JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **93**, 574–592 (1983)

Some Results for Semilinear Differential Equations at Resonance*

SONG-SUN LIN

Department of Applied Mathematics, National Chiao Tung University, Hsin-Chu, Taiwan 300, Republic of China

Submitted by C. L. Dolph

1. INTRODUCTION

In this paper we investigate the existence of a solution of a so-called resonance problem, that is, for an equation of the type

$$Au = F(x, u) \qquad \text{in } \Omega, \tag{1.1}$$

where the linear (differential) operator A is self-adjoint with a nontrivial kernel on $L_2(\Omega)$ and the nonlinear map $F(x, \xi)$ satisfies some growth conditions for large values of $|\xi|$. The starting of the problem is the well-known paper of Landesman and Lazer [1]. They assume that $F(x, \xi) = f(x) + h(x)$ with $f(\xi) \to f(\pm\infty)$ as $\xi \to \pm\infty$ and $f(-\infty) < f(\xi) < f(+\infty)$, and are able to prove necessary and sufficient conditions for (1.1) to be solvable. Since then, many works have been done on the programs. We refer to the extensive bibliographies of the paper by Brézis and Nirenberg [2] and the survey paper by Fucik [3].

This paper is stimulated by the work of Amann and Mancini [4]. In [4], they present a very general existence theorem for the case where the nonlinearity "does not cross an eigenvalue," that is, F satisfies

$$\bar{\lambda} \leqslant \frac{F(x, \xi)}{\xi} \leqslant \hat{\lambda} - \delta \qquad \text{or} \qquad \bar{\lambda} + \delta \leqslant \frac{F(x, \xi)}{\xi} \leqslant \hat{\lambda} \tag{1.2}$$

for all large values of $|\xi|$, where $\delta > 0$ and $\bar{\lambda} < \hat{\lambda}$ are two consecutive eigenvalues of A. In this paper, we assume F satisfies

$$\bar{\lambda} \leqslant \frac{F(x, \xi)}{\xi} \leqslant \hat{\lambda}, \tag{1.3}$$

* This work was partially supported by the National Science Council of the Republic of China.

0022-247X/83 $3.00

the case could be regarded as the most general case, where nonlinearity does not cross eigenvalues. As in [4], we study (1.1) by the perturbation method, but different from [4], we consider the following perturbed equation of (1.1)

$$Au = F_\varepsilon(x, u) \qquad \text{in } \Omega, \tag{1.4}$$

where $F_\varepsilon(x, \xi) = \eta F(x, \xi) + \eta\varepsilon\xi$, $\eta = (\delta - \varepsilon)/(\delta + \varepsilon)$, $\varepsilon > 0$, and $\delta = \hat{\lambda} - \bar{\lambda}$.

In Section 2 we recall an abstract existence theorem of a nonresonance problem established by Amann and Mancini [4] by using the well-known existence theorem for coercive pseudomonotone mappings.

In Section 3 we obtain some a priori estimates of the projections of solutions u_ε of (1.4) on $\ker(A - \bar{\lambda})$, $\ker(A - \hat{\lambda})$, and $\{\ker(A - \bar{\lambda}) \oplus \ker(A - \hat{\lambda})\}^\perp$. These estimates enable us to prove that either $u_j/\|u_j\| \to \phi \in \ker(A - \bar{\lambda})$ or $u_j/\|u_j\| \to \psi \in \ker(A - \hat{\lambda})$ if $\|u_j\| \to \infty$ as $j \to \infty$. Then, the existence theorems of a nonresonance problem are immediate by standard proof.

In Section 4 we consider the resonance case. We decompose F into two parts which are easier to handle. The approach results from De Figueiredo [5] and is simplified by Amann and Mancini [4]. By the estimates obtained in Section 3, we are able to obtain an existence theorem which generalizes most of the known results, where the nonlinearity does not cross eigenvalues.

2. Notation and Preliminaries

In this section, we recall a perturbation lemma and an existence theorem of a nonresonance problem given by Amann and Mancini [4].

Throughout the paper we denote by H a real Hilbert space and by

$$A\colon\ D(A) \subset H \to H$$

a self-adjoint linear operator with dense domain $D(A)$ and closed range $R(A)$. Let $N(A)$ be the kernel of A. Then $R(A) = N(A)^\perp$, which implies that

$$A^{-1} := [A \,|\, D(A) \cap N(A)^\perp]^{-1} : N(A)^\perp \to N(A)^\perp$$

is a continuous linear operator. We always assume that A^{-1} is compact. From these hypotheses, the spectrum $\sigma(A)$ of A is a pure point spectrum. More precisely, every $\lambda \in \sigma(A) - \{0\}$ is an eigenvalue of finite multiplicity, and $\sigma(A) - \{0\}$ has no finite cluster point. Hence $\sigma(A)$ is countable and can be enumerated in the following way:

$$\cdots < \lambda_{-2} < \lambda_{-1} < \lambda_0 := 0 < \lambda_1 < \lambda_2 < \cdots .$$

Clearly, $\lambda_0 \in \sigma(A)$ iff A is not invertible. In this case, λ_0 is an eigenvalue of finite or infinite multiplicity. In the case $\lambda_i \in \sigma(A) - \{0\}$, we denote $N_i = \ker(A - \lambda_i I)$ as finite dimensional. We note that the number of positive or negative eigenvalues can be infinite, finite, or zero.

Recall that a nonlinear operator is called bounded if it maps bounded sets into bounded sets. A map M: $H \to H$ is called monotone if

$$(M(u) - M(v), u - v) \geqslant 0 \qquad \text{for all} \quad u, v \in H.$$

We shall state two existence theorems given in [4] to the nonlinear operator equation

$$Au = B(u), \tag{2.1}$$

where we assume that B: $H \to H$ is continuous and bounded.

We first recall a perturbation lemma which is essentially well known and a complete proof is given in [4].

LEMMA 1. *Suppose that either*

(i) *N is finite dimensional, or*

(ii) *B or $-B$ is monotone.*

Moreover, suppose that there exist a bounded map g: $H \to H$ and a null sequence (ε_j) in R such that

(α) *for every $j \in N$, there exists a $u_j \in D(A)$ such that $Au_j = B(u_j) + \varepsilon_j g(u_j)$, and*

(β)

$$\sup_{j \in N} \|u_j\| < \infty. \tag{2.2}$$

Then Eq. (2.1) *is solvable.*

By applying Lemma 1 and a well-known existence theorem for coercive pseudomonotone mappings (in the sense of Browder and Hess [6]), Amann and Mancini [4] prove the following existence theorem in the case that the nonlinearity "lies between two consecutive eigenvalues."

