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碩 士 論 文

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前言

我們將在本論文中探討正規 Laurent 級數體之下的賦距同步 Diophantine 逼近。在最近的一篇論文中，Kim 和 Nakada 證明了在一維度的正規 Laurent 級數之下，和 Kurzweil 定理相似的一個結果。本論文主要的工作是提供一個新的證明方法，甚至可推廣到同步 Diophantine 逼近。

本文的主要架構如下：我們將在第一章介紹 Diophantine 逼近的背景。此章分為三小節。第一節，我們簡單地回顧 Diophantine 逼近和賦距 Diophantine 逼近的概念，並說明一些在實數體下的結論及在正規 Laurent 級數體下的相似結果。在第二節中，我們將介紹非齊次同步 Diophantine 逼近的概念。此外，我們羅列了一些定義和符號，以及關於所謂的 double-metric 和 single-metric 的結論。最後，第三節包含了我們本論文主要的結果。

在第二章，我們回顧一些在正規 Laurent 級數體之下的基本性質。

而第三、四章包含了 0-1 法則和一連串對於我們在一維、高維度的證明過程中非常重要的引理。而我們主要結果的證明就是根據這些引理得證。事實上，在一維度的結果即是高維度結論中的一個特例，但為了方便閱讀以及為了高維度證明的想法做準備，我們將優先處理一維度的例子。

最後，在第五章，我們將針對本論文做一個總結。

Preface

This thesis is concerned with metric simultaneous Diophantine approximation in the field of formal Laurent series. In a recent paper, Kim and Nakada proved an analogue of Kurzweil's theorem in dimension one for formal Laurent series. The main aim of this thesis is to give a new proof which works for simultaneous Diophantine approximation as well.

An outline of this thesis is as follows. In Chapter 1, we will introduce background on Diophantine approximation. This chapter is split into three sections. In Section 1.1, we will briefly recall Diophantine and metric Diophantine approximation, and state some results in the real case and some analogues over the field of formal Laurent series. Then, in Section 1.2, we will introduce inhomogeneous (simultaneous) Diophantine approximation. Moreover, we will collect notations and results for the so-called double-metric and single-metric cases. Finally, Section 1.3 will contain our main results. In Chapter 2, we will recall some fundamental properties for formal Laurent series. Chapter 3 and Chapter 4 will contain zero-one laws and a series of lemmas which are important for the proof of our results in dimension one and higher dimension, respectively. The proofs will follow from these lemmas. We want to point out that the result in dimension one is in fact only a special case of the higher dimensional result. Nevertheless, for the sake of readability and as a warm-up, we will treat the one-dimensional case separately. Finally, we will end the thesis with some concluding remarks in Chapter 5.

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剛進入交大似乎還記憶猶新，時間卻在不知不覺中飛逝，碩士班生活即將畫下句點。回顧過去這兩年的時光，首先誠摯的感謝指導老師—符麥克教授，讓我在碩士班學習過程得以順利，以及細心地指導我論文寫作。感謝兩位口試委員—楊一帆教授及蕭守仁教授，針對這份論文提出寶貴的建議及通過碩士資格的審核。

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我知道，這不僅為碩士班的日子畫下句點，也是下一個人生目標的起跑點。希望可以秉持著在碩士班學到的那份努力，在未來的道路上盡情揮灑。

最後，謹以此文獻給我摯愛的家人。

書誼　謹誌

Contents

Chapter 1

Introduction and Background

In this chapter, we will give a historical discussion and discuss recent results related to this research.

1.1 (Metric) Diophantine Approximation

It is well-known that the set of rational number $\mathbb{Q}$ is dense in $\mathbb{R}$. This means that for each $x \in \mathbb{R}$, there exists a rational sequence $\{r_n\}$ such that $|x - r_n| < \frac{1}{n}$, $\forall n$. A fundamental task is to approximate real numbers by a rational sequence with good accuracy, where the accuracy is measured in terms of the size of the denominator. The area which is concerned with such investigations is called *Diophantine approximation*. In this area, an important question is as follows: for a fixed irrational number α, which function ψ will make the inequality

$$\left|\alpha - \frac{m}{n}\right| < \frac{\psi(n)}{n},\ m, n \in \mathbb{Z}$$

have infinitely many solutions m and n? The following is a typical result.

Theorem 1.1 (G. L. Dirichlet)**.** *Let α be an irrational number. Then*

$$\left|\alpha - \frac{m}{n}\right| < \frac{1}{n^2},\ \ m, n \in \mathbb{Z} \tag{1.1}$$

has infinitely many solutions m and n.

In this theorem, Dirichlet took $\psi(n) = \frac{1}{n}$ such that (1.1) has infinitely many solutions. Note that the result holds for all irrational numbers α. The subarea called *metric Diophantine approximation*, on the other hand, asks for properties which hold for almost all real numbers α. Here, a property holds *almost all* if the set of elements for which the property does not hold is a null set, that is, a set of Lebesgue measure zero. A famous result in metric Diophantine approximation is the following theorem of Khintchine.

Theorem 1.2 (A. Khintchine)**.** *Let $\psi(x)$ be a positive continuous function and suppose that $x\psi(x)$ is non-increasing. Then*

$$\left|\alpha - \frac{m}{n}\right| < \frac{\psi(n)}{n},\ m, n \in \mathbb{Z}$$

has infinitely many solutions for almost all $\alpha \in \mathbb{R}$ if and only if $\sum_{n=1}^{\infty} \psi(n) = \infty$.

There are many other results in the case of real numbers. In this research, we are concerned with analogues in the field of formal Laurent series. Next, we will fix some notation and introduce Diophantine approximation and metric Diophantine approximation for formal Laurent series.

We denote by $\mathbb{F}_q$ a finite field with q elements, where $q = p^n$, $n \in \mathbb{N}$, $p \in \mathbb{P}$. Moreover, we denote by $\mathbb{F}_q[X]$ the set of polynomials with coefficients in $\mathbb{F}_q$, and by $\mathbb{F}_q(X)$ the quotient set of $\mathbb{F}_q[X]$. The elements in this set are called *rational*. Finally, we denote by

$$\mathbb{F}_q\left(\left(X^{-1}\right)\right) = \left\{ f = \sum_{n=-\infty}^{l} a_n X^n : a_l \neq 0, a_n \in \mathbb{F}_q \right\} \cup \{0\}$$

the set of formal Laurent series. Next, we consider in $\mathbb{F}_q((X^{-1}))$ an addition and multiplication, where both operations are defined as for polynomials. Then, the resulting structure is easily seen to be a field. Moreover, we equip $\mathbb{F}_q((X^{-1}))$ with a norm in the standard way, namely, $|f| = q^{\deg(f)}$ for $f \neq 0$ and $|0| = 0$ (here, $\deg(f)$ denotes the generalized degree function). In the sequel, the following set will be of importance

$$\mathbb{L} = \left\{ f \in \mathbb{F}_q((X^{-1})) : |f| < 1 \right\}.$$

Restricting the above norm to this set gives a compact topological group. Hence, there exists a unique translation-invariant probability measure which we are going to denote by m.

