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Note

Proper interval graphs and the guard problem[1]

Chiuyuan Chen*, Chin-Chen Chang, Gerard J. Chang

Department of Applied Mathematics, National Chiao Tung University, Hsinchu 30050, Taiwan

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Abstract

This paper is a study of the hamiltonicity of proper interval graphs with applications to the guard problem in spiral polygons. We prove that proper interval graphs with $\geqslant 2$ vertices have hamiltonian paths, those with $\geqslant 3$ vertices have hamiltonian cycles, and those with $\geqslant 4$ vertices are hamiltonian-connected if and only if they are, respectively, 1-, 2-, or 3-connected. We also study the guard problem in spiral polygons by connecting the class of nontrivial connected proper interval graphs with the class of stick-intersection graphs of spiral polygons.

Keywords: Proper interval graph; Hamiltonian path (cycle); Hamiltonian-connected; Guard; Visibility; Spiral polygon

1. Introduction

The main purpose of this paper is to study the hamiltonicity of proper interval graphs with applications to the guard problem in spiral polygons. Our terminology and graph notation are standard, see [2], except as indicated.

The *intersection graph* of a family $\mathscr{F}$ of nonempty sets is derived by representing each set in $\mathscr{F}$ with a vertex and connecting two vertices with an edge if and only if their corresponding sets intersect. An *interval graph* is the intersection graph G of a family $\mathscr{I}$ of intervals on a real line. $\mathscr{I}$ is usually called the *interval model* for G. A *proper interval graph* is an interval graph with an interval model $\mathscr{I}$ such that no interval in $\mathscr{I}$ properly contains another. Proper interval graphs are also referred to in the literature as unit interval graphs, indifference graphs, and time graphs.

Bertossi [1] proved that a proper interval graph has a hamiltonian path if and only if it is connected. He also gave a condition under which a proper interval graph will have

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* Correspondence address: 1915 Maple #611, Evanston, IL 60201, USA.

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a hamiltonian cycle. Using this, he proposed an algorithm for finding the hamiltonian cycle of a proper interval graph.

In Section 2, we derive alternative conditions under which a proper interval graph will have a hamiltonian path, have a hamiltonian cycle, and be hamiltonian-connected. Algorithmic results follow easily from these conditions. Recall that a graph is *hamiltonian-connected* if there is a hamiltonian path between any two distinct vertices.

A polygon P is *simple* if no pair of nonconsecutive edges share a point. All polygons discussed in this paper are assumed to be simple. A vertex v of P is *convex* (*concave*) if its interior angle is less than (greater than) 180°. A *convex* (*concave*) *chain* of P is a sequence of consecutive convex (concave) vertices. P is a *spiral polygon* if it has exactly one concave subchain, see Fig. 1 for an example.

A point p in a polygon P is said to *see* or *cover* another point q if the line segment $\overline{pq}$ does not intersect the exterior of P. For example, in Fig. 1, w_1, u_1, and p can see each other, but w_1 cannot see w_4. A set of points (vertices) that cover the interior and the boundary of P are called *point* (*vertex*) *guards*. Vertex guards are also point guards, but the converse is not true. Note that two point guards $\{p, q\}$ are sufficient to cover the polygon P in Fig. 1 but three vertex guards are necessary to cover P. For surveys of the guard problem, refer to [12, 14].

The *visibility graph* G of a polygon P is the graph whose vertices correspond to the vertices of P, and two vertices of G are adjacent if and only if their corresponding vertices in P can see each other. In general, the recognition problem for visibility graphs is unresloved. Everett and Corneil [7] gave a linear-time algorithm for recognizing visibility graphs of spiral polygons, which are interval graphs under certain conditions. Consequently, a minimum set of vertex guards for a spiral polygon can be determined by solving the domination problem of an interval graph. Nilsson and Wood, on the other hand, proposed a linear-time algorithm for finding a minimum set of point guards for a spiral polygon [10, 11].

