Discrete Mathematics 45 (1983) 245–253
North-Holland Publishing Company

ON A MULTIPLICATIVE GRAPH FUNCTION CONJECTURE*

Lih-Hsing HSU

National Chiao Tung University, Department of Applied Mathematics, 1001 Ta Hsueh Road, Hsinchu, Taiwan, Republic of China

Received 16 December 1981
Revised 18 May 1982 and 13 August 1982

For any graph H, the function h_H, defined by setting $h_H(G)$ equal to the number of homomorphisms from G into H, is a multiplicative increasing function. Lovász [2] has asked whether all nonzero multiplicative increasing functions are generated by functions of this type. We show that this is not the case. However, the classification of multiplicative increasing graph functions is still unsolved. We prove several properties of such functions in this paper.

1. Definition and introduction

Let $G=(X,E)$ is called a *graph* if X is a finite set and E is a subset of $\{(a,b) \mid a \neq b, (a,b)$ is an unordered pair of $X\}$. We say $X=V(G)$ is the vertex set of G, $E=E(G)$ is the edge set of G.

Let $G=(X,E)$, $H=(Y,F)$ be two graphs. The product of G and H is the graph $G\times H=(Z,K)$, where $Z=X\times Y$, the Cartesian product of X and Y, and $K=\{((x_1,y_1),(x_2,y_2)) \mid (x_1,x_2)\in E$ and $(y_1,y_2)\in F\}$. We let G^k denote $G\times G\times \cdots \times G$ (k times); the sum of G and H is the graph $G+H=(W,U)$ with $W=X_1\cup Y_1$, $U=E_1\cup F_1$ where $G'=(X_1,E_1)\cong G$, $H_1=(Y_1,F_1)\cong H$ and $X_1\cap Y_1=\emptyset$. A map $\psi: Y\to X$ is called a homomorphism if it satisfies $(y_1,y_2)\in F$ implies $(\psi(y_1),\psi(y_2))\in E$.

For a fixed graph H, we can define h_H from $\mathcal{G}$, the set of all graphs, into R such that $h_H(G)$ equals the number of homomorphisms from H, into G. It is very easy to get the following theorem.

Theorem 1.1. (1) $h_G(A\times B)=h_G(A)h_G(B)$.

(2) *If A is a subgraph of B, then* $h_G(A)\leq h_G(B)$.

(3) $h_{A+B}(G)=h_A(G)h_B(G)$.

(4) *If G is a connected graph,* $h_G(A+B)=h_G(A)+h_G(B)$.

A real-valued function f defined on the set of all graphs $\mathcal{G}$, such that $f(G\times H)=f(G)f(H)$ is called a *multiplicative function*, and a real-valued function f is *increasing* if $f(A)\leq f(B)$ whenever A is a subgraph of B. We use M to denote the set of all multiplicative increasing graph functions.

* This paper is a part of the author's Ph.D. thesis.

0012-365X/83/$03.00 © Elsevier Science Publishers B.V. (North-Holland)

From Theorem 1.1, we know that h_G is an element of M. Note that M is closed under finite product, taking the positive power and pointwise convergence. Hence the following functions are elements of M.

(1) h_G^{α}, $\alpha>0, G\in\mathcal{G}$.

(2) $\prod_{i=1}^{k} h_{G_i}^{\alpha_i}$, $\alpha_i>0, G_i\in\mathcal{G}$.

(3) $\lim_{m\to\infty} f_m$ where f_m is of type (1) or (2).

Lovász [2] observed these facts and he asked whether all nonzero multiplicative increasing functions are of these forms. Let L denote the set of all functions in Lovász's conjecture. We shall exhibit two counterexamples. Before proceeding, we need the following observation.