LEMMA 2. *Suppose that there exist two consecutive eigenvalues $\bar{\lambda} < \hat{\lambda}$ of A and positive constants γ_1, and $\gamma < (\hat{\lambda} - \bar{\lambda})/2$ such that*

$$\left\| B(u) - \frac{\hat{\lambda} + \bar{\lambda}}{2} u \right\| \leqslant \gamma \|u\| + \gamma_1$$

for all $u \in H$. *Moreover, let* $\alpha := \operatorname{sign}(\bar{\lambda} + \hat{\lambda})$ *and suppose that either*

(i) N *is finite dimensional, or*

(ii) αB *is monotone.*

Then Eq. (2.1) *is solvable.*

3. The Nonresonance Case

Throughout the remainder of the paper, we work on $H = L_2(\Omega)$, where Ω is a finite measure space with measure m. Moreover, we suppose that

$$B(u)(x) := F(x, u(x)) \qquad \text{for} \quad \text{a.a. } x \in \Omega,$$

where $F\colon \Omega \times R \to R$ is a Carathédory function, that is, $F(x, \xi)$ is continuous in $\xi \in R$ for a.a. $x \in \Omega$ and measurable in $x \in \Omega$ for every $\xi \in R$.

Amann and Mancini [4] give a sufficient condition (N) for F, which guarantees that B satisfies the hypotheses of Lemma 2 and so resonance is excluded.

(N) There exist two consecutive eigenvalues $\bar{\lambda} < \hat{\lambda}$ of A and numbers $\delta > 0$ and $\sigma, \rho \geqslant 0$ such that

$$\bar{\lambda} + \delta \leqslant \frac{F(x, \xi)}{\xi} \leqslant \hat{\lambda} - \delta \qquad \text{for} \qquad |\xi| \geqslant \rho \quad \text{and} \quad \text{a.a. } x \in \Omega,$$

and

$$|F(x, \xi)| \leqslant \sigma |\xi| + a(x) \qquad \text{for all} \quad \xi \in R \quad \text{and} \quad \text{a.a. } x \in \Omega,$$

where $a \in L_2(\Omega)$.

To the situation where resonance may occur, they impose the following hypotheses:

(H) $F(x, \xi) = \lambda_k \xi + f(x, \xi)$ for some $\lambda_k \in \sigma(A)$, where $\lambda_k \neq 0$ if $\dim N = \infty$, and $\inf \sigma(A) < \lambda_k < \sup \sigma(A)$, and

(H_+) (i) $\xi f(x, \xi) \geqslant -c(x) |\xi| - d(x)$,

(ii) $|f(x, \xi)| \leqslant (\lambda_{k+1} - \lambda_k - \delta) |\xi| + f_0(x)$ for a.a. $x \in \Omega$ and all $\xi \in R$,

where f_0, c, $d \in L_2(\Omega)$ are nonnegative functions, and $\delta > 0$.

In the case $\dim N < \infty$, the perturbed equation

$$Au = B(u) + \varepsilon u \tag{3.1}$$

has a solution u_ε if $0 < \varepsilon < \varepsilon_0 := \delta/2$. To ensure the existence of a priori bounds on u, and so $Au = B(u)$ is solvable, they impose a Landesman–Lazer type condition

(H$_+$) (iii) $\int_\Omega (\bar{f}_+ \phi^+ - \hat{f}_- \bar{\phi}) > 0$ for all $\phi \in N_k - \{0\}$,

where $\bar{f}_\pm(x) := \liminf_{\xi \to \pm\infty} f(x, \xi)$ and $\hat{f}_\pm(x) := \limsup_{\xi \to \pm\infty} f(x, \xi)$, and $\phi^+ := \max\{\phi(x), 0\}$ for a.a. $x \in \Omega$ and $\phi^- := \phi^+ - \phi$. The dual set of hypotheses (H$_+$) has the form

(H$_-$) (i) $\xi f(x, \xi) \leqslant c(x)\,|\xi| + d(x)$,

(ii) $|f(x, \xi)| \leqslant (\lambda_k - \lambda_{k-1} - \delta)\,|\xi| + f_0(x)$ for a.a. $x \in \Omega$ and $\xi \in R$, and

(iii) $\int_\Omega (\hat{f}_+ \phi^+ - \bar{f}_- \phi^-) < 0$ for all $\phi \in N_k - \{0\}$,

and the corresponding perturbed equation is

$$Au = B(u) - \varepsilon u \tag{3.1$'$}$$

with $0 < \varepsilon < \varepsilon_0 = \delta/2$.

In the case $\dim N = \infty$, to ensure αB is monotone, $\alpha := \operatorname{sign}(k)$, they impose some kind of monotone conditions on f, namely, either

(M) (i) $\alpha((f(x, \xi) - f(x, \eta))/(\xi - \eta) + \lambda_k) \geqslant 0$ for a.a. $x \in \Omega$ and all $\xi \neq \eta$, or

(ii) there exists a number $\sigma > 0$ such that

$$\alpha \left(\frac{f(x, \xi) - f(x, \eta)}{\xi - \eta} + \lambda_k \right) \geqslant \sigma \qquad \text{for a.a.} \quad x \in \Omega \quad \text{and all} \quad \xi \neq \eta.$$

Then (2.1) is solvable if any one of the following sets of conditions is satisfied:

(1) (M(i)), $\alpha = 1$ and (H$_+$);

(2) (M(i)), $\alpha = -1$ and (H$_-$); or

(3) (M(ii)) and either (H$_+$), or (H$_-$).

In this paper, we relax the restrictions (H$_+$(ii)) and (H$_-$(ii)) by assuming the following hypotheses:

(H-1) $\xi f(x, \xi) \geqslant -c(x)\,|\xi| - d(x)$,

(H-2) $|f(x, \xi)| \leqslant (\lambda_{k+1} - \lambda_k)\,|\xi| + f_0(x)$ for a.a. $x \in \Omega$ and all $\xi \in R$, where $f_0, c, d \in L_2(\Omega)$ are nonnegative.

