國 立 交 通 大 學

電 子 物 理 研 究 所

碩 士 論 文

BCFW 遞迴關係式與玻色開弦中樹圖的散射振幅

BCFW recursion relation and tree level amplitudes for bosonic open string

研 究 生：張永業
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摘　　要

在本篇論文中，我們將BCFW遞迴關係式的應用從場論散射振幅推廣到弦論。計算弦論的散射振幅困難之處在於需要對無限多個中間物理態 (Intermediate physical state) 做加總。我們將加總的範圍從物理態擴大到全部的 Fock states 解決了這個問題，而且成功地利用此方法計算出四個快子(tachyon)的散射振幅；並更進一步計算出一個任意物理態與三個快子的散射振幅。除此之外，我們了解到利用上述的方法計算散射振幅須要生成函數(generating function)的輔助，且利用路徑積分得到此生成函數的一般結構。

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ABSTRACT

In this thesis, we extend the application of BCFW recursion relation to string tree-level amplitudes. In contrast to the field theory calculation, we encounter the difficulty of summing over all intermediate physical states with infinite tower of mass levels. We develop a method to resolve this difficulty by enlarging the sum over all intermediate physical states to an easier sum over the entire Fock space of string spectrum. The calculation is successfully applied to the 4-tachyon amplitude and then to the cases of one arbitrary higher spin state and 3-tachyon amplitudes. We also figure out a generating function for summing the infinite poles of string spectrum in the BCFW string amplitude calculation. The generic structure of this generating function for higher spin scattering amplitude can be obtained from the standard path integral calculation of string scattering amplitude.

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Contents

Chapter 1

Introduction

The calculation of scattering amplitudes has been a central issue in quantum field theory in which Feynman's rule provides a clear picture in the calculation of scattering processes. However, with the increasing number of external particles, the efficiency of this method was restricted since the number of Feynman diagrams increase tremendously. As a result, the previous statement may need to be changed from *calculating scattering processes* to *how to calculate scattering amplitudes more efficiently*. To do so, many novel theories such as spinor method, color–ordered technique and BCFW on-shell recursion relation popped out one after another during the past few decades.

The BCFW method was initially proposed for gauge field theory. It merely relies on the general complex analytic structures of scattering amplitudes. The original higher point scattering amplitude can then be expressed as sum of products of lower point on-shell scattering amplitudes. As a result, one can recycle the calculation for lower point functions to simplify the calculation for higher point functions.

The success of BCFW calculation of scattering amplitudes in quantum field theory motivates us to extend the calculation to string theory. In this thesis, we extend the application of BCFW recursion relation to string tree-level amplitudes. In contrast to the field theory calculation, we encounter the difficulty of summing over all intermediate physical states with infinite tower of mass levels. We develop a method to resolve this difficulty by enlarging the sum over all intermediate physical states to an easier sum over the entire Fock space of string spectrum. The calculation is successfully applied to the 4-

tachyon amplitude and then to the cases of one arbitrary higher spin state and 3-tachyon amplitudes.

This thesis is organized as following. In Chapter 2, we give a brief introduction to BCFW recursion relation in quantum field theory, and some basics of string theory. Chapter 3 is divided into two parts: the first part contains spinor semiology. Then, in the second part, we adopt the BCFW recursion relation to compute a concrete example, namely, 4-gluon color-ordered scattering amplitude. In Chapter 4, we begin with the familiar four-point Veneziano formula, and demonstrate that how one can extend BCFW method to four-tachyon string scattering amplitude. In Chapter 5, we extend BCFW method to string scattering processes with a higher spin vertex and 3 tachyons. Finally, we give a brief conclusion of this thesis. The last three chapters are mainly based on our paper [1], which has been accepted in January, 2013.

1.1 Literature Reviews

In [2,3], Britto, Cachazo, Feng and Witten (BCFW) proposed a recursion relation for scattering amplitudes of Yang–Mills theory based on deforming the momenta and taking the analytic continuation over the complex plane. After doing so, the amplitude can then be characterized by its poles and the corresponding residues. This feature allowed one to express the higher point scattering amplitude in terms of sum of products of two lower-point on-shell scattering amplitudes.

The extension of BCFW recursion relation from field theory to string theory was initiated by Rutger Boels, Kasper Jens Larsen, Niels A. Obers and Marcel Vonk in [4]. They showed that BCFW technique is applicable for all open 4–point amplitudes in flat space. They also conjectured that BCFW method could be extended to higher point amplitudes and to closed string cases.

In 2010, Clifford Cheung, Donal O'Connell, and Brian Wecht [5] demonstrated that all tree-level amplitudes possessed convergent asymptotic behavior and thus allowed application of BCFW recursion relation. Furthermore, in this paper pole structures were made manifest through binomially expanding the Koba–Nielson formula for tachyon amplitudes.

In [6], Angelos Fotopoulos proposed how to construct the Veneziano amplitude via BCFW procedure by applying conjectured 3-point function with two tachyons and an arbitrary intermediate massive state. Namely, for four-point function, the product of two 3-point tachyon-liked amplitudes can be produced by summing over all massive intermediate states.

Chapter 2

Preliminaries

In this chapter, we provide a very concise introduction to BCFW recursion relation as well as some background knowledge of string theory.

2.1 Review of BCFW recursion relation

BCFW on–shell recursion relation [2,3] allows us to express on–shell amplitudes as sums of products of relatively lower–point on–shell amplitudes. It is known that, from the Feynman's rules, the essential ingredients for tree–level amplitudes are propagators and vertices, which implies that scattering amplitudes are rational functions in terms of kinematic variables. If we shift two of the external momenta into complex plane by

$$\hat{k}_1(z) = k_1 + zq, \qquad \hat{k}_n(z) = k_n - zq. \tag{2.1}$$

Energy-momentum conservation is manifestly preserved, $\hat{k}_1 + \hat{k}_n = k_1 + k_n$. We also need to impose these constraints $q^2 = q \cdot k_1 = q \cdot k_n = 0$ so as to preserve the on–shell conditions $\hat{k_1^2} = k_1^2$ and $\hat{k_2^2} = k_2^2$ of the deformed pair. Amplitudes are now complex functions with simple poles, which is the consequence of the light-like condition of the shifted momentum q, locating at the propagators . An complex amplitude with simple poles can be expresses as

$$A(z) = \sum_a \frac{R_a}{z - z_a}. \tag{2.2}$$

Taking the analytic continuation over the whole complex plane yields

$$\frac{1}{2\pi i}\oint \frac{dz}{z}A(z) = A(z=0) + \sum_{poles\ z_\alpha \neq 0} \mathrm{Res}\left(\frac{A(z)}{z}\right)_{z=z_\alpha}. \tag{2.3}$$

A simple pole manifestly exists at $z = 0$, which reproduces the un–shifted amplitude $A(z = 0)$ while other residues from other finites poles form the products of two relatively lower–point on–shell amplitudes $\mathrm{Res}\left(\frac{A(z)}{z}\right)_{z=z_\alpha} = -A_L(z_a)\frac{1}{P^2}A_R(z_a)$. If we assume no boundary contributions as $z \to \infty$ in the left–handed side of equation (2.3), we can then write down the BCFW on–shell recursion relation for n external particles

$$A_n(1,2,\cdots,n) = \sum_{poles\, z_a}\ \sum_{physcal\, h} A_L(\hat{1},2,\cdots,\hat{P}_a^h)\frac{1}{P_a^2}A_R(\hat{P}_a^h, a+1\cdots,\hat{n}). \tag{2.4}$$

The first summation indicates that we have to sum over all finite poles z_a while the second summation is over all physical intermediate states at a given simple pole z_a.

2.2 String theory

2.2.1 The classical version

Nambu–Goto action

Assume the universe is composed by one dimensional objects, so-called strings, instead of point particles, then we have the corresponding relativistic action

$$\begin{aligned} S_{NG} =& -T\int dA \\ =& -T\int d\sigma d\tau \left[-\det\frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X^\nu}{\partial\sigma^\beta}\,\eta_{\mu\nu}\right]^{\frac{1}{2}} \\ =& -T\int d\sigma d\tau\sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2X'^2}. \end{aligned} \tag{2.5}$$

This is the well–known Nambu-Goto action. $X(\sigma,\tau)$ is the $t$rack swept by the one dimensional object, called $w$orldsheet, which is parametrized by the coordinate σ and by the evolution τ. While A is the area of $X(\sigma,\tau)$ bounded by σ and τ. The notations $\dot{X}$ and X' mean

$$\dot{X} = \frac{\partial X}{\partial\tau} \quad , \quad X' = \frac{\partial X}{\partial\sigma}. \tag{2.6}$$

From the principle of least action $\delta S = 0$, we have the equations of motion by calculating the Euler–Lagrange equation

$$\frac{\partial}{\partial\tau}\left(\frac{\partial L}{\partial \dot{X}^{\mu}}\right) + \frac{\partial}{\partial\sigma}\left(\frac{\partial L}{\partial X'^{\mu}}\right) = 0. \tag{2.7}$$

This method is the same with classical mechanics : we vary the path and fix the initial point and the end point. The canonical momentum conjugated to X^{μ} can be obtained by definition

$$\Pi^{\mu} = \frac{\partial L}{\partial \dot{X}_{\mu}} = -T\,\frac{(\dot{X}\cdot X')X'^{\mu} - (X')^2\dot{X}^{\mu}}{\left[(X'\cdot\dot{X})^2 - (\dot{X})^2(X')^2\right]^{\frac{1}{2}}}. \tag{2.8}$$

Clearly to see that due to the square root appears in the denominator, quantizing this theory is rather complicated.

Polyakov action

In order to avoid the difficulty in quantizing the Nambu–Goto action, the Polyakov action S_P is therefore proposed, which is given by

$$S_P = -\frac{T}{2}\int d\sigma d\tau \sqrt{h} h^{\alpha\beta}\partial_{\alpha}X^{\mu}\partial_{\beta}X^{\nu}\eta_{\mu\nu}, \tag{2.9}$$

where $h \equiv -\mathrm{det}h_{\alpha\beta}$. From $\delta S = 0$, It is easy to get the equations of motion with respect to the variation of $\delta(X^{\mu})$ and $\delta h^{\alpha\beta}$,

$$\text{w.r.t.}\quad \delta X^{\mu} \Rightarrow \partial_{\alpha}\left(\sqrt{h}h^{\alpha\beta}\partial_{\beta}X^{\mu}\right) = 0, \tag{2.10}$$

$$\text{w.r.t.}\quad \delta h^{\alpha\beta} \Rightarrow \partial_{\alpha}X^{\mu}\partial_{\beta}X_{\mu} - \frac{1}{2}h_{\alpha\beta}h^{\gamma\delta}\partial_{\gamma}X^{\mu}\partial_{\delta}X_{\mu} = 0. \tag{2.11}$$

Equation (2.11) comes from the variantion with respect to the induced metric $h_{\alpha\beta}$, i.e. $\partial L/\partial h_{\alpha\beta} = 0$. Thus, [7]

$$h_{\alpha\beta} = \frac{2\partial_{\alpha}X^{\mu}\partial_{\beta}X_{\mu}}{h^{\gamma\delta}\partial_{\gamma}X^{\mu}\partial_{\delta}X_{\mu}}. \tag{2.12}$$

Substitute the metric tensor (2.12) above back to the Polyakov action, we are able to re–derive the Nambu–Goto action.

The action (2.9) is left invariant with 3 symmetries: Poincare, reparametrization and Weyl rescaling. Using these degrees of freedom, the induced metric $h^{\alpha\beta}$ can be simplified to the two–dimensional Minkowski metric $\eta^{\alpha\beta}$

$$h^{\alpha\beta} = \eta^{\alpha\beta} = diag\,(-1, 1). \tag{2.13}$$

Replace $h^{\alpha\beta}$ with $\eta^{\alpha\beta}$ in equation (2.10), the E.O.M. becomes a wave equation, namely

$$\partial_\alpha \partial^\alpha X^\mu = \left(\partial_\tau^2 - \partial_\sigma^2\right) X^\mu = 4\partial_+\partial_- X^\mu = 0, \tag{2.14}$$

where

$$\partial_+ = \frac{\partial}{\partial \sigma^+} \quad , \quad \partial_- = \frac{\partial}{\partial \sigma^-} \tag{2.15}$$

with $\sigma^\pm \equiv \tau \pm \sigma$.

To solve for X^μ for open strings, boundary conditions have to be imposed at the endpoints, i.e. $X^\mu(\tau, \sigma = 0) = X^\mu(\tau, \sigma = \pi) = 0$ and $X'^\mu(\tau, \sigma = 0) = X'^\mu(\tau, \sigma = \pi) = 0$. The equation of motion (2.14) is the two–dimensional wave equation with d'Alembert's solution of wave equation, i.e.

$$\begin{aligned} X^\mu(\sigma^+, \sigma^-) &= X_R^\mu(\sigma^-) + X_L^\mu(\sigma^+) \\ &= x^\mu - ip^\mu \ln z + i \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu z^{-n}. \end{aligned} \tag{2.16}$$

Note that we have set the Regge slope $\alpha' = 1/2$, $\sigma = 0$ [1] [8] and change the variable $z \equiv \exp(i\tau)$ in the second line of the above equation. The requirement for the reality of X^μ implies

$$\alpha_{-n}^\mu = \left(\alpha_n^\mu\right)^\dagger. \tag{2.17}$$

The Poisson brackets of X^μ and the canonical momentum $\Pi^\mu \equiv \partial L/\partial \dot{X}_\mu = T\dot{X}^\mu$ [2]

[1]This assumption is often used for vertex operators. It means that the emission of a state at the end of the string $\sigma = 0$ at the proper time τ. See GSW book, chapter 7.

[2]By taking the conformal gauge, $\eta^{\alpha\beta} = h^{\alpha\beta} = diag(-1, 1)$. Hence, the Lagrangian is simply $L = (-T/2)\partial_\alpha X^\mu \partial^\alpha X_\mu$. The canonical momentum could be easily carried out.

are defined in that way similar to classical mechanics

$$[X^\mu(\sigma), \Pi^\nu(\sigma')]_{P.B.} = -\eta^{\mu\nu}\delta(\sigma - \sigma'). \tag{2.18}$$

The subscripts "P.B." denotes the Poisson bracket. This definition quickly leads to the Poisson brackets for the position x^μ and momentum p^ν in the C.M. frame, and for the Fourier modes α_n^μ of X^μ, i.e.

$$\begin{aligned} &[\alpha_m^\mu, \alpha_n^\nu]_{P.B.} = im\delta_{m+n}\eta^{\mu\nu}, \\ &[x^\mu, p^\nu]_{P.B.} = -\eta^{\mu\nu}. \end{aligned} \tag{2.19}$$

Now let us turn to the constraint equations (2.11) of α_m^μ's. Equation (2.11) demands that all of the Fourier modes of world sheet $X^\mu(\tau, \sigma)$ have to obey $T_{++} = T_{--} = 0$ in classical level. For open string, we are able to do the Fourier transformation to express T_{++} and T_{--} in terms of the ladder operators, yielding

$$\begin{aligned} L_m &= T\int_0^\pi \left(e^{im\sigma}T_{++} + e^{-im\sigma}T_{--}\right) d\sigma \\ &= \frac{T}{4}\int_{-\pi}^{\pi} e^{im\sigma}\left(\dot{X} + X'\right)^2 d\sigma \\ &= \frac{1}{2}\sum_{n=-\infty}^{\infty} \alpha_{m-n} \cdot \alpha_n \\ &= 0. \end{aligned} \tag{2.20}$$

2.2.2 The quantum version

The first quantization of open bosonic strings is presented in this subsection. It is known that one standard method of getting into the quantum physics from classical is to promote the physical quantities and Fourier modes to operators. This is equivalent by the substitution: replace the classical Poisson brackets with commutator, i.e. $[\cdots]_{P.B.} \to -i[\cdots]$. Thus, equation (2.18) and (2.19) need to be rewritten as follows

$$\begin{aligned} &[X^\mu(\sigma), \Pi^\nu(\sigma')] = i\eta^{\mu\nu}\delta(\sigma - \sigma'), \\ &[\alpha_m^\mu, \alpha_n^\nu] = m\delta_{m+n}\eta^{\mu\nu}, \\ &[x^\mu, p^\nu] = i\eta^{\mu\nu}. \end{aligned} \tag{2.21}$$

Similar analogy are able to be made with simple harmonic oscillators in quantum mechanics. If we normalize α_m^μ's such that $a_m^\mu \equiv \alpha_m^\mu/\sqrt{m}$, then $[a_m^\mu, a_n^{\nu\dagger}] = \eta^{\mu\nu}\delta_{m-n}$. The physical interpretation of a_m^μ's is also very much similar to that in simple harmonic oscillators. For α_m^μ with $m > 0$, it lowers a physical state and as a result $a_{m>0}^\mu|0\rangle = 0$. In contrast, an operator a_{-m}^μ with $m > 0$ rises the level of a physical state. Since the world sheet contains momentum which does not share the same Hilbert space with the oscillation operators. Therefore, a completely ground state for an bosonic open string can be denoted as $|0;\, p\rangle$ satisfies

$$\alpha_{m>0}^\mu|0;\, p\rangle = 0, \tag{2.22}$$

$$\hat{p^\mu}|0;\, p\rangle = p^\mu|0;\, p\rangle. \tag{2.23}$$

The constraints of the classical theory correspond to the vanishing of the energy momentum tensors T_{++} and T_{--} as shown in equation (2.20). In quantum level, the vanishing of L_m in classical theory should be replaced by the positive frequency modes annihilate a physical state $|\psi\rangle$, that is

$$L_{m>0}|\psi\rangle = 0. \tag{2.24}$$

This is much like the Gupta–Bleuler treatment in quantizing the E.M. theory. But L_0 should be discussed independently since there exists an ordering ambiguity due to normal ordering. The normal–ordered expression of L_0 is

$$L_0 = \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n \tag{2.25}$$

up to a to–be–determined constant say a. We include a and demand that a physical state $|\psi\rangle$ must satisfy

$$(L_0 - a)|\psi\rangle = 0. \tag{2.26}$$

Choose $a = 1$ to avoid ghosts. In addition, equation (2.26) carries the information of mass M of open strings. From $M^2 = -p^2$ and the number operator $N \equiv \sum_k \alpha_{-k} \cdot \alpha_k$, we have

$$\begin{aligned} M^2 =& -2 + 2\sum_{k=1}^{\infty} \alpha_{-k} \cdot \alpha_k \\ =& \; 2(N-1). \end{aligned} \tag{2.27}$$

For example, the scalar ground state ($N = 0$) with $M^2 = -2$ is tachyon. In contrast, the first excited state $N = 1$ is given by $\epsilon \cdot \alpha_{-1}|0; p\rangle$, which has $M^2 = 0$ and thus is a massless vector particle with polarization ϵ.

