + \phi_1(A')] + N_2(A') \cos [2\omega_i t + \angle L(j2\omega_i) + \phi_2(A')] + \cdots \tag{18}$$

where $A' = A \cdot |L(j\omega_i)|$. $y(t)$ can be approximated by a finite Fourier series as (5). Two methods for identifying $L(s)$ in the *LN* model are as follows:

Method I. Based on amplitude frequency response alone

If input frequency ω_i is fixed, then the gain $|L(j\omega_i)|$ is a constant for arbitrary input amplitude A, and the amplitude of fundamental output is $N_1(A')$, where $A' = A|L(j\omega_i)|$. This corresponds to modifying the nonlinearity N by a constant factor $|L(j\omega_i)|$. Hence, the amplitude relationship between the input signal $u(t)$ and the fundamental output of the nonlinearity can be obtained by varying the values of A. Now modify the original *LN* model to an $L'N'$ model with $L'(s) = [1/|L(j\omega_i)|]L(s)$ and $N' = |L(j\omega_i)|N$. If the nonlinearity is independent of frequency, $L'(s)$ can be identified from the following procedure:

Step 1. Fix $\omega_i = \omega_0$, apply different amplitudes of input signal as in the $S_A = \{A_1, A_2, \ldots, A_k\}$, and measure the corresponding amplitude of fundamental output as the set $S_N = \{N_1(A_1 \cdot |L(j\omega_0)|), N_1(A_2 \cdot |j\omega_0|), \ldots, N_1(A_k \cdot |L(j\omega_0)|)\}$. These two sets of data reveal the property of modified nonlinearity.

Step 2. Change the input frequency to ω_i, $i = 1, 2, \ldots$. Properly adjust the input amplitude A so that the amplitude of the fundamental output α_1 falls in the interval of the data set S_N. Then the input amplitude of the modified nonlinearity can be estimated by the interpolation method based on two data sets S_A and S_N. Once the input amplitude of the modified nonlinearity is found, the amplitude ratio of modified $L(s)$ can be obtained easily. Then, the modified transfer function $L(s)$ can be obtained easily by Jong and Shanmugam's method (12). Finally $\hat{L}(s)$ can be found by normalizing the dc gain of identified transfer function $L'(s)$ to unity.

Method II. Based on both amplitude and phase frequency responses

The method for identifying an *LN* model is similar to that for the *NL* model as described in Section II. The only difference for identifying the *LN* model is that we need to adjust the input signal $u(t) = A \cos (\omega_i t + \theta_i)$ to generate the same output wave form,

$$y(t) = \frac{\alpha_0}{2} + \alpha_1 \cos (\omega_i t) + \cdots,$$

for different frequencies ω_i, $i = 1, 2, \ldots, k$. Then the fundamental frequency response data of the LN model can be obtained and the normalized linear system $L(s)$ can be identified easily from method II as discussed in Section II.

Once the normalized linear system $L(s)$ is identified, the modified nonlinearity can be determined by comparing $\hat{x}(t)$ and $y(t)$ graphically, where

$$\hat{x}(t) = A_i \cdot |\hat{L}(j\omega_i)| \cdot \cos [\omega_i t + \theta_i + \angle \hat{L}(j\omega_i)].$$

IV. Identification of LNL *Model*

In this section, a model consisting of a nonlinearity sandwiched in between two linear systems is considered (Fig. 1). The nonlinearity N is restricted to being memoryless. $L_1(s)$ and $L_2(s)$ are normalized as follows:

$$L_1(s) = \frac{(1+b_1 s)(1+b_2 s)\dots(1+b_m s)}{(1+a_1 s)(1+a_2 s)\dots(1+a_n s)}, \tag{19}$$

$$L_2(s) = \frac{(1+d_1 s)(1+d_2 s)\dots(1+d_{\bar{m}} s)}{(1+c_1 s)(1+c_2 s)\dots(1+c_{\bar{n}} s)} \tag{20}$$