Similar to the real case, we can study now Diophantine approximation and metric Diophantine approximation in the field of formal Laurent series, where elements of $\mathbb{F}_q[X]$ play the role of integers. $\mathbb{F}_q(X)$ is dense in $\mathbb{F}_q((X^{-1}))$, *i.e.*, for each $f \in \mathbb{F}_q((X^{-1}))$, there exists a sequence $\{r_n\} \subseteq \mathbb{F}_q(X)$ such that

$$|f - r_n| < \frac{1}{q^n}, \ \forall n.$$

Again as in the real case, an important task is to approximate the value of $f \in \mathbb{F}_q((X^{-1}))$ by $\{r_n\}$ with good accuracy, where the accuracy is measured in terms of the size of the denominator. This area is called *Diophantine approximation in the field of formal Laurent series.* In particular, the analogue to the problem above is as follows: for fixed $f \in \mathbb{L}$, which function ψ will make the Diophantine inequality

$$\left| f - \frac{P}{Q} \right| < \frac{\psi(|Q|)}{|Q|}, \ P, Q \in \mathbb{F}_q[X] \tag{1.2}$$

have infinitely many solutions P and Q? The following result is an analogue of Dirichlet theorem for formal Laurent series.

Theorem 1.3 (Analogue of Dirichlet's Theorem for Formal Laurent Series)**.** *We have,*

$$\left| f - \frac{P}{Q} \right| < \frac{1}{|Q|^2}, \ P, Q \in \mathbb{F}_q[X]$$

has infinitely many solutions P and Q.

We will prove Theorem 1.3 in the next chapter. The subarea called *metric Diophantine approximation in the field of formal Laurent series* asks for properties which hold for almost all $f \in \mathbb{L}$. In this setting, an analogue of Khintchine's theorem for formal Laurent series was proved by Fuchs in [1].

Theorem 1.4 (M. Fuchs [1])**.** *Let $\psi : \{q^t : t \in \mathbb{Z}_{\geq 0}\} \to \{q^t : t \in \mathbb{Z}\}$ be a function with $|Q|\,\psi(|Q|)$ non-increasing. Then the inequality (1.2) has infinitely many solutions P and Q for almost all $f \in \mathbb{L}$, if and only if*

$$\sum_{k=0}^{\infty} q^k \psi(|X|^k) = \infty.$$

Moreover, in [4], Inoue and Nakada improved this by dropping the monotonicity condition " $|Q|\,\psi(|Q|)$ non-increasing" .

Theorem 1.5 (K. Inoue and H. Nakada [4]). *Let* $\psi : \{q^t : t \in \mathbb{Z}_{\geq 0}\} \to \{q^t : t \in \mathbb{Z}\}$ *be a function. Then for any set* S *of positive integers, the inequality (1.2) has infinitely many solutions* P *and* Q *for almost all* $f \in \mathbb{L}$*, if and only if*

$$\sum_{k \in S} q^k \psi(|X|^k) = \infty.$$

In analogy with the integer part of real numbers, we denote by $[\, g\,]$ the polynomial part of g for all $g \in \mathbb{F}_q((X^{-1}))$, *i.e.*, the part of the expansion for which no negative exponents occur. And we denote by $\{g\} = g - [\, g\,]$ the fractional part of g. Note that $|\{g\}| \leq 1$. Then, the inequality (1.2) can be rewritten to

$$|Qf - P| < \psi(|Q|),\ P, Q \in \mathbb{F}_q[X]$$

which, if $\psi(|Q|) \leq 1$, is equivalent to

$$|\{Qf\}| < \psi(|Q|),\ Q \in \mathbb{F}_q[X].$$

So far, what we have discussed the so-called homogeneous case. The major investigations in this research will, however, be for the inhomogeneous case. Thus, we will introduce metric inhomogeneous Diophantine approximation in the field of formal Laurent series next.

1.2 Double-metric and Single-metric Inhomogeneous Diophantine Approximation

Here, we will introduce the metric inhomogeneous Diophantine approximation. Let us consider the Diophantine inequality

$$|\{Qf\} - g| < \psi(|Q|),\ Q \in \mathbb{F}_q[X], \tag{1.3}$$

where $f, g \in \mathbb{L}$, and ψ is a $\{q^t : t \in \mathbb{Z}_{\geq 0}\} \to \{q^t : t \in \mathbb{Z}\}$ function. We will be concerned with the question of the existence of infinitely many solutions to (1.3) as well as the asymptotic number of solutions as $|Q|$ grows. This area is called *metric inhomogeneous Diophantine approximation for formal Laurent series*. In

[9], Ma and Su investigated the problem of (1.3) if f and g are both chosen randomly. Let

$$W(\psi) = \left\{(f,g) \in \mathbb{L}^2 : (1.3) \text{ has infinitely many solutions } Q \in \mathbb{F}_q[X]\right\}.$$

Then, Ma and Su proved the following result.

Theorem 1.6 (C. Ma and W.-Y. Su [9])**.** *Let* $\psi : \{q^t : t \in \mathbb{Z}_{\geq 0}\} \to \{q^t : t \in \mathbb{Z}\}$ *be non-increasing. Then, we have*

$$(m \times m)\left(W(\psi)\right) = \begin{cases} 0, & \text{if } \sum_{Q \in \mathbb{F}_q[X]} \psi(|Q|) < \infty, \\ 1, & \text{if } \sum_{Q \in \mathbb{F}_q[X]} \psi(|Q|) = \infty. \end{cases}$$

Moreover, let us consider the inequality whose $\psi(|Q|)$ is equal to q^{-n-l_n} with $l_n \geq 0$ such that

$$|\{Qf\} - g| < \frac{1}{q^{n+l_n}}, \ Q \in \mathbb{F}_q[X], \ Q \text{ monic}, \ n = \deg(Q), \tag{1.4}$$

where $f, g \in \mathbb{L}$. In [2], Fuchs investigated the problem of (1.4) and derived strong laws of large numbers with error terms for the number of solutions Q of this inequality with $\deg(Q) \leq N$. In order to state his result, define

$$\Psi(N) := \sum_{n \leq N} \frac{1}{q^{l_n}}.$$

Then, his result reads as follows.

Theorem 1.7 (M. Fuchs [2])**.** *For almost all* $(f,g) \in \mathbb{L}^2$*, the number of solutions of (1.4) with* $0 \leq \deg(Q) \leq N$ *satisfies*

$$\Psi(N) + \mathcal{O}\left((\Psi(N))^{\frac{1}{2}} (\log \Psi(N))^{\frac{3}{2}+\epsilon}\right),$$

where $\epsilon > 0$ *is an arbitrary constant.*

These results are for f and g both random. This is the so-called *double-metric* case. Moreover, the following two *single-metric* cases have been considered in the field of formal Laurent series.