Theorem 1.2. *If $f\in L$ and $f(P_1)=2$, then $f(P_2)\geqslant 3$ and $f(P_2+K_1)\geqslant 4$.*

Proof. By Theorem 1.1(3), L is in fact generated by h_G with G is connected. Let G be a connected graph. If $G=K_1$, then $h_G(K_1)=1$, $h_G(P_1)=2$, $h_G(P_2)=3$, $h_G(P_2+K_1)=4$, if G is bipartite and $G\neq K_1$, then $h_G(K_1)=0$, $h_G(P_1)=2$, $h_G(P_2)\geqslant 4$, $h_G(P_2+K_1)\geqslant 4$; and if G is not bipartite, then $h_G(K_1)=h_G(P_1)=h_G(P_2)=h_G(P_2+K_1)=0$. Thus, the theorem follows. □

2. Generalized homomorphism functions

For every graph G and integer $m>0$, let G_m be the induced subgraph of G such that $x\in G_m$ if and only if x is in an m-clique of G. For every fixed $H\in\mathcal{G}$, we define a function $h_{m,H}:\mathcal{G}\to R$ by

$$h_{m,H}(G)=h_H(G_m).$$

Since $h_H=h_{1,H}$ for every graph H, we call $h_{m,H}$ the *generalized homomorphism function.*

Theorem 2.1. *$h_{m,K}\in M$ for every $m\in N$, $K\in\mathcal{G}$*

Proof. Observe that $(x,y)\in(G\times H)_m$ if and only if x is in G_m and y is in H_m. Therefore $(G\times H)_m=G_m\times H_m$. Then we have $h_k((G\times H)_m)=h_K(G_m\times H_m)=h_K(G_m)h_K(H_m)$. This implies $h_{m,K}(G\times H)=h_{m,K}(G)h_{m,K}(H)$. If $G\subset H$, we have $G_m\subset H_m$, then $h_K(G_m)\leqslant h_K(H_m)$. That is to say $h_{m,K}(G)\leqslant h_{m,K}(H)$. Thus $h_{m,K}$ is a multiplicative increasing function.

Notice that not every generalized homorphism function can be generated by homomorphism functions, for example, $h_{2,K_1}\in M-L$.

Theorem 2.2. $h_{2,K_1} \in M - L$.

Proof. From Theorem 2.1, we know $h_{2,K_1} \in M$. Observe that $h_{2,K_1}(P_1) = 2$ but $h_{2,K_1}(P_2 + K_1) = 3 < 4$. By Theorem 1.2, we get $h_{2,K_1} \notin L$. □

3. Another counterexample

Definition. A bipartite graph is a graph whose vertex set $V(G)$ can be partitioned into two subsets A and B such that every edge of G joins A with B and vice versa. If G is connected bipartite, such a partition is unique; we say such G is of (r, s) type if $|A| = r$ and $|B| = s$. For an arbitrary bipartite graph G with connected components $C_1, C_2, \ldots, C_m$, where each C_i is bipartite, we say G is of $\sum_{i=1}^{m} (r_i, s_i)$ *type* if C_i is of (r_i, s_i) type for every i.

Theorem 3.1. *If G is a bipartite graph of (r, s) type and H is a bipartite graph of (t, u) type, then $G \times H$ is a bipartite graph of $(rt, su) + (ru, st)$ type.*

Proof. Let G be a bipartite graph with partition A and B; and H is a bipartite graph with partition C and D. Note that $(x, y) \in A \times C$ can only adjacent to $(z, w) \in B \times D$ and vice versa. Similarly, $(x, y) \in A \times D$ can only adjacent to $(z, w) \in B \times C$ and vice versa. It is easy to check that the subgraph generated by $(A \times C) \cup (B \times D)$ and the subgraph generated by $(A \times D) \cup (B \times C)$ are connected. Then the theorem follows. □

Corollary 3.1. *If G is of $\sum_{i=1}^{n} (r_i, s_i)$ type and H is of $\sum_{i=1}^{m} (t_j, u_j)$ type, then $G \times H$ is of $\sum_{i,j} (r_i t_j, s_i u_j) + \sum_{i,j} (r_i u_j, s_i t_j)$ type.*

Instead of considering the set of all graphs, we concentrate only on the set of all bipartite graphs, $\mathbb{B}$. First, let us consider the function $\theta : \mathbb{B} \to R$ defined by

$$\theta(G) = 2\left(\sum_{i=1}^{n} (r_i s_i)^{\frac{1}{2}}\right)$$

where G is of $\sum_{i=1}^{n} (r_i, s_i)$ type.