Condition (H-2) allows that the nonlinearity "can touch but not cross two eigenvalues" and then causes some difficulties to find solutions u_ε of (3.1) or (3.1$'$) and to obtain a priori bounds on them. Instead of investigating the perturbed equation (3.1) or (3.1′) of (2.1), we consider the perturbed equations

$$Au = \lambda_k u + \eta f(u) + \eta\varepsilon u, \tag{3.2}$$

where $\eta = (\delta - \varepsilon)/(\delta + \varepsilon)$, $\varepsilon > 0$, $\delta = \lambda_{k+1} - \lambda_k$ and $f(u)(x) := f(x, u(x))$ for a.a. $x \in \Omega$.

We first prove the following existence theorem of solution for (3.2).

LEMMA 3. *Let hypotheses* (H), (H-1), and (H-2) *be satisfied. Then there exists* $\varepsilon_0 > 0$ *such that* (3.2) *has a solution* u_ε *for all* $0 < \varepsilon < \varepsilon_0$ *if* $\dim N < \infty$ *or* $\dim N = \infty$ *and* (M(i)) *holds.*

Proof. We note that (H-1) and (H-2) imply that

$$-f_1(x) \leqslant f(x, \xi) \leqslant \delta\xi + f_1(x) \qquad \text{for a.a.} \quad x \in \Omega \quad \text{and} \quad \xi \geqslant 0, \tag{3.3}$$

and

$$\delta\xi - f_1(x) \leqslant f(x, \xi) \leqslant f_1(x) \qquad \text{for a.a.} \quad x \in \Omega \quad \text{and} \quad \xi \leqslant 0, \tag{3.4}$$

where $f_1 \in L_2(\Omega)$ is nonnegative (for example, $f_1 = \delta + f_0 + c + d$). Conversely, (3.3) and (3.4) imply (H-1) and (H-2) with $c = f_1$, $d = 0$, and $f_0 = f_1$.

Let $F_\varepsilon(x, \xi) = \lambda_k \xi + \eta f(x, \xi) + \eta\varepsilon\xi$. Then (3.3) and (3.4) imply

$$\left| F_\varepsilon(x, \xi) - \left(\frac{\lambda_{k+1} + \lambda_k}{2}\right)\xi \right| \leqslant \left(\frac{\lambda_{k+1} - \lambda_k}{2} - \eta\varepsilon\right)|\xi| + f_1(x).$$

If $\dim N = \infty$,

$$\begin{aligned}
&\alpha(F_\varepsilon(x, \xi) - F_\varepsilon(x, \zeta))(\xi - \zeta) \\
&\quad = (\xi - \zeta)^2\, \alpha \left\{\lambda_k + \eta\frac{f(x, \xi) - f(x, \zeta)}{\xi - \zeta} + \eta\varepsilon\right\} \\
&\quad \geqslant (\xi - \zeta)^2\, \alpha\{(1 - \eta)\,\lambda_k + \eta\varepsilon\} \qquad \text{by (M(i))}.
\end{aligned}$$

If $\alpha = 1$, then $\lambda_k > 0$, $\alpha\{(1 - \eta)\,\lambda_k + \eta\varepsilon\} \geqslant 0$. If $\alpha = -1$, then $\lambda_k < \lambda_{k+1} \leqslant 0$,

$$\alpha\{(1 - \eta)\,\lambda_k + \eta\varepsilon\} = \frac{\eta\varepsilon}{\delta - \varepsilon}\{-(\lambda_{k+1} + \lambda_k) + \varepsilon\} \geqslant 0.$$

Hence $\alpha B_\varepsilon(u)$, $B_\varepsilon(u)(x) := F_\varepsilon(x, u(x))$, is monotone. Therefore, Lemma 3 follows from Lemma 2.

In the remainder of the paper, we always assume that hypotheses (H), (H-1), and (H-2) are satisfied. If $\dim N = \infty$, we also assume $\lambda_{k+1} \neq 0$ and (M(i)) holds.

Before we can derive a priori bounds on solutions u_ε of (3.2), we need the following estimates:

LEMMA 4. $\sup_{0<\varepsilon\leqslant\varepsilon_0} \varepsilon \|u_\varepsilon\| < \infty$, *where* u_ε *is a solution of* (3.2).

Proof. By decomposing u into $u = v + z$ with $v \in N_k$ and $z \in N_k^\perp \cap D(A)$, (3.2) takes the form

$$Lz = \eta f(u) + \eta\varepsilon u, \tag{3.5}$$

where $L := A - \lambda_k I$.

Recall that for every $u \in D(A)$,

$$\|Lu\|^2 \geqslant \delta(Lu, u); \tag{3.6}$$

for the proof see [4].

Then, following a device of Brézis and Nirenberg [2], (H-1) implies

$$\begin{aligned} \xi f(\cdot, \xi) &= |\xi f(\cdot, \xi) + c|\xi| + d| - c|\xi| - d \\ &\geqslant |\xi|\,|f(\cdot, \xi)| - 2c|\xi| - 2d. \end{aligned}$$

Moreover, (H-2) implies

$$\xi f(\cdot, \xi) \geqslant \delta^{-1}|f(\cdot, \xi)|^2 - \delta^{-1} f_0 |f(\cdot, \xi)| - 2c|\xi| - 2d.$$

Therefore

$$\|f(u)\| \leqslant \delta\|u\| + \gamma, \tag{3.7}$$

and

$$(f(u), u) \geqslant \delta^{-1}\|f(u)\|^2 - \gamma(\|u\| + 1), \tag{3.8}$$

for all $u \in H$, where γ is a generic constant, independent of ε but not necessarily the same in different formulas. Hence

$$\begin{aligned} \|Lz\|^2 &\geqslant \delta(Lz, z) = \delta(Lz, u) \\ &= \delta\eta(f(u), u) + \delta\eta\varepsilon\|u\|^2 \\ &\geqslant \eta\|f(u)\|^2 + \delta\eta\varepsilon\|u\|^2 - \gamma(\|u\| + 1). \end{aligned}$$

On the other hand,

$$\|Lz\|^2 \leqslant \eta^2\|f(u)\|^2 + 2\eta^2\varepsilon\|u\|\,\|f(u)\| + \eta^2\varepsilon^2\|u\|^2. \tag{3.9}$$

Therefore

$$\gamma(\|u\| + 1) \geqslant \eta(1 - \eta)\|f(u)\|^2 - 2\eta^2\varepsilon\|u\|\,\|f(u)\| + \eta\varepsilon(\delta - \eta\varepsilon)\|u\|^2.$$