(1) fix g and choose a random $f \in \mathbb{L}$,

(2) fix f and choose a random $g \in \mathbb{L}$.

In [2], Fuchs proved the following result for case (1).

Theorem 1.8 (M. Fuchs [2]). *For almost all $f \in \mathbb{L}$, the number of solutions of (1.4) with $0 \leq \deg(Q) \leq N$ satisfies*

$$\Psi(N) + \mathcal{O}\left(\Psi(N)^{1/2} (\log \Psi(N))^{2+\epsilon}\right),$$

where $\epsilon > 0$ is an arbitrary constant.

Moreover, Fuchs also obtained generalizations of the above result in [2]. On the other hand, in this research, we will concerned with the problem of (1.4) for case (2). We will obtain a necessary and sufficient condition such that (1.4) has infinitely many solutions. Moreover, we also generalize this result to simultaneous Diophantine approximation. Therefore, we will discuss simultaneous Diophantine approximation next.

Let r, s be positive integers. We denote by $\mathbb{F}_q[X]^r$ the r-fold Cartesian product of $\mathbb{F}_q[X]$. Moreover, we denote by $\mathbb{F}_q(X)^r$ and $\mathbb{F}_q((X^{-1}))^r$ the vector spaces over $\mathbb{F}_q(X)$ and $\mathbb{F}_q((X^{-1}))$, respectively. Let $\mathbf{f} = [f_1, f_2, \ldots, f_r]$ be an element of $\mathbb{F}_q((X^{-1}))^r$. Then,

$$\deg(\mathbf{f}) = \max_{j=1,\ldots,r} \deg(f_j), \text{ and } \deg(\mathbf{0}) = -\infty.$$

Define $\|\cdot\|$ a norm with domain $\mathbb{F}_q((X^{-1}))^r$ and range $\mathbb{R}^+ \cup \{0\}$ such that $\|\mathbf{f}\| = q^{\deg(\mathbf{f})}$. In the sequel, the following sets will be of importance

$$\mathbb{L}^r = \left\{\mathbf{f} \in \mathbb{F}_q\left(\left(X^{-1}\right)\right)^r : \|\mathbf{f}\| < 1\right\}$$

and

$$\mathbb{L}^{r\times s} = \{\text{matrix } A \text{ of size } r \times s : \text{all the elements in } A \text{ are belonging to } \mathbb{L}\}.$$

We equip $\mathbb{L}^r$ with the r-fold product measure of $\mathbb{L}$ which we also denote by m. Now, we consider a Diophantine inequality

$$\|\{\mathbf{q}A\} - \mathbf{g}\| < \psi(\|\mathbf{q}\|), \ \mathbf{q} \in \mathbb{F}_q[X]^r, \tag{1.5}$$

where ψ is a $\{q^t : t \in \mathbb{Z}_{\geq 0}\} \to \{q^t : t \in \mathbb{Z}\}$ function, and

$$\mathbf{q} = \begin{bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_r \end{bmatrix}^{\mathsf{T}} \in \mathbb{F}_q[X]^r, \ A = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1s} \\ f_{21} & f_{22} & \cdots & f_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ f_{r1} & f_{r2} & \cdots & f_{rs} \end{bmatrix} \in \mathbb{L}^{r\times s}, \ \mathbf{g} = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_s \end{bmatrix}^{\mathsf{T}} \in \mathbb{L}^s.$$

In fact, the inequality (1.5) is equivalent to the following system of inequalities

$$\begin{gathered} |\{Q_1 f_{11} + Q_2 f_{21} + \cdots + Q_r f_{r1}\} - g_1| < \psi(\|\mathbf{q}\|) \\ |\{Q_1 f_{12} + Q_2 f_{22} + \cdots + Q_r f_{r2}\} - g_2| < \psi(\|\mathbf{q}\|) \\ \vdots \\ |\{Q_1 f_{1s} + Q_2 f_{2s} + \cdots + Q_r f_{rs}\} - g_s| < \psi(\|\mathbf{q}\|). \end{gathered}$$

will be again concerned with the question of the existence of infinitely many solutions to (1.5) as well as the asymptotic number of solutions to the equation as $\|\mathbf{q}\|$ grows. This subarea is called *metric inhomogeneous simultaneous Diophantine approximation.* In fact, we say that (1.5) is in dimension one if $r = s = 1$, and in higher dimension if r or s is more than 1. Similar to the result in dimension one, Kristensen investigated the problem of (1.5) if A and $\mathbf{g}$ are chosen randomly. Let

$$W_{r,s}(\psi) = \{(A, \mathbf{g}) \in \mathbb{L}^{r\times s} \times \mathbb{L}^s : (1.5) \text{ has infinitely many solutions } \mathbf{q} \in \mathbb{F}_q[X]^r\}$$

Then, Kristensen proved the following result in [6].

Theorem 1.9 (S. Kristensen [6]). *Let $\psi : \{q^t : t \in \mathbb{Z}_{\geq 0}\} \to \{q^t : t \in \mathbb{Z}\}$ be non-increasing. Then, we have*

$$m(W_{r,s}(\psi)) = \begin{cases} 0, & \text{if } \sum_{\mathbf{q}\in\mathbb{F}_q[X]^r} \psi(\|\mathbf{q}\|)^s < \infty, \\ 1, & \text{if } \sum_{\mathbf{q}\in\mathbb{F}_q[X]^r} \psi(\|\mathbf{q}\|)^s = \infty. \end{cases}$$

Moreover, let

$$\Psi(N) := \sum_{\|\mathbf{q}\|\leq q^N} \psi(\|\mathbf{q}\|)^s.$$

Then, in [6], Kristensen obtained the following result for the number of solutions of (1.5).

Theorem 1.10 (S. Kristensen [6]). *Let $\psi : \{q^t : t \in \mathbb{Z}_{\geq 0}\} \to \{q^t : t \in \mathbb{Z}\}$ be non-increasing. Then, for almost all $(A, \mathbf{g}) \in \mathbb{L}^{r\times s} \times \mathbb{L}^s$, the number of solutions of (1.5) with $0 \leq \deg(\mathbf{q}) \leq N$ satisfies*

$$\Psi(N) + \mathcal{O}\left(\Psi(N)^{\frac{1}{2}}\left(\log \Psi(N)^{\frac{3}{2}+\epsilon}\right)\right),$$

where $\epsilon > 0$ is an arbitrary constant.

This situation is again called the *double-metric* case. Moreover, the following two *single-metric* cases are considered in simultaneous Diophantine approximation.

(1) fix $\mathbf{g} \in \mathbb{L}^s$ and choose a random $A \in \mathbb{L}^{r\times s}$;

(2) fix $A \in \mathbb{L}^{r\times s}$ and choose a random $\mathbf{g} \in \mathbb{L}^s$.