Equation (2.24) and (2.26) form the essential conditions for physical states. L_m and L_0 are the so–called Virasoro generators of bosonic open strings satisfying the the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{D(m^3-m)}{12}\delta_{m+n}, \tag{2.28}$$

where D means the dimension of the space–time, which is 26 if we choose $a = 1$.

Chapter 3

Spinor semiology and the application of BCFW rercursion relation

In this Chapter, a brief introduction to the spinor notations [9–11] is given by solving Dirac equation of a massless particle. After these kind of notations have been introduced, some Lorentz invariant quantities are created in terms of these notations. Having the above preparations, those quantities will be used to build up the 4–gluon scattering amplitudes by employing the BCFW technique. This calculation is provided in the last section.

3.1 Spinor notations

In this section, we would like to introduce the spinor notations. Throughout the whole section, the Lorentz signature $\eta_{\mu\nu} = diag(-1, 1, 1, 1)$ is used. At first, consider a spin-1/2 particle with momentum p. Its behaviors can be understood by solving the Dirac equation

$$\gamma \cdot p\, \psi(p) = 0 \tag{3.1}$$

with $p^2 = 0$. The gamma matrices in the Dirac representation (or standard representation) are

$$\left(\gamma^0\right)_D = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad (\gamma_i)_D = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \tag{3.2}$$

with $\sigma_i,\ i = 1,\, 2,\, 3$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.3}$$

Those are the well–known Pauli matrices.

In order to distinguish the Dirac solutions from the Weyl solutions, we add "W " to denote the case in Weyl representation and "D" for Dirac. The solution of Dirac equation (3.1) are often written as

$$\psi(p) = \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}. \tag{3.4}$$

Expanding equation (3.1) yields

$$-p_0\psi_A + \vec{\sigma}\cdot\vec{p}\psi_B = 0, \quad -\vec{\sigma}\cdot\vec{p}\psi_A + p_0\psi_B = 0. \tag{3.5}$$

Above equations (3.5) give us two choices for the solutions of ψ_A and ψ_B. They are respectively $\psi_A = \psi_B, \quad \psi_A = -\psi_B$. For positive energy $p_0 > 0$, we have

$$\psi_A = \psi_B \Rightarrow \psi_+(p) = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{p_+} \\ \sqrt{p_-}e^{i\phi_p} \\ \sqrt{p_+} \\ \sqrt{p_-}e^{i\phi_p} \end{pmatrix}, \tag{3.6}$$

where the subscript " $+$ " on $\psi(p)$ means positive helicity and $p_+ = p_0 + p_3, \quad p_- = p_0 - p_3$ and

$$e^{i\phi_p} = \frac{p_1 + ip_2}{\sqrt{p_1^2 + p_2^2}} = \frac{p_1 + ip_2}{\sqrt{p_+p_-}}. \tag{3.7}$$

For the case of $\psi_A = -\psi_B$, the corresponding wave function should be

$$\psi_A = -\psi_B \Rightarrow \psi_-(p) = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{p_-}e^{-i\phi_p} \\ -\sqrt{p_+} \\ -\sqrt{p_-}e^{-i\phi_p} \\ \sqrt{p_+} \end{pmatrix}. \tag{3.8}$$

Besides the Dirac representation, the so–called Weyl representation is also common to see.

In Weyl representation, the gamma matrices are

$$\gamma_\mu = \begin{pmatrix} 0 & -\left(\bar{\sigma}_\mu\right)^{\dot{\alpha}\beta} \\ -\left(\sigma_\mu\right)_{\alpha\dot{\beta}} & 0 \end{pmatrix}. \tag{3.9}$$

σ_μ and $\bar{\sigma}_\mu$ are defined in the following

$$\sigma_\mu = \left(I,\ \vec{\sigma}\right), \quad \bar{\sigma}_\mu = \left(I,\ -\vec{\sigma}\right). \tag{3.10}$$

$\vec{\sigma}$ are still the three Pauli matrices.

The off-diagonal gamma matrices in (3.9) imply $\psi(p)$ could be divided into the combination of two 2-component spinors obey different kinds of transformation, i.e.

$$\psi(p) = \begin{pmatrix} \xi^{\dot{\beta}} \\ \eta_\alpha \end{pmatrix}. \tag{3.11}$$

The lower undotted index α and the upper dotted one $\dot{\beta}$ label the components of spinors η and ξ with both of the indices running from 1 to 2. The transformations of η and ξ are assigned in the following: If we denote $\tilde{\eta} = \eta^\dagger$, $\quad \tilde{\xi} = \xi^\dagger$, and the $SL(2, C)$ transformation matrix A with $det(A) = 1$, then we have

$$\eta' = A\eta \to \eta_\alpha, \tag{3.12}$$

$$\tilde{\eta}' = (\eta')^\dagger = \tilde{\eta}A^\dagger \to \tilde{\eta}_{\dot{\alpha}}, \tag{3.13}$$

$$\xi' = \xi A^{-1} \to \xi^\alpha, \tag{3.14}$$

$$\tilde{\xi}' = (\xi')^\dagger = \left(A^{-1}\right)^\dagger \tilde{\xi} \to \tilde{\xi}^{\dot{\alpha}}. \tag{3.15}$$

The consistency of the index structure implies following indices assignments for the transformation matrices:

$$\begin{aligned} &A \to A_\alpha{}^\beta, \quad A^\dagger \to \left(A^\dagger\right)^{\dot{\beta}}{}_{\dot{\alpha}}, \\ &A^{-1} \to \left(A^{-1}\right)_\beta{}^\alpha, \quad \left(A^{-1}\right)^\dagger \to \left(\left(A^{-1}\right)^\dagger\right)^{\dot{\alpha}}{}_{\dot{\beta}} \end{aligned} \tag{3.16}$$

For the reason that the Lagrangian of Dirac equation have to be Lorentz invariant, σ_μ, $\bar{\sigma_\mu}$, A and the Lorentz transformation $L^\mu{}_\nu$ are required to satisfy the following relations:

$$A\sigma_\mu A^\dagger = L_\mu{}^\nu \sigma_\nu \tag{3.17}$$

$$\left(A^\dagger\right)^{-1} \bar{\sigma_\mu} A^{-1} = L_\mu{}^\nu \bar{\sigma_\nu}. \tag{3.18}$$

Consistency requires the following indices assignments for σ_μ and $\bar{\sigma_\mu}$, i.e.

$$\sigma_\mu \to (\sigma_\mu)_{\alpha\dot{\beta}}\,, \quad \bar{\sigma_\mu} \to (\bar{\sigma_\mu})^{\dot{\alpha}\beta}\,. \tag{3.19}$$

If we define a 2×2 matrix ϵ as the following

$$\epsilon \equiv i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\epsilon^{-1}. \tag{3.20}$$

where σ_2 is the Pauli matrix. We can soon find out, by direct calculation, that

$$\epsilon\left(\sigma_\mu\right)^T \epsilon^{-1} = \bar{\sigma}_\mu. \tag{3.21}$$

If we add the indices into the above equation (3.21), it leads to

$$\begin{aligned} &\epsilon^{\dot{\alpha}\dot{\gamma}} \left(\sigma_\mu\right)_{\dot{\gamma}\delta} \left(\epsilon^{-1}\right)^{\delta\beta} = \left(\bar{\sigma}_\mu\right)^{\dot{\alpha}\beta} \\ \Rightarrow &\left(\sigma_\mu\right)_{\dot{\beta}\alpha} = \left(\epsilon^{-1}\right)_{\dot{\beta}\dot{\gamma}} \left(\bar{\sigma}_\mu\right)^{\dot{\gamma}\delta} \epsilon_{\delta\alpha}. \end{aligned} \tag{3.22}$$

From equation (3.22), we can immediately see that ϵ changes a upper undotted (a lower dotted) index into an lower undotted (an upper dotted). In contrast to ϵ, ϵ^{-1} changes an upper dotted (a lower undotted) index into a lower dotted (an upper undotted) . Therefore, the index structures for ϵ and ϵ^- are

$$\epsilon_{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}}, \quad \left(\epsilon^{-1}\right)^{\alpha\beta} = \left(\epsilon^{-1}\right)_{\dot{\alpha}\dot{\beta}} \tag{3.23}$$

such that

$$\eta_\alpha = \epsilon_{\alpha\beta}\,\eta^\beta, \quad \eta^\alpha = (\epsilon^{-1})^{\alpha\beta}\,\eta_\beta. \tag{3.24}$$

ϵ and ϵ^{-1} here are very much similar to the metric tensor $g_{\mu\nu}$.

Expanding the Dirac equation (3.1), η and ξ satisfy

$$\frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\xi = -\xi, \quad \frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\eta = \eta. \tag{3.25}$$

For positive energy $|\vec{p}| = p_0 > 0$, the equation for η means the solution has positive helicity or right–handed while negative helicity or left–handed for ξ. The Weyl representation is related to Dirac representation by a similarity transformation

$$(\gamma_\mu)^W = S\,(\gamma_\mu)^D\,S^{-1},$$

where S is the transformation matrix given by

$$S = \frac{1}{\sqrt{2}}\begin{pmatrix} I & -I \\ I & I \end{pmatrix}. \tag{3.26}$$

This transformation also implies that the solutions of Dirac equation could be transformed to each other by

$$\psi^{(W)}(p) = S\psi^{(D)}(p). \tag{3.27}$$

Thus, the Weyl solutions could be obtained through (3.27)

$$\psi_+^{(W)}(p) = S\psi_+^{(D)}(p) = \begin{pmatrix} 0 \\ 0 \\ \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \sqrt{p_+} \\ \sqrt{p_-}e^{i\phi_p} \end{pmatrix},$$
$$\psi_-^{(W)}(p) = S\psi_-^{(D)}(p) = \begin{pmatrix} \xi^{\dot{1}} \\ \xi^{\dot{2}} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{p_-}e^{-i\phi_p} \\ -\sqrt{p_+} \\ 0 \\ 0 \end{pmatrix}. \tag{3.28}$$

Lorentz scalars could be represented as $\bar{\psi}\psi$, where $\bar{\psi} \equiv \psi^\dagger\gamma^0$. The two Weyl basis η and ξ indicate that there may be two kinds of Lorentz scalars. They could be defined in the way

$$\langle p_i p_j\rangle \equiv \bar{\psi}_-^{(D)}(p_i)\,\psi_+^{(D)}(p_j) = \eta(p_i)^\beta \eta(p_j)_\beta = -\langle p_j p_i\rangle, \tag{3.29}$$

$$[p_i p_j] \equiv \bar{\psi}_+^{(D)}(p_i)\,\psi_-^{(D)}(p_j) = \xi(p_i)_{\dot\beta}\xi(p_j)^{\dot\beta} = -[p_j p_i]. \tag{3.30}$$

They are simply called the *angle bracket* for $\langle\cdots\rangle$ and *square bracket* for $[\cdots]$. In the above, we have taken

$$|p_i\rangle = \eta(p_i)_\beta, \quad \langle p_j| = \eta(p_j)^\beta, \tag{3.31}$$

$$|p_i] = \xi(p_i)^{\dot\beta}, \quad [p_j| = \xi(p_j)_{\dot\beta}. \tag{3.32}$$

The spinors are also related to its null momentum by the identities

$$P_{\dot b\dot a} \equiv (\sigma\cdot p)_{\dot b a} = |p\rangle[p|, \tag{3.33}$$

$$P^{\dot a b} \equiv (\bar\sigma\cdot p)^{\dot a b} = -|p]\langle p|. \tag{3.34}$$

Furthermore, some properties of these brackets could be derived by directly calculation:

$$[p_i p_j] = \langle p_j p_i\rangle^*, \tag{3.35}$$

$$\langle p_i p_j\rangle[p_j p_i] = -2p_i\cdot p_j. \tag{3.36}$$

So far, some useful Lorentz scalars have been made by contracting the an upper dotted index with a lower dotted index spinor, or an upper undotted index with a lower undotted index such as in (3.29) and (3.30). As mentioned at the beginning of this section, we would like to use these Lorentz invariances as building blocks to build up scattering amplitudes. If we consider amplitudes include massless vector bosons, polarizations also require to be rewritten in the language of spinors. Polarization vectors with definite helicities for bosons can be represented as [12,13]

$$\epsilon(k,\,q)_+^\mu = \frac{\langle q|\sigma^\mu|k]}{\sqrt{2}\langle qk\rangle}, \quad \epsilon(k,\,q)_-^\mu = -\frac{[q|\bar\sigma^\mu|k\rangle}{\sqrt{2}[qk]}. \tag{3.37}$$

Here q is some chosen light–like momentum, called *reference momentum.*

3.2 Calculating color–ordered 4–gluon amplitude through BCFW recursion relation technique

Before starting calculating the color–ordered amplitude, we would like to introduce how does "color–ordered" amplitude come from [11].

The gauge transformation for QCD for a spin-1/2 particles is

$$\psi' = e^{i\sum_a T_a \lambda^a}\psi. \tag{3.38}$$

The generators T_a obey the commutative relation

$$[T_a, T_b] = i\sqrt{2} f_{abc} T_c. \tag{3.39}$$

If we are primarily interested in the $SU(N_c)$, T_a can be used as basis for any $N_c \times N_c$ matrices. T_a's also satisfy the relation

$$\sum_{a=1}^{N_c^2-1} (T_a)_{i_1}^{j_1} (T_a)_{i_2}^{j_2} = \delta_{i_1}^{j_2}\delta_{i_2}^{j_1} - \frac{1}{N_c}\delta_{i_1}^{j_1}\delta_{i_2}^{j_2}. \tag{3.40}$$

The structure constants (color factors) f_{abc} can be obtained from

$$f_{abc} = -\frac{i}{\sqrt{2}} Tr\left([T_a, T_b]\, T_c\right). \tag{3.41}$$

Let us now consider the amplitude for four incoming gluons. With the coupling constant suppressed, the scattering amplitude M is composed of s, t and u channels

$$M = M_s + M_t + M_u \tag{3.42}$$

with

$$M_s = f_{a_1a_2b} f_{a_3a_4b}\, A_s, \quad M_u = f_{a_1a_3b} f_{a_2a_4b}\, A_u, \; M_t = f_{a_1a_4b} f_{a_3a_2b}\, A_t. \tag{3.43}$$

From equation (3.41) and (3.40), product of two color factors, for example $f_{a_1a_2b}f_{a_3a_4b}$, can be decomposed as sum of traces of product of T_a's, i.e.

$$f_{a_1a_2b}f_{a_3a_4b} = -\frac{1}{2}\left[Tr(T_{a_1}T_{a_2}T_{a_3}T_{a_4}) - Tr(T_{a_1}T_{a_2}T_{a_4}T_{a_3}) - Tr(T_{a_1}T_{a_3}T_{a_4}T_{a_2}) + Tr(T_{a_1}T_{a_4}T_{a_3}T_{a_2})\right]. \tag{3.44}$$

This color factor is indeed belonged to the s-channel. Similar results can be obtained for both the t and u channels, they are respectively

$$f_{a_1a_3b}f_{a_2a_4b} = -\frac{1}{2}\left[Tr(T_{a_1}T_{a_3}T_{a_2}T_{a_4}) - Tr(T_{a_1}T_{a_3}T_{a_4}T_{a_2}) - Tr(T_{a_1}T_{a_2}T_{a_4}T_{a_3}) + Tr(T_{a_1}T_{a_4}T_{a_2}T_{a_3})\right], \tag{3.45}$$

$$f_{a_1a_4b}f_{a_3a_2b} = -\frac{1}{2}\left[Tr(T_{a_1}T_{a_4}T_{a_3}T_{a_2}) - Tr(T_{a_1}T_{a_4}T_{a_2}T_{a_3}) - Tr(T_{a_1}T_{a_3}T_{a_2}T_{a_4}) + Tr(T_{a_1}T_{a_2}T_{a_3}T_{a_4})\right]. \tag{3.46}$$

With the color decomposition, we can write amplitude M in (3.42) as

$$M = \sum_{j\neq k\neq l}^{2,3,4} M\left(1jkl\right) Tr(T_{a_1}T_{a_j}T_{a_k}T_{a_l}). \tag{3.47}$$

$M\left(1jkl\right)$ inside the summation in the above equation are called the "color-striped" amplitudes since the color factors f_{abc} has been striped away. We find

$$\begin{aligned}
M\left(1432\right) = M\left(1234\right) &= -\frac{1}{2}\left(A_s + A_t\right),\\
M\left(1243\right) = M\left(1342\right) &= \frac{1}{2}\left(A_s + A_u\right),\\
M\left(1324\right) = M\left(1432\right) &= -\frac{1}{2}\left(-A_u + A_t\right).
\end{aligned} \tag{3.48}$$

We can immediately find out that the sum of the right hand sides of the above three equations add up to zero, which means that there are only two independent color–striped amplitudes.

Now that most of the preliminaries have been developed, we shall start calculating the color–ordered amplitude $M\left(1^-2^-3^+4^+\right) = -\frac{1}{2}\left(A_s + A_t\right)$. By choosing the reference momenta which satisfy $q_1 = q_2 = p_4$, $q_3 = q_4 = p_1$, the four-gluon vertex amplitude and the t-channel A_t give zero contribution, only A_s survives. One can carry out this example through standard Feynman's rule, and it turns out the answer is

$$M\left(1^-2^-3^+4^+\right) = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}. \tag{3.49}$$

We next directly borrow the above result and use BCFW recursion relation to re–construct this amplitude. We can use contour integral to write

$$\frac{1}{2\pi i}\oint \frac{dz}{z} M\left(\hat{1}^-2^-3^+\hat{4}^+\right) = M\left(1^-2^-3^+4^+\right)|_{z=0} + \sum_{poles,\, z_a} Res\left[\frac{M\left(\hat{1}^-2^-3^+\hat{4}^+\right)}{z}\right]|_{z=z_a}. \tag{3.50}$$

If the boundary terms vanish as $z \to 0$, we have

$$M\left(1^-2^-3^+4^+\right)|_{z_a=0} = -\sum_{poles,\, z_a\neq 0} Res\left[\frac{M\left(\hat{1}^-2^-3^+\hat{4}^+\right)}{z}\right]\Big|_{z=z_a}. \tag{3.51}$$

The momenta deformation in equation (2.1) is equivalent to deform the spinors in the way that

$$\begin{aligned} |\hat{p_1}\rangle &= |p_1\rangle, \quad [\hat{p_1}| = [p_1| + z[p_4|, \\ |\hat{p_4}\rangle &= |p_4\rangle - z|p_1\rangle, \quad [\hat{p_4}| = [p_4|. \end{aligned} \tag{3.52}$$

It is easy to verify that momentum is conserved by

$$\sum_{i=1}^{4} |p_i\rangle[p_i| = \sum_{i=1}^{4} \sigma\cdot p_i = \sigma\cdot(\hat{p_1}+p_2+p_3+\hat{p_4}) = \sigma\cdot(p_1+p_2+p_3+p_4).$$

Thus,

$$\hat{p_1}+p_2+p_3+\hat{p_4} = p_1+p_2+p_3+p_4 = 0.$$

Obviously, a simple pole arises from the propagator when the intermediate vector $\hat{p}^2 = (\hat{p_1}+p_2)^2$ goes on-shell, that is

$$-(\hat{p_1}+p_2)^2 = 0 = \langle\hat{p_1}p_2\rangle[p_2\hat{p_1}] = \langle p_1p_2\rangle[p_2p_1] + z\langle p_1p_2\rangle[p_2p_4], \tag{3.53}$$

and the pole occurs at

$$z = z_a = -\frac{[p_2p_1]}{[p_2p_4]}. \tag{3.54}$$

It is convenient to express the propagator as

$$\frac{1}{(\hat{p_1}+p_2)^2} = \frac{R_a}{z-z_a} \tag{3.55}$$

in which $R_a = -1/\langle p_1p_2\rangle[p_2p_4]$.