By dividing the last inequality by $\varepsilon \|u\|^2$, we have

$$\gamma/\varepsilon \|u\| + \gamma/\varepsilon \|u\|^2 \\ \geqslant (2/(\delta+\varepsilon)) \|f(u)\|^2/\|u\|^2 - 2\eta \|f(u)\|/\|u\| + (\delta - \eta\varepsilon).$$

Suppose that there exists a sequence (u_j) in $D(A)$ such that

$$\varepsilon_j \|u_j\| \to \infty \quad \text{and} \quad \varepsilon_j \to \varepsilon' \geqslant 0 \quad \text{as} \quad j \to \infty$$

and

$$Lu_j = \eta_j f(u_j) + \eta_j \varepsilon_j f(u_j).$$

Let $\alpha = \liminf_{j\to\infty} \|f(u_j)\|/\|u_j\|$. If $\alpha = \infty$, a contradiction is obvious. If $\alpha < \infty$, then

$$0 \geqslant \frac{1}{\delta + \varepsilon'} \{(\alpha - \delta)^2 + (\alpha + \varepsilon')^2\} > 0,$$

a contradiction. Lemma 4 is proved.

We further decompose u into

$$u = v + w + y,$$

where $v \in N_k$, $w \in N_{k+1}$, and $y \in (N_R \oplus N_{k+1})^\perp \cap D(A)$.

We note that for all $y \in (N_k \oplus N_{k+1})^\perp \cap D(A)$,

$$\|Ly\|^2 \geqslant \beta(Ly, y), \tag{3.10}$$

where $\beta := \lambda_{k+2} - \lambda_k$. In fact, let P_j be the orthogonal projection of H onto eigenspace N_j, $N_j = \ker(A - \lambda_j I)$. Then for all $y \in (N_k \oplus N_{k+1})^\perp \cap D(A)$

$$\begin{aligned} \|Ly\|^2 &= \sum_{j \neq k, k+1} (\lambda_j - \lambda_k)^2 \|P_j y\|^2 \\ &\geqslant \sum_{j \geqslant k+2} (\lambda_j - \lambda_k)^2 \|P_j y\|^2 \\ &\geqslant (\lambda_{k+2} - \lambda_k) \sum_{j \geqslant k+2} (\lambda_j - \lambda_k) \|P_j y\|^2 \\ &\geqslant (\lambda_{k+2} - \lambda_k) \left\{ \sum_{j \geqslant k+2} (\lambda_j - \lambda_k) \|P_j y\|^2 + \sum_{j<k} (\lambda_j - \lambda_k) \|P_j y\|^2 \right\} \\ &= (\lambda_{k+2} - \lambda_k) \sum_{j \neq k, k+1} (\lambda_j - \lambda_k) \|P_j y\|^2 \\ &= (\lambda_{k+2} - \lambda_k)(Ly, y). \end{aligned}$$

Since for all $w \in N_{k+1}$,

$$Lw = (A - \lambda_{k+1})\, w + (\lambda_{k+1} - \lambda_k)\, w = \delta w,$$

(3.5) takes the form

$$Ly + \delta w = \eta f(u) + \eta \varepsilon u. \tag{3.11}$$

The following estimates play a crucial role in deriving a priori bounds on solutions u_ε of (3.2).

LEMMA 5. *If u is a solution of* (3.11), *then*

$$\|Ly\| \leqslant \gamma(\|u\|^{1/2} + 1), \tag{3.12}$$

and

$$\|f(u) - \delta w\| \leqslant \gamma(\|u\|^{1/2} + 1), \tag{3.13}$$

where γ is a generic constant, independent of ε.

Proof. By (3.10) and (3.11), we have

$$\|Ly\|^2 \geqslant \beta(Ly, y) = \beta\eta(f(u), u) + \beta\eta\varepsilon \|u\|^2 - \beta\delta \|w\|^2.$$

Furthermore, by (3.7), (3.8), and Lemma 4, we obtain

$$\|Ly\|^2 \geqslant \beta\delta^{-1} \|f(u)\|^2 - \beta\delta \|w\|^2 - \gamma(\|u\| + 1).$$

On the other hand, by (3.7), (3.11), and Lemma 4,

$$\begin{aligned}\|Ly\|^2 &\leqslant \eta^2 \|f(u)\|^2 + 2\eta^2\varepsilon \|u\| \, \|f(u)\| + \eta^2\varepsilon^2 \|u\|^2 - \delta^2 \|w\|^2 \\ &\leqslant \|f(u)\|^2 - \delta^2 \|w\|^2 + \gamma(\|u\| + 1).\end{aligned}$$

Therefore

$$\|f(u)\|^2 - \delta^2 \|w\|^2 \leqslant \gamma(\|u\| + 1).$$

Hence

$$\|Ly\|^2 \leqslant \gamma(\|u\| + 1).$$

Finally, (3.13) follows from the last estimate and Lemma 4.

Now suppose that (2.2) is false. Then there exist a null sequence (ε_j) in $(0, \varepsilon_0]$ and a sequence (u_j) in $D(A)$ such that

$$\|u_j\| \to \infty \qquad \text{as} \quad j \to \infty,$$

and

$$Ly_j + \delta w_j = \eta_j f(u_j) + \eta_j \varepsilon_j u_j, \tag{3.14}$$

where $u_j = v_j + w_j + y_j$ with $v_j \in N_k$, $w_j \in N_{k+1}$, and $y_j \in (N_k \oplus N_{k+1})^\perp \cap D(A)$.

We note that there exists a constant $c > 0$ such that

$$\|Ly\| \geqslant c\,\|y\| \tag{3.15}$$

for all $y \in (N_k \oplus N_{k+1})^\perp \cap D(A)$. Since $\dim N_k < \infty$ and $\dim N_{k+1} < \infty$, we can assume (by passing to an appropriate subsequence if necessary) that

$$\begin{aligned} u_j/\|u_j\| &= v_j/\|u_j\| + w_j/\|u_j\| + y_j/\|u_j\| \\ &\to \phi + \psi \in N_k \oplus N_{k+1} - \{0\}, \end{aligned}$$

and that

$$u_j/\|u_j\| \to \phi + \psi \qquad \text{almost everwhere in } \Omega. \tag{3.16}$$

Moreover, we can prove that the limit does not mix in $N_k \oplus N_{k+1}$, that is, either $\phi = 0$ or $\psi = 0$, if the following condition (E) is satisfied:

(E) For all $\phi \in N_k - \{0\}$ and $\psi \in N_{k+1} - \{0\}$, $m\{x \in \Omega \mid \phi(x)\,\psi(x) \neq 0\} > 0$.