where $m, n, \bar{m}$ and $\bar{n}$ are known *a priori*. Due to the presence of the nonlinearity, there is a phase shift $k_1\pi$ between $x(t)$ and fundamental component $z(t)$ (see Appendix). Then, the phase between $u(t)$ and fundamental component of $y(t)$ is determined by the phase characteristic of $L_1(s) \cdot L_2(s)$ and modified by $k_1\pi$ only. If the input of the LNL model is $u(t) = A\cos(\omega_i t)$, the steady-state response of $y(t)$ can be approximated by a finite Fourier series as (5). After collecting k phase data between $u(t)$ and the fundamental component of approximated $y(t)$ of (5) at different input frequencies $\omega = \omega_1, \omega_2, \dots, \omega_k$, the product of $L_1(s) \cdot L_2(s)$ can be found by Jong's method (**14**) as

$$\hat{L}_1(s) \cdot \hat{L}_2(s) = \frac{(1+B_1 s)(1+B_2 s)\dots(1+B_{\bar{m}+m} s)}{(1+A_1 s)(1+A_2 s)\dots(1+A_{\bar{n}+n} s)} \tag{21}$$

if there is no pole-zero cancellation in $L_1(s) \cdot L_2(s)$. (The order test which is similar to that discussed in Section II can be used to indicate this.) Note that $k_1\pi$ contributes nothing when Jong's method is used to find (21). The next step is to separate $\hat{L}_1(s)$ and $\hat{L}_2(s)$ from $\hat{L}_1(s) \cdot \hat{L}_2(s)$. From the characteristic of the memoryless nonlinearity (see Appendix) and the LNL model, it is easily seen that

$$\arg[L_1(j\omega_i)] + \arg[L_2(j\omega_i)] + k_1\pi = \theta_1, \tag{22}$$

$$l \arg[L_1(j\omega_i)] + \arg[L_2(jl\omega_i)] + k_l\pi = \theta_l. \tag{23}$$

Subtracting (23) from l times (22), we obtain

$$\arg[L_2(jl\omega_i)] - l\arg[L_2(j\omega_i)] + \bar{K}\pi = \theta_l - l\theta_1 \tag{24}$$

where $\bar{K} = k_l - lk_1$. Consider the following equation:

$$\sum_{l_1=1}^{m+\bar{m}} \{\arg(1+jB_{l_1} l\omega_i)\} C_{l_1} - \sum_{l_2=1}^{n+\bar{n}} \{\arg(1+jA_{l_2} l\omega_i)\} D_{l_2}$$
$$- \sum_{l_1=1}^{m+\bar{m}} \{l \arg(1+jB_{l_1}\omega_i)\} C_{l_1} + \sum_{l_2=1}^{n+\bar{n}} \{l\arg(1+jA_{l_2}\omega_i)\} D_{l_2} + \bar{K}\pi = \theta_l - l\theta_1. \tag{25}$$

It is clear that the above equation is formed by (24) and (21) if the rules of selecting C_{l_1}, D_{l_2} are as follows:

$C_{l_1} = 0$ if $(1+B_{l_1}s)$ is not in the numerator of $\hat{L}_2(s)$;

$C_{l_1} = 1$ if $(1+B_{l_1}s)$ is in the numerator of $\hat{L}_2(s)$;

$D_{l_2} = 0$ if $(1+A_{l_2}s)$ is not in the denominator of $\hat{L}_2(s)$;

and

$D_{l_2} = 1$ if $(1+A_{l_2}s)$ is in the denominator of $\hat{L}_2(s)$.

For $\omega_i = \omega_1, \omega_2, \ldots, \omega_k$, k linear equations can be obtained from (25). Then, C_{l_1}, D_{l_2} and $\bar{K}$ can be solved in the sense of least squares error if $k \geqslant m+\bar{m}+n+\bar{n}+1$. Of course, the obtained values of C_{l_1} and D_{l_2} are only approximations of either 1 or 0. Thus $\hat{L}_2(s)$ is obtained, and so is $\hat{L}_1(s)$.