After doing the analytic continuation, the residue (3.52) gives rise from this pole is

$$\text{Res}\left(\frac{A(z)}{z}\right)\Big|_{z=z_a} = M(\hat{1}^-2^-\hat{p}^+)\frac{R_a}{z_a}M(\hat{p}^-3^+\hat{4}^+). \tag{3.56}$$

If momentum p's are allowed to be complex, three–point functions can be defined, they are

$$M\left(1^-2^-3^+\right) = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}, \quad M\left(1^+2^+3^-\right) = \frac{[12]^4}{[12][23][31]}. \tag{3.57}$$

Explicit calculation of equation (3.57) are offered in the Appendix A. Then from (3.57), we have

$$M(\hat{1}^-2^-\hat{p}^+)M(\hat{p}^-3^+\hat{4}^+) = \frac{\langle p_1p_2\rangle^3[p_3p_4]^3}{\langle p_2\hat{p}\rangle[\hat{p}p_3]\,[p_4\hat{p}]\langle\hat{p}p_1\rangle}. \tag{3.58}$$

Put the shifted momenta into (3.58), its denominator goes like

$$\begin{aligned}
&\langle p_2\hat{p}\rangle[\hat{p}p_3]\,[p_4\hat{p}]\langle\hat{p}p_1\rangle \\
&= \{\langle p_2|\,(\,|p_1\rangle\,(\,[p_1|+z[p_4|\,)+|p_2\rangle[p_2|\,)\,|p_4]\}\cdot\{\langle p_1|\,(|p_1\rangle\,([p|+z[p_4|)+|p_2\rangle[p_2|)\,|p_3\rangle\} \\
&= \langle p_2p_1\rangle[p_1p_4]\langle p_1p_2\rangle[p_2p_3].
\end{aligned} \tag{3.59}$$

From the fact that $\langle p_ip_i\rangle = [p_ip_i] = 0$, some of the terms in the denominator of equation (3.58) will vanish. Finally, $M(\hat{1}^-2^-\hat{p}^+)M(\hat{p}^-3^+\hat{4}^+)$ could be found out to be independent of z, i.e.

$$M(\hat{1}^-2^-\hat{p}^+)M(\hat{p}^-3^+\hat{4}^+) = \frac{\langle p_1p_2\rangle^3[p_3p_4]^3}{\langle p_2p_1\rangle[p_1p_4]\langle p_1p_2\rangle[p_2p_3]}. \tag{3.60}$$

Use the result of R_a as well as equation (3.54) and (3.60), the residue (3.56) becomes

$$\frac{[p_2p_4]}{[p_2p_1]}\cdot\frac{1}{\langle p_1p_2\rangle[p_2p_4]}\cdot\frac{\langle p_1p_2\rangle^3[p_3p_4]^3}{\langle p_2p_1\rangle[p_1p_4]\langle p_1p_2\rangle[p_2p_3]}. \tag{3.61}$$

After applying the following formulae,

$$\begin{aligned}
&\langle p_2p_1\rangle[p_1p_4] = -\langle p_2p_3\rangle[p_3p_4], \\
&\langle p_1p_2\rangle[p_2p_3] = -\langle p_1p_4\rangle[p_4p_3], \\
&\langle p_ip_j\rangle = -\langle p_jp_i\rangle, \\
&[p_ip_j] = -[p_jp_i].
\end{aligned}$$

, then equation (3.56) becomes

$$-\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}. \tag{3.62}$$

Put a minus sign in the above equation, we can soon recover the result of equation (3.49).

Chapter 4

Four-tachyon scattering amplitude

When applying BCFW on-shell recursion relation to string amplitudes, pole structures become obscure if we formulate the amplitudes by say Koba–Nielson formulas. We begin with the familiar four–point Koba–Nielson formula, and review how the pole structures are made manifest through binomially expanding this integral formula in [5]. Later, we use our algorithm to solve for the difficulty of summing over infinite number of physical states by enlarging the sum over all physical states to over the completely Fock states [1]. This algorithm comes from the inspiration from the Ward identity in field theory. Finally, we find a mathematical connection between the residue prescribed from BCFW and the generating function for Stirling number of the first kind. This connection is quite useful for the further evaluation when we extend the application of our algorithm to the amplitudes containing higher spin particles.

4.1 Poles extraction

The Koba–Nielson formula for four-tachyon scattering amplitude is given by

$$A(1234) = \int_0^1 dz_2 (1-z_2)^{k_2\cdot k_3}\, z_2^{k_1\cdot k_2}, \tag{4.1}$$

where we use gauge fixing to set $z_1 = 0$, $z_3 = 1$ and $z_4 = +\infty$. For the purpose of extracting poles, we need the following binomial expansion

$$(x-y)^a = \sum_{N=0}^{\infty} \binom{a}{N} x^{a-N}(-y)^N, \tag{4.2}$$

where the combinatorial factor $\binom{a}{N}$ is defined to be

$$\binom{a}{N} \equiv \frac{a(a-1)(a-2)\cdots(a-N+1)}{N!}. \tag{4.3}$$

Inserting the result in equation (4.2) to (4.1), the Koba–Nielson formula becomes

$$A(1234) = \sum_{N=0}^{\infty} \binom{k_2 \cdot k_3}{N} (-1)^N \int_0^1 z_2^{k_1 \cdot k_2 + N}\, dz_2. \tag{4.4}$$

Carry out the world sheet integral over z_2 and use the mass-shell conditions for tachyons $k_1^2 = k_2^2 = -M^2 = +2$, we have $k_1 \cdot k_2 = (k_1+k_2)^2/2 - 2$. After doing so, the s–channel propagator emerges [5]

$$A(1234) = \sum_{N=0}^{\infty} \binom{k_2 \cdot k_3}{N} (-1)^N \frac{2}{(k_1+k_2)^2 + 2(N-1)}. \tag{4.5}$$

Having extracted the propagator $2/\left[(k_1+k_2)^2 + 2(N-1)\right]$ from the tree–level tachyon amplitude, we next would like to re–construct (4.5) by BCFW technique. Manually choose the pair of deformation to be k_1 and k_4, i.e.

$$\hat{k_1}(z) = k_1 + zq, \quad \hat{k_4}(z) = k_4 - zq \tag{4.6}$$

with $q^2 = k_1 \cdot q = k_4 \cdot q = 0$. Assume there is no boundary contributions when z approaches to infinity, (4.5) could be expressed by the BCFW recursion relation

$$A\left(1,2,3,4\right) = \sum_{poles\ z_N} \sum_{physical\ h} A_L\left(\hat{1},\, 2,\, \hat{P}^h\right) \frac{2}{(k_1+k_2)^2 + 2(N-1)} A_R\left(\hat{P}^h,\, 3,\, \hat{4}\right). \tag{4.7}$$

Obviously, there are infinite number of poles exist in the denominator due to infinite tower of mass levels of the intermediate states. For an arbitrary mass level N, they are

$$z_N = -\frac{\left(k_1+k_2\right)^2 + 2\left(N-1\right)}{2q \cdot k_2}, \quad N = 0,\, 1,\, \cdots. \tag{4.8}$$

Compare equation (4.5) and (4.7) at each level N and we have

$$\sum_{physical\ h} A_L(\hat{1},\ 2,\ \hat{P}^h)\, A_R(\hat{P}^h,\ 3,\ \hat{4}) = (-)^N \begin{pmatrix} k_2 \cdot k_3 \\ N \end{pmatrix}. \tag{4.9}$$

Thus, in order to verify the validity of BCFW in the tree level string amplitudes, we have to be able to handle the scalar residue at the left–handed side of equation (4.9) as sum over all intermediate physical states at a given mass level N and prove the equality of (4.9).

4.2 Summing over all physical states

Before straightly doing the summation over all intermediate states in equation (4.9), let us first take a look at a scattering process such as $e^+e^- \to e^+e^-$ in field theory . The propagator of this process is gauge boson. If we shift the first and the fourth particles, the BCFW recursion relation reads

$$A \sim \sum_{state\ h} A_L^\mu(\hat{1},\ 2,\ \hat{P}^h)\eta_{\mu\nu} A_R^\nu(\hat{P}^{-h},\ 3,\ \hat{4}). \tag{4.10}$$

The intermediate states are composed of massless bosons. In (3+1)D flat space–time, we have [14]

$$\eta_{\mu\nu} = \epsilon_\mu^+\epsilon_\nu^- + \epsilon_\mu^-\epsilon_\nu^+ + \epsilon_\mu^T\epsilon_\nu^L + \epsilon_\mu^L\epsilon_\nu^T, \tag{4.11}$$

where ϵ_μ^+ and ϵ_μ^- are the two transverse polarizations with definite helicities while ϵ_μ^T and ϵ_μ^L are respectively the time—like and longitudinal 4–vector. We can replace $\eta_{\mu\nu}$ in equation (4.10) with that in (4.11). But since the Ward identity of gauge theory governs that, for a scattering of n particles, if all $(n-1)$ particles are physical polarized while the n-th particle carries unphysical polarization, the amplitude vanishes. Thus, we have

$$\begin{aligned} A &\sim A_L^\mu(\hat{1},\ 2,\ \hat{P}^h)\left(\epsilon_\mu^+\epsilon_\nu^- + \epsilon_\mu^-\epsilon_\nu^+ + \epsilon_\mu^T\epsilon_\nu^L + \epsilon_\mu^L\epsilon_\nu^T\right) A_R^\nu(\hat{P}^{-h},\ 3,\ \hat{4}) \\ &= A_L^\mu(\hat{1},\ 2,\ \hat{P}^h)\left(\epsilon_\mu^+\epsilon_\nu^- + \epsilon_\mu^-\epsilon_\nu^+\right) A_R^\nu(\hat{P}^{-h},\ 3,\ \hat{4}). \end{aligned} \tag{4.12}$$

Originally in (4.10), we sum over all intermediate states. It turns out that it is equivalent to summing only physical states since the time–like and longitudinal polarization are clearly unphysical.

Having had a glimpse at the scattering amplitude in field theory, we then come to the main task of converting the scalar residue (4.9) as a summation over physical states. Firstly, we have to understand how to construct an arbitrary state from ladder operators with general mass level N in the Fock space [15]. In general, a Fock state can be built up by successively acting creation operators α^{μ}_{-m} with $m > 0$ on a ground state $|0;\, P\rangle$, that is

$$|\,\{N_m\}\,;\, P\rangle = \left[\prod_{m=1}^{\infty} \frac{(\alpha^{\mu}_{-m})^{N_m}}{\sqrt{N_m!\, m^{N_m}}}\right] |0;\, P\rangle. \tag{4.13}$$

The above Fock state $|\,\{N_m\}\,;\, P\rangle$ carries N_1–multiple of α^{μ}_{-1} mode operators and N_2–multiple of α^{μ}_{-2} mode operators and so on. We use $\{N_m\}$ to label the normalized Fock state and, for simplicity, denote $\alpha^{\mu_1}_{-m}\,\alpha^{\mu_2}_{-m}\cdots\cdots\alpha^{\mu_{N_m}}_{-m} = \left(\alpha^{\mu}_{-m}\right)^{N_m}$. The number N_m of the m-th mode operators must satisfy

$$N = \sum_{m=1}^{\infty} m\, N_m. \tag{4.14}$$

It should be emphasized that different tensor indices of α_{-m} are treated as different operators although they have the same mode. However, generic Fock states include "*ghosts*" such as $\alpha^{0\dagger}_{m}|0\rangle$ since it has negative norm, i.e. $\langle 0\,|\alpha^{0}_{m}\alpha^{0\dagger}_{m}|\,0\rangle < 0$. Thus, there comes a problem, we need to get rid of these kind of unphysical states from Fock space while summing over all intermediate states. It turns out to be a difficult objective since we do not know so far how to construct the polarization tensor for a general mass level N. With the implication from the previous discussion of the electron–positron ($e^+e^- \to e^+e^-$) scattering process at the beginning of this subsection, we are able to avoid this difficulty by enlarging the range of summation from the physical states to the entire Fock space of string spectrum. This fact is guaranteed by the so–called "*No-Ghost theorem*". Hence, equation (4.5) can be written as

$$A_n\,(1,2,3,4) = \sum_{poles\ z_N}\ \sum_{Fock} A_L\left(\hat{1},\, 2,\, \hat{P}\right)\frac{1}{P^2+M^2}A_R\left(\hat{P},\, 3,\, \hat{4}\right). \tag{4.15}$$

With this understanding, we are able to write the left–handed side of the residue equation (4.9) as

$$I_N \equiv \sum_{\{\sum mN_m=N\}} \left\langle -\hat{k}_1;0\,\middle|\,V_0(k_2)\,\middle|\,\{N_m\}\,;\hat{P}\right\rangle \mathcal{T}_{\{N_m\}} \left\langle\,\{N_m\}\,;\hat{P}\,\middle|\,V_0(k_3)\,\middle|\,\hat{k}_4;0\,\right\rangle\Bigg|_{z_2=1} \tag{4.16}$$

for a given mass level N. $\mathcal{T}_{\{N_m\}} = \left(\eta_{\mu_1\nu_1}\eta_{\mu_2\nu_2}\cdots\eta_{\mu_{N_m}\nu_{N_m}}\right)$ is the polarization tensor for a set of intermediate states $\{N_m\}$ with definite mass level N, which satisfies equation (4.14). Thus, we can see a subscript $\{N_m\}$ at the bottom of $\mathcal{T}_{\{N_m\}}$. Equation (4.16) is expected to equal

$$\binom{k_2 \cdot k_3}{N}(-1)^N. \tag{4.17}$$

This equality will be explicitly proved in the following subsection 4.4. We can similarly generalize equation (4.7) to an arbitrary n-point function. If we choose k_1 and k_n to be the deformation pair, then the BCFW recursion relation for this scattering amplitude can be calculated by

$$A_n(1,\,\cdots,\,n) = \sum_i \sum_N \sum_{\{N=\sum_m mN_m\}} \langle -\hat{k}_1 | V_2(k_2)\cdots V_{i-1}(k_{i-1})\,|\,\{N_m\}\,;\,\hat{P}_i\rangle \frac{2\mathcal{T}_{\{N_m\}}}{P_i^2 + 2(N-1)}$$
$$\langle\{N_m\}\,;\,\hat{P}_i|\,V_i(k_i)\cdots V_{n-1}(k_{n-1})|\,\hat{k}_n\rangle, \tag{4.18}$$

where $P_i = \sum_i k_i$ is the momenta flow from the adjacent external particles.

4.3 Level matching

In order to quickly match the residue (4.17) getting from the Koba–Nielson formula with that obtaining from BCFW prescription (4.16), one can do level matching to see whether our result is correct or not. For future reference, the explicit expressions of the first 4 levels from equation (4.17) are provided in the following

$\mathbf{N=0} \Rightarrow I_0 = 1,$

$\mathbf{N=1} \Rightarrow I_1 = -k_2\cdot k_3,$

$$\mathbf{N=2} \Rightarrow I_2 = \frac{k_2\cdot k_3\,(k_2\cdot k_3 - 1)}{2!},$$

$$\mathbf{N=3} \Rightarrow I_3 = -\frac{k_2\cdot k_3\,(k_2\cdot k_3-1)\,(k_2\cdot k_3-2)}{3!}.$$

- **Level** $N=0$**:** For $N=0$, all $N_m=0$. We have $\mathcal{T}=1$. Thus, the contribution from this level is simply

$$I_0 = \sum \langle -\hat{k}_1; 0 \,|\, V_0(k_2) \,|0; \hat{p}\rangle \times 1 \times \langle 0; \hat{p} \,|\, V_0(k_3) \,|\, \hat{k}_4; 0\rangle \Big|_{z_2=1} = 1. \tag{4.19}$$

As was expected!

- **Level** $N = 1$**:** In this case, there is only one intermediate state:

$$\{\, \alpha_{-1} \}.$$

$\mathcal{T} = \eta_{\mu\nu}$. Put α^{μ}_{-1} into (4.16), residue I_1 of this level could be easily worked out

$$\begin{aligned} I_1 &= \langle -\hat{k}_1; 0 \,|\, V_0(k_2)\, (\alpha^{\mu}_{-1}) \,|0; \hat{p}\rangle\, \eta_{\mu\nu}\, \langle 0; \hat{p} \,|\, (\alpha^{\nu}_{1})\, V_0(k_3) \,|\, \hat{k}_4; 0\rangle \Big|_{z_2=1} \\ &= -\, k_2 \cdot k_3. \end{aligned} \tag{4.20}$$

Agrees with our argument!