LEMMA 6. *Let condition* (E) *be satisfied. If* $\|u_j\| \to \infty$ *as* $j \to \infty$, *then either*

$$u_j/\|u_j\| \to \phi \in N_k - \{0\},$$

or

$$u_j/\|u_j\| \to \psi \in N_{k+1} - \{0\}.$$

Proof. Let $v_j/\|u_j\| = \phi_j$ and $w_j/\|u_j\| = \psi_j$. Then, (3.13) implies

$$\lim_{j \to \infty} \int_\Omega |f(u_j)/\|u_j\| - \delta\psi_j|^2 = 0. \tag{3.17}$$

Suppose that $\psi \neq 0$, we shall prove $\phi = 0$.

We first prove that

$$[\psi > 0] = [\psi > 0,\ \psi + \phi > 0] \qquad \text{and} \qquad [\psi < 0] = [\psi < 0,\ \psi + \phi < 0],$$

where $[\psi > 0] = \{x \in \Omega \mid \psi(x) > 0\}$, $[\psi > 0,\ \psi + \phi > 0] = \{x \in \Omega \mid \psi(x) > 0$, and $\psi(x) + \phi(x) > 0\}$. Since

$$[\psi > 0] = [\psi > 0,\ \psi + \phi > 0] \cup [\psi > 0,\ \psi + \phi \leqslant 0].$$

Suppose that $m[\psi > 0,\ \psi + \phi \leqslant 0] > 0$. By using Egoroff's theorem, there exist a subset Ω' of $[\psi > 0,\ \psi + \phi \leqslant 0]$ with $m(\Omega') > 0$, and numbers $N > 0$ and $\gamma > 0$ such that for all $j \geqslant N$ and a.a. $x \in \Omega'$,

$$u_j(x)/\|u_j\| \leqslant \gamma/2 \qquad \text{and} \qquad \psi_j(x) \geqslant \gamma.$$

If $u_j(x) \geqslant 0$, by (3.3),

$$\begin{aligned}\delta\psi_j(x) - f(x, u_j(x))/\|u_j\| &\geqslant \delta\gamma - \delta u_j(x)/\|u_j\| - f_1(x)/\|u_j\| \\ &\geqslant \delta\gamma/2 - f_1(x)/\|u_j\|\end{aligned}$$

and if $u_j(x) \leqslant 0$, by (3.4),

$$\delta\psi_j(x) - f(x, u_j(x))/\|u_j\| \geqslant \delta\gamma - f_1(x)/\|u_j\|$$

for all $j \geqslant N$ and a.a. $x \in \Omega'$. Hence

$$\lim_{j \to \infty} \int_{[\psi > 0,\, \psi + \phi \leqslant 0]} |\delta\psi_j - f(u_j)/\|u_j\||^2 \geqslant m(\Omega')(\delta\gamma/2)^2 > 0,$$

which contradicts (3.17). Similarly, we can prove $[\psi < 0] = [\psi < 0,\ \psi + \phi < 0]$ by (3.3) and (3.4).

We next prove that condition (E) implies that $\phi = 0$ if $m[\psi > 0,\ \phi < 0] = 0$ and $m[\psi < 0,\ \phi > 0] = 0$. In fact,

$$\begin{aligned}\int_\Omega \psi\phi &= \int_{[\psi > 0]} \psi\phi + \int_{[\psi < 0]} \psi\phi \\ &= \int_{[\psi > 0,\, \phi > 0]} \psi\phi + \int_{[\psi < 0,\, \phi < 0]} \psi\phi,\end{aligned}$$

if $m[\psi > 0,\ \phi < 0] = 0$ and $m[\psi > 0,\ \phi < 0] = 0$. Hence

$$\int_\Omega \psi\phi > 0$$

if (E) holds and $\phi \neq 0$, which contradicts $\int_\Omega \psi\phi = 0$.

Now suppose that $\phi \neq 0$. Then $m[\psi > 0,\ \phi < 0] > 0$ or $m[\psi < 0,\ \phi > 0] > 0$. Assume $m[\psi > 0,\ \phi < 0] > 0$. Since

$$[\psi > 0,\ \phi < 0] \subset [\psi > 0] = [\psi > 0,\ \psi + \phi > 0],$$

by (3.16) and by using Egoroff's theorem, there exist a subset Ω' of $[\psi > 0,$

$\phi < 0]$ with $m(\Omega') > 0$, and numbers $N > 0$ and $\gamma > 0$ such that for all $j \geqslant N$ and a.a. $x \in \Omega'$,

$$u_j(x) \geqslant 0 \qquad \text{and} \qquad \phi_j(x) \leqslant -\gamma.$$

By (3.3),

$$\begin{aligned} &f(x, u_j(x))/\|u_j\| - \delta\psi_j(x) \\ &\qquad \leqslant \delta\phi_j(x) + f_1(x)/\|u_j\| \leqslant -\delta\gamma + f_1(x)/\|u_j\|, \end{aligned}$$

for all $j \geqslant N$ and a.a. $x \in \Omega'$. Hence

$$\lim_{j\to\infty} \int_{[\psi>0]} |f(u_j)/\|u_j\| - \delta\psi_j|^2 \geqslant m(\Omega')(\delta\gamma)^2 > 0,$$

which contradicts (3.17).

Similarly, if $m[\psi < 0,\ \phi > 0] > 0$, by (3.4), it leads to a contradiction to (3.17).

Hence, if $\psi \neq 0$, then $\phi = 0$. Lemma 6 is proved.

In application, condition (E) should not cause severe restriction. In the remainder of the paper, we always assume that condition (E) holds.

Let

$$l_\pm(x) := \liminf_{\xi\to\pm\infty} \frac{f(x, \xi)}{\xi} \qquad \text{and} \qquad h_\pm(x) := \limsup_{\xi\to\pm\infty} \frac{f(x, \xi)}{\xi}.$$

We can now prove an existence theorem where resonance is excluded, by only considering the limiting functions $l_\pm$ and $k_\pm$.