In [6], Kristensen also investigated the problem of (1.5) for case (1). Let

$$W_{r\times s}(\psi, \mathbf{g}) := \{A \in \mathbb{L}^{r\times s} : (1.5) \text{ has infinitely many solutions } \mathbf{q} \in \mathbb{F}_q[X]^r\}.$$

Here, Kristensen obtained the following result.

Theorem 1.11 (S. Kristensen [6]). *Let $r \geq 2$ and let $\psi : \{q^t : t \in \mathbb{Z}_{\geq 0}\} \to \{q^t : t \in \mathbb{Z}\}$ be non-increasing. Then, for any $\mathbf{g} \in \mathbb{L}^s$,*

$$m\left(W_{r\times s}(\psi, \mathbf{g})\right) = \begin{cases} 0, & \text{if } \sum_{\mathbf{q}\in\mathbb{F}_q[X]^r} \psi(\|\mathbf{q}\|)^s < \infty, \\ 1, & \text{if } \sum_{\mathbf{q}\in\mathbb{F}_q[X]^r} \psi(\|\mathbf{q}\|)^s = \infty. \end{cases}$$

Moreover, Kristensen also obtained a result for the number of solutions of (1.5) in case (1) as well.

Theorem 1.12 (S. Kristensen [6]). *Let $r \geq 2$, and $\psi : \{q^t : t \in \mathbb{Z}_{\geq 0}\} \to \{q^t : t \in \mathbb{Z}\}$ be non-increasing. Then, for almost all $A \in \mathbb{L}^{r\times s}$, the number of solutions of (1.5) with $0 \leq \deg(\mathbf{q}) \leq N$ satisfies*

$$\Psi(N) + \mathcal{O}\left(\Psi(N)^{\frac{1}{2}}\left(\log \Psi(N)\right)^{\frac{3}{2}+\epsilon}\right),$$

where $\epsilon > 0$ is an arbitrary constant.

Note that Kristensen's result holds for the number of solutions to (1.5) in case (1) as $r \geq 2$, and Fuchs obtained the result for the remaining case of $r = 1$ in [2]. Therefore, this subarea is complete. On the other hand, we will be concerned with the question of the existence of infinitely many solutions to (1.5) in case (2). More specifically, we will find a necessary and sufficient condition such that (1.5) has infinitely many solutions. In the next section, we will introduce the main results of this research.

1.3 Kurzweil's Theorem in the Field of Formal Laurent Series

We consider the Diophantine inequality

$$|\{Qf\} - g| < \frac{1}{q^{n+l_n}},\ Q \in \mathbb{F}_q[X],\ n = \deg(Q), \tag{1.6}$$

where f is fixed and g is chosen randomly in $\mathbb{L}$. Set

$$W(l_n, f) := \{g \in \mathbb{L} :\ (1.6) \text{ has infinitely many solutions}\}.$$

By the Borel-Cantelli lemma, we obtain that $\sum_n \frac{1}{q^{l_n}} < \infty$ implies $m(W(l_n, f)) = 0$. However, the other direction, namely, $m(W(l_n, f)) = 1$ if $\sum_n \frac{1}{q^{l_n}} = \infty$ is not necessarily true for all sequence l_n. Consequently, an important question is as follows: for which f is $m(W(l_n, f)) = 1$ or 0 according to $\sum_n q^{-l_n}$ converges or not? In [5], Kim and Nakada obtained a characterization of these f. In order to state their result, define

$$S := \left\{ f \in \mathbb{L} : \forall l_n \text{ with } \sum_n \frac{1}{q^{l_n}} = \infty, \right.$$
$$\left. (1.6) \text{ has infinitely many solutions for almost all } g \right\}.$$

Moreover, we need the following notation.

Definition 1.1. f *is called* ***badly approximable*** *if there exists a constant* $c > 0$ *such that for all* $Q \in \mathbb{F}_q[X],\ Q \neq 0$ *with* $n = \deg(Q)$,

$$|\{Qf\}| > \frac{1}{q^{n+c}}.$$

Then, Kim and Nakada proved the following result in [5].

Theorem 1.13 (D. H. Kim and H. Nakada [5])**.** *We have,*

$$S = \{f \in \mathbb{L} : f \textit{ is badly approximable}\}.$$

As for the method of proof, Kim and Nakada used continued fraction expansion in $\mathbb{F}_q((X^{-1}))$. Thus, their method cannot be extended to simultaneous Diophantine approximation. Here, we will reprove their result with a method closer to the one of Kurzweil who proved the analogue of the above result in the real number field. This new approach not only works in dimension one but also works in higher dimension. Therefore, we consider the Diophantine inequality

$$\|\{\mathbf{q}A\} - \mathbf{g}\| < \frac{1}{q^{\lfloor \frac{nr}{s} \rfloor + l_n}}, \ \mathbf{q} \in \mathbb{F}_q[X]^r, \ n = \deg(\mathbf{q}), \tag{1.7}$$

where A is fixed and $\mathbf{g}$ is chosen randomly in $\mathbb{L}^s$. Let us again define a set by

$$S_{r\times s} := \left\{ A \in \mathbb{L}^{r\times s} : \forall l_n \text{ with } \sum_n \frac{1}{q^{sl_n}} = \infty, \right.$$
$$\left. \text{(1.7) has infinitely many solutions for almost all } \mathbf{g} \in \mathbb{L}^s \right\}.$$

Moreover, similar as above, we need the following notation.

Definition 1.2. *$A \in \mathbb{L}^{r\times s}$ is called* ***badly approximable*** *if there exists a constant $c > 0$ such that for all $\mathbf{q} \in \mathbb{F}_q[X]^r, \mathbf{q} \neq 0$ with $\deg(\mathbf{q}) = n$,*

$$\|\{\mathbf{q}A\}\| > \frac{1}{q^{\lfloor \frac{nr}{s} \rfloor + c}}.$$

Then, the main result in this research is the following theorem.

Theorem 1.14. *We have,*

$$S_{r\times s} = \left\{A \in \mathbb{L}^{r\times s} : A \textit{ is badly approximable}\right\}.$$

In the next chapter, we will introduce some properties in the field of formal Laurent series which we will use in the proof. Then, we are going to prove Theorem 1.13 in Chapter 3, and Theorem 1.14 in Chapter 4.

Chapter 2

Preliminaries

In this chapter, we will collect some results that we are going to use.

2.1 Fundamental Properties in Dimension One

We start by recalling some results which were already briefly mentioned in the introduction. First, we have the following property (see [8] for a proof).

Proposition 2.1.1. $(\mathbb{F}_q((X^{-1})), +, \cdot)$ *is a field.*

Next, $|\cdot|$ is an ultra-metric norm.

Proposition 2.1.2. *Let* $f, g \in \mathbb{F}_q((X^{-1}))$, *then* $|\cdot|$ *satisfies the following:*

(1) $|f| = 0 \Leftrightarrow f = 0.$

(2) $|fg| = |f|\,|g|.$

(3) $|f+g| \leq \max\{|f|, |g|\}.$

Proof.