- **Level** $N = 2$**:** In this case, the intermediate states for $N = 2$ are shown as follows:

$$\{\, \frac{\alpha_{-2}}{\sqrt{2}}, \frac{\alpha^2_{-1}}{\sqrt{2!}} \}.$$

$$I_2 = T_1 + T_2. \tag{4.21}$$

I_2 is respectively constituted of 2 components from contracting the right-handed and left-handed 3–amplitudes with 2 distinct intermediate states. For T_1, the tensor structure is $\mathcal{T} = \eta_{\mu\nu}$.

$$\begin{aligned} T_1 &= \langle -\hat{k}_1; 0 \,|\, V_0(k_2)\, (\frac{\alpha^{\mu}_{-2}}{\sqrt{2}}) \,|0; \hat{p}\rangle\, \eta_{\mu\nu}\, \langle 0; \hat{p} \,|\, (\frac{\alpha^{\nu}_{2}}{\sqrt{2}})\, V_0(k_3) \,|\, \hat{k}_4; 0\rangle \,|_{z_2=1} \\ &= \langle -\hat{k}_1; 0 \,|\, e^{-\sum_{n=1}^{\infty} \frac{1}{n} k_2 \cdot \alpha_n} \left(\frac{\alpha^{\mu}_{-2}}{\sqrt{2}}\right) |0; \hat{p}\rangle\, \eta_{\mu\nu}\, \langle 0; \hat{p} \,|\, \left(\frac{\alpha^{\nu}_{2}}{\sqrt{2}}\right) e^{\sum_{n=1}^{\infty} \frac{1}{n} k_3 \cdot \alpha_{-n}} \,|\, \hat{k}_4; 0\, \rangle \\ &= -\, \frac{k_2 \cdot k_3}{2}. \end{aligned} \tag{4.22}$$

As for T_2, $\mathcal{T} = \eta_{\mu_1\nu_1}\eta_{\mu_2\nu_2}$ since this state contains two operators and of course, two sets of tensor indices are required. Thus, its contribution reads

$$\begin{aligned}T_2 &= \langle -\hat{k}_1; 0 \,|\, e^{\sum_{n=1}^{\infty}\frac{1}{n}k_2\cdot\alpha_{-n}}\, e^{-\sum_{n=1}^{\infty}\frac{1}{n}k_2\cdot\alpha_n} \left(\frac{\alpha_{-1}^{\mu_1}\alpha_{-1}^{\mu_2}}{\sqrt{2}}\right) |0;\hat{p}\rangle\, \eta_{\mu_1\nu_1}\eta_{\mu_2\nu_2} \\ &\quad \langle 0;\hat{p}\,| \left(\frac{\alpha_1^{\nu_1}\alpha_1^{\nu_2}}{\sqrt{2}}\right) e^{\sum_{n=1}^{\infty}\frac{1}{n}k_3\cdot\alpha_{-n}} |\, \hat{k}_4; 0\,\rangle \\ &= \frac{k_2^{\mu_1}k_2^{\mu_2}}{\sqrt{2}}\eta_{\mu_1\nu_1}\eta_{\mu_2\nu_2}\frac{k_3^{\nu_1}k_3^{\nu_2}}{\sqrt{2}} \\ &= \frac{(k_2\cdot k_3)^2}{2}.\end{aligned} \tag{4.23}$$

Frankly, the sum of the two components reveals our prediction!

$$\begin{aligned}I_2 &= T_1 + T_2 \\ &= -\frac{k_2\cdot k_3}{2} + \frac{(k_2\cdot k_3)^2}{2} \\ &= \frac{k_2\cdot k_3\,(k_2\cdot k_3 - 1)}{2!}.\end{aligned} \tag{4.24}$$

- **Level** $N = 3$**:** Three intermediate states are needed to be taken into account:

$$\{\frac{\alpha_{-3}}{\sqrt{3}}, \frac{\alpha_{-2}\alpha_{-1}}{\sqrt{2}}, \frac{\alpha_{-1}^3}{\sqrt{3!}}\}.$$

$$I_3 = T_1 + T_2 + T_3. \tag{4.25}$$

T_1 comes from the contribution of $\alpha_{-3}/\sqrt{3}$, and the tensor structure is simply $\mathcal{T} = \eta_{\mu\nu}$.

$$\begin{aligned}T_1 &= \langle -\hat{k}_1; 0 \,|\, V_0(k_2)\, (\frac{\alpha_{-3}^{\mu}}{\sqrt{3}})\, |0;\hat{p}\rangle\, \eta_{\mu\nu}\, \langle 0;\hat{p}\,|\, (\frac{\alpha_3^{\nu}}{\sqrt{3}})\, V_0(k_3)\,|\,\hat{k}_4; 0\rangle\,|_{z_2=1} \\ &= \langle -\hat{k}_1; 0 \,|\, e^{-\sum_{n=1}^{\infty}\frac{1}{n}k_2\cdot\alpha_n}\, (\frac{\alpha_{-3}^{\mu}}{\sqrt{3}})\, |0;\hat{p}\rangle\, \eta_{\mu\nu}\, \langle 0;\hat{p}\,| \left(\frac{\alpha_3^{\nu}}{\sqrt{3}}\right) e^{\sum_{n=1}^{\infty}\frac{1}{n}k_3\cdot\alpha_{-n}}\,|\,\hat{k}_4; 0\rangle \\ &= -\frac{k_2\cdot k_3}{3}.\end{aligned} \tag{4.26}$$

As for T_2, the tensor structure is $\mathcal{T} = \eta_{\mu_1\nu_1}\eta_{\mu_2\nu_2}$. So its contribution reads

$$
\begin{aligned}
T_2 = & \langle -\hat{k}_1; 0 \,|\, e^{\sum_{n=1}^{\infty} \frac{1}{n} k_2 \cdot \alpha_{-n}} \, e^{-\sum_{n=1}^{\infty} \frac{1}{n} k_2 \cdot \alpha_n} \left(\frac{\alpha_{-2}^{\mu_1} \alpha_{-1}^{\mu_2}}{\sqrt{2}} \right) |0; \hat{p}\rangle \, \eta_{\mu_1\nu_1}\eta_{\mu_2\nu_2} \\
& \langle 0; \hat{p} \,|\, \left(\frac{\alpha_2^{\nu_1} \alpha_1^{\nu_2}}{\sqrt{2}} \right) e^{\sum_{n=1}^{\infty} \frac{1}{n} k_3 \cdot \alpha_{-n}} \,|\, \hat{k}_4; 0\rangle \\
= & \langle -\hat{k}_1; 0 \,|\, \frac{(k_2 \cdot \alpha_1)(k_2 \cdot \alpha_2)}{2!} \left(\frac{\alpha_{-2}^{\mu_1} \alpha_{-1}^{\mu_2}}{\sqrt{2}} \right) |0; \hat{p}\rangle \, \eta_{\mu_1\nu_1}\eta_{\mu_2\nu_2} \\
& \langle 0; \hat{p} \,|\, \left(\frac{\alpha_2^{\nu_1} \alpha_1^{\nu_2}}{\sqrt{2}} \right) \frac{(k_2 \cdot \alpha_{-1})(k_2 \cdot \alpha_{-2})}{2!} \,|\, \hat{k}_4; 0\rangle \\
= & \frac{(k_2 \cdot k_3)^2}{2}.
\end{aligned} \tag{4.27}
$$

Furthermore, T_3 could be also computed, and its tensor structure is obviously $\mathcal{T} = \eta_{\mu_1\nu_1}\eta_{\mu_2\nu_2}\eta_{\mu_3\nu_3}$.

$$
\begin{aligned}
T_3 = & \left\langle -\hat{k}_1; 0 \,\middle|\, e^{\sum_{n=1}^{\infty} \frac{1}{n} k_2 \cdot \alpha_{-n}} \, e^{-\sum_{n=1}^{\infty} \frac{1}{n} k_2 \cdot \alpha_n} \left(\frac{\alpha_{-1}^{\mu_1} \alpha_{-1}^{\mu_2} \alpha_{-1}^{\mu_3}}{\sqrt{3!}} \right) \middle| 0; \hat{p} \right\rangle \eta_{\mu_1\nu_1}\eta_{\mu_2\nu_2}\eta_{\mu_3\nu_3} \\
& \left\langle 0; \hat{p} \,\middle|\, \left(\frac{\alpha_1^{\nu_1} \alpha_1^{\nu_2} \alpha_1^{\nu_3}}{\sqrt{3!}} \right) e^{\sum_{n=1}^{\infty} \frac{1}{n} k_3 \cdot \alpha_{-n}} \,\middle|\, \hat{k}_4; 0 \right\rangle \\
= & \left\langle -\hat{k}_1; 0 \,\middle|\, \frac{(-k_2 \cdot \alpha_1)^3}{3!} \left(\frac{\alpha_{-1}^{\mu_1} \alpha_{-1}^{\mu_2} \alpha_{-1}^{\mu_3}}{\sqrt{3!}} \right) \middle| 0; \hat{p} \right\rangle \eta_{\mu_1\nu_1}\eta_{\mu_2\nu_2}\eta_{\mu_3\nu_3} \\
& \left\langle 0; \hat{p} \,\middle|\, \left(\frac{\alpha_1^{\nu_1} \alpha_1^{\nu_2} \alpha_1^{\nu_3}}{\sqrt{3!}} \right) \frac{(k_2 \cdot \alpha_{-1})^3}{3!} \,\middle|\, \hat{k}_4; 0 \right\rangle \\
= & -\frac{(k_2 \cdot k_3)^3}{3!}.
\end{aligned} \tag{4.28}
$$

Thus, sum T_1, T_2 and T_3 together

$$
\begin{aligned}
I_3 &= T_1 + T_2 + T_3 \\
&= -\frac{k_2 \cdot k_3}{3} + \frac{(k_2 \cdot k_3)^2}{2} - \frac{(k_2 \cdot k_3)^3}{3!} \\
&= -\frac{k_2 \cdot k_3 \, (k_2 \cdot k_3 - 1) \, (k_2 \cdot k_3 - 2)}{3!}.
\end{aligned} \tag{4.29}
$$

Same as we have claimed before.

4.4 Explicit calculation of residue

Level matching method is restricted in verifying *only* the first few levels since the calculation becomes more complicated while the mass level of the intermediate states getting higher although it seems to be straightforward. What we are going to do in this section is to demonstrate the explicit calculation of equation (4.16) for an arbitrary mass level N, and prove the equality of (4.16) and equation (4.17).

At first, recall the tachyon vertex operator

$$V_0(k,\, z) =: e^{ik\cdot X(Z)} := Z_0 W_0, \tag{4.30}$$

where

$$Z_0 = e^{ik\cdot x + k\cdot\hat{p}\ln z} = z^{k\cdot\hat{p}-1} e^{ik\cdot x}. \tag{4.31}$$

$\hat{p}$ in equation (4.31) is the momentum operator. W_0 is the pure oscillation part of V_0, which is

$$W_0 = e^{\sum_{n=1}^{\infty}\frac{z^n}{n}k_2\cdot\alpha_{-n}}\, e^{-\sum_{n=1}^{\infty}\frac{z^{-n}}{n}k_2\cdot\alpha_n}. \tag{4.32}$$

Those conventions and definitions could be found in Green, Schwarz and Witten's book [8].

4.4.1 Explicit calculation of 3–point amplitude

Considering the on-shell amplitude, z_2 can be set to be 1. Thus, we just have to take care of the oscillation part W_0. We use A_R and A_L to denoted the right and left 3–amplitudes

$$A_L = A_L(\hat{k}_1, k_2, \hat{P}) = \Big\langle -\hat{k}_1; 0 \Big| V_0(k_2) \Big| \{N_m\}\,; \hat{P} \Big\rangle \Big|_{z_2=1}, \tag{4.33}$$

$$A_R = A_R(-\hat{P}, k_3, \hat{k}_4) = \Big\langle \{N_m\}\,; \hat{P} \Big| V_0(k_3) \Big| \hat{k}_4; 0 \Big\rangle \Big|_{z_2=1}. \tag{4.34}$$

Calculating A_R

Put the definition of $|\,\{N_m\}\,; P\rangle$ and V_0 into A_R, we have

$$\begin{aligned} A_R &= \Big\langle 0; \hat{P} \Big| \left(\prod_{m=1}^{\infty} \frac{(\alpha_m^{\nu})^{N_m}}{\sqrt{m^{N_m}}\, N_m!} \right) e^{\sum_{n=1}^{\infty}\frac{1}{n}k_3\cdot\alpha_{-n}} \Big| \hat{k}_4; 0 \Big\rangle \\ &= \Big\langle 0; \hat{P} \Big| \prod_{m=1}^{\infty} \frac{(\alpha_m^{\nu})^{N_m}}{\sqrt{m^{N_m}}\, N_m!}\, e^{\frac{1}{m}k_3\cdot\alpha_{-m}} \Big| \hat{k}_4; 0 \Big\rangle \\ &= \Big\langle 0; \hat{P} \Big| \prod_{m=1}^{\infty} \frac{(\alpha_m^{\nu})^{N_m}}{\sqrt{m^{N_m}}\, N_m!} \cdot \frac{1}{N_m!} \left(\frac{k_3\cdot\alpha_{-m}}{m} \right)^{N_m} \Big| \hat{k}_4; 0 \Big\rangle. \end{aligned} \tag{4.35}$$

Using the commutative relation $[\alpha_m^\mu, \alpha_n^\nu] = m\delta_{m+n}\eta^{\mu\nu}$ and the identity $\langle 0|\alpha_n^a \alpha_{-n}^a|0\rangle = a!n^a$, where a denotes the power of α_n not a tensor index, yields

$$A_R = \prod_{m=1}^{\infty} \frac{\left(k_3^\nu\right)^{N_m}}{\sqrt{m^{N_m}\, N_m!}}. \tag{4.36}$$

Calculating A_L

A_L can also be able to carry out by following the same manners

$$\begin{aligned} A_L &= \Big\langle -\hat{k}_1; 0 \Big| e^{-\sum_{n=1}^{\infty}\frac{1}{n}k_2\cdot\alpha_n} \left(\prod_{m=1}^{\infty} \frac{(\alpha_{-m}^\mu)^{N_m}}{\sqrt{m^{N_m}\, N_m!}}\right) \Big| 0; \hat{P} \Big\rangle \\ &= \Big\langle -\hat{k}_1; 0 \Big| \prod_{m=1}^{\infty} e^{-\frac{1}{m}k_2\cdot\alpha_m} \frac{(\alpha_{-m}^\mu)^{N_m}}{\sqrt{m^{N_m}\, N_m!}} \Big| 0; \hat{P} \Big\rangle. \end{aligned} \tag{4.37}$$

For the reason that the number of creation and annihilation operators inside the Dirac bracket must be the same otherwise it will vanish. Use the Taylor expansion to expand the exponential part and only the N_m-th order term survives. Then,

$$\begin{aligned} A_L &= \Big\langle -\hat{k}_1; 0 \Big| \frac{1}{N_m!}\left(-\frac{1}{m}k_2\cdot\alpha_m\right)^{N_m} \frac{(\alpha_{-m}^\mu)^{N_m}}{\sqrt{m^{N_m}\, N_m!}} \Big| 0; \hat{P} \Big\rangle \\ &= \prod_{m=1}^{\infty} \frac{\left(-k_2^\mu\right)^{N_m}}{\sqrt{m^{N_m}\, N_m!}}. \end{aligned} \tag{4.38}$$

4.4.2 Contracting A_R and A_L

Using equation (4.16), (4.36) and (4.38), it is easy to calculate I_N for a general mass level N.

$$\begin{aligned} I_N &= \sum_{\{\sum mN_m = N\}} \prod_{m=0}^{\infty} \frac{\left(-k_2^\mu\right)^{N_m}}{\sqrt{m^{N_m}\, N_m!}} \eta_{\mu_1\nu_1}\eta_{\mu_2\nu_2}\cdots\eta_{\mu_{N_m}\nu_{N_m}} \frac{\left(k_3^\nu\right)^{N_m}}{\sqrt{m^{N_m}\, N_m!}} \\ &= \sum_{\left\{\sum_m mN_m = N\right\}} \prod_{m=1}^{\infty} \frac{(-k_2\cdot k_3)^{N_m}}{N_m!\, m^{N_m}}. \end{aligned} \tag{4.39}$$

Notice that N_m and m satisfy the following two relations:

$$\sum_m mN_m = N, \qquad \sum_m N_m = J. \tag{4.40}$$

Using equation (B.9) and the definition (B.8) of Stirling number of the first kind in the Appendix as a generating function , I_N can be rewritten as

$$I_N = \sum_{J=0}^{N} \frac{|s(N,J)|}{N!} (-k_2 \cdot k_3)^J = \binom{k_2 \cdot k_3}{N} (-)^N. \tag{4.41}$$

This is the result we have expected!

Chapter 5

Scattering with higher spin particles

Having a great success in the full–tachyon amplitude, now let us generalize our algorithm to the scattering amplitude contains an arbitrary spin state and three tachyons. As a warmup exercise, we first demonstrate how to explicitly calculate the residue from BCFW prescription of the scattering amplitude of one vector and 3 tachyons by using our algorithm. The corresponding generating function can be found as well but which is slightly different from the four–tachyon case. Then, through path integral approach, the generic structure of generating function of scattering amplitude of an arbitrary spin vertex and 3 tachyons can be systematically worked out [1].

5.1 Scattering amplitude of one vector and 3 tachyons

5.1.1 Algebraic calculation

The vertex operator of a massless vector is

$$V(k,z) =: \epsilon \cdot \dot{X} e^{ik\cdot X(z)} : . \tag{5.1}$$

The 1–vector 3–tachyon scattering amplitude is given by the following integration

$$
\begin{aligned}
A(1\bar{2}34) &= \int_0^1 \langle -k_1; 0|\, \epsilon_2 \cdot \dot{X}(z_2) V_0(k_2, z_2)\, V_0(k_3, z_3)\, |k_4; 0\rangle dz_2 \\
&= (\epsilon_2 \cdot k_3) \int_0^1 dz_2\, |1 - z_2|^{k_2 \cdot k_3 - 1} |z_2|^{k_1 \cdot k_2} - (\epsilon_2 \cdot k_1) \int_0^1 dz_2\, |1 - z_2|^{k_2 \cdot k_3} |z_2|^{k_1 \cdot k_2 - 1},
\end{aligned}
\tag{5.2}
$$

where $\bar{2}$ means the second particle is chosen to be a vector. ϵ_2 here is the polarization of the incoming vector particle carrying momentum k_2 which obey $k_2^2 = k_2 \cdot \epsilon_2 = 0$. By applying gauge fixing, we set $z_1 = 0$, $z_3 = 1$ and $z_4 = \infty$. As the same with the pure tachyon scattering case, we expand $(1 - z_2)^{k_2 \cdot k_3 - 1}$ and $(1 - z_2)^{k_2 \cdot k_3}$ by using equation (4.2), and integrate over z_2, which yields

$$
\begin{aligned}
A(1\bar{2}34) = \quad & (\epsilon_2 \cdot k_3) \sum_{N=1}^{\infty} \binom{k_2 \cdot k_3 - 1}{N - 1} (-)^{N-1} \frac{2}{(k_1 + k_2)^2 + 2(N - 1)} \\
- \quad & (\epsilon_2 \cdot k_1) \sum_{N=0}^{\infty} \binom{k_2 \cdot k_3}{N} (-)^{N} \frac{2}{(k_1 + k_2)^2 + 2(N - 1)},
\end{aligned}
\tag{5.3}
$$

where N is the mass level of the intermediate states. The propagator $2/[\,(k_1+k_2)^2+2(N-1)\,]$ has now been extracted. It is obvious that the residue of equation (5.3) is consisted of two terms, one of the term is proportional to $\epsilon_2 \cdot k_1$ and the other one is proportional to $\epsilon_2 \cdot k_3$. The term proportional to $\epsilon_2 \cdot k_1$, i.e. $\binom{k_2 \cdot k_3}{N}(-1)$, is simply corresponding to the 4-tachyon amplitude. But, a new term comes from the other one proportional to $\epsilon_2 \cdot k_3$.