THEOREM 1. *Let hypotheses* (H), (H-1), (H-2), *and* (E) *be satisfied. If*

$$\int_\Omega (l_+\phi^+)^2 + (l_-\phi^-)^2 > 0 \quad \textit{and} \quad \int_\Omega [(\delta - k_+)\,\psi^+]^2 + [(\delta - k_-)\,\psi^-]^2 > 0,$$

for all $\phi \in N_k - \{0\}$ *and* $\psi \in N_{k+1} - \{0\}$, *then* (2.1) *is solvable.*

Proof. Suppose that $\|u_j\| \to \infty$ as $j \to \infty$. Then it leads to a contradiction as follows: By Lemma 6, either $u_j/\|u_j\| \to \phi \in N_k - \{0\}$ or $u_j/\|u_j\| \to \psi \in N_{k+1} - \{0\}$.

If $u_j/\|u_j\| \to \phi \in N_k - \{0\}$, by (3.13),

$$\lim_{j\to\infty} \|f(u_j)\|/\|u_j\| = 0.$$

On the other hand, by applying Fatou's lemma,

$$\lim_{j\to\infty} \|f(u_j)\|^2/\|u_j\|^2 \geqslant \int_\Omega \liminf_{j\to\infty} |f(x, u_j(x))|^2/\|u_j\|^2$$

$$= \int_{[\phi>0]} \liminf_{j\to\infty} f(x, u_j(x))^2/\|u_j\|^2$$

$$+ \int_{[\phi<0]} \liminf_{j\to\infty} f(x, u_j(x))^2/\|u_j\|^2$$

$$\geqslant \int_{[\phi>0]} (l_+ \phi^+)^2 + \int_{[\phi<0]} (l_- \phi^-)^2 > 0,$$

a contradiction.

If $u_j/\|u_j\| \to \psi \in N_{k+1} - \{0\}$, by (3.13),

$$\lim_{j\to\infty} \|f(u_j)/\|u_j\| - \delta\psi_j\| = 0.$$

Again, by applying Fatou's lemma,

$$\lim_{j\to\infty} \|f(u_j)/\|u_j\| - \delta\psi_j\|^2$$

$$\geqslant \int_{[\psi\geqslant 0]} \liminf_{j\to\infty} |f(x, u_j(x))/\|u_j\| - \delta\psi_j(x)|^2$$

$$+ \int_{[\psi<0]} \liminf_{j\to\infty} |f(x, u_j(x))/\|u_j\| - \delta\psi_j(x)|^2$$

$$\geqslant \int_{[\psi>0]} \liminf_{j\to\infty} (\delta - f(x, u_j(x))/u_j(x))^2\, \psi^2(x)$$

$$+ \int_{[\psi<0]} \liminf_{j\to\infty} (\delta - f(x, u_j(x))/u_j(x))^2\, \psi^2(x)$$

$$\geqslant \int_\Omega [(\delta - k_+)\,\psi^+]^2 + [(\delta - k_-)\,\psi^-]^2 > 0,$$

a contradiction.

Hence $\sup_{0<\varepsilon\leqslant\varepsilon_0} \|u_\varepsilon\| < \infty$, the theorem follows from Lemma 1.

COROLLARY. *Suppose that constants do not being to $N_k \oplus N_{k+1} - \{0\}$. Then* (2.1) *is solvable if*

(i) $m[l_+ = 0] = 0$ *or* $m[l_- = 0] = 0$, *and*

(ii) $m[k_+ = \delta] = 0$ *or* $m[k_- = \delta] = 0$.

Theorem 1 generalizes the result of Amann and Mancini [4] in the case where F satisfies condition (N).

Let $\bar{f}_{\pm}(x) := \liminf_{\xi \to \pm\infty} f(x, \xi)$ and $\hat{f}_{\pm}(x) := \limsup_{\xi \to \pm\infty} f(x, \xi)$. Moreover, let

$$h(x, \xi) := \delta\xi - f(x, \xi),$$

and let $\bar{h}_{\pm}$ and $\hat{h}_{\pm}$ be defined as above.

To prove the existence theorem for (2.1) by considering the limiting functions $\bar{f}_{\pm}$, we need the following estimates which complement Lemma 5.

LEMMA 7. *Suppose that* $\|u_j\| \to \infty$ *as* $j \to \infty$. *Then*

(i) *if* $u_j/\|u_j\| \to \phi \in N_k - \{0\}$, *then* $\|w_j\| \leqslant \gamma \|u_j\|^{1/2}$, *and*

(ii) *if* $u_j/\|u_j\| \to \psi \in N_{k+1} - \{0\}$, *then* $\|v_j\| \leqslant \gamma \|u_j\|^{1/2}$,

where $u_j = v_j + w_j + y_j$ *with* $v_j \in N_k$, $w_j \in N_{k+1}$ *and* $y_j \in (N_k \oplus N_{k+1})^{\perp} \cap D(A)$.

Proof. (i) If $u_j/\|u_j\| \to \phi \in N_k - \{0\}$, and suppose that

$$\sup_j \|w_j\|/\|u_j\|^{1/2} = \infty.$$

By (3.12), we can assume (by passing to an appropriate subsequence if necessary) that

$$w_j/\|w_j\| \to \tilde{\psi} \in N_{k+1} - \{0\} \qquad \text{and} \qquad Ly_j/\|w_j\| \to 0,$$

and that the convergences are almost everywhere in Ω.

Divided (3.14) by $\|w_j\|$, by Lemma 4, for a.a. $x \in \Omega$,

$$\lim_{j\to\infty} f(x, u_j(x))/\|w_j\| = \lim_{j\to\infty} \delta w_j(x)/\|w_j\| = \delta\tilde{\psi}(x).$$

On the other hand, by (3.3) and (3.4), for a.a. $x \in \Omega$,

$$\liminf_{j\to\infty} f(x, u_j(x))\,\phi(x)/\|w_j\| \geqslant 0.$$

Hence

$$\phi(x)\,\tilde{\psi}(x) \geqslant 0 \qquad \text{for a.a.} \quad x \in \Omega,$$

a contradiction $\int_\Omega \phi\tilde{\psi} = 0$ if (E) holds.

Condition (ii) can be proved by a similar argument.