In Section 3, we consider the problem of finding a minimum set of point guards for a spiral polygon. We first prove that the class of stick-intersection graphs associated with spiral polygons equals the class of nontrival, connected, proper interval graphs. We then give alternative verification of the validity of Nilsson and Wood's algorithm.

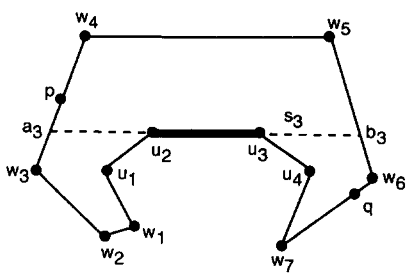


Fig. 1. A spiral polygon in standard form, concave edge $\overline{u_2u_3}$, and stick $s_3 = \overline{a_3b_3}$.

2. Hamiltonicity in proper interval graphs

The hamiltonicity of proper interval graphs is addressed in this section.

The *closed neighborhood* $N[v]$ of a vertex v is the set of vertices adjacent to v along with v itself. An ordering $[v_1, v_2, \ldots, v_n]$ of the vertices of G is a *consecutive ordering* if for every i, $N[v_i]$ is consecutive; i.e., $N[v_i] = \{v_t : i_1 \leqslant t \leqslant i_2\}$ for some $i_1 \leqslant i_2$. Note that, for $i < j$, we have $i_1 \leqslant j_1$ and $i_2 \leqslant j_2$. Consequently, $[v_1, v_2, \ldots, v_n]$ is a consecutive ordering of $G = (V, E)$ if and only if

$$i < j < k \text{ and } v_i v_k \in E \quad \text{imply} \quad v_i v_j \in E \text{ and } v_j v_k \in E.$$

Also, if $[v_1, v_2, \ldots, v_n]$ is a consecutive ordering, then so is $[v_n, v_{n-1}, \ldots, v_1]$.

Roberts [13] proved that G is a proper interval graph if and only if its augmented adjacency matrix, which is the adjacency matrix plus the identity matrix, satisfies the consecutive 1's property for columns; see also [6]. This fact can be restated as

Theorem 1. *A graph $G = (V, E)$ is a proper interval graph if and only if G has a consecutive ordering.*

Booth and Leuker's [3] consecutive 1's testing algorithm provides a way to determine whether a graph is a proper interval graph, and gives a consecutive ordering if the answer is positive. Corneil et al. [4] and de Figueiredo et al. [5] proposed simpler methods for accomplishing these by using a breadth-first search and a lexicographic breadth-first search, respectively.

We are now ready to study the hamiltonicity of proper interval graphs.

Theorem 2. *For any positive integer k and any proper interval graph $G = (V, E)$ of $n \geqslant k + 1$ vertices with a consecutive ordering $[v_1, v_2, \ldots, v_n]$, G is k-connected if and only if $v_i v_j \in E$ whenever $1 \leqslant |i - j| \leqslant k$.*

Proof. ($\Rightarrow$) Suppose G is k-connected and $1 \leqslant |i - j| \leqslant k$. Without loss of generality, we may assume that $i < j \leqslant i + k$. Since G is k-connected and $S = \{v_t : i < t < j\}$ has at most $k - 1$ vertices, $G - S$ is connected. There is a shortest v_i–v_j path $P = \langle v_{i_1}, v_{i_2}, \ldots, v_{i_r} \rangle$ in $G - S$, where $v_{i_1} = v_i$ and $v_{i_r} = v_j$. Let i_p (i_q) be the minimum (maximum) index in $\{i_1, i_2, \ldots, i_r\}$. Since $[v_1, v_2, \ldots, v_n]$ is a consecutive ordering of G, if $1 < p < r$ ($1 < q < r$), then $v_{i_{p-1}} v_{i_{p+1}} \in E$ ($v_{i_{q-1}} v_{i_{q+1}} \in E$). This contradicts the assumption that P is a shortest path in $G - S$. Therefore, $\{i_p, i_q\} = \{i_1, i_r\} = \{i, j\}$. Since P is a path in $G - S$, P contains no vertex v_t such that $i < t < j$. Consequently, $r = 2$ and $v_i v_j \in E$.