Theorem 3.2. *θ is a multiplicative increasing function on the set of all bipartite graphs.*

Proof. Let G, H be bipartite graphs of $\sum_{i=1}^{n} (r_i, s_i)$ type and $\sum_{i=1}^{m} (t_j, u_j)$ type, respectively. By *Corollary* 3.1, $G \times H$ is of $\sum_{i,j} (r_i t_j, s_i u_j) + \sum_{i,j} (r_i u_j, s_i t_j)$ type. Thus

$$\theta(G \times H) = 2\left[\sum_{i,j} (r_i t_j s_i u_j)^{\frac{1}{2}} + \sum_{i,j} (r_i u_j s_i t_j)^{\frac{1}{2}}\right] = \left[2 \sum_{i} (r_i s_i)^{\frac{1}{2}}\right]\left[2 \sum_{j} (t_j u_j)^{\frac{1}{2}}\right] = \theta(G)\theta(H).$$

If H is a subgraph of G and G is of $\sum_{i=1}^{n}(r_i, s_i)$ type, then H is of $\sum_i(\sum_j(a_{ij}, b_{ij}))$ type where $\sum_j a_{ij} \leq r_i$ and $\sum_j b_{ij} \leq s_i$. Without loss of generality, we may assume $\sum_j a_{ij} = r_i$ and $\sum_j b_{ij} = s_i$ for every i. For each i we have

$$2\sum_j (a_{ij}b_{ij})^{\frac{1}{2}} \leq 2\left[\left(\sum_j a_{ij}\right)\left(\sum_j b_{ij}\right)\right]^{\frac{1}{2}} = 2(r_i s_i)^{\frac{1}{2}}.$$

Summing over all i, we get $\theta(G) \geq \theta(H)$. Therefore θ is multiplicative and increasing. □

Now we want to extend θ to the set of all graph, $\mathcal{G}$. Let us define $\delta : \mathcal{G} \to R$ by $\delta(G) = \frac{1}{2}\theta(G \times P_1)$. Note that for every graph G, $G \times P_1$ is bipartite. Therefore δ is well defined. Moreover, we have $\theta = \delta$ in the set of all bipartite graphs. Also, it is easy to see that for nonbipartite connected graph, $\delta(G) = v$.

Theorem 3.3. $\delta \in M - L$.

Proof. Since $\theta(P_1) = 2$,

$$\delta(G \times H) = \tfrac{1}{2}\theta(G \times H \times P_1) = \tfrac{1}{4}\theta(G \times H \times P_1)\theta(P_1)$$
$$= \tfrac{1}{4}\theta(G \times H \times P_1 \times P_1) = [\tfrac{1}{2}\theta(G \times P_1)][\tfrac{1}{2}\theta(H \times P_1)] = \delta(G)\,\delta(H).$$

And, if H is a subgraph of G, then $H \times P_1$ is also a subgraph of $G \times P_1$. Thus $\theta(G \times P_1) \geq \theta(H \times P_1)$. We get $\delta(G) \geq \delta(H)$. Thus $\delta \in M$. Note that $\delta(P_1) = 2$ and $\delta(P_2) = 2\sqrt{2} < 3$. By Theorem 1.2, we get $\delta \notin H$. □

4. The number of disjoint edges in a graph

The function δ that was studied in Section 3 gives rise to the following question: Find

$$\inf\{f(P_2) \mid f(P_1) = 2, f \in M\}.$$

For $f \in M$, $f(P_1) = 2$, we have $f(P_2) \geq f(P_1) = 2$. Therefore the infimum exists, say c_0. Moreover, since $\delta \in M$, $\delta(P_1) = 2$ and $\delta(P_2) = 2\sqrt{2}$, we have $2 \leq c_0 \leq 2\sqrt{2}$.

We shall use the following theorem.

Theorem 4.1. *If $f \in M$, $f(P_1) = 2$ and $f(G) = g$, then $f(mG) = mg$. In particular $f(mP_1) = 2m$.*

Proof. We prove the theorem through the following steps.

(1) By induction, it can be easily proved that $(mP_1)^k = 2^{k-1}m^kP_1$.

(2) From (1), we know $2^sP_1 = P_1^{s+1}$ for every integer $s \geq 0$. Thus, we have $f(2^sP_1) = f[(P_1)^{s+1}] = [f(P_1)]^{s+1} = 2^{s+1}$.

(3) We prove that $f(nP_1) = 2n$.