If the order test indicates that the order of $L_1(s)\cdot L_2(s)$ is not $n+\bar{n}$, the pole-zero cancellation exists in $L_1(s)\cdot L_2(s)$. The identification method can be achieved by considering (24) which means that

$$\angle L''(s) = -\bar{K}\pi + \theta_l - l\theta_1 \tag{26}$$

where

$$L''(s) = \frac{(1+ld_1 s)\ldots(1+ld_{\bar{m}}s)(1+c_1 s)^l \ldots (1+c_{\bar{n}}s)^l}{(1+lc_1 s)\ldots(1+lc_{\bar{n}}s)(1+d_1 s)^l \ldots (1+d_{\bar{m}}s)^l}. \tag{27}$$

If there is no pole-zero cancellation in (27), $L''(s)$ can be identified from Jong's method. Let the identified model be denoted by $\hat{L}''(s)$. If there is a factor $(1+xs)/[(1+x_1 s)(1+x_2 s)\ldots(1+x_l s)]$ in $\hat{L}''(s)$ for $x_1 \approx x_2 \approx \cdots \approx x_l \approx x/l$, then $-l/x$ is a zero of $\hat{L}_2(s)$; if there is a factor $[(1+y_1 s)(1+y_2 s)\ldots(1+y_l s)]/(1+ys)$ in $\hat{L}''(s)$ for $y_1 \approx \cdots \approx y_l \approx y/l$, then $-l/y$ is a pole of $\hat{L}_2(s)$. From this rule, all poles and zeros of $\hat{L}_2(s)$ can be obtained. If there is pole-zero cancellation in $\hat{L}''(s)$, the order test of $L''(s)$ is needed. The procedure is similar to that discussed in Section II, Method II. Once $\hat{L}_2(s)$ is found, the phase information of $L_1(s)$ can be obtained from (22) as

$$\arg[L_1(j\omega_i)] = \theta_1 - \arg[\hat{L}_2(j\omega_i)] - k_1\pi \tag{28}$$

then $L_1(s)$ can be identified from Jong's method. It is clear that the choice of l in (24) is that the lth harmonic is present in the approximated steady-state output response (5). If the nonlinearity is symmetric, the even-order harmonics are absent.

Once the linear systems $L_1(s)$ and $L_2(s)$ are found, $x(t)$ and $z(t)$ can be estimated as

$$\hat{x}(t) = A\,|\hat{L}_1(j\omega_i)|\,\cos[\omega_i t + \angle L_1(j\omega_i)] \tag{29}$$

and

$$\hat{z}(t) = \frac{\alpha_0}{2} + \sum_{l=1}^{\bar{k}} \frac{\alpha_l}{|\hat{L}_2(jl\omega_i)|}\cos[l\omega_i t + \theta_l - \angle \hat{L}_2(jl\omega_i)]. \tag{30}$$

Then, the nonlinearity N can be found by comparing $\hat{x}(t)$ and $\hat{z}(t)$ graphically.

V. Applications

1. Describing function evaluation

The describing function is a quasi-linear representation of a nonlinear element when the input of the nonlinearity is a sinusoidal function. Since the negative reciprocal of the describing function is widely used for the stability analysis of a complex control system (**15**), it is interesting to identify the value of describing function instead of finding the actual form of the nonlinearity. From the discussions in the previous sections, the waveforms of the input to the nonlinearity and the fundamental output of the nonlinearity can be easily determined. Then, the value of the describing function can be obtained by comparing the amplitude and phase of both the input and the output of the nonlinearity. Varying the input, a different value of describing function at different input amplitudes can also be determined.