($\Leftarrow$) On the other hand, suppose $v_i v_j \in E$ whenever $1 \leqslant |i - j| \leqslant k$. For any subset $S \subseteq V$ of size $|S| < k$, remove all vertices of S from $[v_1, v_2, \ldots, v_n]$ to get a subsequence $[v_{i_1}, v_{i_2}, \ldots, v_{i_m}]$. For each p with $1 \leqslant p \leqslant m - 1$, since $|S| < k$, $|i_p - i_{p+1}| \leqslant k$ and so $v_{i_p} v_{i_{p+1}} \in E$. Therefore, $G - S$ is connected and so, G is k-connected. $\square$

Theorem 3 (Bertossi [1]). *For any proper interval graph $G=(V,E)$ of $n \geqslant 2$ vertices, G has a hamiltonian path if and only if G is 1-connected.*

Proof. ($\Rightarrow$) If G has a hamiltonian path, then G is certainly 1-connected.

($\Leftarrow$) Suppose G is 1-connected. By Theorem 2, for any consecutive ordering $[v_1, v_2, \ldots, v_n]$ of G, $v_iv_j \in E$ whenever $|i-j|=1$. Thus, $\langle v_1, v_2, \ldots, v_n \rangle$ is a hamiltonian path of G. $\square$

Theorem 4. *For any proper interval graph $G=(V,E)$ of $n \geqslant 3$ vertices, G has a hamiltonian cycle if and only if G is 2-connected.*

Proof. ($\Rightarrow$) Suppose G has a hamiltonian cycle. For any $|S| \leqslant 1$, $G-S$ has a hamiltonian path and hence is connected. This proves that G is 2-connected.

($\Leftarrow$) Suppose G is 2-connected. By Theorem 2, for any consecutive ordering $[v_1, v_2, \ldots, v_n]$ of G, $v_iv_j \in E$ whenever $1 \leqslant |i-j| \leqslant 2$. Thus, $\langle v_1, v_3, v_5, v_7, \ldots, v_{n-2}, v_n, v_{n-1}, v_{n-3}, v_{n-5}, \ldots, v_4, v_2, v_1 \rangle$ is a hamiltonian cycle of G if n is odd, and $\langle v_1, v_3, v_5, v_7, \ldots, v_{n-1}, v_n, v_{n-2}, v_{n-4}, \ldots, v_4, v_2, v_1 \rangle$ is a hamiltonian cycle of G if n is even. $\square$

Theorem 5. *For any proper interval graph $G = (V,E)$ of $n \geqslant 4$ vertices, G is hamiltonian-connected if and only if G is 3-connected.*

Proof. ($\Rightarrow$) Suppose G is hamiltonian-connected. For any $|S| \leqslant 2$, choose two distinct vertices u and v such that $S \subseteq \{u,v\}$. Since there is a hamiltonian path from u to v in G, $G-S$ has a hamiltonian path and hence is connected. This proves that G is 3-connected.