(1) $|f| = 0 \Leftrightarrow \deg(f) = -\infty \Leftrightarrow f = 0.$

(2) $|fg| = q^{\deg(fg)} = q^{\deg(f)+\deg(g)} = q^{\deg(f)}q^{\deg(g)} = |f|\,|g|.$

(3) $$|f+g| = q^{\deg(f+g)} = q^{\max\{\deg(f),\deg(g)\}} = \max\{q^{\deg(f)}, q^{\deg(g)}\} = \max\{|f|, |g|\}.$$ ∎

Next, recall

$$\mathbb{L} = \left\{ f \in \mathbb{F}_q((X^{-1})) : |f| < 1 \right\}$$

which we have equipped with the normalized Haar measure m. For all $g \in \mathbb{L}$, $d \geq 1$, we define

$$B\left(g, \frac{1}{q^d}\right) = \left\{ f \in \mathbb{L} : |f - g| < \frac{1}{q^d} \right\}.$$

Then, we have the following important properties.

Proposition 2.1.3. *Each two balls in $\mathbb{L}$ are either disjoint or one is contained in the other.*

Proof. Let $B\left(f, q^{-d}\right)$, $B\left(g, q^{-e}\right)$ be two balls with centers f, g and radii q^{-d}, q^{-e}. Without loss of generality, we suppose $d > e$. Assume that they are not disjoint, then we have to prove that one is contained in the other. First, we estimate the distance of the two centers f and g. Let h be in the intersection of $B\left(f, q^{-d}\right)$ and $B\left(g, q^{-e}\right)$. Then,

$$|f - g| = |f - h + h - g| \leq \max\left\{|f - h|, |h - g|\right\} < \frac{1}{q^e}.$$

This means that $f \in B\left(g, q^{-e}\right)$. Next, we claim that $B\left(f, q^{-d}\right)$ belongs to $B\left(g, q^{-e}\right)$. Assume that this is wrong. Then, there exists h in $B\left(f, q^{-d}\right) \setminus B\left(g, q^{-e}\right)$. Now,

$$|h - g| = |h - f + f - g| \leq \max\left\{|h - f|, |f - g|\right\} < \frac{1}{q^e}.$$

This implies that $h \in B\left(g, q^{-e}\right)$, a contradiction. Hence, we obtain $B\left(f, q^{-d}\right) \subseteq B\left(g, q^{-e}\right)$, which means that one is contained in the other. ∎

Proposition 2.1.4. *Fix $b_1, b_2, \ldots, b_d \in \mathbb{F}_q$, $g \in \mathbb{L}$ and $d \geq 1$. Then, we have*

$$m\left(\left\{ f : f = \sum_{i=1}^{d} b_i X^{-i} + \sum_{i=d+1}^{\infty} a_i X^{-i}, \forall a_i \in \mathbb{F}_q \right\}\right) = \frac{1}{q^d},$$

and

$$m\left(B\left(g, \frac{1}{q^d}\right)\right) = \frac{1}{q^d}.$$

Proof. Assume that $h = b_1 X^{-1} + b_{-2} X^{-2} + \cdots + b_d X^{-d}$. Thus, $f = \sum_{i=1}^{\infty} a_i X^{-i} \in B\left(h, q^{-d}\right)$ if and only if $a_i = b_i$ for all i with $1 \leq i \leq d$. Consequently,

$$\left\{f : f = \sum_{i=1}^{d} b_i X^{-i} + \sum_{i=d+1}^{\infty} a_i X^{-i}, \forall a_i \in \mathbb{F}_q\right\} = B\left(h, \frac{1}{q^d}\right).$$

Next, observe that

$$\begin{aligned}
1 &= m\left(\mathbb{L}\right) \\
&= m\left(\bigcup_{c_1,\ldots,c_d \in \mathbb{F}_q} \left\{f : f = \sum_{i=1}^{d} c_i X^{-i} + \sum_{i=d+1}^{\infty} a_i X^{-i}, \forall a_i \in \mathbb{F}_q\right\}\right) \\
&= \sum_{c_1,\ldots,c_d \in \mathbb{F}_q} m\left(\left\{f : f = \sum_{i=1}^{d} c_i X^{-i} + \sum_{i=d+1}^{\infty} a_i X^{-i}, \forall a_i \in \mathbb{F}_q\right\}\right) \\
&= q^d\, m\left(\left\{f : f = \sum_{i=1}^{d} c_i X^{-i} + \sum_{i=d+1}^{\infty} a_i X^{-i}, \forall a_i \in \mathbb{F}_q\right\}\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
m\left(B\left(h, \frac{1}{q^d}\right)\right) &= m\left(\left\{f : f = \sum_{i=1}^{d} b_i X^{-i} + \sum_{i=d+1}^{\infty} a_i X^{-i}, \forall a_i \in \mathbb{F}_q\right\}\right) \\
&= \frac{1}{q^d}
\end{aligned}$$

which proves the first result. Since m is a translation-invariant measure, we have that for any $g \in \mathbb{L}$, the measures of $B\left(h, q^{-d}\right)$ and $B\left(g, q^{-d}\right)$ are the same. So, we get

$$m\left(\left(g, q^{-d}\right)\right) = q^{-d}$$

for any $g \in \mathbb{L}$. ∎

We conclude this subsection by recalling Dirichlet's theorem and providing a proof.

Theorem 2.1 (Analogue of Dirichlet's Theorem for Formal Laurent series)**.** *We have that*

$$|\{Qf\}| < \frac{1}{|Q|},\ Q \in \mathbb{F}_q[X] \tag{2.1}$$

has infinitely many solutions.

Proof. Note that the claimed result is trivial if f is not irrational. Therefore, we can assume that f is irrational. Now, we need to prove the following claim: for all $N \in \mathbb{N}$, there exists a non-zero polynomial Q with $\deg(Q) \leq N$ such that

$$|\{Qf\}| < \frac{1}{q^N}. \tag{2.2}$$

First, we know that the number of $Q \neq 0$ with $\deg(Q) \leq N$ is $q^{N+1} - 1$. We divide $\mathbb{L}$ into q^N balls such that

$$\mathbb{L} = \bigcup_{b_1,\ldots,b_N \in \mathbb{F}_q} \left\{ f : f = \sum_{i=1}^{N} b_i X^{-i} + \sum_{i=N+1}^{\infty} a_i X^{-i}, \forall a_i \in \mathbb{F}_q \right\}.$$

Then there exist at least two different nonzero $Q_1, Q_2 \in \mathbb{F}_q[X]$ with $\deg(Q_1), \deg(Q_2) \leq N$ such that $|\{Q_1 f\} - \{Q_2 f\}| < q^{-N}$(if not, then the number of $Q \neq 0$ with $\deg(Q) \leq N$ is at most q^N which is a contradiction). Hence,

$$\frac{1}{q^N} > |\{Q_1 f\} - \{Q_2 f\}| = |\{(Q_1 - Q_2) f\}|.$$

So, $(Q_1 - Q_2)$ is a solution of (2.2). This proves our claim. Moreover, our claim clearly implies that (2.1) has infinitely many solutions. ∎

2.2 Fundamental Properties in Higher Dimension

In this section, we will show that all properties from the previous section hold in higher dimension as well.