2. Identification of nonlinear feedback system

Consider the basic nonlinear feedback control system as shown in Fig. 6. Assume that the higher order harmonics generated by the nonlinearity can be significantly attenuated by the linear system. If $R(t) = A \cos \omega_i t$, and the corresponding output $C(t)$ is approximated to $\bar{k}$th harmonic by the following finite Fourier series:

$$\hat{C}(t) = \frac{1}{2}\alpha_0 + \sum_{l=1}^{\bar{k}} (\alpha_l \cos l\omega_i t + \beta_l \sin l\omega_i t). \tag{31}$$

Then, by dropping the higher order harmonic terms, we have

$$\begin{aligned} e(t) &\approx A \cos \omega_i t - \frac{1}{2}\alpha_0 - \alpha_1 \cos \omega_i t - \beta_1 \sin \omega_i t \\ &= -\frac{1}{2}\alpha_0 + [(A-\alpha_1)^2 + \beta_1^2]^{1/2} \cos\left(\omega_i t + \tan^{-1}\frac{\beta_1}{A-\alpha_1}\right). \end{aligned} \tag{32}$$

Since the fundamental component of the output $C(t)$ is $(\alpha_1^2+\beta_1^2)^{1/2} \cos(\omega_i t + \theta_1)$, where $\theta_1 = \tan^{-1}(-\beta_1/\alpha_1)$, the phase response of $G(s)$ at frequency ω_i is

$$\arg[G(j\omega_i)] = \theta_1 - \tan^{-1}\frac{\beta_1}{A-\alpha_1} + k_1\pi. \tag{33}$$

By varying the frequency ω_i, we obtain a set of phase response data. For the same reason discussed in the previous section, we normalize the dc gain K of the linear system to unity as follows:

$$G'(s) = s^{\bar{l}}G(s) = \frac{1+b_1 s+\cdots+b_m s^m}{1+a_1 s+\cdots+a_n s^n}. \tag{34}$$

Then, from (33), the phase of $G'(j\omega_i)$ can be determined from the following equation:

$$\arg[G'(j\omega_i)] = \arg[G(j\omega_i)] + \bar{l}\pi/2. \tag{35}$$

Thus, a set of the phase data of $G'(s)$ can be generated, hence a_i and b_i can be estimated from the Jong's method (**14**) if $G'(s)$ is of minimum phase. Let $\hat{G}(s)$ be the identified model of $G(s)$, then the signal $e_N(t)$ can be computed approximately from the same method discussed in the previous section as $\hat{e}(t)$. Then, the nonlinearity N can be identified graphically by comparing $e(t)$ and $\hat{e}_N(t)$ or determined according to Section V.1.

VI. Numerical Examples

Example 1

Consider the NL model as shown in Fig. 1, where the linear model is $L(s) = (s-1)/[(s+1)(s+2)]$, and the nonlinearity is shown in Fig. 2. If a sinusoidal input of amplitude $A = 1$ is applied, the amplitude and phase of the fundamental component at the output of the nonlinearity is $0.6086 \angle -6.0039°$. This means that $N_1(A) = 0.6086$ and $\phi_1(A) = -6.0039°$. Hence, the noise corrupted fundamental frequency response at $\omega = 0.1, 0.2, \ldots, 4.0$ can be generated from (8) by rounding the amplitude ratio (db) and phase angle (degree) to one decimal place. For $r = 2$, the fundamental frequency responses at $\omega = 0.2, 0.4, \ldots, 8.0$ are also generated from (8) in the same way. Then the frequency response of $L'(s)$ defined by (13) can be obtained from (12). In order to test the pole-zero cancellation of $L'(s)$, e_J defined in (15) is computed as

$$e_3 = 0.00181,$$

$$e_2 = 0.00183,$$

$$e_1 = 3.2453.$$

Thus, the order of $L'(s)$ is two and $\hat{L}'(s)$ is

$$\hat{L}'(s) = \frac{1.0051(1+1.9909s)(1-1.0042s)}{(1+0.4988s)(1-2.0134s)}.$$

Thus $(1+z_1 s)$ can be chosen as $(1+0.9965s)$ to retain the form of (13). Finally, we find

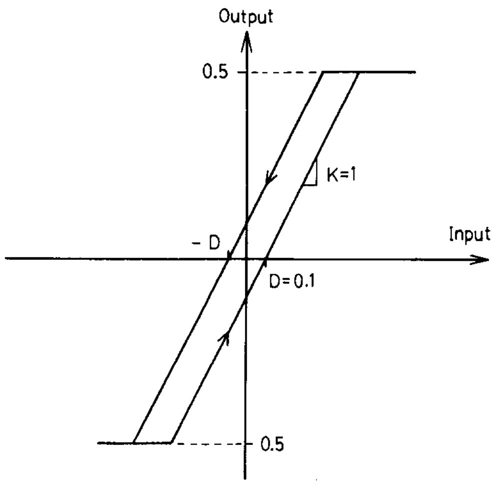

FIG. 2. Nonlinearity of NL model in Example 1.