($\Leftarrow$) Suppose G is 3-connected. By Theorem 2, for any consecutive ordering $[v_1, v_2, \ldots, v_n]$ of G, $v_iv_j \in E$ whenever $1 \leqslant |i-j| \leqslant 3$. Suppose v_ℓ and v_m, $\ell < m$, are two arbitrary distinct vertices of G. A hamiltonian path from v_ℓ to v_m can be constructed as follows. First let $P_1 = \langle v_\ell, v_{\ell-2}, v_{\ell-4}, v_{\ell-6}, \ldots, v_1, v_2, v_4, v_6, \ldots, v_{\ell-1} \rangle$ when ℓ is odd and let $P_1 = \langle v_\ell, v_{\ell-2}, v_{\ell-4}, v_{\ell-6}, \ldots, v_2, v_1, v_3, v_5, v_7, \ldots, v_{\ell-1} \rangle$ when ℓ is even. P_1 is then a v_ℓ–$v_{\ell'}$ path with $\ell - 1 \leqslant \ell' \leqslant \ell$ and passing through every vertex in $\{v_1, v_2, \ldots, v_\ell\}$ exactly once. Similarly, there is a $v_{m'}$–v_m path P_2 with $m \leqslant m' \leqslant m+1$ and passing through every vertex in $\{v_m, v_{m+1}, \ldots, v_n\}$ exactly once. Thus, $\langle P_1, v_{\ell+1}, v_{\ell+2}, \ldots, v_{m-1}, P_2 \rangle$ is a hamiltonian path from v_ℓ to v_m. Therefore G is hamiltonian-connected. Note that for the case in which $\ell' = \ell - 1$ and $\ell = m - 1$ and $m' = m + 1$, we do use the condition that $v_iv_j \in E$ whenever $|i-j| = 3$. $\square$

3. Guard problem in spiral polygons

In this section, we consider the problem of finding a minimum set of point guards for a spiral polygon. We first prove that the class of stick-intersection graphs associated with spiral polygons equals the class of nontrivial, connected, proper interval graphs.

We then give alternative verification of the validity of Nilsson and Wood's algorithm [10, 11] for resolving the point guard problem with respect to spiral polygons.

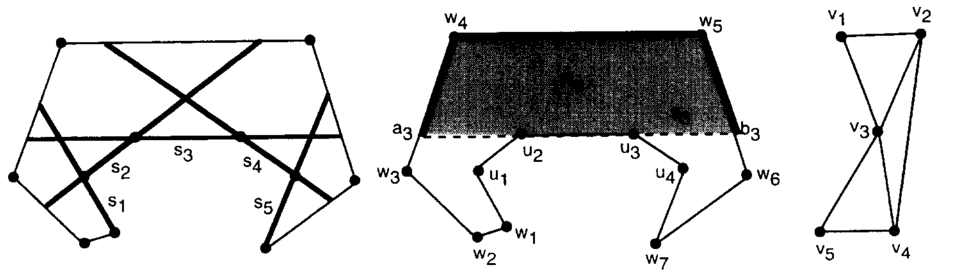
We assume that a spiral polygon P is given in *standard form*: the vertices of P are listed as a concave chain $[u_1, u_2, \ldots, u_m]$ in clockwise order ($m \geqslant 1$), and a convex chain $[w_1, w_2, \ldots, w_n]$ in clockwise order such that u_1 and w_1 are adjacent and u_m and w_n are adjacent; see Fig. 1. An edge e of P is called *concave* if it contains at least one concave vertex. A *stick* s of P is a longest line segment containing a concave edge and lying inside P. Denote $s_1 = \overline{a_1b_1}$, $s_2 = \overline{a_2b_2}, \ldots, s_{m+1} = \overline{a_{m+1}b_{m+1}}$ as the sticks containing concave edges $\overline{w_1u_1}, \overline{u_1u_2}, \ldots, \overline{u_mw_n}$, respectively; see Fig. 1 and Fig. 2(a). For each i, R_i denotes the region bounded by stick s_i and the convex chain of P and $\widehat{a_ib_i}$ denotes the boundary of R_i extending from a_i to b_i in clockwise order; see Fig. 2(b). It is easy to see that s_i intersects s_j if and only if R_i intersects R_j if and only if $\widehat{a_ib_i}$ intersects $\widehat{a_jb_j}$.