THEOREM 2. *Let hypotheses* (H), (H-1), (H-2), *and* (E) *be satisfied. If* $\int_\Omega (\bar{f}_+ \phi^+ - \hat{f}_- \phi^-) = \infty$ *and* $\int_\Omega (\bar{h}_+ \psi^+ - \hat{h}_- \psi^-) = \infty$ *for all* $\phi \in N_k - \{0\}$ *and* $\psi \in N_{k+1} - \{0\}$, *then* (2.1) *is solvable.*

Proof. Suppose that $\|u_j\| \to \infty$ as $j \to \infty$. We may assume that for a.a. $x \in \Omega$ and $j \in N$

$$|u_j(x)/\|u_j\|| \leqslant f_2(x)$$

with an appropriate $f_2 \in L_2(\Omega)$. If $u_j/\|u_j\| \to \phi \in N_k - \{0\}$, then

$$\sup_j \, (f(u_j), u_j/\|u_j\|) < \infty$$

by Lemmas 4, 5, and 7. On the other hand, for a.a. $x \in \Omega$

$$\begin{aligned} u_j(x) f(x, u_j(x))/\|u_j\| &\geqslant -c(x)\,|u_j(x)|/\|u_j\| - d(x)/\|u_j\| \\ &\geqslant -c(x) f_2(x) - d(x)/\|u_j\|. \end{aligned}$$

By applying Fatou's lemma,

$$\begin{aligned} \liminf_{j \to \infty} (f(u_j), u_j/\|u_j\|) &\geqslant \int_\Omega \liminf_{j \to \infty} u_j(x) f(x, u_j(x))/\|u_j\| \\ &\geqslant \int_\Omega (\bar{f}_+ \phi^+ - \hat{f}_- \phi^-) = \infty, \end{aligned}$$

a contradiction.

If $u_j/\|u_j\| \to \psi \in N_{k+1} - \{0\}$, then

$$\sup_j \, (\delta u_j - f(u_j), u_j/\|u_j\|) < \infty.$$

A similar argument leads to a contradiction

$$\int_\Omega (\bar{h}_+ \psi^+ - \hat{h}_- \psi^-) = \infty.$$

The theorem is proved.

We remark that Theorem 2 generalizes Theorem 1.

4. The Resonance Case

We are now in a position to study the resonance problem in the case $\int_\Omega (\bar{f}_+ \phi^+ - \hat{f}_- \phi^-) < \infty$ or $\int_\Omega (\bar{h}_+ \psi^+ - \hat{h}_- \psi^-) < \infty$. We shall decompose the function f into two parts, which are easier to handle. The approach relies on a device of De Figueiredo [5] and simplified by Amann and Mancini [4].

We first recall some notation and statements in [4, Appendix, (iii)].

We note that Eq. (3.2) can be written as

$$Lu = \eta f(u) + \eta\varepsilon u, \tag{4.1}$$

or

$$Mu = \eta(f(u) - \delta u) - \varepsilon u, \tag{4.2}$$

where $L = A - \lambda_k I$ and $M = A - \lambda_{k+1} I$, and

$$\|Lu\|^2 \geqslant \delta(Lu, u) \qquad \text{and} \qquad \|Mu\|^2 \geqslant -\delta(Mu, u) \tag{4.3}$$

for all $u \in D(A)$, (see [4, (A.1)]).

If u_j are solutions of (4.1) and $\|u_j\| \to \infty$ as $j \to \infty$. Then, by Lemma 6, either $u_j/\|u_j\| \to \phi \in N_k - \{0\}$ or $u_j/\|u_j\| \to \psi \in N_{k+1} - \{0\}$. Equation (4.1) is used in the former case and (4.2) in the latter. Let $G(x, \xi) = f(x, \xi)$ and $\alpha' = 1$ if (4.1) is used, and $G(x, \xi) = f(x, \xi) - \delta\xi$ and $\alpha' = -1$ if (4.2) is used. We also remark that (3.3) and (3.4) imply that

(H-1′) $\xi(f(x, \xi) - \delta\xi) \leqslant f_1(x)\,|\xi|$,
(H-2′) $|f(x, \xi) - \delta\xi| \leqslant \delta\,|\xi| + f_1(x)$.

For every fixed $r > 0$, we define functions

$$\hat{g}_r \colon \Omega \times \{\xi \mid |\xi| \geqslant 1\} \to R$$

by

$$\begin{aligned} \hat{g}_r(\cdot, \xi) &= G(\cdot, \xi), && \text{if} \quad \xi \geqslant 1 \quad \text{and} \quad \alpha' G(\cdot, \xi) \leqslant r, \\ &= \alpha' r, && \text{if} \quad \xi \geqslant 1 \quad \text{and} \quad \alpha' G(\cdot, \xi) \geqslant r, \\ &= G(\cdot, \xi), && \text{if} \quad \xi \leqslant -1 \quad \text{and} \quad \alpha' G(\cdot, \xi) \geqslant -r, \\ &= -\alpha' r, && \text{if} \quad \xi \leqslant -1 \quad \text{and} \quad \alpha' G(\cdot, \xi) \leqslant -r, \end{aligned}$$

and $G_r \colon \Omega \times R \to R$ by

$$\begin{aligned} G_r(\cdot, \xi) &= G(\cdot, \xi) - \hat{g}_r(\cdot, \xi), && \text{for} \quad |\xi| \geqslant 1, \\ &= \xi[G(\cdot, 1) - \hat{g}_r(\cdot, 1)], && \text{for} \quad 0 \leqslant \xi \leqslant 1, \\ &= \xi[G(\cdot, -1) - \hat{g}_r(\cdot, -1)], && \text{for} \quad -1 \leqslant \xi \leqslant 0, \end{aligned}$$

and let $g_r = G - G_r$.

Then G_r and g_r are Carathédory functions, and

$$\alpha' \xi G_r(\cdot, \xi) \geqslant 0 \qquad \text{for all} \quad \xi \in R. \tag{4.4}$$

Moreover,

$$\sup_{u\in H} \| g_r(\cdot, u(\cdot))\| \leqslant \gamma_r < \infty, \tag{4.5}$$

where constant γ_r depends only on r, (see [4, (A.17)]).
We shall prove the following estimate on $(\alpha' g_r(u), u)$, which is bounded above by assuming [4, $(\mathrm{H}_+(\mathrm{ii}))$ or $(\mathrm{H}_-(\mathrm{ii}))$].

LEMMA 8. *For every fixed* $r > 0$,

$$(\alpha' g_r(u), u) \leqslant C_r(\| G(u)\| + 1)$$

for all solution u *of* (4.1) (*and* (4.2)), *where* C_r *depends only on* r.

Proof. Since

$$(\alpha' g_r(u), u) = (\alpha' G(u), u) - (\alpha' G_r(u), u),$$

we shall give an upper bound on $(\alpha' G(u), u)$ and a lower bound on $(\alpha' G_r(u), u)$.