The residue of 1–vector 3–tachyon amplitude from BCFW prescription can be written as

$$
\sum_{\{\sum_m mN_m = N\}} \langle 0; -k_1 \,|\, (\epsilon_2 \cdot \dot{X})\, V\,(k_2, z_2)\,|\,\{N_m\}\,;\, p\rangle \mathcal{T}_{\{N_m\}} \langle \{N_m\}\,;\, p|\, V\,(k_3, z_3)\,|0;\, k_4\rangle|_{z_2 = z_3 = 1}.
\tag{5.4}
$$

Differ from the 4–tachyon amplitude, an extra term $\epsilon_2 \cdot \dot{X}$ locates at the second particle. Algebraically, the term proportional to $\epsilon_2 \cdot k_1$ in equation (5.3) can be obtained by acting the operator $\epsilon_2 \cdot \hat{p}$ in $\epsilon_2 \cdot \dot{X} = \epsilon_2 \cdot (\hat{p} + \sum_{n=1} \alpha_n e^{-in\tau})$ [1] to the bra $\langle -k_1;\, 0|$, which reproduces

[1] $\sum_{n=1} \alpha_{-n} e^{in\tau}$ gives no contribution since $\langle 0 \,|\, \alpha_{-n} = 0$, for $n > 0$. Thus, $\epsilon_2 \cdot \dot{X}$ could be equivalently written in that way.

$-\epsilon_2 \cdot k_1$. The rest of its kinematic dependence could be easily shown to be the pure tachyon residue as in equation (4.16), that is

$$\sum_{\{\sum mN_m\}} \langle -k_1;\, 0|\, V\,(k_2, z_2\,)\,|\,\{N_m\}\,;\, p\rangle\, \mathcal{T}_{\{N_m\}}\, \langle\, \{N_m\}\,;\, p|\, V\,(k_3, z_3)\,|0;\, k_4\,\rangle,$$

which confirms our prediction. And yet, the term proportional to $\epsilon_2 \cdot k_3$ is

$$I_N = \sum_{\{\sum mN_m = N\}} \Big\langle -k_1; 0 \Big| \Big(\sum_{n=1}^{\infty} \epsilon_2 \cdot \alpha_n\, z_2^{-n}\Big) V_0(k_2) \Big| \{N_m\}\,; p \Big\rangle\, \mathcal{T}_{\{N_m\}} \Big\langle \{N_m\}\,; p \Big| V_0(k_3) \Big| k_4; 0 \Big\rangle \Big|_{z_2=1}. \tag{5.5}$$

Residue (5.5) has to be equal to

$$(\epsilon_2 \cdot k_3) \sum_{N=1}^{\infty} \binom{k_2 \cdot k_3 - 1}{N-1} (-)^{N-1} \tag{5.6}$$

in equation (5.3). The next step is to show the equality between equation (5.5) and (5.6). For this example, the level matching and evaluation for the generating function are left in the appendix C.

5.1.2 Explicit derivation for the related residue of interest

Having done the basic algebraic calculation in the previous subsection, we would like to explicitly deduce the equality between equation (5.5) and (5.6). For a general mass level N, this kinematic dependence I_N can be derived from gluing two 3-point amplitudes, that is

$$I_N = \sum_{\{\sum mN_m = N\}} \Big\langle -k_1; 0 \Big| \Big(\sum_{n=1}^{\infty} \epsilon_2 \cdot \alpha_n\Big) V_0(k_2) \Big| \{N_m\}\,; p \Big\rangle\, \mathcal{T}_{\{N_m\}} \Big\langle \{N_m\}\,; p \Big| V_0(k_3) \Big| k_4; 0 \Big\rangle \Big|_{z_2=1}.$$

We denote the two 3-point on–shell amplitudes (the left part and right one) as

$$A_L = A_L(k_1, k_2, P) = \Big\langle -k_1; 0 \Big| \Big(\sum_{n=1}^{\infty} \epsilon_2 \cdot \alpha_n\Big) V_0(k_2) \Big| \{N_m\}\,; p \Big\rangle \Big|_{z_2=1},$$
$$A_R = A_R(-P, k_3, k_4) = \Big\langle \{N_m\}\,; p \Big| V_0(k_3) \Big| k_4; 0 \Big\rangle \Big|_{z_2=1}.$$

The explicit calculation of 3-point amplitudes

A_R has been obtained before, which is

$$A_R = \prod_{m=1}^{\infty} \frac{\left(k_3^{\nu}\right)^{N_m}}{\sqrt{m^{N_m}\, N_m!}}. \tag{5.7}$$

On the other hand, the algebraic structure of A_L is new to us, which is given by

$$\begin{aligned} A_L &= \left\langle 0 \middle| \left(\sum_{n=1}^{\infty} \epsilon_2 \cdot \alpha_n \right) V_0(k_2) \middle| \{N_m\} \right\rangle \Big|_{z_2=1} \\ &= \sum_{n=1}^{\infty} \left\langle 0 \middle| \left(\epsilon_2 \cdot \alpha_n \right) V_0(k_2) \prod_{m=1}^{\infty} \frac{\left(\alpha_{-m}^{\mu}\right)^{N_m}}{\sqrt{m^{N_m}\, N_m!}} \middle| 0 \right\rangle \Bigg|_{z_2=1}. \end{aligned} \tag{5.8}$$

After setting $z_2 = 1$, the tachyonic vertex $V_0(k_2)$ becomes

$$V_0(k_2) = \exp\left[\sum_{p=1}^{\infty} \frac{k_2 \cdot \alpha_{-p}}{p} \right] \exp\left[- \sum_{p=1}^{\infty} \frac{k_2 \cdot \alpha_p}{p} \right].$$

When moving $\epsilon_2 \cdot \alpha_n$ to the right, it will first encounter the term $\exp\left[\, \sum_{p=1}^{\infty} (k_2 \cdot \alpha_{-p})/p \right]$. But remember that the polarization ϵ_2 and the momentum k_2 are orthogonal; therefore, only the identity of this term survives after α_n contracting with α_{-p}. Thus

$$\begin{aligned} A_L &= \sum_{n=1}^{\infty} \left\langle 0 \middle| \left(\epsilon_2 \cdot \alpha_n \right) \prod_{m=1}^{\infty} e^{-\frac{k_2 \cdot \alpha_m}{m}} \frac{\left(\alpha_{-m}^{\mu}\right)^{N_m}}{\sqrt{m^{N_m}\, N_m!}} \middle| 0 \right\rangle \\ &= \sum_{n=1}^{\infty} \left\langle 0 \middle| \left(\epsilon_2 \cdot \alpha_n \right) \left[e^{-\frac{k_2 \cdot \alpha_n}{n}} \frac{\left(\alpha_{-n}^{\mu}\right)^{N_n}}{\sqrt{n^{N_n}\, N_n!}} \right] \prod_{m=1, m\neq n}^{\infty} e^{-\frac{k_2 \cdot \alpha_m}{m}} \frac{\left(\alpha_{-m}^{\mu}\right)^{N_m}}{\sqrt{m^{N_m}\, N_m!}} \middle| 0 \right\rangle. \end{aligned} \tag{5.9}$$

Using Taylor expansion to expand the exponential part inside the square bracket in the equation (5.9) and moving those things to the left, we see that only the $(N_n - 1)$-th order term do not vanish. Thus,

$$\begin{aligned} A_L = \sum_{n=1}^{\infty} \Bigg\langle 0 \Bigg| \left(\epsilon_2 \cdot \alpha_n \right) &\left[\frac{1}{(N_n - 1)!} \left(- \frac{k_2 \cdot \alpha_n}{n} \right)^{N_n - 1} \frac{\left(\alpha_{-n}^{\mu}\right)^{N_n}}{\sqrt{n^{N_n}\, N_n!}} \right] \\ &\prod_{m=1, m\neq n}^{\infty} e^{-\frac{k_2 \cdot \alpha_m}{m}} \frac{\left(\alpha_{-m}^{\mu}\right)^{N_m}}{\sqrt{m^{N_m}\, N_m!}} \Bigg| 0 \Bigg\rangle. \end{aligned} \tag{5.10}$$

Using the commutative relation $[\,\alpha_m^\mu\ ,\ \alpha_n^\nu\,] =\ m\delta_{m+n}\eta^{\mu\nu}$, we will reach at

$$A_L \;=\; \sum_{n=1}^{\infty}\left\{\left[\frac{(-)^{N_n-1}\, n\, N_n\, \epsilon_2^\mu\,(k_2^\mu)^{N_n-1}}{\sqrt{n^{N_n}\,N_n!}}\right]\prod_{m=1,m\neq n}^{\infty}\frac{\left(-\,k_2^\mu\right)^{N_m}}{\sqrt{m^{N_m}\,N_m!}}\right\}. \tag{5.11}$$

Gluing A_L and A_R

Combine the left 3-point amplitude A_L in equation (5.11) with the right one A_R in (5.7), we obtain

$$\begin{aligned}
&\Big\langle -k_1;0\Big|\Big(\sum_{n=1}^{\infty}\epsilon_2\cdot\alpha_n\Big)\,V_0(k_2)\Big|\{N_m\}\,;p\Big\rangle\,\mathcal{T}_{\{N_m\}}\Big\langle\{N_m\}\,;p\Big|V_0(k_3)\Big|k_4;0\Big\rangle\\
&=\sum_{n=1}^{\infty}\left\{\left[\frac{(-)^{N_n-1}\,nN_n\,(\epsilon_2\cdot k_3)\,(k_2\cdot k_3)^{N_n-1}}{n^{N_n}\,N_n!}\right]\prod_{m=1,m\neq n}^{\infty}\frac{\left(-\,k_2\cdot k_3\right)^{N_m}}{m^{N_m}\,N_m!}\right\}\\
&=\sum_{n=1}^{\infty}\left[-\,\frac{nN_n\,(\epsilon_2\cdot k_3)}{k_2\cdot k_3}\right]\prod_{m=1}^{\infty}\frac{\left(-\,k_2\cdot k_3\right)^{N_m}}{m^{N_m}\,N_m!}.
\end{aligned}\tag{5.12}$$

For any given mass level N, we have

$$N=\sum_{n=1}^{\infty}nN_n.$$

Thus, the result (5.12) we just obtain above can be written as

$$-\,N\,\frac{\epsilon_2\cdot k_3}{k_2\cdot k_3}\prod_{m=1}^{\infty}\frac{\left(-\,k_2\cdot k_3\right)^{N_m}}{m^{N_m}\,N_m!}. \tag{5.13}$$

Summing over all physical states, we have the residue I_N:

$$\boxed{I_N=\sum_{\left\{N=\sum_m mN_m\right\}}(-)\,N\,\frac{\epsilon_2\cdot k_3}{k_2\cdot k_3}\prod_{m=1}^{\infty}\frac{\left(-\,k_2\cdot k_3\right)^{N_m}}{m^{N_m}\,N_m!}.}$$

5.1.3 Recover the result

The generating function for the residue proportional to $\epsilon_2 \cdot k_3$ is

$$\left(\frac{\epsilon_2 \cdot k_3}{k_2 \cdot k_3}\right) z \frac{d}{dz} e^{k_2 \cdot k_3 \ln(1-z)}$$
$$= (\epsilon_2 \cdot k_3)\, z \frac{d}{dz} [\ln(1-z)]\, e^{k_2 \cdot k_3 \ln(1-z)}. \tag{5.14}$$

The quick evaluation of this generating function is left in Appendix C.2. Use the relations (B.3) and (B.4) in the appendix and I_N can be rewritten as

$$I_N = (-)^{N-1} N \frac{\epsilon_2 \cdot k_3}{k_2 \cdot k_3} \sum_{J=0}^{N} \frac{s(N,J)}{N!} (k_2 \cdot k_3)^J \tag{5.15}$$
$$= \epsilon_2 \cdot k_3 \sum_{J=1}^{N} \frac{s(N,J)}{N!} (-)^{N-1} N (k_2 \cdot k_3)^{J-1}. \tag{5.16}$$

Refering to (B.10) in the appendix and setting $X = k_2 \cdot k_3$ in that relation, we soon recover the result:

$$\boxed{I_N = \epsilon_2 \cdot k_3 \, (-)^{N-1} \binom{k_2 \cdot k_3 - 1}{N-1}.}$$

5.2 Generating function from path integral approach

As we shall see in the previous examples of full–tahcyon and 1–vector 3–tachyon amplitude, we can always find a generating function related to the residue prescribed by BCFW. In the following, instead of the operator method used previously, we adopt path integral approach [16] to calculate the generating functions. We first take the 1–vector 3–tachyon amplitude as an example to illustrate how to re–derive its generating function (5.14) from path integral formalism.

Note that the amplitude can be written as

$$A(1\bar{2}34) = \int \prod_{i=1}^{4} dz_i \langle : e^{ik_1 \cdot X(z_1)} :: \epsilon_2 \cdot \dot{X} e^{ik_2 \cdot X(z_2)} :: e^{ik_3 \cdot X(z_3)} :: e^{ik_4 \cdot X(z_4)} : \rangle, \tag{5.17}$$

where $\langle \cdots \rangle$ is short for $\langle 0 | \cdots | 0 \rangle$. For convenience, one can exponentiate $\epsilon_2 \cdot k_3$ up to the exponent and take the linear term in ϵ_2 at the end of calculation. Here, one also need the identity [8] $\langle : e^{A_1} :: e^{A_2} : \cdots : e^{A_M} : \rangle = \exp\left[\sum_{i<j}\langle A_i A_j\rangle\right]$. Use the world sheet $SL(2,R)$ to set $z_1 = 0$, $z_3 = 1$ and $z_4 = \infty$. We have [16]

$$A(1\bar{2}34) = \int \prod_{i=1}^{4} dz_i \langle : e^{ik_1\cdot X(z_1)} :: e^{ik_2\cdot X(z_2)+\epsilon_2\cdot\dot{X}} :: e^{ik_3\cdot X(z_3)} :: e^{ik_4\cdot X(z_4)} : \rangle|_{linear\ in\ \epsilon_2} \tag{5.18}$$

$$= \int \prod_{i=1}^{4} dz_i e^{-\sum_{j<l} k_{j\mu}k_{l\nu}\langle X^\mu(z_j)X^\nu(z_l)\rangle + i\sum_{j\neq 2}\epsilon_{2\mu}k_{j\nu}\langle \dot{X}(z_2)X(z_j)\rangle}|_{linear\ in\ \epsilon_2} \tag{5.19}$$

$$= \int_0^1 dz_2\,(1-z_2)^{k_2\cdot k_3}\, z_2^{k_1\cdot k_2}\left[\frac{\epsilon_2\cdot k_1}{z_2} - \frac{\epsilon_2\cdot k_3}{1-z_2}\right]. \tag{5.20}$$

The propagator in the above equation is $\langle X^\mu(x)X^\nu(y)\rangle = -\eta^{\mu\nu}\ln(x-y)$. The term proportional to $\epsilon_2\cdot k_1$ has been considered before, which is the same as in the case of pure tachyon amplitude. Thus, we just neglect the analysis on it. The term of our interest is the one proportional to $\epsilon_2\cdot k_3$ in equation (5.20). Please note that, in the previous discussion we binomially expand $(1-z)^{k_2\cdot k_3}$ and take the world sheet integral to obtain the propagator. Obviously, $z^{k_1\cdot k_2}$ has been integrated away, hence does not involve in the generating function. This implies that the generating function should be

$$\begin{aligned} G_1 &= e^{\{k_2\cdot k_3\ln(1-z_2)\}} e^{\left\{\epsilon_2\cdot k_3\, z_2 \frac{d}{dz_2}\ln(1-z_2)\right\}}|_{linear\ in\ \epsilon_2} \\ &= (\epsilon_2\cdot k_3)\, z_2\frac{d}{dz_2}\left[\ln(1-z_2)\right] e^{[k_2\cdot k_3\ln(1-z_2)]}, \end{aligned} \tag{5.21}$$

which is exactly the same with (5.14). Please note that there is an extra term $z\frac{d}{dz}\ln(1-z)$ in equation (5.21) if we compare this equation with the generating function for the pure tachyon case. This result can be viewed from the term $\epsilon_2\cdot\dot{X}$ in the vertex operator, which results in $\langle \dot{X}^\mu(x)X^\nu(y)\rangle$ in equation (5.19),

$$\langle \dot{X}^\mu(x)X^\nu(y)\rangle = ix\frac{d}{dx}\langle X^\mu(x)X^\nu(y)\rangle = -i\eta^{\mu\nu}x\frac{d}{dx}\ln(x-y). \tag{5.22}$$

Following the same spirit, one can generalize this method to a more complicated case such as a vertex containing arbitrarily n-multiple of $\epsilon\cdot\dot{X}$'s, that is

$$V(k_2,\, z_2) =: \left(\epsilon_2^{(1)}\cdot\dot{X}\right)\left(\epsilon_2^{(2)}\cdot\dot{X}\right)\times\cdots\times\left(\epsilon_2^{(n)}\cdot\dot{X}\right)e^{ik_2\cdot X(z_2)} : . \tag{5.23}$$

This higher spin particle has mass level n, therefore, $k_2^2 = -M^2 = -2(n-1)$. From the path integral formalism, the amplitude should be

$$A(1\bar{2}34) = \int \prod_{i=1}^{4} dz_i \langle : e^{ik_1\cdot X(z_1)} :: \left(\epsilon_2^{(1)}\cdot\dot{X}\right)\left(\epsilon_2^{(2)}\cdot\dot{X}\right)\times\cdots\times\left(\epsilon_2^{(n)}\cdot\dot{X}\right) e^{ik_2\cdot X(z_2)} : \\ : e^{ik_3\cdot X(z_3)} :: e^{ik_4\cdot X(z_4)} :\rangle \tag{5.24}$$

$$= \int \prod_{i=1}^{4} dz_i \langle e^{ik_1\cdot X(z_1)} :: e^{ik_2\cdot X(z_2)+\epsilon_2^{(1)}\cdot\dot{X}+\epsilon_2^{(2)}\cdot\dot{X}+\cdots+\epsilon_2^{(n)}\cdot\dot{X}} : \\ : e^{ik_3\cdot X(z_3)} :: e^{ik_4\cdot X(z_4)} :\rangle|_{linear\ in\ \epsilon_2^{(1)},\epsilon_2^{(2)},\cdots,\epsilon_2^{(n)}} \tag{5.25}$$

$$= \int \prod_{i=1}^{4} dz_i \exp\{-\sum_{j<l} k_{j\mu}k_{l\nu}\langle X^\mu(z_j)X^\nu(z_l)\rangle + i\sum_{j\neq 2}\epsilon_{2\mu}^{(1)}k_{j\nu}\langle\dot{X}^\mu(z_2)X^\nu(z_j)\rangle + \\ i\sum_{j\neq 2}\epsilon_{2\mu}^{(2)}k_{j\nu}\langle\dot{X}^\mu(z_2)X^\nu(z_j)\rangle + \cdots + i\sum_{j\neq 2}\epsilon_{2\mu}^{(n)}k_{j\nu}\langle\dot{X}^\mu(z_2)X^\nu(z_j)\rangle\}|_{multilinear\ in\ \epsilon_2^{(1)},\epsilon_2^{(2)},\cdots,\epsilon_2^{(n)}} \tag{5.26}$$

$$= \int_0^1 dz\,(1-z)^{k_2\cdot k_3} z^{k_1\cdot k_2}\left[\frac{\epsilon_2^{(1)}\cdot k_1}{z} - \frac{\epsilon_2^{(1)}\cdot k_3}{1-z}\right]\left[\frac{\epsilon_2^{(2)}\cdot k_1}{z} - \frac{\epsilon_2^{(2)}\cdot k_3}{1-z}\right]\times \\ \cdots\times\left[\frac{\epsilon_2^{(n)}\cdot k_1}{z} - \frac{\epsilon_2^{(n)}\cdot k_3}{1-z}\right]. \tag{5.27}$$