that

$$\hat{L}(s) = \frac{(1-1.0042s)}{(1+0.9965s)(1+0.4988s)}.$$

Since the steady-state response of this system to the input $u(t) = \sin t$ is approximated by (5) from 60 data points (each data is rounded to three decimal places) of a complete cycle as

$$\begin{aligned} y(t) = {} & 0.22937 \cos t - 0.02309 \cos 3t - 0.0047 \cos 5t + 0.00138 \cos 7t \\ & + 0.00116 \cos 9t + 0.14662 \sin t + 0.03023 \sin 3t + 0.00157 \sin 5t \\ & + 0.00054 \sin 7t + 0.00085 \sin 9t \end{aligned}$$

and $\hat{x}(t)$ is obtained from (16), the nonlinearity N can be found graphically as shown in Fig. 3. If the nonlinearity $\hat{N}$ is in cascade with a gain factor -2.0, it is close to the nonlinearity shown in Fig. 2.

Example 2

Consider the *LNL* model shown in Fig. 1, where $L_1(s) = 1/(s+1)$, $L_2(s) = 1/(s+2)$ and the nonlinearity is shown in Fig. 4. The noise corrupted phase information are shown in Table I by retaining the three significant digits of the actual values.

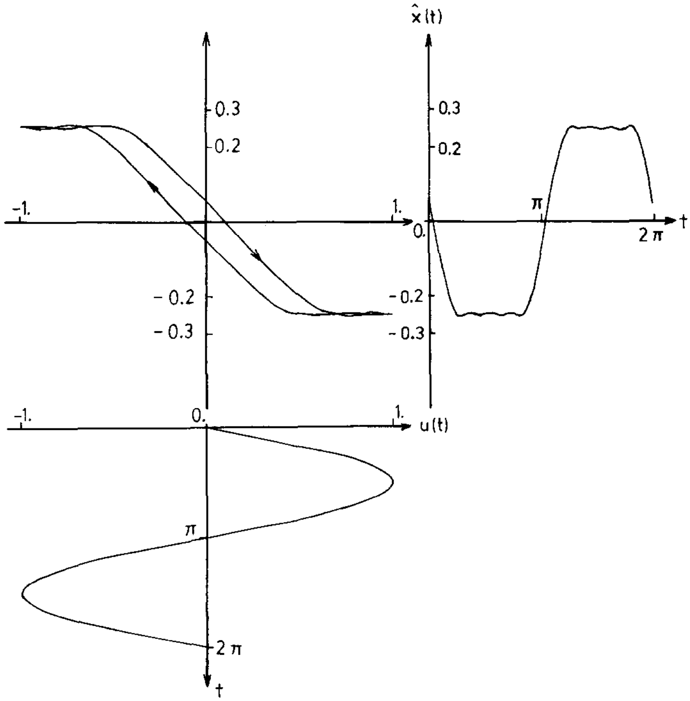

FIG. 3. Determination of the nonlinearity $\hat{N}$ in Example 1.

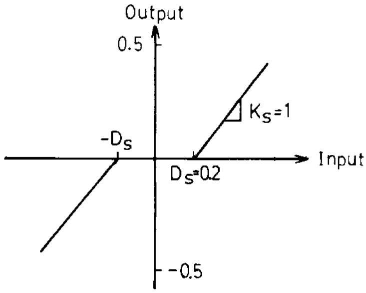

FIG. 4. Dead-space nonlinearity.