Lemma 6. *The intersection graph $G_P(S)$ of $S = \{s_1, s_2, \ldots, s_{m+1}\}$ equals the intersection graph $G_P(R)$ of $R = \{R_1, R_2, \ldots, R_{m+1}\}$ and also equals the intersection graph $G_P(B)$ of $B = \{\widehat{a_1b_1}, \widehat{a_2b_2}, \ldots, \widehat{a_{m+1}b_{m+1}}\}$.*

The intersection graphs $G_P(S), G_P(R), G_P(B)$, which are equal by Lemma 6, are called respectively, the *stick, region*, and *boundary intersection graphs* associated with P. Note that the intersection graph of the sticks in Fig. 2(a) equals the proper interval graph in Fig. 2(c), in which vertex v_i corresponds to stick s_i. This is not an accident, since we have the following theorem.

Theorem 7. *The class of stick-intersection graphs associated with spiral polygons equals the class of nontrivial, connected, proper interval graphs. Moreover, for a spiral polygon P, $[R_1, R_2, \ldots, R_{m+1}]$ is a consecutive ordering of $G_P(R)$.*

Proof. ($\Rightarrow$) Suppose P is a spiral polygon with m concave vertices. Since $m \geqslant 1$ and $G_P(S)$ has $m+1$ vertices, $G_P(S)$ is nontrivial. Since sticks s_i and s_j intersect whenever



(a) Sticks. (b) Region R_3 (shaded) and boundary $\widehat{a_3b_3}$ (bold). (c) A proper interval graph.

Fig. 2.

$|i-j|=1$, $G_P(S)$ is connected. By Lemma 6, $G_P(S)=G_P(R)=G_P(B)$. Consider the convex chain of P as 'embedded' in a real line, and $\{\widehat{a_1b_1}, \widehat{a_2b_2}, \ldots, \widehat{a_{m+1}b_{m+1}}\}$ as a set of intervals on a real line. Since b_i lies to the left of b_j whenever a_i lies to the left of a_j, no interval properly contains another. Hence $G_P(R)$ has the consecutive ordering $[R_1, R_2, \ldots, R_{m+1}]$, and so, is a proper interval graph.

($\Leftarrow$) Suppose $G=(V,E)$ is a nontrivial, connected, proper interval graph of n vertices. By Theorems 1 and 2, G has a consecutive ordering $[v_1, v_2, \ldots, v_n]$ such that $v_iv_j \in E$ whenever $|i-j|=1$. We shall construct a spiral polygon whose stick-intersection graph is G by means of a unit circle.

For any two points a and b on the unit circle C, denote $\widehat{ab}$ as the arc from a to b in clockwise order and denote $|\widehat{ab}|$ as the length of $\widehat{ab}$. We shall construct a set of chords $\{\overline{a_1b_1}, \overline{a_2b_2}, \ldots, \overline{a_nb_n}\}$ on C such that $\overline{a_ib_i}$ corresponds to v_i as follows; see Fig. 3(a) for an illustration of constructing a spiral polygon from Fig. 2(c).

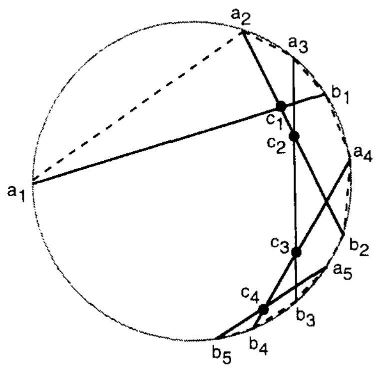
(1) Choose a chord $\overline{a_1b_1}$ on C such that $|\widehat{a_1b_1}| < \frac{1}{2}(2\pi)$.

(2) Since $v_1 \in N[v_2]$, choose a chord $\overline{a_2b_2}$ on C such that $\overline{a_2b_2}$ intersects $\overline{a_1b_1}$ at c_1 and $|\widehat{b_1b_2}| < (\frac{1}{2})^2(2\pi)$.