Let us fix positive numbers r and s. Then, the norm $\|\cdot\|$ on $\mathbb{F}_q((X^{-1}))^r$ from the introduction has the following properties.

Proposition 2.2.1. *Let* $\mathbf{f}, \mathbf{g} \in \mathbb{F}_q\left((X^{-1})\right)^r$, *then* $\|\cdot\|$ *satisfies the following:*

(1) $\|\mathbf{f}\| = 0 \Leftrightarrow \mathbf{f} = \mathbf{0}$.

(2) $\|\mathbf{f} + \mathbf{g}\| \leq \max\{\|\mathbf{f}\|, \|\mathbf{g}\|\}$.

Proof.

(1) $\|\mathbf{f}\| = 0 \Leftrightarrow \deg(\mathbf{f}) = -\infty \Leftrightarrow \mathbf{f} = \mathbf{0}$.

(2) $$\begin{aligned}\|\mathbf{f}+\mathbf{g}\| = q^{\deg(\mathbf{f}+\mathbf{g})} &= q^{\max\{\deg(f_1+g_1),\deg(f_2+g_2),\dots,\deg(f_r+g_r)\}} \\ &\leq q^{\max\{\deg(f_1),\dots,\deg(f_r),\deg(g_1),\dots,\deg(g_r)\}} \\ &= q^{\max\{\deg(\mathbf{f}),\deg(\mathbf{g})\}} \\ &= \max\{\|\mathbf{f}\|, \|\mathbf{g}\|\}. \quad \blacksquare\end{aligned}$$

Recall

$$\mathbb{L}^r = \left\{\mathbf{f} \in \mathbb{F}_q((X^{-1}))^r : \|\mathbf{f}\| < 1\right\}$$

which we have equipped with the product measure of $\mathbb{L}$ (also denoted by m). Moreover, as before, for all $\mathbf{g} = [g_1, \dots, g_r] \in \mathbb{L}^r$, $d \geq 1$, we define

$$\begin{aligned}B\left(\mathbf{g}, \frac{1}{q^d}\right) &= \left\{\mathbf{f} \in \mathbb{L}^r : \|\mathbf{f}-\mathbf{g}\| < \frac{1}{q^d}\right\} \\ &= \prod_{i=1}^{r} B\left(g_i, \frac{1}{q^d}\right).\end{aligned}$$

As in the one-dimensional case, we again have the following important properties.

Proposition 2.2.2. *Each two balls in $\mathbb{L}^r$ are either disjoint or one is contained in the other.*

Proof. Let $B\left(\mathbf{f}, q^{-d}\right)$, $B\left(\mathbf{g}, q^{-e}\right)$ be two balls with centers $\mathbf{f} = [f_1, \dots, f_r]$, $\mathbf{g} = [g_1, \dots, g_r]$ and radii q^{-d}, q^{-e}. Without loss of generality, we suppose $d > e$. Assume that they are not disjoint. We know that

$$B\left(\mathbf{f}, \frac{1}{q^d}\right) = \prod_{i=1}^{r} B\left(f_i, \frac{1}{q^d}\right) \text{ and } B\left(\mathbf{g}, \frac{1}{q^e}\right) = \prod_{i=1}^{r} B\left(g_i, \frac{1}{q^e}\right).$$

Then, by Proposition 2.1.3, we have

$$B\left(f_i, \frac{1}{q^d}\right) \subseteq B\left(g_i, \frac{1}{q^e}\right), \ \forall i = 1, \dots, r.$$

This implies that

$$\prod_{i=1}^{r} B\left(f_i, \frac{1}{q^d}\right) \subseteq \prod_{i=1}^{r} B\left(g_i, \frac{1}{q^e}\right).$$

Hence, we obtain $B\left(\mathbf{f}, q^{-d}\right) \subseteq B\left(\mathbf{g}, q^{-e}\right)$ which means that one is contained in the other. $\blacksquare$

Proposition 2.2.3. *Let d be a positive integer. Fix $b_i^{(j)} \in \mathbb{F}_q$ for all $i = 1, \ldots, d$ and $j = 1, \ldots, r$. Then, we have*

$$m\left(\left\{\mathbf{f} = [f_1, \ldots, f_r] : f_j = \sum_{i=1}^{d} b_i^{(j)} X^{-i} + \sum_{i=d+1}^{\infty} a_i^{(j)} X^{-i}, \forall a_i^{(j)} \in \mathbb{F}_q, \forall j\right\}\right) = \frac{1}{q^{rd}}$$

and

$$m\left(B\left(\mathbf{g}, \frac{1}{q^d}\right)\right) = \frac{1}{q^{rd}}.$$

Proof. Assume that $\mathbf{h} = [h_1, \ldots, h_r]$, where $h_j = \sum_{i=1}^{d} b_i^{(j)} X^{-i}$ for all j. Then, for any $\mathbf{f} = [f_1, \ldots f_r]$ with $f_j = \sum_{i=1}^{\infty} a_i^{(j)} X^{-i}$,

$$\mathbf{f} \in B\left(\mathbf{h}, \frac{1}{q^d}\right) \text{ iff } a_i^{(j)} = b_i^{(j)},\ 1 \leq i \leq d, \forall j.$$

This implies that

$$\left\{\mathbf{f} = [f_1, \ldots, f_s] : f_j = \sum_{i=1}^{d} b_i^{(j)} X^{-i} + \sum_{i=d+1}^{\infty} a_i^{(j)} X^{-i}, \forall a_i^{(j)} \in \mathbb{F}_q, \forall j\right\} = B\left(\mathbf{h}, \frac{1}{q^d}\right).$$

Consequently,

$$\begin{aligned}
& m\left(\left\{\mathbf{f} = [f_1, \ldots, f_s] : f_j = \sum_{i=1}^{d} b_i^{(j)} X^{-i} + \sum_{i=d+1}^{\infty} a_i^{(j)} X^{-i}, \forall a_i^{(j)} \in \mathbb{F}_q, \forall j\right\}\right) \\
&= m\left(B\left(\mathbf{h}, \frac{1}{q^d}\right)\right) = m\left(\prod_{i=1}^{r} B\left(h_i, \frac{1}{q^d}\right)\right) \\
&= \prod_{i=1}^{r} m\left(B\left(h_i, \frac{1}{q^d}\right)\right) = \prod_{i=1}^{r} \frac{1}{q^d} = \frac{1}{q^{rd}}.
\end{aligned}$$

Since m is a translation invariant measure, we have

$$m\left(B\left(\mathbf{h}, \frac{1}{q^d}\right)\right) = m\left(B\left(\mathbf{g}, \frac{1}{q^d}\right)\right)$$

for any $\mathbf{g} \in \mathbb{L}^r$. So, we get

$$m\left(B\left(\mathbf{g}, \frac{1}{q^d}\right)\right) = \frac{1}{q^{rd}} \text{ for any } \mathbf{g} \in \mathbb{L}^r. \quad \blacksquare$$

Next, we need the following notation.

