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The Space BV Is Not Enough for Hyperbolic Conservation Laws*

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In this paper we discuss the function space of the solutions of the hyperbolic conservation law

$$\begin{aligned} u_t + f(u)_x = 0, \qquad & t > 0, \quad -\infty < x < \infty, \\ u(x, 0) = u_0(x), \qquad & -\infty < x < \infty. \end{aligned} \tag{1}$$

We assume that the function $f(\cdot)$ is smooth and $f''(\cdot)$ does not vanish identically on any interval.

It is well known that, in general, the initial value problem for (1) does not have global smooth solutions even if the initial data $u_0(\cdot)$ are smooth. Hence we look for weak solutions. Solutions in the class of piecewise smooth functions with jump discontinuities across smooth curves admit a natural interpretation, the lines of discontinuities being interpreted as trajectories of propagating shock waves. Unfortunately, the class of piecewise smooth functions is too narrow to encompass all solutions of (1).

Conway and Smoller [1] were the first to recognize the relevance of the class of functions of locally bounded variation in the sense of Tonelli and Cesari. Volpert [2] called this class of functions the space BV and considered solutions of (1) in this space. For the case when $f''(\cdot)$ is positive for any finite interval, Lax [3] (see also Dafermos [4]) establishes that for bounded measurable initial data $u_0(\cdot)$, $u(\cdot, t)$ is in BV for all $t > 0$. This nice regularity result naturally leads one to conjecture that it still holds even when $f''(\cdot)$ vanishes at isolated points [5]. If this conjecture were true, then membership in the space BV would provide maximal information on the structure of solutions of (1). Unfortunately, it is the purpose of this paper to show that this conjecture is not true. We give two examples to show that the

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space BV is not enough for hyperbolic conservation laws. The first example uses nonconvex $f(u)=u^3/3$ and the second one uses strictly (but not uniformly) convex $f(u)=u^4/4$. Although $u_0(\cdot)\in L_\infty$ does not imply $u(\cdot, t)\in \mathrm{BV}$ for $t>0$, we expect that $f'(u(\cdot, t))\in \mathrm{BV}$ for all $t>0$. We still cannot give a complete proof for this conjecture.

EXAMPLE 1. In (1), let $f(u)=u^3/3$, $g(x)=x\sin(1/x)$, and

$$\begin{aligned} u_0(x) &= g(x_0), && \text{if} \quad x\leqslant -x_0, \\ &= g(x), && \text{if} \quad -x_0<x\leqslant x_0, \\ &= g(x_0), && \text{if} \quad x_0<x, \end{aligned} \tag{2}$$

where $x_0\in(\frac{1}{3}\pi, \frac{1}{2}\pi)$ and x_0 is the local maximum point of $g(x)$. Then

$$\begin{aligned} u(x,t) &= g(x_0), && \text{if} \quad x\leqslant(-x_0)+(g(x_0))^2\,t, \\ &= g(y), && \text{if} \quad x=y+(g(y))^2\,t, \quad -x_0<y\leqslant x_0, \\ &= g(x_0), && \text{if} \quad x_0+(g(x_0))^2\,t<x, \end{aligned} \tag{3}$$

is a solution for (1) in the strip $0\leqslant t\leqslant\frac{1}{2}$. The function $u_0(\cdot)$ is bounded and measurable. The function $u(\cdot, t)$ is not of locally bounded variation for each $0\leqslant t\leqslant\frac{1}{2}$.

Proof of Example 1. The function $u_0(\cdot)$ is obviously bounded and measurable, but $u_0(\cdot)$ is not of bounded varation on the interval $[-x_0, x_0]$. Let

$$F(y)=y+(g(y))^2\,t.$$

We have

$$\begin{aligned} F'(y) &= 1, && \text{if} \quad y=0, \\ &= 1+[2y\sin^2(1/y)-2\sin(1/y)\cos(1/y)]\,t, && \text{if} \quad y\neq 0. \end{aligned}$$

Since

$$2y\sin^2(1/y)-2\sin(1/y)\cos(1/y)>-2$$

for $y\in[-x_0, x_0]$, we have

$$F'(y)>0$$

for all $y\in[-x_0, x_0]$ and all $t\in[0,\frac{1}{2}]$.

Hence

$$y\mapsto y+(g(y))^2\,t$$

is a homeomorphism between $[-x_0, x_0]$ and $[(-x_0) + (g(x_0))^2 t, x_0 + (g(x_0))^2 t]$ for each $t \in [0, \frac{1}{2}]$. This proves that the $u(x, t)$ in (3) is a solution of (1) in the strip $0 \leqslant t \leqslant \frac{1}{2}$ and that $u(\cdot, t)$ is not of bounded variation on the interval $[(-x_0) + (g(x_0))^2 t,\ x_0 + (g(x_0))^2 t]$ for each $t \in [0, \frac{1}{2}]$. The proof is complete.

EXAMPLE 2. In (1), let $f(u) = u^4/4$; $g(\cdot)$ and $u_0(\cdot)$ are the same as in Example 1. Then

$$\begin{aligned} u(x, t) &= g(x_0), &&\text{if } x \leqslant (-x_0) + (g(x_0))^3 t, \\ &= g(y), &&\text{if } x = y + (g(y))^3 t, \quad -x_0 < y \leqslant x_0, \\ &= g(x_0), &&\text{if } x_0 + (g(x_0))^3 t < x, \end{aligned}$$

is a solution for (1) in the strip $0 \leqslant t \leqslant \frac{2}{5}$. Furthermore, $u(\cdot, t)$ is not of bounded variation for each $0 \leqslant t \leqslant \frac{2}{5}$ on the interval $[(-x_0) + (g(x_0))^3 t, x_0 + (g(x_0))^3 t]$.

The proof of Example 2 is similar to that of Example 1. We omit it.

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