Having the above experience, some other terms can be obtained from amplitudes containing relative lower spin vertex. Thus, we know the new term is the one proportional to $\left(\epsilon_2^{(1)}\cdot k_3\right)\left(\epsilon_2^{(2)}\cdot k_3\right)\times\cdots\times\left(\epsilon_2^{(n)}\cdot k_3\right)$, and thus it is

$$\int_0^1 dz\,(1-z)^{k_2\cdot k_3} z^{k_1\cdot k_2}\left\{(-)^n\frac{\left(\epsilon_2^{(1)}\cdot k_3\right)\left(\epsilon_2^{(2)}\cdot k_3\right)\times\cdots\times\left(\epsilon_2^{(n)}\cdot k_3\right)}{(1-z)^n}\right\} \\ = (-)^n\left(\epsilon_2^{(1)}\cdot k_3\right)\left(\epsilon_2^{(2)}\cdot k_3\right)\times\cdots\times\left(\epsilon_2^{(n)}\cdot k_3\right)\int_0^1 dz_2\,(1-z_2)^{k_2\cdot k_3-n} z_2^{k_1\cdot k_2}. \tag{5.28}$$

Binomially expand $(1-z_2)^{k_2\cdot k_3-n}$ at the end of equation (5.28), we reach

$$\sum_{a=0}^{\infty}\int_0^1 dz_2\begin{pmatrix} k_2\cdot k_3-n \\ a\end{pmatrix}(-)^a z_2^{k_1\cdot k_2+a} \\ = \sum_{a=n}^{\infty}\begin{pmatrix} k_2\cdot k_3-n \\ a-n\end{pmatrix}(-)^{a-n}\frac{2}{(k_1+k_2)^2+2(a-1)}. \tag{5.29}$$

From the previous experience, throw the propagator in the above equation (5.29) away, and the rest should be the residue from BCFW prescription. The generating function G_n for this new term can be seen from equation (5.26)

$$G_n = e^{\left\{\epsilon_2^{(1)}\cdot k_3\, z\frac{d}{dz}\ln(1-z)\right\}} e^{\left\{\epsilon_2^{(2)}\cdot k_3\, z\frac{d}{dz}\ln(1-z)\right\}} \times \cdots$$
$$\times\, e^{\left\{\epsilon_2^{(n)}\cdot k_3\, z\frac{d}{dz}\ln(1-z)\right\}} e^{\{k\cdot k_3 \ln(1-z)\}}|_{multilinear\ in\ \epsilon_2^{(1)},\epsilon_2^{(2)},\cdots,\epsilon_2^{(n)}} \tag{5.30}$$
$$= \left[\left(\epsilon_2^{(1)}\cdot k_3\right) z\frac{d}{dz}\ln(1-z)\right]\left[\left(\epsilon_2^{(2)}\cdot k_3\right) z\frac{d}{dz}\ln(1-z)\right]\times\cdots$$
$$\times\left[\left(\epsilon_2^{(n)}\cdot k_3\right) z\frac{d}{dz}\ln(1-z)\right] e^{k_2\cdot k_3\ln(1-z)} \tag{5.31}$$
$$= \left(\epsilon_2^{(1)}\cdot k_3\right)\left(\epsilon_2^{(2)}\cdot k_3\right)\times\cdots\times\left(\epsilon_2^{(n)}\cdot k_3\right)\sum_{a=n}^{\infty}\begin{pmatrix} k_2\cdot k_3 - n \\ a-n\end{pmatrix}(-)^a z^a. \tag{5.32}$$

After setting $z=1$ in equation (5.32), this is exactly the residue from BCFW prescription of the new term in equation (5.29). Equation (5.31) contains n derivative terms $z(d/dz)\ln(1-z)$. They can be traced back to the n–multiple of $\epsilon\cdot\dot{X}$'s in the vertex operator.

From the above arguments, we can conclude that the derivative term $z\,(d/dz)\ln(1-z)$ appears in the generating function can be traced back to the term $\epsilon\cdot\dot{X}$ of the vertex operator. This term $\epsilon\cdot\dot{X}$ will later results in $\langle\dot{X}^\mu(x)X^\nu(y)\rangle\propto x\,(d/dx)\ln(x-y)$ and thus exists in the generating function. The key feature connecting the generating function with vertex operator has been made manifest now, generalization to more complicated configuration of vertex operators can be established. For example, consider this emission vertex for the second excited state

$$V(k_2,z_2) =: \epsilon_2\cdot\ddot{X}(z_2)e^{ik_2\cdot X(z_2)}:. \tag{5.33}$$

Having the previous experience, to obtain the generating function of the vertex (5.33), we have to calculate $\langle\ddot{X}(x)X(y)\rangle$

$$\langle\ddot{X}^\mu(x)X^\nu(y)\rangle = -\eta^{\mu\nu}x\frac{d}{dx}x\frac{d}{dx}\ln(x-y). \tag{5.34}$$

Multiplying $z(d/dz)z(d/dz)\ln(1-z)$ by $\epsilon_2\cdot k_3$ and the factor $\exp\left[k_2\cdot k_3\ln(1-z)\right]$ leads to the generating function of the vertex (5.33)

$$G_2 = (\epsilon_2\cdot k_3)\,z\frac{d}{dz}z\frac{d}{dz}\left[\ln(1-z)\right]e^{k_2\cdot k_3\ln(1-z)}. \tag{5.35}$$

The above calculation can then be generalized to the vertex operator having arbitrarily n-th order derivative of $X^\mu(\tau)$, namely

$$V(k_2,z_2) =: \epsilon_2\cdot\left(\partial_{\tau_2}^{(n)}X\right)e^{ik_2\cdot X(z_2)}:. \tag{5.36}$$

Following the same procedures, we first evaluate

$$\langle \frac{\partial^n X^\mu(x)}{\partial \tau_x^n} X^\nu(y) \rangle = \eta^{\mu\nu} \left(x \frac{d}{dx} \right)^n \ln(x-y), \tag{5.37}$$

and the generating function of the scattering amplitude with vertex (5.36) and three tachyons can be easily calculated by

$$G_n = (\epsilon_2 \cdot k_3) \left(z \frac{d}{dz} \right)^n \ln(1-z) e^{k_2 \cdot k_3 \ln(1-z)}. \tag{5.38}$$

Combining the algebraic structures of equation (5.31) and (5.38), we are able to obtain the generating function of a scattering amplitude of 3 tachyons and one arbitrary spin state, for example

$$\left(\epsilon_1 \cdot \dot{X} \right) \left(\epsilon_2 \cdot \ddot{X} \right) \left(\epsilon_3 \cdot \ddot{X} \right) \cdots : e^{ik \cdot X} :, \tag{5.39}$$

is given by

$$G = \left(Y_1 \, z \frac{d}{dz} \ln(1-z) \right) \left(Y_2 \, z \frac{d}{dz} z \frac{d}{dz} \ln(1-z) \right) \left(Y_3 \, z \frac{d}{dz} z \frac{d}{dz} \ln(1-z) \right) \cdots e^{X \ln(1-z)}, \tag{5.40}$$

where $Y_1 = \epsilon_1 \cdot k$, $Y_2 = \epsilon_2 \cdot k$ and $Y_3 = \epsilon_3 \cdot k$.

5.3 Brief summary

In the first section of this chapter, we first generalize our algorithm from four–tachyon amplitude to an amplitude of one vector and three tachyons. Generating function for this case still exists. In section 5.2, we show how to adopt path integral approach to systematically find out the generating function of a scattering amplitude of one arbitrary string state and 3 tachyons.

Chapter 6

Conclusion

Let us summarize what we have achieved in the thesis. Motivated by the success of BCFW recursion relation in field theory, we are interested in applying this method to string theory. In field theory, Feynman's rule helps us to write down scattering amplitude systematically. In BCFW calculation of field theory, the BCFW poles are determined by Feynman propagators. However, when applying BCFW method to string amplitudes, the BCFW poles for string amplitude such as Veneziano amplitude are not manifest. In [5], Clifford Cheung et at binomially expanded the integrand of Veneziano amplitude to extract the pole structure of string amplitude.

In this thesis, we extend the application of BCFW recursion relation to string tree-level amplitudes. In contrast to the field theory calculation, we encounter the difficulty of summing over all intermediate physical states with infinite tower of mass levels. We develop a method to resolve this difficulty by enlarging the sum over all intermediate physical states to an easier sum over the entire Fock space of string spectrum. The zero contributions of extra states are guaranteed by the no-ghost theorem in the open bosonic string theory. In this calculation, we do produce the conjectured scalar-behaved residue observed in [5]. The calculation is successfully applied to the 4-tachyon amplitude, and then to the cases of one arbitrary higher spin state and 3-tachyon amplitudes. For the cases of higher spin scatterings, we figure out a generating function for summing the infinite poles of string spectrum in the BCFW string amplitude calculation. We also find out that we can use identities of the Stirling number of the first kind to sum over the string

Fock space. The generic structure of this generating function for higher spin scattering amplitude can be obtained from the standard path integral calculation of string scattering amplitude.

Our work has lead to several interesting topics for further study. So far, we just considered the string scattering amplitude of 3 tachyons and one arbitrary string state. It would be natural to extend to 4–point amplitudes containing more than one arbitrary spin state and to scattering amplitudes of more than 4 particles. It would be also worthy to explore the string amplitudes beyond bosonic open string. Since our algorithm describes a sum over infinite intermediate states, studying the sub–leading terms in high energy limit might be achieved.

Appendix A

Calculation of 3–point functions with complex momenta

Here we provide the detailed calculation of 3–point functions, namely equation (3.57) in Section 3.2. Part of the calculations are referred to [10,11].

Momentum conservation implies that $p_1 + p_2 + p_3 = 0$ for three–gluon vertex. Hence, we have

$$p_1^2 = 0 = (p_2 + p_3)^2 = -2\, p_2 \cdot p_3. \tag{A.1}$$

Consequently, the angle bracket and the square bracket are zero if we confine ourselves to real momenta, i.e.

$$|\langle p_i p_j \rangle|^2 = |[p_i p_j]|^2 = 0, \quad i,\, j \in \{1,\, 2,\, 3\}. \tag{A.2}$$

As a result, three–point functions are ill–defined. However, if momenta are allowed to be complex, then equation (3.35) collapses since it is valid for real momenta. We thus can choose $[p_i p_j] = 0$ but $\langle p_i p_j \rangle \neq 0$, which gives meaning to the three–point function $M(1^- 2^- 3^+)$; while the other choice $[p_i p_j] \neq 0$ but $\langle p_i p_j \rangle = 0$ gives meaning to $M(1^+ 2^+ 3^-)$.

With this idea in mind, we are going to compute the color–ordered amplitude $M(1^- 2^- 3^+)$. Slightly different from standard Feynman's rule for QCD, here we employ the color–striped 3–gluon vertex since the color dependences in the 4–gluon scattering amplitude have been striped away, which is

$$V_3(p_1,\, p_2,\, p_3)^{\mu_1\mu_2\mu_3} = \frac{1}{\sqrt{2}}\left[g^{\mu_1\mu_2}(p_1-p_2)^{\mu_3} + g^{\mu_2\mu_3}(p_2-p_3)^{\mu_1} + g^{\mu_3\mu_1}(p_3-p_1)^{\mu_2}\right] \quad \text{(A.3)}$$

with the coupling constant suppressed. Through Feynman's rule, the on–shell amplitude for 3 incoming gluons is

$$\begin{aligned} M(1^-2^-3^+) =& \epsilon_{1\,\mu_1}\epsilon_{2\,\mu_2}\epsilon_{3\,\mu_3}V_3(p_1,\, p_2,\, p_3)^{\mu_1\mu_2\mu_3} \\ =& \frac{1}{\sqrt{2}}\left[\epsilon_1\cdot\epsilon_2\,(p_1-p_2)\cdot\epsilon_3 + \epsilon_2\cdot\epsilon_3\,(p_2-p_3)\cdot\epsilon_1 + \epsilon_3\cdot\epsilon_1\,(p_3-p_1)\cdot\epsilon_2\right]. \end{aligned} \quad \text{(A.4)}$$

Choose for the polarization vectors to be

$$\epsilon(p_1)^-_\mu = -\frac{1}{\sqrt{2}}\frac{[q|\bar{\sigma}_\mu|\,p_1\rangle}{[qp_1]}, \quad \epsilon(p_2)^-_\mu = -\frac{1}{\sqrt{2}}\frac{[q|\bar{\sigma}_\mu|\,p_2\rangle}{[qp_2]}, \quad \epsilon(p_3)^+_\mu = \frac{1}{\sqrt{2}}\frac{\langle s|\sigma_\mu|\,p_3]}{\langle sp_3\rangle} \quad \text{(A.5)}$$

with reference momenta q and s. This choice of polarizations leads to $\epsilon_1\cdot\epsilon_2 = 0$, which can be easily verified by

$$\epsilon_1\cdot\epsilon_2 \propto [q|\bar{\sigma}_\mu|\,p_1\rangle[q|\bar{\sigma}^\mu|\,p_2\rangle = -\langle p_1|\sigma_\mu|q][q|\bar{\sigma}^\mu|\,p_2\rangle = 2\langle p_1p_2\rangle[qq] = 0.$$

Equation (A.4) then becomes

$$M(1^-2^-3^+) = \frac{2}{\sqrt{2}}\left[(\epsilon_3\cdot\epsilon_1)(p_3\cdot\epsilon_2) - (\epsilon_3\cdot\epsilon_2)(p_3\cdot\epsilon_1)\right]. \quad \text{(A.6)}$$

Each terms in the above equation can be also computed separately by using the contraction rules of spinors and the Fierz identity, yielding

$$\begin{aligned} \epsilon_3\cdot\epsilon_2 &= -\frac{\langle sp_2\rangle[qp_3]}{[qp_2]\langle sp_3\rangle}, \quad \epsilon_1\cdot p_3 = -\frac{1}{\sqrt{2}}\cdot\frac{[qp_3]\langle p_3p_1\rangle}{[qp_1]}; \\ \epsilon_3\cdot\epsilon_1 &= -\frac{\langle sp_1\rangle[qp_3]}{[qp_1]\langle sp_3\rangle}, \quad \epsilon_2\cdot p_3 = -\frac{1}{\sqrt{2}}\cdot\frac{[qp_3]\langle p_3p_2\rangle}{[qp_2]}. \end{aligned} \quad \text{(A.7)}$$

So far, we have expressed those kinematic dependences (A.7) of the three–point amplitude in terms of the angle brackets and the square brackets; then equation (A.6) evaluates to

$$\begin{aligned} M(1^-2^-3^+) =& \frac{[qp_3]}{[qp_1][qp_2]\langle sp_3\rangle}\left([qp_3]\langle p_3p_1\rangle\langle sp_2\rangle - [qp_3]\langle p_3p_2\rangle\langle sp_1\rangle\right) \\ =& \frac{\langle p_1p_2\rangle[qp_3]^2}{[qp_1][qp_2]}. \end{aligned} \quad \text{(A.8)}$$

Note that we have used equation (3.34) twice to get the above result. When multiply the numerator and denominator by $\langle p_1p_2\rangle^2$ and use equation (3.34) again, the q dependent factors of $\langle qp_3\rangle$ cancel, and we obtain

$$M\left(1^-2^-3^+\right) = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}. \quad \text{(A.9)}$$

Similarly, we can also calculate the other three–point amplitude $M(1^+2^+3^-)$ by following the above procedures. But for this case, we must assume $\langle p_ip_j\rangle = 0$ but $[p_ip_j] \neq 0$.

Appendix B

Stirling number of the first kind

The unsigned Stirling number $|s(N,J)|$ and signed (ordinary) Stirling number $s\,(N,J)$ can be obtained from the rising factorial and falling factorial, respectively, i.e.

$$x(x-1)\cdots(x-N+1) = \sum_{J=0}^{N} s(N,J)x^J, \tag{B.1}$$

$$x(x+1)\cdots(x+N-1) = \sum_{J=0}^{N} |s(N,J)|\, x^J. \tag{B.2}$$

The relation between the unsigned and signed Stirling numbers could be easily proved as

$$\big|s(N,J)\big| = (-)^{N-J}\, s(N,J). \tag{B.3}$$

Besides, the unsigned Stirling number of the first kind also has its meaning in combinatorics, which counts the number of permutations of N elements with J disjoint cycles [17], namely

$$|s(N,J)| = \sum_{\left\{J=\sum_m N_m\right\}} \prod_{m=1}^{\infty} \frac{N!}{N_m!\, m^{N_m}}, \tag{B.4}$$

where $N,\ J,\ m$ and N_m obey the following two relations

$$\sum_m mN_m = N, \qquad \sum_m N_m = J. \tag{B.5}$$

Let's consider the following binomial expansion of $(1-z)^X$,

$$(1-z)^X = \exp\Big[\, X\ln(1-z)\Big] = \sum_{N=0}^{\infty} \binom{X}{N} (-)^N z^N. \tag{B.6}$$

The combinatorial factor $\binom{X}{N}$ is defined as

$$\binom{X}{N} = \frac{x(x-1)\cdots(x-N+1)}{N!}. \tag{B.7}$$

It is easy to see that, from equations (B.1) and (B.3), the binomial expansion (B.6) can be rewritten as

$$\begin{aligned}(1-z)^X &= \sum_{N=0}^{\infty}\sum_{J=0}^{N}\frac{s(N,J)}{N!}(-)^N X^J z^N \\ &= \sum_{N=0}^{\infty}\sum_{J=0}^{N}\frac{|s(N,J)|}{N!}(-X)^J z^N. \end{aligned} \tag{B.8}$$

After getting rid of the z dependent terms in both (B.6) and (B.8) formulas and applying the equation (B.4) , one could further express the binomial factor $(-)^N\binom{X}{N}$ as

$$\binom{X}{N}(-)^N = \sum_{J=0}^{N}\sum_{\{N_m\}}\prod_{m=1}^{\infty}\frac{(-X)^{N_m}}{N_m!\,m^{N_m}}, \tag{B.9}$$

which leads to a crucial mathematical identity for the full-tachyon scattering amplitude.

Furthermore, we can perform the operation $\frac{1}{X}\left(z\,\frac{d}{dz}\right)$ to both the left-handed and the right-handed sides of equation (B.8). Then, compare the coefficients of the powers of z, one obtain an useful identity

$$\binom{X-1}{N-1}(-)^N = \left(-\frac{N}{X}\right)\sum_{J=1}^{N}\sum_{\{N_m\}}\prod_{m=1}^{\infty}\frac{(-X)^{N_m}}{N_m!\,m^{N_m}}. \tag{B.10}$$

This identity would be applied to the residue of 1–vector 3–tachyon amplitude.