Definition 2.1. *A $r \times s$ matrix A is called **irrational** if $\mathbf{q}A$ does not belong to $\mathbb{F}_q[X]^s$ for all $\mathbf{q} \in \mathbb{F}_q[X]^r$ with $\mathbf{q} \neq \mathbf{0}$.*

As in dimension one, we conclude by stating and proving Dirichlet's theorem.

Theorem 2.2 (Analogue of Dirichlet's Theorem for Formal Laurent Series)**.** *We have that*

$$\|\{\mathbf{q}A\}\| < \frac{1}{q^{\lfloor \frac{nr}{s} \rfloor}}, \ \mathbf{q} \in \mathbb{F}_q[X]^r, \ \deg(\mathbf{q}) = n \tag{2.3}$$

has infinitely many solutions.

Proof. Note that the claimed result is trivial if A is not irrational. Therefore, we can assume that A is irrational. Then, similar as in the one-dimensional case, we need to prove the following claim: for all $N \in \mathbb{N}$, there exists a non-zero polynomial vector $\mathbf{q}$ with $\deg(\mathbf{q}) \leq N$ such that

$$\|\{\mathbf{q}A\}\| < \frac{1}{q^{\lfloor \frac{Nr}{s} \rfloor}} \tag{2.4}$$

First, we know that the number of $\mathbf{q} \neq \mathbf{0}$ with $\deg(\mathbf{q}) \leq N$ is $q^{(N+1)r} - 1$. We divide $\mathbb{L}$ into $q^{\lfloor \frac{Nr}{s} \rfloor}$ balls as in the proof of Dirichlet's theorem in dimension one. This yields that a subdivision of $\mathbb{L}^s$ into $q^{\lfloor \frac{Nr}{s} \rfloor s}$ balls. Then, there are two different nonzero polynomial vectors $\mathbf{q}_1$,$\mathbf{q}_2$ with $\deg(\mathbf{q}_1)$, $\deg(\mathbf{q}_2) \leq N$ such that $\|\{\mathbf{q}_1 A\} - \{\mathbf{q}_2 A\}\| < q^{-\lfloor \frac{Nr}{s} \rfloor}$
(if not, then the number of $\mathbf{q}$ with $\deg(\mathbf{q}) \leq N$ is at most $q^{\lfloor \frac{Nr}{s} \rfloor s} \leq q^{\frac{Nr}{s} s} = q^{Nr}$, a contradiction). Hence,

$$\frac{1}{q^{\lfloor \frac{Nr}{s} \rfloor}} > \|\{\mathbf{q}_1 A\} - \{\mathbf{q}_2 A\}\| = \|\{(\mathbf{q}_1 - \mathbf{q}_2)A\}\|$$

So, $(\mathbf{q}_1 - \mathbf{q}_2)$ is a solution of (2.4). This proves our claim. Moreover, due to the irrationality of A, our claim implies that (2.3) has infinitely many solutions. ∎

Chapter 3

Kurzweil's Theorem in Dimension One

Here, we are going to prove Theorem 1.13 from the introduction. Therefore, fix a $f = f_1 X^{-1} + f_2 X^{-2} + \cdots$. For the next three lemmas, we assume that f is irrational.

Lemma 3.1. *$\{\{Qf\} : Q \in \mathbb{F}_q[X]\}$ is dense in $\mathbb{L}$.*

Proof. Let us fix $n \in \mathbb{N}$ and $g = g_1 X^{-1} + g_2 X^{-2} + \cdots$, where $g_i \in \mathbb{F}_q$. Then we claim: there exists Q with $\deg(Q) = N$ such that $|\{Qf\} - g| < q^{-n}$. In order to prove this, we consider $\mathbf{g} = \mathbf{a}A$, where

$$\mathbf{g} = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix}^\intercal, \mathbf{a} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_N \end{bmatrix}^\intercal, A = \begin{bmatrix} f_1 & f_2 & \cdots & f_n \\ f_2 & f_3 & \cdots & f_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ f_{N+1} & f_{N+2} & \cdots & f_{N+n} \end{bmatrix}.$$

Then we claim: $\text{rank}(A) = n$ as N is large enough. We need to prove that the column vectors of A are linear independent. Suppose that this is wrong. Then,

$$\alpha_1(f_1, f_2, \ldots, f_{N+1}) + \cdots + \alpha_n(f_n, f_{n+1}, \ldots, f_{N+n}) = (0, 0, \ldots, 0)$$

with α_i not all zero. Let $P(X) = \alpha_1 + \alpha_2 X + \cdots + \alpha_n X^{n-1}$, $P(X) \neq 0$. Then, $|\{Pf\}| < q^{-N-1}$. On the other hand, the number of P is finite (since n is fixed), which means that $\min|\{Pf\}| \geq q^{-N-1}$ for N large enough. This gives a contradiction. Therefore, $P(X)$ must be 0, which implies that the column vectors of A are linear independent. Then for all $\mathbf{g}$, there exists $\mathbf{a}$ such that $\mathbf{a}A = \mathbf{g}$. This implies that $Q = a_0 + a_1 X + \cdots + a_N X^N$ satisfies $|\{Qf\} - g| < q^{-n}$. Hence, $\{\{Qf\} : Q \in \mathbb{F}_q[X]\}$ is dense in $\mathbb{L}$. ∎

Lemma 3.2 (0-1 law)**.** *Let a measurable set E in $\mathbb{L}$ be invariant under the action $\cdot + \{Qf\}$ for all $Q \in \mathbb{F}_q[X]$. Then, we have $m(E) = 0$ or 1.*

Proof. Suppose that $m(E) > 0$. By [3], for all $\epsilon > 0$, there exists a radius q^{-d} such that

$$\int \left| \chi_E(g) - \frac{m\left(E \cap \left(B\left(g, \frac{1}{q^d}\right) + \{Qf\}\right)\right)}{m\left(B\left(g, \frac{1}{q^d}\right) + \{Qf\}\right)} \right| dm < \epsilon\, m(E)$$

for all $Q \in \mathbb{F}_q[X]$. Consequently,

$$\int_E \left[1 - \frac{m\left(E \cap \left(B\left(g, \frac{1}{q^d}\right) + \{Qf\}\right)\right)}{m\left(B\left(g, \frac{1}{q^d}\right) + \{Qf\}\right)} \right] dm < \epsilon\, m(E).$$