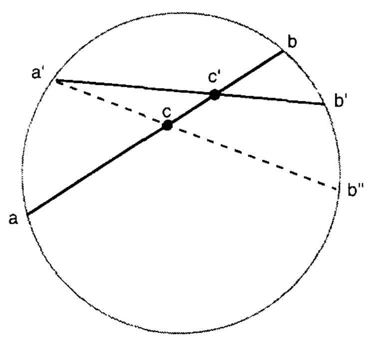
To choose $\overline{a_3b_3}, \ldots, \overline{a_nb_n}$ on C, we use the following lemma, which is clearly valid, repeatedly.

Lemma 8. *Suppose $a' \in \widehat{ab}$ and $c \in \overline{ab}$. For any $\varepsilon > 0$, there is a point b' on the unit circle such that a, a', b, b' are in clockwise order, $\overline{a'b'}$ intersects $\overline{cb}$ at c' and $|\widehat{bb'}| < \varepsilon$.* (See Fig. 3(b).)

Assume that $\overline{a_1b_1}, \overline{a_2b_2}, \ldots, \overline{a_{i-1}b_{i-1}}$ have already been chosen. We choose $\overline{a_ib_i}$ as follows, where $3 \leqslant i \leqslant n$. Assume that $N[v_i]=\{v_t : i_1 \leqslant t \leqslant i_2\}$. Note that $i_1 < i$. List $a_1, b_1, \ldots, a_{i-1}, b_{i-1}$ starting from a_1 and ending at b_{i-1} in clockwise order, say, $a_1, \ldots, x, b_{i_1}, \ldots, b_{i-1}$, where x is the point immediately preceding b_{i_1}. Choose a point a_i from the open arc $\widehat{xb_{i_1}}$. By Lemma 8, we can choose b_i on C such that $a, \ldots, x, a_i, b_{i_1}, \ldots, b_{i-1}$,



(a) Constructing a spiral polygon.



(b) Lemma 9 illustrated.

Fig. 3.

b_i are in clockwise order, chords $\overline{a_ib_i}$ and $\overline{c_{i-2}b_{i-1}}$ intersect at c_{i-1}, and $|\widehat{b_{i-1}b_i}| < (\frac{1}{2})^i(2\pi)$.

It is not difficult to verify that if $N[v_i] = \{v_t : i_1 \leqslant t \leqslant i_2\}$, then $\overline{a_ib_i}$ intersects $\overline{a_{i_1}b_{i_1}}, \ldots, \overline{a_{i-1}b_{i-1}}, \overline{a_{i+1}b_{i+1}}, \ldots, \overline{a_{i_2}b_{i_2}}$. Therefore the intersection graph for $\{\overline{a_1b_1}, \overline{a_2b_2}, \ldots, \overline{a_nb_n}\}$ equals G.

Consider a spiral polygon P whose concave chain is $[c_1, c_2, \ldots, c_{n-1}]$ and whose convex chain contains all a_i's and b_i's in clockwise order on C; see Fig. 3(a). Then $\overline{a_ib_i}$'s are precisely the sticks of P. Therefore G equals the stick-intersection graph of the spiral polygon P. □

Recall that Nilsson and Wood's algorithm [10, 11] for finding a minimum set of point guards can be stated as follows. Let P be a spiral polygon with m concave vertices in standard form. First find the $m+1$ sticks $s_1 = \overline{a_1b_1}$, $s_2 = \overline{a_2b_2}, \ldots, s_{m+1} = \overline{a_{m+1}b_{m+1}}$ of P. Then find a maximal sequence of points $b_{i_1}, b_{i_2}, \ldots, b_{i_t}$ in the following manner: $b_{i_1} = b_1$; b_{i_j} is the first point in $[b_1, b_2, \ldots, b_{m+1}]$, and follows $b_{i_{j-1}}$ such that stick s_{i_j} does not intersect stick $s_{i_{j-1}}$. A minimum set of point guards for P is $\{b_{i_1}, b_{i_2}, \ldots, b_{i_t}\}$. We give an alternative proof for the correctness of the algorithm by means of an argument for the maximum independent set problem in chordal graphs.