Appendix C

1–vector 3-tachyon amplitude

C.1 Level matching of the first 3 mass levels

Before giving the exact proof of the equality between equation (5.6) and (5.5), we are going to use the level matching approach to check whether the first few mass levels are right or not. In the following, we list the first three mass levels of equation (5.6) for future reference:

$$\mathbf{N=1} \Rightarrow I_1 = \epsilon_2 \cdot k_3,$$

$$\mathbf{N=2} \Rightarrow I_2 = -\left(\epsilon_2 \cdot k_3\right)\left(k_2 \cdot k_3 - 1\right),$$

$$\mathbf{N=3} \Rightarrow I_3 = \left(\epsilon_2 \cdot k_3\right)\left[\frac{\left(k_2 \cdot k_3 - 1\right)\left(k_2 \cdot k_3 - 2\right)}{2!}\right] = \frac{\epsilon_2 \cdot k_3}{2!}\left[\left(k_2 \cdot k_3\right)^2 - 3\left(k_2 \cdot k_3\right) + 2\right].$$

Please note that $k_2 \cdot \epsilon_2 = 0$ and the normalized Fock state is given by

$$|\{N_m\}; P\rangle = \prod_{m=1}^{\infty} \frac{\left(\alpha_{-m}^{\mu}\right)^{N_m}}{\sqrt{m^{N_m} N_m!}} |0; P\rangle. \tag{C.1}$$

In the presence of the term $\sum_n \epsilon_2 \cdot \alpha_n$ in A_L, things become a little be complicated.

- **Level** $N = 1$**:** For $N = 1$, there is just one intermediate state:

$$\{\alpha_{-1}\}.$$

We expect that the residue (5.5) for this level should be $I_1 = \epsilon_2 \cdot k_3$.

$$I_1 = \Big\langle -k_1;0 \Big| \Big(\sum_{n=1}^{\infty} \epsilon_2 \cdot \alpha_n\Big) e^{\sum_{n=1}^{\infty}\frac{1}{n}k_2\cdot\alpha_{-n}} e^{-\sum_{n=1}^{\infty}\frac{1}{n}k_2\cdot\alpha_n} \alpha_{-1}^{\mu}\Big|0;P\Big\rangle \eta_{\mu\nu} \Big\langle 0;P \Big| \alpha_1^{\nu} e^{\sum_{n=1}^{\infty}\frac{1}{n}k_3\cdot\alpha_{-n}} \Big| k_4;0\Big\rangle \tag{C.2}$$

$$= \Big\langle -k_1;0 \Big| \Big(\sum_{n=1}^{\infty} \epsilon_2 \cdot \alpha_n\Big)\alpha_{-1}^{\mu}\Big|0;P\Big\rangle \eta_{\mu\nu} \Big\langle 0;P \Big| \alpha_1^{\nu} (k_3\cdot\alpha_{-1}) \Big| k_4;0\Big\rangle$$

$$= \eta_{\mu\nu}\epsilon_2^{\mu}k_3^{\nu} = \epsilon_2 \cdot k_3. \tag{C.3}$$

As was expected!

- **Level $N=2$:** In this case, we have two intermediate states:

$$\{\frac{\alpha_{-2}}{\sqrt{2}}, \quad \frac{\alpha_{-1}\alpha_{-1}}{\sqrt{2!}}\}.$$

Residue I_2 of this level is consisted of two parts T_1 and T_2 corresponding to the two different intermediate states, which is

$$I_2 = T_1 + T_2, \tag{C.4}$$

where T_1 and T_2 are

$$T_1 = \Big\langle -k_1;0 \Big| \Big(\sum_{n=1}^{\infty} \epsilon_2 \cdot \alpha_n\Big) e^{\sum_{n=1}^{\infty}\frac{1}{n}k_2\cdot\alpha_{-n}} e^{-\sum_{n=1}^{\infty}\frac{1}{n}k_2\cdot\alpha_n} \Big(\frac{\alpha_{-2}^{\mu}}{\sqrt{2}}\Big) \Big|0;P\Big\rangle g_{\mu\nu} \Big\langle 0;P \Big| \Big(\frac{\alpha_2^{\nu}}{\sqrt{2}}\Big) e^{\sum_{n=1}^{\infty}\frac{1}{n}k_3\cdot\alpha_{-n}} \Big| k_4;0\Big\rangle$$

$$= \epsilon_2 \cdot k_3, \tag{C.5}$$

and

$$T_2 = \Big\langle -k_1;0 \Big| \Big(\sum_{n=1}^{\infty} \epsilon_2 \cdot \alpha_n\Big) e^{\sum_{n=1}^{\infty}\frac{1}{n}k_2\cdot\alpha_{-n}} e^{-\sum_{n=1}^{\infty}\frac{1}{n}k_2\cdot\alpha_n} \Big(\frac{\alpha_{-1}^{\mu_1}\alpha_{-1}^{\mu_2}}{\sqrt{2!}}\Big) \Big|0;P\Big\rangle \eta_{\mu_1\nu_1}\eta_{\mu_2\nu_2} \Big\langle 0;P \Big| \Big(\frac{\alpha_1^{\nu_1}\alpha_1^{\nu_2}}{\sqrt{2!}}\Big) e^{\sum_{n=1}^{\infty}\frac{1}{n}k_3\cdot\alpha_{-n}} \Big| k_4;0\Big\rangle$$

$$= -(\epsilon_2 \cdot k_3)(k_2 \cdot k_3). \tag{C.6}$$

Thus,

$$I_2 = T_1 + T_2 = -(\epsilon_2 \cdot k_3)(k_2 \cdot k_3 - 1). \tag{C.7}$$

- **Level $N = 3$:** For $N = 3$, there are three distinct Fock states as shown in the following:

$$\{ \frac{\alpha_{-3}}{\sqrt{3}}, \quad \frac{\alpha_{-2}\alpha_{-1}}{\sqrt{2}}, \quad \frac{\alpha_{-1}^3}{\sqrt{3!}} \}.$$

$$I_3 = T_1 + T_2 + T_3. \tag{C.8}$$

I_3 is respectively constituted of 3 contributions of the three intermediate states. Carry out the tedious calculation of T_1, T_2 and T_3,

$$\begin{aligned} T_1 = &\Big\langle -k_1;0 \Big| \Big(\sum_{n=1}^{\infty} \epsilon_2\cdot\alpha_n\Big) e^{\sum_{n=1}^{\infty}\frac{1}{n}k_2\cdot\alpha_{-n}}\, e^{-\sum_{n=1}^{\infty}\frac{1}{n}k_2\cdot\alpha_n} \left(\frac{\alpha_{-3}^{\mu}}{\sqrt{3}}\right) \Big|0;P\Big\rangle \eta_{\mu\nu} \\ &\Big\langle 0;P \Big| \left(\frac{\alpha_3^{\nu}}{\sqrt{3}}\right) e^{\sum_{n=1}^{\infty}\frac{1}{n}k_3\cdot\alpha_{-n}} \Big| k_4;0 \Big\rangle \\ = &\ \epsilon_2\cdot k_3, \end{aligned} \tag{C.9}$$

$$\begin{aligned} T_2 = &\Big\langle -k_1;0 \Big| \Big(\sum_{n=1}^{\infty} \epsilon_2\cdot\alpha_n\Big) e^{\sum_{n=1}^{\infty}\frac{1}{n}k_2\cdot\alpha_{-n}}\, e^{-\sum_{n=1}^{\infty}\frac{1}{n}k_2\cdot\alpha_n} \left(\frac{\alpha_{-2}^{\mu_1}\alpha_{-1}^{\mu_2}}{\sqrt{2}}\right) \Big|0;P\Big\rangle \eta_{\mu_1\nu_1}\eta_{\mu_2\nu_2} \\ &\Big\langle 0;P \Big| \left(\frac{\alpha_2^{\nu_1}\alpha_1^{\nu_2}}{\sqrt{2}}\right) e^{\sum_{n=1}^{\infty}\frac{1}{n}k_3\cdot\alpha_{-n}} \Big| k_4;0 \Big\rangle \\ = &\ -\frac{3}{2}(\epsilon_2\cdot k_3)(k_2\cdot k_3), \end{aligned} \tag{C.10}$$

and

$$\begin{aligned} T_3 = &\Big\langle -k_1;0 \Big| \Big(\sum_{n=1}^{\infty} \epsilon_2\cdot\alpha_n\Big) e^{\sum_{n=1}^{\infty}\frac{1}{n}k_2\cdot\alpha_{-n}}\, e^{-\sum_{n=1}^{\infty}\frac{1}{n}k_2\cdot\alpha_n} \left(\frac{\alpha_{-1}^{\mu_1}\alpha_{-1}^{\mu_2}\alpha_{-1}^{\mu_3}}{\sqrt{3!}}\right) \Big|0;P\Big\rangle \\ &\eta_{\mu_1\nu_1}\eta_{\mu_2\nu_2}\eta_{\mu_3\nu_3}\Big\langle 0;P \Big| \left(\frac{\alpha_1^{\nu_1}\alpha_1^{\nu_2}\alpha_1^{\nu_3}}{\sqrt{3!}}\right) e^{\sum_{n=1}^{\infty}\frac{1}{n}k_3\cdot\alpha_{-n}} \Big| k_4;0 \Big\rangle \\ = &\ \frac{1}{2}(\epsilon_2\cdot k_3)(k_2\cdot k_3)^2, \end{aligned} \tag{C.11}$$

we then have

$$\begin{aligned} I_3 &= T_1 + T_2 + T_3 \\ &= \frac{\epsilon_2\cdot k_3}{2!}\left[(k_2\cdot k_3)^2 - 3\,(k_2\cdot k_3) + 2\right] \\ &= \epsilon_2\cdot k_3 \times \frac{(k_2\cdot k_3)(k_2\cdot k_3 - 1)}{2!}. \end{aligned} \tag{C.12}$$

Frankly, the sum of these three components confirms our prediction!

C.2 Quick examination of the generating function

In Chapter 5, we have claimed a generating function to help us work out the residue of the 1–vector 3–tachyon amplitude. In this section, we are going to demonstrate how to quickly examine the correctness of this generating function. At first we can choose an arbitrary intermediate state, for example $|\{N_m\}\,;P\,\rangle = \frac{\alpha_{-q}^{\mu_1}}{\sqrt{q}}\frac{\alpha_{-r}^{\mu_2}}{\sqrt{r}}\,|\,0;P\rangle$; equation (5.5) reads

$$\begin{aligned}\langle 0;\,-k_1|\left(\sum_{n=1}^{\infty}\epsilon_2\cdot\alpha_n\,z_2^{-n}\right)\,e^{-\frac{1}{n}k_2\cdot\alpha_n z_2^{-n}}\left(\frac{\alpha_{-q}^{\mu_1}\alpha_{-r}^{\mu_2}}{\sqrt{q}\sqrt{r}}\right)|0;\,P\rangle\eta_{\mu_1\nu_1}\eta_{\mu_2\nu_2}\\ \langle 0;\,P\,|\left(\frac{\alpha_{q}^{\nu_1}\alpha_{r}^{\nu_2}}{\sqrt{q}\sqrt{r}}\right)\,e^{-\frac{1}{n}k_3\cdot\alpha_{-n}z_3^{n}}|0;\,k_4\rangle.\end{aligned} \tag{C.13}$$

Equation (C.13) can be easily carried out, yielding

$$\Rightarrow \frac{(\epsilon_2\cdot k_3)\times q}{q}\left(\frac{z_3}{z_2}\right)^q\frac{(k_2\cdot k_3)}{r}\left(\frac{z_3}{z_2}\right)^r+\frac{(k_2\cdot k_3)}{q}\left(\frac{z_3}{z_2}\right)^q\frac{(\epsilon_2\cdot k_3)\times r}{r}\left(\frac{z_3}{z_2}\right)^r. \tag{C.14}$$

Please notice that, while deriving for the residue (5.5), we act the operator $\frac{1}{X}\left(z\,\frac{d}{dz}\right)$ with $X = k_2\cdot k_3$ upon $(1-z)^X = \exp[X\ln(1-z)]$, and the coefficients of the powers of z in $z\frac{d}{dz}\,[\ln(1-z)]\,e^{X\ \ln(1-z)}$ just right provide a connection between the combinatorial factor (5.6) and the residue (5.5). Therefore, residue proportional to $\epsilon_2\cdot k_3$ at a given level $N = q+r$ is determined by the z^{q+r} term expansion coefficient of the generating function

$$Y\cdot z\frac{d}{dz}\,[\ln(1-z)]\,e^{X\ \ln(1-z)} \tag{C.15}$$

$$\begin{aligned}&=\left(-\sum_{n=1}^{\infty}\frac{Y\cdot n}{n}\,z^n\right)\,e^{-X\left(z+\frac{z^2}{2}+\frac{z^3}{3}+\cdots\cdots\right)}\\ &=(\cdots+\frac{Y\cdot q}{q}\,z^q+\cdots+\frac{Y\cdot r}{r}\,z^r+\cdots)(\cdots+\frac{X}{q}\,z^q+\cdots+\frac{X}{r}\,z^r+\cdots)\\ &=\cdots+\left(\frac{Y\cdot q}{q}\,z^q\frac{X}{r}\,z^r+\frac{Y\cdot r}{r}\,z^r\frac{X}{q}\,z^q\right)+\cdots.\end{aligned} \tag{C.16}$$

Setting $Y = \epsilon_2\cdot k_3$ and $z = z_3/z_2$, equation (C.14) agrees with the result in (C.16). We can therefore conclude that equation (C.15) is indeed the generating function of the term proportional to $\epsilon_2\cdot k_3$ in the 1–vector 3–tachyons amplitude.

Appendix D

Amplitude of one rank–two tensor and 3 tachyons

In the following, we provide an explicit calculation on one rank–two tensor and three tachyons amplitude, but fail to prove the equality of Eq.(D.13) and Eq.(D.15). Instead, we do level matching for up to $N = 5$.

D.1 Path integral

The vertex operator of rank–two tensor is given by

$$V(k, z) =: \epsilon_{\mu\nu} \dot{X}^{\mu} \dot{X}^{\nu} \, e^{ik\cdot X(z)} : . \tag{D.1}$$

The formalism for the scattering amplitude of one rank–two tensor with 3 tachyons is shown as follows

$$A(1234) = \int \prod_{i=1}^{4} dz_i \langle -k_1; 0| \, \epsilon_{\mu\nu} \, \partial X^{\mu}(z_2) \partial X^{\nu}(z_2) V_0(k_2) \, V_0(k_3) \, |k_4; 0\rangle, \tag{D.2}$$

where $\epsilon_{\mu\nu}$ is the polarized tensor which obeys $k_{\mu}\epsilon^{\mu\nu} = \text{tr}\epsilon = 0$. Thus, some of the terms of $\epsilon_{\mu\nu} \, \dot{X}^{\mu} \dot{X}^{\nu}$ would vanish. As a result, $\epsilon_{\mu\nu} \, \dot{X}^{\mu} \dot{X}^{\nu}$ of equation (D.2) can be equivalently written as

$$
\begin{aligned}
&\left(\epsilon_1 \cdot \dot{X}\right)\left(\epsilon_2 \cdot \dot{X}\right) \\
&= \left[\frac{\epsilon_1 \cdot \hat{p}}{z_2} + \frac{\epsilon_2}{z_2} \cdot \sum_m \left(\alpha_{-m} z_2^m + \alpha_m z_2^{-m}\right)\right] \times \left[\frac{\epsilon_2 \cdot \hat{p}}{z_2} + \frac{\epsilon_2}{z_2} \cdot \sum_n \left(\alpha_{-m} z_2^n + \alpha_n z_2^{-n}\right)\right] \\
&= \left(\frac{\epsilon_1 \cdot \hat{p}}{z_2}\right)\left(\frac{\epsilon_2 \cdot \hat{p}}{z_2}\right) + \frac{1}{z_2^2}\left[\left(\epsilon_1 \cdot \hat{p}\right)\epsilon_2 \cdot \sum_{m=1}^{\infty} \alpha_m z_2^{-m} + \left(\epsilon_2 \cdot \hat{p}\right)\epsilon_1 \cdot \sum_{m=1}^{\infty} \alpha_m z_2^{-m}\right] + \\
&\quad \frac{1}{z_2^2}\left(\epsilon_1 \cdot \sum_{m=1}^{\infty} \alpha_m z_2^m\right)\left(\epsilon_2 \cdot \sum_{n=1}^{\infty} \alpha_n z_2^n\right). \qquad \text{(D.3)}
\end{aligned}
$$

Moving $\hat{p}$ in (D.3) to the right and acting on the bra $\langle -k_1; 0|$, we can quickly find out that the residues proportional to $k_1^\mu k_1^\nu$ and $k_1^\mu k_3^\nu$ have been observed previously for the calculation of four–tachyon and one–vector three–tachyon amplitudes. The only new residue is the one proportional to $k_3^\mu k_3^\nu$, which is

$$
\begin{aligned}
I_N = \sum_{\{\sum aN_a = N\}} &\left\langle -k_1; 0 \right| \sum_{m,n} (\epsilon_1 \cdot \alpha_m)(\epsilon_2 \cdot \alpha_n)\, V_0(k_2) \left| \{N_a\}; P \right\rangle \mathcal{T}_{\{N_a\}} \\
&\left\langle \{N_a\}; P \right| V_0(k_3) \left| k_4; 0 \right\rangle \Big|_{z_2=1}. \qquad \text{(D.4)}
\end{aligned}
$$

D.2 Explicit derivation of the residue of interest

In this section, we aim to analytically calculate equation (D.4) by using our algorithm. As usual, we first calculate A_R and A_L separately. A_R is the same as before, which is

$$
\begin{aligned}
A_R &= \left\langle \{N_m\}\, ;\, P \right| V_0(k_3) \left| 0\, ;\, k_4 \right\rangle \Big|_{z_2=1} \\
&= \prod_{m=1}^{\infty} \frac{\left(k_3^\nu\right)^{N_m}}{\sqrt{m^{N_m}\, N_m!}}. \qquad \text{(D.5)}
\end{aligned}
$$

While the left 3–point amplitude A_L is given by

$$
A_L = \left\langle -k_1; 0 \right| \sum_{m,n} (\epsilon_1 \cdot \alpha_m)(\epsilon_2 \cdot \alpha_n)\, V_0(k_2) \left| \{N_a\}; P \right\rangle. \qquad \text{(D.6)}
$$

Here, the double summation over m and n can be rewritten as a sum of two parts, that is

$$
\sum_{m,n} (\epsilon_1 \cdot \alpha_m)(\epsilon_2 \cdot \alpha_n) = \sum_{n=1}^{\infty} (\epsilon_1 \cdot \alpha_n)(\epsilon_2 \cdot \alpha_n) + \sum_{m \neq n} (\epsilon_1 \cdot \alpha_m)(\epsilon_2 \cdot \alpha_n). \qquad \text{(D.7)}
$$