Suppose $f(nP_1)=2m\neq 2n$ for some n. First assume $m>n$. Because $m/n>1$, we have $\lim_{m\to\infty}(m/n)^k=\infty$. We can always find t and s such that $n^t\leq 2^s<m^t$. Now consider $(nP_1)^t=2^{t-1}n^tP_1$. Since $2^{t-1}n^t<2^{t+s-1}$, we get $(nP_1)^t$ is a subgraph of $2^{t+s-1}P_1$. Therefore

(a) $f[(nP_1)^t]<f(2^{t+s-1}P_1)=2^{t+s}$.

However

(b) $f[(nP_1)^t]=[f(nP_1)]^t=(2m)^t>2^{t+s}$.

(a) and (b) contradict each other. Therefore it is impossible that $f(nP_1)=2m\neq 2n$ with $m>n$. We can use the same method to show it is also impossible for $f(nP_1)=2m$ with $m<n$. Therefore $f(nP_1)=2n$ for every $n>0$.

(4) If G is a bipartite graph and m is an integer, then $f(mG)=mg$.

First, if $m=2k$ is an even integer, we have $mG=2kG=G\times kP_1$. Then $f(mG)=f(G\times kP_1)=f(G)\ f(kP_1)=g2k=mg$.

If m is an odd number, $2m$ is an even integer. From the above discussion, we know that $f(2mG)=2mg$. But $2mG=mG\times P_1$. We have $f(mG)f(P_1)=f(2mG)=2mg)$ i.e., $2f(mG)=2mg$. Therefore $f(mG)=mg$.

(5) If G is an arbitrary graph, $G\times P_1$ is bipartite graph. Since $mG\times P_1=m(G\times P_1)$, we have

$$f(mG)f(P_1)=f(mG\times P_1)=mf(G)f(P_1)=2mg.$$

But since $f(P_1)=2$, we have $f(mG)=mg$. □

Let G be an arbitrary graph G with vertex set $\{x_1,x_2,\ldots,x_v\}$, $v=|V(G)|$. Using $\gamma(G)$ to denote the maximal number of disjoint edges of G, we can define $P(G)=\lim_{m\to\infty}[\gamma(G^m)]^{1/m}$. Then, as $\gamma(G^m)P_1\subset G^m$, if f is in M and $f(P_1)=2$, then by Theorem 4.1, $f(\gamma(G^m)P_1)=2\gamma(G^m)$. We get

$$f(G)=f(G^m)^{1/m}\geq f(\gamma(G^m)P_1)^{1/m}=(2\gamma(G^m))^{1/m}.$$

Therefore

$$f(G)\geq\lim_{m\to\infty}(2\gamma(G^m))^{1/m}=\lim_{m\to\infty}(\gamma(G^m))^{1/m}=P(G).$$

In particular $c_0\geq P(P_2)$. Later we will show that $P(P_2)=2\sqrt{2}$. Thus, we get $c_0=2\sqrt{2}$.

We are going to give a formula for the calculation of $P(G)$. However, we need some probabilistic results.

Let $D=\{(a_1,a_2,\ldots,a_v)\mid a_i\geq 0,\sum_{i=1}^n a_i=1\}$. Let $H:D\to R$ be a function defined by:

$$H(\boldsymbol{a})=\sum_{i=1}^{n}-a_i\log_v a_i\quad\text{where } \boldsymbol{a}=(a_1,a_2,\ldots,a_v)$$

The function H is called the *entropy function*. It is well known that the entropy function satisfies

(i) $\lim_{m\to\infty}\binom{m}{a_1m, a_2m, \ldots, a_vm}^{1/m} = v^{H(a)}$ where $a_i m \in N$ for every i.

(ii) $H(\frac{1}{2}(\boldsymbol{a}+\boldsymbol{b})) \geq \min(H(\boldsymbol{a}), H(\boldsymbol{b}))$.

Let $A = A(G)$ be the subset of D defined by

$$A(G) = \Big\{\boldsymbol{a}\in D \mid \exists z_{ij},\ 1\le i, j\le v,\ 0\le z_{ij}\le 1,\ z_{ij}=0 \text{ if } (x_i, x_j)\notin E(G),$$
$$z_{ij} = z_{ji} \text{ and } \sum_j z_{ij} = a_i\Big\}.$$

Since $A(G)$ is defined by equations and $0\le z_{ij}\le 1$, $A(G)$ is a compact subset of D. We can find $\boldsymbol{c}$ in $A(G)$ such that

$$H(\boldsymbol{c}) = \max\{H(\boldsymbol{a}) \mid \boldsymbol{a}\in A(G)\}.$$

Our main result is the following theorem.