It is easy to verify that

$$|G_r(x, \xi)| \leqslant \delta |\xi| + 2f_1(x)$$

for all $\xi \in R$ and a.a. $x \in \Omega$. By (4.4),

$$\begin{aligned}(\alpha' G_r(u), u)) &= \int_\Omega |G_r(u)|\,|u| \\ &\geqslant (1/\delta)\, \| G_r(u)\|^2 - \gamma\, \| G_r(u)\|\end{aligned}$$

for all $u \in H$. Since

$$\begin{aligned}\| G_r(u)\|^2 &= \| G(u) - g_r(u)\|^2 \\ &\geqslant \| G(u)\|^2 - 2\, \| G(u)\|\ \| g_r(u)\| + \| g_r(u)\|^2 \\ &\geqslant \| G(u)\|^2 - 2\gamma_r\, \| G(u)\|\end{aligned}$$

and

$$\begin{aligned}\| G_r(u)\| &\leqslant \| G(u)\| + \| g_r(u)\| \\ &\leqslant \| G(u)\| + \gamma_r\end{aligned}$$

by (4.5). Hence

$$(\alpha' G_r(u), u) \geqslant (1/\delta)\, \| G(u)\|^2 - C_r'(\| G(u)\| + 1),$$

where constant C_r' depends only on γ_r and so on r.

We next give an upper bound on $(\alpha' G(u), u)$. If $\alpha' = 1$, by (4.1),

$$\begin{aligned}(\alpha' G(u), u) &= (f(u), u)\\ &= (Lu, u) - \eta\varepsilon \|u\|^2 + (1-\eta)(f(u), u)\\ &\leqslant (1/\delta)\|Lu\|^2 + (1-\eta)\|u\|\,\|f(u)\|\\ &\leqslant (1/\delta)\|f(u)\|^2 + \gamma(\|f(u)\| + 1)\end{aligned}$$

by (4.3), (3.9), and Lemma 4. Similarly, if $\alpha' = -1$, by (4.2),

$$\begin{aligned}(\alpha' G(u), u) &= -(f(u) - \delta u, u)\\ &= -(Mu, u) - \varepsilon\|u\|^2 - (1-\eta)(f(u) - \delta u, u)\\ &\leqslant (1/\delta)\|Mu\|^2 + (1-\eta)\|u\|\,\|f(u) - \delta u\|\\ &\leqslant (1/\delta)\|f(u) - \delta u\|^2 + \gamma(\|f(u) - \delta u\| + 1).\end{aligned}$$

Hence, in both cases, we have

$$(\alpha' G(u), u) \leqslant (1/\delta)\|G(u)\|^2 + \gamma(\|G(u)\| + 1).$$

Therefore,

$$(\alpha' g_r(u), u) \leqslant C_r(\|G(u)\| + 1),$$

where constant C_r depends only on r.

THEOREM 3. *Let hypotheses* (H), (H-1), (H-2), (E), *and*

(H-3) $\int_\Omega (\bar{f}_+ \phi^+ - \hat{f}_- \phi^-) > 0$ *and* $\int_\Omega (\bar{h}_+ \psi^+ - \hat{h}_- \psi^-) > 0$

for all $\phi \in N_k - \{0\}$ *and* $\psi \in N_{k+1} - \{0\}$, *be satisfied. Then* (2.1) *is solvable.*

Proof. Suppose that $\|u_j\| \to \infty$ as $j \to \infty$. It is easy to see that

$$\|G(u_j)\| \leqslant \gamma(\|u_j\|^{1/2} + 1).$$

In fact, if $u_j/\|u_j\| \to \phi \in N_k - \{0\}$,

$$\|f(u_j)\| \leqslant \gamma(\|u_j\|^{1/2} + 1),$$

and, if $u_j/\|u_j\| \to \psi \in N_{k+1} - \{0\}$,

$$\begin{aligned}\|f(u_j) - \delta u_j\| &\leqslant \|f(u_j) - \delta w_j\| + \delta\|v_j\| + \delta\|y_j\|\\ &\leqslant \gamma(\|u_j\|^{1/2} + 1)\end{aligned}$$

by Lemmas 5 and 7. By Lemma 8,

$$\liminf_{j \to \infty} (\alpha' g_r(u_j), u_j)/\|u_j\| \leqslant 0.$$

Moreover, by the construction of g_r and by assuming $|u_j(x)|/\|u_j\| \leqslant f_2(x)$ for a.a. $x \in \Omega$, all $j \in N$, and an appropriate $f_2 \in L_2(\Omega)$, $\alpha' u_j g_r(u_j)/\|u_j\|$ is bounded below by an integrable function, see [4] for details. Therefore, by applying Fatou's lemma,

$$\int_\Omega \liminf_{j \to \infty} \alpha' g_r(u_j) u_j/\|u_j\| \leqslant 0.$$

By letting $r \to \infty$ and using B. Levi's theorem, we have

$$\int_\Omega (\bar{f}_+ \phi^+ - \hat{f}_- \phi^-) \leqslant 0 \qquad \text{if} \quad \alpha' = 1,$$

and

$$\int_\Omega (\bar{h}_+ \psi^+ - \hat{h}_- \psi^-) \leqslant 0 \qquad \text{if} \quad \alpha' = -1,$$

which contradict (H-3). The theorem is proved.

We remark that the results obtained in this paper can be applied to semilinear elliptic boundary value problems as in [2, 4, 5] and semilinear wave equations as in [2, 4].

REFERENCES

1. E. LANDESMAN AND A. LAZER, Nonlinear perturbations of linear elliptic boundary value problems at resonance, *J. Math. Mech.* **19** (1970), 609–623.
2. H. BRĚZIS AND L. NIRENBERG, Characterizations of the ranges of some nonlinear operators and applications to boundary value problems, *Ann. Scuola. Norm. Sup. Pisa* **5** (1978), 225–326.
3. S. FUCIK, "Ranges of Nonlinear Operators," Lectures Notes, Prague, Charles University, 1977.
4. H. AMANN AND G. MANCINI, Some applications of monotone operator theory to resonance problems, *Nonlinear Anal.* **3** (1977), 815–830.
5. D. G. DE FIGUEIREDO, Semilinear elliptic equations at resonance: higher eigenvalues and unbounded nonlinearities, to appear.
6. F. BROWDER AND P. HESS, Nonlinear mappings of monotone type in Banach spaces, *J. Funct. Anal.* **11** (1972), 251–249.