Plug equation (D.7) into A_L, yielding

$$
\begin{aligned}
A_L &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Big\langle 0\Big|\Big(\epsilon_1\cdot\alpha_m\Big)\Big(\epsilon_2\cdot\alpha_n\Big)\prod_{m=1}^{\infty} e^{-\frac{k_2\cdot\alpha_m}{m}}\frac{\big(\alpha_{-m}^{\mu}\big)^{N_m}}{\sqrt{m^{N_m}\,N_m!}}\Big|0\Big\rangle \\
&= \Big\langle 0\Big|\Big[\sum_{m\neq n}\big(\epsilon_1\cdot\alpha_m\big)\big(\epsilon_2\cdot\alpha_n\big)+\sum_{n=1}^{\infty}\big(\epsilon_1\cdot\alpha_n\big)\big(\epsilon_2\cdot\alpha_n\big)\Big]\prod_{a=1}^{\infty} e^{-\frac{k_2\cdot\alpha_a}{a}}\frac{\big(\alpha_{-a}^{\mu}\big)^{N_a}}{\sqrt{a^{N_a}\,N_a!}}\Big|0\Big\rangle.
\end{aligned}
\tag{D.8}
$$

Following the same manners, A_L can be calculated by

$$
\begin{aligned}
A_L = &\sum_{m\neq n}\Big\langle 0\Big|\Big[(\epsilon_1\cdot\alpha_m)e^{-\frac{k_2\cdot\alpha_m}{m}}\frac{\big(\alpha_{-m}^{\mu}\big)^{N_m}}{\sqrt{m^{N_m}\,N_m!}}\Big]\Big[(\epsilon_2\cdot\alpha_n)\,e^{-\frac{k_2\cdot\alpha_n}{n}}\frac{\big(\alpha_{-n}^{\mu}\big)^{N_n}}{\sqrt{n^{N_n}\,N_n!}}\Big]\times \\
&\prod_{a=1,a\neq m\neq n}^{\infty} e^{-\frac{k_2\cdot\alpha_a}{a}}\frac{\big(\alpha_{-a}^{\mu}\big)^{N_a}}{\sqrt{a^{N_a}\,N_a!}}\Big|0\Big\rangle+\sum_{n=1}^{\infty}\Big\langle 0\Big|\Big[\big(\epsilon_1\cdot\alpha_n\big)\big(\epsilon_2\cdot\alpha_n\big)\,e^{-\frac{k_2\cdot\alpha_n}{n}}\frac{\big(\alpha_{-n}^{\mu}\big)^{N_n}}{\sqrt{n^{N_n}\,N_n!}}\Big]\times \\
&\prod_{a=1,a\neq n}^{\infty} e^{-\frac{k_2\cdot\alpha_a}{a}}\frac{\big(\alpha_{-a}^{\mu}\big)^{N_a}}{\sqrt{a^{N_a}\,N_a!}}\Big|0\Big\rangle.
\end{aligned}
\tag{D.9}
$$

Applying the commutative relation of ladder operators $[\alpha_m^{\mu},\,\alpha_n^{\nu}]=m\delta_{m+n}\eta^{\mu\nu}$, we reach

$$
\begin{aligned}
A_L = &\sum_{n\neq m}\Big[\frac{(-)^{N_m-1}\;mN_m\;\epsilon_1^{\mu}\,\big(k_2^{\mu}\big)^{N_m-1}}{\sqrt{N_m!\;m^{N_m}}}\Big]\Big[\frac{(-)^{N_n-1}\;nN_n\;\epsilon_2^{\mu}\,\big(k_2^{\mu}\big)^{N_n-1}}{\sqrt{N_n!\;n^{N_n}}}\Big]\prod_{a=1,a\neq m\neq n}^{\infty}\frac{\big(-k_2^{\mu}\big)^{N_a}}{\sqrt{N_a!\;a^{N_a}}} \\
&+\sum_{n=1}^{\infty}\frac{(-)^{N_n-2}\;n^2N_n(N_n-1)\;\epsilon_1^{\mu_1}\,\epsilon_2^{\mu_2}\,\big(k_2^{\mu}\big)^{N_n-2}}{\sqrt{N_n!\;n^{N_n}}}\prod_{a=1,a\neq n}^{\infty}\frac{\big(-k_2^{\mu}\big)^{N_a}}{\sqrt{N_a!\;a^{N_a}}}.
\end{aligned}
\tag{D.10}
$$

After contracting with A_R, finally we obtain I_N

$$
\begin{aligned}
I_N = &\sum_{n\neq m}\Big[\frac{(-)^{N_m-1}\;mN_m\;\big(\epsilon_1\cdot k_3\big)\,\big(k_2\cdot k_3\big)^{N_m-1}}{N_m!\;m^{N_m}}\Big]\Big[\frac{(-)^{N_n-1}\;nN_n\;\big(\epsilon_2\cdot k_3\big)\,\big(k_2\cdot k_3\big)^{N_n-1}}{N_n!\;n^{N_n}}\Big] \\
&\prod_{a=1,a\neq m\neq n}^{\infty}\frac{\big(-k_2\cdot k_3\big)^{N_a}}{N_a!\;a^{N_a}}+\sum_{n=1}^{\infty}\frac{(-)^{N_n-2}\;n^2N_n(N_n-1)\;\big(\epsilon_1\cdot k_3\big)\,\big(\epsilon_2\cdot k_3\big)\,\big(k_2\cdot k_3\big)^{N_n-2}}{N_n!\;n^{N_n}}\times \\
&\prod_{a=1,a\neq n}^{\infty}\frac{\big(-k_2\cdot k_3\big)^{N_a}}{N_a!\;a^{N_a}}.
\end{aligned}
$$

But as we shall see, I_N looks very "ugly". The next step is to make it clean. Recall that in order to calculate A_L, we draw out the terms of $a = m, n$ inside the product notation and let them contract with α_m and α_n outside. We then put them back into the product notation, *i.e.*

$$\Rightarrow \left\{ \sum_{n\neq m} \frac{mN_m \, nN_n \big(\epsilon_1 \cdot k_3\big) \big(\epsilon_2 \cdot k_3\big)}{(k_2 \cdot k_3)^2} + \sum_{n=1}^{\infty} \frac{n^2 N_n (N_n - 1) \big(\epsilon_1 \cdot k_3\big) \big(\epsilon_2 \cdot k_3\big)}{(k_2 \cdot k_3)^2} \right\} \prod_{a=1}^{\infty} \frac{\big(-k_2 \cdot k_3\big)^{N_a}}{N_a! \; a^{N_a}}. \tag{D.11}$$

Equation(D.11) can be further simplified by the following identity

$$\begin{aligned} N^2 &= \sum_m mN_m \sum_n nN_n \\ &= \sum_{m\neq n} mN_m \, nN_n + \sum_{n=1}^{\infty} n^2 N_n^2. \end{aligned} \tag{D.12}$$

Finally, I_N can be calculated by

$$I_N = \frac{\big(\epsilon_1 \cdot k_3\big)\big(\epsilon_2 \cdot k_3\big)}{\big(k_2 \cdot k_3\big)^2} \sum_{\{N = \sum_m mN_m\}} \left[N^2 - \sum_n n^2 N_n \right] \prod_m \frac{\big(-k_2 \cdot k_3\big)^{N_m}}{m^{N_m} \, N_m!}. \tag{D.13}$$

Alternatively, by using path integral, the term proportional to $k_3^\mu k_3^\nu$ in the scattering amplitude is given by

$$\begin{aligned} &\big(\epsilon_1 \cdot k_3\big) \big(\epsilon_2 \cdot k_3\big) \int_0^1 dz_2 \, |1 - z_2|^{k_2 \cdot k_3 - 2} \, |z_2|^{k_1 \cdot k_2} \\ =&\big(\epsilon_1 \cdot k_3\big) \big(\epsilon_2 \cdot k_3\big) \sum_{N=2}^{\infty} \binom{k_2 \cdot k_3 - 2}{N-2} (-)^{N-2} \, \frac{2}{(k_1 + k_2)^2 + 2(N-1)}. \end{aligned} \tag{D.14}$$

From the previous experience, we know that I_N is the rest of the part of equation (D.14) without the propagator $2/[(k_1 + k_2)^2 + 2(N-1)]$. Therefore, we expect that I_N must equal to

$$\binom{k_2 \cdot k_3 - 2}{N-2} (-)^{N-2} \, (\epsilon_1 \cdot k_3) \, (\epsilon_2 \cdot k_3) \, . \tag{D.15}$$

So far, we cannot directly prove Eq.(D.13) and Eq.(D.15) are equal. But, we indirectly show that they should be equal through level matching, which is given in the next subsection.

D.2.1 Level matching

In the following, we do level matching for up to $N = 5$ to see whether equation (D.13) and (D.15) are equal to each other or not. For convenience, we set $X = k_2 \cdot k_3$, $Y_1 = \epsilon_1 \cdot k_3$ and $Y_2 = \epsilon_2 \cdot k_3$.

- **Level $N = 2$:** The intermediate states for this level is shown as follows

$$\{ \frac{\alpha_{-2}}{\sqrt{2}}, \quad \frac{\alpha_{-1}\alpha_{-1}}{\sqrt{2!}} \}.$$

$$\begin{aligned} I_2 &= \frac{Y_1 Y_2}{X^2} \left\{ 4 \times \left(-\frac{X}{2} + \frac{X^2}{2!} \right) - \left[4 \times \left(-\frac{X}{2} \right) + 2 \times \left(\frac{X^2}{2!} \right) \right] \right\} \\ &= Y_1 Y_2. \end{aligned} \tag{D.16}$$

- **Level $N = 3$:** We have three intermediate states for $N = 3$. They are

$$\{ \frac{\alpha_{-3}}{\sqrt{3}}, \quad \frac{\alpha_{-2}\alpha_{-1}}{\sqrt{2}}, \quad \frac{\alpha_{-1}^3}{\sqrt{3!}} \}.$$

$$\begin{aligned} I_3 &= \frac{Y_1 Y_2}{X^2} \left\{ 9 \times \left(-\frac{X}{3} + \frac{X^2}{2} - \frac{X^3}{3!} \right) - \left[9 \times \left(-\frac{X}{3} \right) + 5 \times \left(\frac{X^2}{2!} \right) - 3 \times \left(\frac{X^3}{3!} \right) \right] \right\} \\ &= -Y_1 Y_2 \left(X - 2 \right). \end{aligned} \tag{D.17}$$

As was expected!

- **Level $N = 4$:** For $N = 4$,

$$\{ \frac{\alpha_{-4}}{\sqrt{4}}, \quad \frac{\alpha_{-3}\alpha_{-1}}{\sqrt{3}}, \quad \frac{\alpha_{-2}^2}{\sqrt{2^2 \cdot 2!}}, \quad \frac{\alpha_{-1}^2\alpha_{-2}}{\sqrt{2 \cdot 2!}}, \quad \frac{\alpha_{-1}^4}{\sqrt{4!}} \}.$$

$$\begin{aligned} I_4 =& \frac{Y_1 Y_2}{X^2} \left\{ 16 \times \left(-\frac{X}{4} + \frac{X^2}{2^2 \cdot 2!} + \frac{X^2}{3 \cdot 1} - \frac{X^3}{2 \cdot 1 \cdot 2!} + \frac{X^4}{4!} \right) - \left[16 \times \left(-\frac{X}{4} \right) + \right. \right. \\ & \left. \left. 8 \times \left(\frac{X^2}{2^2 \cdot 2!} \right) + 10 \times \left(\frac{X^2}{3 \cdot 1} \right) - 6 \times \left(\frac{X^3}{2 \cdot 1 \cdot 2!} \right) + 4 \times \left(\frac{X^4}{4!} \right) \right] \right\} \\ =& Y_1 Y_2 \left[\frac{(X-2)(X-3)}{2!} \right]. \end{aligned} \tag{D.18}$$

- **Level** $N = 5$**:** There are 7 intermediates states needed to be taken into account

$$\{\frac{\alpha_{-5}}{\sqrt{5}},\quad \frac{\alpha_{-1}\alpha_{-4}}{\sqrt{4}},\quad \frac{\alpha_{-2}\alpha_{-3}}{\sqrt{2\cdot 3}},\quad \frac{\alpha_{-2}^2\alpha_{-1}}{\sqrt{2!\cdot 2^2}},\quad \frac{\alpha_{-1}^2\alpha_{-3}}{\sqrt{2!\cdot 3}},\quad \frac{\alpha_{-1}^3\alpha_{-2}}{\sqrt{3!\cdot 2}}\quad \frac{\alpha_{-1}^5}{\sqrt{5!}}\}.$$

$$\begin{aligned} I_5 = &-\frac{Y_1Y_2}{X^2}\Bigg\{25\times\Big(-\frac{X}{5}+\frac{X^2}{4\cdot 1}+\frac{X^2}{3\cdot 2}-\frac{X^3}{2!\cdot 3}-\frac{X^3}{2!\cdot 2^2}+\frac{X^4}{3!\cdot 2}-\frac{X^5}{5!}\Big)-\Big[25\times \\ &\Big(-\frac{X}{5}\Big)+17\times\Big(\frac{X^2}{4\cdot 1}\Big)+13\times\Big(\frac{X^2}{3\cdot 2}\Big)-11\times\Big(\frac{X^3}{2!\cdot 3}\Big)-9\times\Big(\frac{X^3}{2!\cdot 2^2}\Big)+ \\ &7\times\Big(\frac{X^4}{3!\cdot 2}\Big)-5\times\Big(\frac{X^5}{5!}\Big)\Big]\Bigg\} \\ = &-Y_1Y_2\Big[\frac{(X-2)(X-3)(X-4)}{3!}\Big]. \end{aligned} \tag{D.19}$$

Agrees with equation (D.15).

D.3 Generating function verification

In this section, we would like to verify the generating function for one rank–two tensor, 3 tachyons amplitude. All the procedures are the same as that in the previous case. We first choose a particular Fock state as intermediate state, for example $|\{N_m\};P\rangle = \frac{\alpha_{-q}^{\mu_1}}{\sqrt{q}}\frac{\alpha_{-r}^{\mu_2}}{\sqrt{r}}\frac{\alpha_{-l}^{\mu_3}}{\sqrt{l}}|0;P\rangle$ which has mass level of $N = q+r+l$, equation (D.4) reads

$$\begin{aligned} &\Big\langle 0;-k_1\Big|\Big(\sum_{m=1}^{\infty}\epsilon_1\cdot\alpha_m\Big)\Big(\sum_{n=1}^{\infty}\epsilon_2\cdot\alpha_n\Big)e^{-\frac{1}{a}k_2\cdot\alpha_a z_2^{-a}}\Big(\frac{\alpha_{-q}^{\mu_1}}{\sqrt{q}}\frac{\alpha_{-r}^{\mu_2}}{\sqrt{r}}\frac{\alpha_{-l}^{\mu_3}}{\sqrt{l}}\Big)\Big|0;P\Big\rangle\eta_{\mu_1\nu_1}\eta_{\mu_2\nu_2}\eta_{\mu_3\nu_3} \\ &\Big\langle 0;P\Big|\Big(\frac{\alpha_q^{\nu_1}}{\sqrt{q}}\frac{\alpha_r^{\nu_2}}{\sqrt{r}}\frac{\alpha_l^{\nu_3}}{\sqrt{l}}\Big)e^{\frac{1}{a}k_3\cdot\alpha_{-a}z_3^a}\Big|0;k_4\Big\rangle. \end{aligned}$$

Setting $X = k_2\cdot k_3$, $Y_1 = \epsilon_1\cdot k_3$, $Y_2 = \epsilon_2\cdot k_3$ and $z = z_3/z_2$, equation (D.20) can be expressed as

$$\begin{aligned} \Rightarrow &\frac{(Y_1\times q)}{q}z^q\frac{(Y_2\times r)}{r}z^r\frac{X}{l}z^l+(q\longleftrightarrow r)+\frac{(Y_1\times q)}{q}z^q\frac{(Y_2\times l)}{l}z^l\frac{X}{r}z^r+(q\longleftrightarrow l) \\ &+\frac{(Y_1\times l)}{l}z^l\frac{(Y_2\times r)}{r}z^r\frac{X}{q}z^q+(l\longleftrightarrow r). \end{aligned} \tag{D.20}$$

Residue (D.4) proportional to $(\epsilon_1\cdot k_3)(\epsilon_2\cdot k_3)$ at level $N = q+r+l$ is given by the expansion coefficient of the $(q+l+r)$-th order term of the following function

$$G = Y_1\,z\frac{d}{dz}ln(1-z)\,Y_2\,z\frac{d}{dz}ln(1-z)\,e^{X\,ln(1-z)}. \tag{D.21}$$

One can further expand Eq.(D.21) to examine the argument

$$
\begin{aligned}
G = & \Big(\cdots + \frac{Y_1 \cdot q}{q} z^q + \cdots + \frac{Y_1 \cdot r}{r} z^r + \cdots + \frac{Y_1 \cdot l}{l} z^l + \cdots \Big) \\
& \Big(\cdots + \frac{Y_2 \cdot q}{q} z^q + \cdots + \frac{Y_2 \cdot r}{r} z^r + \cdots + \frac{Y_2 \cdot l}{l} z^l + \cdots \Big) \\
& \Big(\cdots + \frac{X}{q} z^q + \cdots + \frac{X}{r} z^r + \cdots + \frac{X}{l} z^l + \cdots \Big) \qquad \text{(D.22)}
\end{aligned}
$$

$$
\begin{aligned}
= & \Big[\cdots + \frac{\big(Y_1 \times q\big)}{q} z^q \, \frac{\big(Y_2 \times r\big)}{r} z^r \, \frac{X}{l} z^l + (q \longleftrightarrow r) + \frac{\big(Y_1 \times q\big)}{q} z^q \, \frac{\big(Y_2 \times l\big)}{l} z^l \, \frac{X}{r} z^r + (q \longleftrightarrow l) \\
& + \frac{\big(Y_1 \times l\big)}{l} z^l \, \frac{\big(Y_2 \times r\big)}{r} z^r \, \frac{X}{q} z^q + (l \longleftrightarrow r) + \cdots \Big]. \qquad \text{(D.23)}
\end{aligned}
$$

It is obvious that the residue (D.20) contributed from the Fock state $|\{N_m\}\,; P\,\rangle = \frac{\alpha_{-q}^{\mu_1}}{\sqrt{q}} \frac{\alpha_{-r}^{\mu_2}}{\sqrt{r}} \frac{\alpha_{-l}^{\mu_3}}{\sqrt{l}} \,|\, 0; P\rangle$ of level $N = q + r + l$ is indeed equal to the z^{q+r+l} term expansion coefficient of equation (D.21).

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