A graph is *chordal* if every cycle of length greater than 3 possesses a chord, which is an edge joining two nonconsecutive vertices of the cycle. It was proved in [8] that G is chordal if and only if G has a *perfect elimination scheme*, which is an ordering $[v_1, v_2, \ldots, v_n]$ of vertices such that

$$i < j < k, \quad v_iv_j \in E \text{ and } v_iv_k \in E \quad \text{imply} \quad v_jv_k \in E.$$

Since a consecutive ordering is a perfect elimination scheme, a proper interval graph is chordal.

A *clique cover* of a graph $G = (V, E)$ is a partition of the vertex set $V = A_1 + A_2 + \cdots + A_t$ such that each A_i induces a clique of G. A *minimum clique cover* of G is a clique cover of minimum cardinality. Gavril [9] proposed an algorithm for finding a maximum independent set and a minimum clique cover of a chordal graph. By Theorem 7, $G_P(R)$ is a proper interval graph in which $[R_1, R_2, \ldots, R_{m+1}]$ is a consecutive ordering and also a perfect elimination scheme. Therefore Nilsson and Wood's algorithm is in fact a slight modification of Gavril's algorithm for finding a maximum independent set and a minimum clique cover of $G_P(R)$ in terms of the ordering $[R_1, R_2, \ldots, R_{m+1}]$: Inductively define a maximal sequence of regions $R_{i_1}, R_{i_2}, \ldots, R_{i_t}$ such that $R_{i_1} = R_1$ and R_{i_j} is the first region in the sequence $[R_1, R_2, \ldots, R_{m+1}]$ and follows $R_{i_{j-1}}$ but is not in $N^+[R_{i_{j-1}}]$, where $N^+[R_i] = \{R_j : j \geqslant i \text{ and } R_j \cap R_i \neq \emptyset\}$. Since the ordering $[R_1, R_2, \ldots, R_{m+1}]$ is consecutive, $R_{i_j} \notin N^+[R_{i_{j-1}}]$ implies that $R_{i_j} \notin N^+[R_{i_1}] \cup N^+[R_{i_2}] \cup \cdots \cup N^+[R_{i_{j-1}}]$. Hence, $N^+[R_{i_1}] \cup N^+[R_{i_2}] \cup \cdots \cup N^+[R_{i_t}] = \{R_1, R_2, \ldots, R_{m+1}\}$. By the arguments in [9], we have:

Lemma 9. *The set* $\{R_{i_1}, R_{i_2}, \ldots, R_{i_t}\}$ *is a maximum independent set of* $G_P(R)$ *and* $\{N^+[R_{i_1}], N^+[R_{i_2}], \ldots, N^+[R_{i_t}]\}$ *is a minimum clique cover of* $G_P(R)$.

Lemma 10. *P requires at least t point guards.*

Proof. For each i, denote p_i as the middle of the concave edge containing stick s_i. Since only points in R_{i_j} can cover p_{i_j} and $\{R_{i_1}, R_{i_2}, \ldots, R_{i_t}\}$ is an independent set, no single point of P can cover two distinct points in $\{p_{i_1}, p_{i_2}, \ldots, p_{i_t}\}$. This proves the lemma. □

Theorem 11. *$\{b_{i_1}, b_{i_2}, \ldots, b_{i_t}\}$ is a minimum set of point guards for P.*

Proof. Since P is spiral, b_i covers R_j for all $R_j \in N^+[R_i]$. By Lemma 9, $N^+[R_{i_1}] \cup N^+[R_{i_2}] \cup \cdots \cup N^+[R_{i_t}] = \{R_1, R_2, \ldots, R_{m+1}\}$. Therefore, $\{b_{i_1}, b_{i_2}, \ldots, b_{i_t}\}$ covers P. This, together with Lemma 10, proves the theorem. □

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