Theorem 4.2. *$P(G) = v^{H(G)}$ where $H(G) = \max\{H(\boldsymbol{a}) \mid \boldsymbol{a}\in A(G)\}$.*

We will prove the theorem in the next section.

5. Proof of Theorem 4.2

On D, we can define a relation '$\to_G$' by $\boldsymbol{a}\to_G \boldsymbol{b}$, if there exists z_{ij}, $1\le i, j\le v$, such that

(1) $0\le z_{ij}\le 1$.
(2) $z_{ij}=0$ if $(x_i, x_j)\notin E(G)$.
(3) $\sum_j z_{ij} = a_i$ for every i.
(4) $\sum_i z_{ij} = b_j$ for every j.

We would like to explain the concept '$\to_G$' more closely. Let m be a positive integer. Let $\boldsymbol{y} = (y_1, y_2, \ldots, y_m)$ be a vertex in G^m. We call $\boldsymbol{a} = (a_1, a_2, \ldots, a_v)$ with $a_i = |\{y_j \mid y_j = x_i\}|/m$ the distribution of $\boldsymbol{y}$. Let $\boldsymbol{y}, \boldsymbol{z}$ be two vertices in G^m such that $(\boldsymbol{y}, \boldsymbol{z})$ is an edge in G^m. Then their respective distributions $\boldsymbol{a}, \boldsymbol{b}$ satisfies $\boldsymbol{a}\to_G \boldsymbol{b}$. Conversely, given $\boldsymbol{a}, \boldsymbol{b}\in D$, where ma_i, mb_i are integers for every i and $\boldsymbol{a}\to_G \boldsymbol{b}$, then there exists $\boldsymbol{y}, \boldsymbol{z}$ vertices in G^m with their respective distributions $\boldsymbol{a}$, $\boldsymbol{b}$ such that $(\boldsymbol{y}, \boldsymbol{z})$ is an edge in G^m. We say $(\boldsymbol{y}, \boldsymbol{z})\in E(G^m)$ is of $\boldsymbol{a}\to_G \boldsymbol{b}$ type if their distributions are $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively.

Lemma 5.1. *If $\boldsymbol{a}\to_G \boldsymbol{b}$, we have $\frac{1}{2}(\boldsymbol{a}+\boldsymbol{b})\to_G \frac{1}{2}(\boldsymbol{a}+\boldsymbol{b})$. Moreover, $A(G) = \{\boldsymbol{a}\in D \mid \boldsymbol{a}\to_G \boldsymbol{a}\}$.*

Proof. Let z_{ij} satisfy (1), (2), (3) and (4). Consider $z'_{ij} = \frac{1}{2}(z_{ij}+z_{ji})$, we then get $\frac{1}{2}(\boldsymbol{a}+\boldsymbol{b})\to_G \frac{1}{2}(\boldsymbol{a}+\boldsymbol{b})$. $\square$

By Lemma 5.1 and $H(\frac{1}{2}(\boldsymbol{a}+\boldsymbol{b}))\geqslant\min(H(\boldsymbol{a}),H(\boldsymbol{b}))$, we see that the following theorem will imply Theorem 4.2.

Theorem 5.1

$$\log P(G)=\max_{\boldsymbol{a}\to_G\boldsymbol{b}}\min(H(\boldsymbol{a}),H(\boldsymbol{b}))\log v=\max_{\boldsymbol{a}\to_G\boldsymbol{a}}H(\boldsymbol{a})\log v.$$

Proof. Since $A(G)$ is defined by rational linear functions, $A(G)$ can be parametrized by rational linear equations. Therefore for a given $\vec{\boldsymbol{a}}\in A$, we can find a sequence $\{\vec{\boldsymbol{r}}_i\}_{i=1}^{\infty}$ such that $\lim_{i\to\infty}\vec{\boldsymbol{r}}_i=\vec{\boldsymbol{a}}$ and for each $\vec{\boldsymbol{r}}_i$, all its components are rational numbers.