From the method discussed in Section IV, we obtain

$$\hat{L}_1(s)\cdot\hat{L}_2(s) = \frac{1}{(1+0.9971s)(1+0.5026s)}.$$

From (25), we obtain the following equations for $A_1 = 0.9971$, $A_2 = 0.5026$ and $l = 3$:

$$-30.89D_1 - 16.78D_2 + 33.83D_1 + 17.22D_2 + 180\bar{K} = 0.4,$$

$$-50.11D_1 - 31.10D_2 + 65.23D_1 + 34.10D_2 + 180\bar{K} = 2.9,$$

$$-60.87D_1 - 42.14D_2 + 92.67D_1 + 50.34D_2 + 180\bar{K} = 8.1,$$

$$-67.32D_1 - 50.34D_2 + 115.7D_1 + 65.71D_2 + 180\bar{K} = 15.5,$$

$$-71.52D_1 - 56.45D_2 + 134.8D_1 + 80.05D_2 + 180\bar{K} = 23.8.$$

The least squares solution of above linear equations are $D_1 = -0.00067$, $D_2 = 1.0036$ and $\bar{K} = 0.000018$. So, from what we discussed, it is easily seen that

$$\hat{L}_1(s) = \frac{1}{(1+0.9971s)},$$

$$\hat{L}_2(s) = \frac{1}{(1+0.5026s)}.$$

Since the steady-state response of this system to the input $u(t) = \cos t$ is approximated by (5) from 60 data points (each data is rounded to three decimal places)

TABLE I
The phase information in Example 2

ω	θ_1	θ_3
0.2	−17.0°	−50.6°
0.4	−33.1°	−96.4°
0.6	−47.7°	−135°
0.8	−60.5°	−166°
1.0	−71.6°	−191°

of a complete cycle as

$$\hat{y}(t) = 0.0644 \cos t - 0.0204 \cos 3t - 0.0026 \cos 5t + 0.0019 \cos 7t$$
$$+ 0.1934 \sin t - 0.0040 \sin 3t + 0.0060 \sin 5t + 0.0011 \sin 7t$$

the signal $\hat{x}(t)$ and $\hat{z}(t)$ can be found from (29) and (30). Then the nonlinearity $\hat{N}$ can be determined by comparing $\hat{x}(t)$ and $\hat{z}(t)$ as shown in Fig. 5. The determined nonlinearity is nearly the same as the nonlinearity shown in Fig. 4 except the gain factor 2.

Example 3

Consider the ideal relay control system as shown in Fig. 6 where $G(s) = 1/(1+s)s$, and

$$f(\cdot) = \begin{cases} 1, & e(t) > 0 \\ -1, & e(t) < 0 \end{cases}$$

which has the jump characteristic. If $R_1(t) = \cos t$ and $R_2(t) = \cos 2t$, the steady-state responses of $C_1(t)$ and $C_2(t)$, respectively, can be approximated as

$$\hat{C}_1(t) = -0.0667 \cos t - 0.8978 \sin t - 0.0346 \cos 3t - 0.0284 \sin 3t + \cdots,$$
$$\hat{C}_2(t) = -0.2364 \cos 2t + 0.1587 \sin 2t + 0.0099 \cos 6t - 0.0061 \sin 6t + \cdots$$

from 60 data points (each data is rounded to three decimal places for the noise corrupted measurement) of a complete cycle. Then, the corresponding error signals

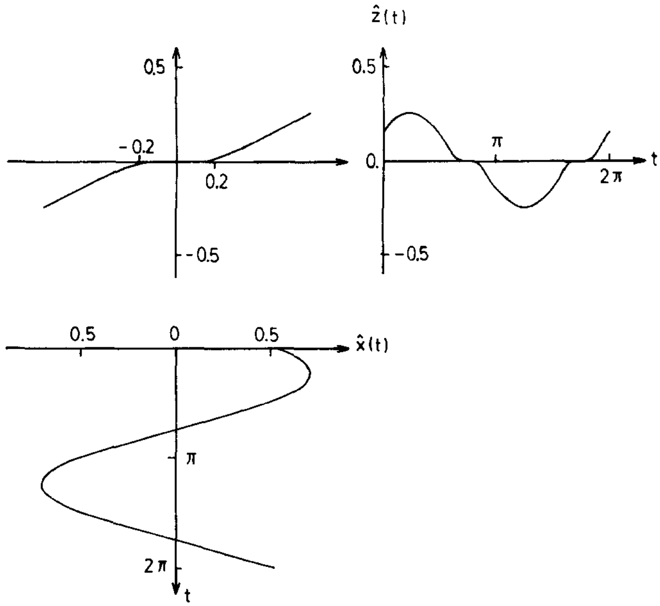

FIG. 5. Determination of the nonlinearity $\hat{N}$ in Example 2.

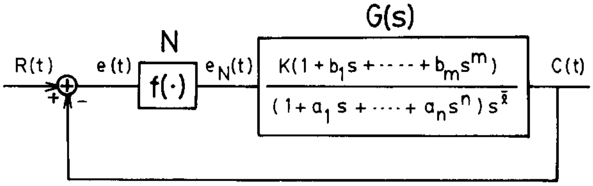

FIG. 6. Nonlinear feedback control system.

are

$$e_1(t) \approx 1.3942 \cos(t+0.6996),$$

$$e_2(t) \approx 1.2465 \cos(2t+0.1277).$$

From what we discussed in Section V.2, $G(s)$ can be identified as

$$\hat{G}(s) = \frac{1}{(1+0.9886s)s}.$$

Then, the values of the describing function of $f(\cdot)$ can be estimated as 0.9079 for the input amplitude 1.3942 (the actual value is 0.9132) and 1.0121 for the input amplitude 1.2465 (the actual value is 1.0215).

VII. Conclusion

The identification of a complex system is easily accomplished by using the input and output data. Special forms of the nonlinear systems (*LNL*, *NL* and *LN* model) are used to identify the coupling system from the steady-state response. In contrast to previous work which has assumed that the nonlinearity in the system is single-valued, the methods described in this paper for the identification of a nonlinear system are applicable to multiple-valued nonlinearities. The methods developed for *LN* and *NL* models are applicable for systems with memory-type nonlinearity, and a method developed for the *LNL* model is applicable with a memoryless nonlinearity, but may exhibit jump phenomena. It is well known that the values of a describing function can be used to estimate the limit-cycle of a complex feedback system. If the components of this system are of the *LNL*, *LN*, or *NL* form, the methods discussed in this paper can be used to estimate the values of a describing function of modified nonlinearity by identifying the components individually without the need to find the nonlinearity itself. The development of this paper is essentially based on the deterministic data. Little attention is given to the noise corrupted measurements and their minimization.

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Appendix

If the input of a memoryless nonlinearity is

$$I(t) = A_D + A_E \cos(\omega t + \theta) \tag{A1}$$

and the memoryless nonlinearity $f(\cdot)$ can be expressed by

$$f[I(t)] = \sum_{i=0}^{\infty} v_i I^i(t), \tag{A2}$$

the output of the nonlinearity is

$$\begin{aligned} O(t) &= f[I(t)] \\ &= f[A_D + A_E \cos(\omega t + \theta)] \\ &= \sum_{i=0}^{\infty} v_i [A_D + A_E \cos(\omega t + \theta)]^i \\ &= \sum_{i=0}^{\infty} v_i [A_D + A_E \cos(\omega \tau)]^i \end{aligned} \tag{A3}$$

where $\tau \equiv t+\theta/\omega$. It is clear that $O(\cdot)$ is an even periodic function in τ with a period $2\pi/\omega$. Hence it can be expressed in terms of Fourier series as

$$O(t) = \sum_{i=0}^{\infty} v_i' \cos(i\omega\tau). \tag{A4}$$

Substitution of $\tau = t+\theta/\omega$ into (A4) yields

$$O(t) = \sum_{i=0}^{\infty} v_i' \cos(i\omega t + i\theta). \tag{A5}$$

Thus, the fundamental component of $O(t)$, $v_1' \cos(\omega t+\theta)$, differs in phase from $I(t)$ by $k_1\pi$, $k_1 = 0, \pm 1$. And a phase $i\theta + k_i\pi$ in the ith order harmonics, $k_i = 0, \pm 1$.