For each $\boldsymbol{r}_i=(r_{i_1},r_{i_2},\ldots,r_{i_v})$, we can find an integer m such that $m\boldsymbol{r}_i=(mr_{i_1},mr_{i_2},\ldots,r_{i_v})$ with all mr_{i_j} integers. In G^m, let $S(m,\boldsymbol{r}_i)=\{\boldsymbol{y}\in G^m \mid \text{the distribution of } \boldsymbol{y} \text{ is } \boldsymbol{r}_i\}$. Then the induced subgraph $G^m|_{S(m,\boldsymbol{r}_i)}$ is a vertex-transitive graph without isolated vertices. To be more precise, if $\boldsymbol{y}$, $\boldsymbol{z}$ are adjacent in $G^m|_{S(m,\boldsymbol{r}_i)}$, then for every $\sigma\in S_m$, the symmetric group on m letters, $(y_{\sigma(1)},y_{\sigma(2)},\ldots,y_{\sigma(m)})$ is adjacent to $(z_{\sigma(1)},z_{\sigma(2)},\ldots,z_{\sigma(m)})$. Moreover, since $\boldsymbol{r}_i\to_G\boldsymbol{r}_i$, for every $\boldsymbol{y}\in S(m,\boldsymbol{r}_i)$ there exists $\boldsymbol{z}\in S(m,\boldsymbol{r}_i)$ such that $\boldsymbol{y}$ is adjacent to $\boldsymbol{z}$ in G^m.

Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be adjacent in $G^m|_{S(m,\boldsymbol{r}_i)}$; there exists $\sigma\in S_m$ such that $\sigma(\boldsymbol{x})=\boldsymbol{y}$. Let $\boldsymbol{x}$ be fixed and consider the set $S_0=\{\boldsymbol{x}_1=\boldsymbol{x},\boldsymbol{x}_2=\sigma(\boldsymbol{x}_1),\ldots,\boldsymbol{x}_k=\sigma(\boldsymbol{x}_{k-1})\}$, where $\sigma(\boldsymbol{x}_k)=\boldsymbol{x}_1$. Then $G^m|_{S_0}$ forms a cycle or an edge (if $k=2$). $S_0,S_1,\ldots,S_t$ having been contructed, if there exists $\boldsymbol{z}\notin S_0\cup S_1\cup\cdots\cup S_t$, then there is a $\pi_t\in S_m$ such that $\pi_t(\boldsymbol{x})=\boldsymbol{z}$. Then set $S_{t+1}=\{\boldsymbol{z},\pi_t\sigma\pi_t^{-1}(\boldsymbol{z}),\ldots,\pi_t\sigma^{k-1}\pi_t^{-1}(\boldsymbol{z})\}$. Then S_{t+1} is disjoint from $S_0\cup S_1\cup\cdots\cup S_t$.

For this reason, we get $S_0,S_1,\ldots,S_{d-1}$ a partition of $S(m,\boldsymbol{r}_i)$ for some d. For each i, $0\leqslant i\leqslant d-1$, C_k is a subgraph of S_i. (We let $C_2=P_1$.) Thus $\gamma(S_i)\geqslant[\frac{1}{2}k]$. But as $kd=|S(m,\boldsymbol{r}_i)|$. We obtain

$$\gamma(G^m)\geqslant[\tfrac{1}{2}k]d=[\tfrac{1}{2}k]\times\frac{|S(m,\boldsymbol{r}_i)|}{k}\geqslant\tfrac{1}{3}|S(m,\boldsymbol{r}_i)|$$
$$=\frac{1}{3}\binom{m}{mr_{i_1},mr_{i_2},\ldots,mr_{i_v}}.$$

[We simply use the fact that each cycle C_k has $[\frac{1}{2}k]$ disjoint edges and $k=3$ is the worst case for the inequality.]

Thus, we have

$$P(G)=\lim_{m\to\infty}[\gamma(G_m)]^{1/m}$$
$$\geqslant\lim_{m\to\infty}\left[\frac{1}{3}\binom{m}{mr_{i_1},mr_{i_2},\ldots,mr_{i_v}}\right]^{1/m}=v^{H(\boldsymbol{r}_i)}$$

But since H is a continuous function, we get

(c) $P(G)\geqslant v^{H(\boldsymbol{a})}$ for every $\boldsymbol{a}\in A(G)$.

On the other hand, let $\boldsymbol{c}$ be the point in $A(G)$ such that $H(\boldsymbol{c}) = \max\{H(\boldsymbol{a}) \mid \boldsymbol{a} \in A(G)\}$.

Note that in G^m, there are at most $\binom{m+v-1}{m}$ different distributions, and therefore there are at most m^{2v} different types $\boldsymbol{a} \to_G \boldsymbol{b}$ where $\boldsymbol{a}$, $\boldsymbol{b}$ are distributions of some vertices in G^m. Let M be a set of disjoint edges in G^m with $|M| = \gamma(G^m)$. Define an equivalence relation in $E(G^m)$ by $(\boldsymbol{x}, \boldsymbol{y}) \sim (\boldsymbol{z}, \boldsymbol{w})$ is they are of the same type. By Pigeon Hole Principle, there exists some $\boldsymbol{a} \to_G \boldsymbol{b}$ such that the set $K = \{\boldsymbol{x}, \boldsymbol{y}) \mid (\boldsymbol{x}, \boldsymbol{y})$ is of type $\boldsymbol{a} \to_G \boldsymbol{b}\}$ satisfies $|K| \geq m^{-2v}|M| = m^{-2v}\gamma(G^m)$. Therefore

$$m^{-2v}\gamma(G^m) \leq \min\{|S(m, \boldsymbol{a})|, |S(m, \boldsymbol{b})|\}.$$

Thus

$$r(G^m) \leq \max_{\boldsymbol{a} \to \boldsymbol{b}} \min\{|S(m, \boldsymbol{a})|, |S(m, \boldsymbol{b})|\}.$$

This implies

$$\begin{aligned} P(G) &= \lim_{m \to \infty} \gamma(G^m)^{1/m} \\ &\leq \max_{\boldsymbol{a} \to \boldsymbol{b}} \min\left\{\lim_{m \to \infty} (m^{2v}|S(m, \boldsymbol{a})|)^{1/m}, \lim_{m \to \infty} (m^{2v}|S(m, \boldsymbol{b})|)^{1/m}\right\} \\ &= \max_{\boldsymbol{a} \to \boldsymbol{b}} \min\{v^{H(\boldsymbol{a})}, v^{H(\boldsymbol{b})}\} \\ &= \max_{\boldsymbol{a} \to \boldsymbol{a}} v^{H(\boldsymbol{a})} \\ &= v^{H(\boldsymbol{c})} \qquad \text{(d)} \end{aligned}$$

From (c) and (d), we have $P(G) = v^{H(\boldsymbol{c})}$ □

As mentioned above, we know $P(G) \leq f(G)$ for every $f \in M$ with $f(P_1) = 2$. Therefore $P(G)$ can be viewed as a lower bound for multiplicative increasing function. However, we do not know whether P itself is multiplicative or not. Our conjecture is that P is not. In general $P(G)$ is very difficult to calculate. In [3], Hsu gives a list of $P(G)$ for some graph G. Here we present an exampie.

Theorem 5.2. $P(P_2) = 2\sqrt{2}$.

Proof. Let $V(P_2) = \{0, 1, 2\}$, $E(P_2) = \{(0, 1), (1, 2)\}$. Consider $z_{0,1} = z_{1,0} = z_{2,1} = z_{1,2} = \frac{1}{4}$ and $z_{ij} = 0$ for other (i, j). We have $(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}) \to P_2(\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$ and $3^{H(\frac{1}{4}, \frac{1}{2}, \frac{1}{4})} = 2\sqrt{2}$. However $\delta \in M$ and $\delta(P_2) = 2\sqrt{2}$. However $\delta \in M$ and $\delta(P_2) = 2\sqrt{2}$. We get $2\sqrt{2} \leq P(P_2) \leq \delta(P_2) = 2\sqrt{2}$. Therefore $P(P_2) = 2\sqrt{2}$. □

Remark. In [3], Hsu has proved that $\delta \neq P$.

References

[1] L. Lovász, Operation with structures, Acta Math. Acad. Sci Hung. 18 (1967) 321–328.
[2] L. Lovász, A personal communication between Lovász and Spencer.
[3] Lih-Hsing Hsu, Monotone multiplicative graph functions and dependence of tree copy functions, Ph.D. Dissertation (1981).