This implies that there exists a $g \in \mathbb{L}$ with

$$1 - \frac{m\left(E \cap \left(B\left(g, \frac{1}{q^d}\right) + \{Qf\}\right)\right)}{m\left(B\left(g, \frac{1}{q^d}\right) + \{Qf\}\right)} < \epsilon.$$

Thus

$$\frac{m\left(E \cap \left(B\left(g, \frac{1}{q^d}\right) + \{Qf\}\right)\right)}{m\left(B\left(g, \frac{1}{q^d}\right) + \{Qf\}\right)} > 1 - \epsilon.$$

Since $\{\{Qf\} : Q \in \mathbb{F}_q[X]\}$ is dense in $\mathbb{L}$, we get the inequality

$$m(E) > 1 - \epsilon$$

for all $\epsilon > 0$. Hence, we obtain the result $m(E) = 1$. ∎

Lemma 3.3. *Let*

$$E := \left\{ g \in \mathbb{L} : |\{Qf\} - g| < \frac{1}{q^{n+l_n}} \text{ with } n = \deg(Q) \text{ has infinitely many solutions} \right\}.$$

Then, E is invariant under the action $\cdot + \{Qf\}$ for all $Q \in \mathbb{F}_q[X]$ and hence $m(E) = 0$ or 1.

Proof. Fix a polynomial Q'. Let $g \in E$. Then, we can find infinitely many Q with $\deg(Q) > \deg(Q')$ such that

$$|\{(Q - Q')f\} - g| = |\{Qf\} - (g + \{Q'f\})| < \frac{1}{q^{n+l_n}}.$$

So, we get $E + \{Q'f\} \subseteq E$. Conversely, since $|\{(Q + Q')f\} - g| < q^{-n-l_n}$ has infinitely many solutions, we get that $|\{Qf\} - (g - \{Q'f\})| < q^{-n-l_n}$ has infinitely many solutions. Thus, $g - \{Q'f\} \in E$. Then, $g = g - \{Q'f\} + \{Q'f\} \in E + \{Q'f\}$, this means $E \subseteq E + \{Q'f\}$. So, we obtain $E = E + \{Q'f\}$. Consequently, E is invariant under the action $\cdot + \{Qf\}$ for all $Q \in \mathbb{F}_q[X]$ and hence $m(E) = 0$ or 1. ∎

For the next two lemmas, f is assumed to be badly approximable. Thus, there exists a constant $c > 0$ such that for all $Q \in \mathbb{F}_q[X], Q \neq 0$ with $n = \deg(Q)$,

$$|\{Qf\}| > \frac{1}{q^{n+c}}.$$

Lemma 3.4. *Let $g \in \mathbb{L}$. Then, the number of $\{Qf\}$ with $\deg(Q) \leq N$ belonging to $B\left(g, q^{-d}\right)$ is at most $\max\{q^{N+c-d}, 1\}$.*

Proof. First, we need the following claim: define $g = g_1X^{-1} + g_2X^{-2} + \cdots + g_dX^{-d} + \cdots$, where $d > 0$. Then, the number of $\{Qf\}$ with $\deg(Q) \leq N$ belonging to $B\left(g, q^{-d}\right)$ is either q^σ or 0, where $\sigma \geq 0$. Let a_i be the coefficient of X^i of Q, $\forall i = 1, 2, \ldots, N$. Define

$$\mathbf{a} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_N \end{bmatrix}^\intercal, \mathbf{b} = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_d \end{bmatrix}^\intercal, A = \begin{bmatrix} f_1 & f_2 & \cdots & f_d \\ f_2 & f_3 & \cdots & f_{d+1} \\ \vdots & \vdots & \ddots & \vdots \\ f_{N+1} & f_{N+2} & \cdots & f_{N+d} \end{bmatrix},$$

Let us consider the linear system

$$\mathbf{a}A = \mathbf{b}.$$

We have to discuss the number of solutions (which is equal to the number of $\{Qf\}$ with $\deg(Q) \leq N$ belonging to $B\left(g, q^{-d}\right)$). There are two cases:

1. If the linear system has no solution, then the number of $\{Qf\}$ which belong to $B\left(g, q^{-d}\right)$ is 0.

2. If the linear system has a particular solution $\mathbf{a} = [a_0, a_1, \ldots, a_N]$, then $\mathbf{a} + \ker A$ is the set of all solutions of the linear system. Hence the number of solutions is $|\ker A| = q^{\dim \ker A} = q^{\sigma}$.

By the above cases, we obtain our claim. Next, we suppose q^{σ} is the number of $\{Qf\}$ with $\deg(Q) \leq N$ belonging to $B\left(g, q^{-d}\right)$. We consider the following cases:

1. If $\sigma = 0$, then the number of $\{Qf\}$ belonging to $B\left(g, q^{-d}\right)$ is 1.

2. If $\sigma > 0$, then there exist two points $\{Q_1 f\}$, $\{Q_2 f\} \in B\left(g, q^{-d}\right)$ such that $|\{Q_1 f\} - \{Q_2 f\}| < q^{-d-(\sigma-1)}$, where $\deg(Q_1), \deg(Q_2) \leq N$. Because f is badly approximable, we get

$$\begin{aligned}\frac{1}{q^{d+(\sigma-1)}} &> |\{Q_1 f\} - \{Q_2 f\}| \\ &= |\{(Q_1 - Q_2) f\}| > \frac{1}{q^{\deg(Q_1 - Q_2) + c}} \geq \frac{1}{q^{N+c}}.\end{aligned}$$

   This implies that $q^{\sigma} \leq q^{N+c-d}$.

By the two cases, we obtain that the number of $\{Qf\}$ with $\deg(Q) \leq N$ belonging to $B\left(g, q^{-d}\right)$ is at most $\max\{q^{N+c-d}, 1\}$. ∎

We give a second method of proof which is in fact easier.

Second Method of Proof. Let Q, Q' be two different polynomials with $\deg(Q), \deg(Q') \leq N$. Because f is badly approximable, we have

$$|\{Qf\} - \{Q'f\}| = |\{(Q - Q')f\}| > \frac{1}{q^{\deg(Q-Q')+c}} \geq \frac{1}{q^{N+c}}.$$

This means that the distance between two points $\{Qf\}$, $\{Q'f\}$ with $\deg(Q)$, $\deg(Q') \leq N$ is at least q^{-N-c}. Now, we consider two cases:

1. If $q^{-N-c} \geq q^{-d}$, then there is at most one point in $B(g, q^{-d})$.

2. If $q^{-N-c} < q^{-d}$, then the number of points in $B(g, q^{-d})$ is at most

$$\frac{q^{-d}}{q^{-N-c}} = q^{N-d+c} .$$

Hence, the number of $\{Qf\}$ with $\deg(Q) \leq N$ belonging to $B\left(g, q^{-d}\right)$ is at most $\max\left\{q^{N-d+c}, 1\right\}$. ∎

Lemma 3.5. *Let l_n be a sequence with $\sum \frac{1}{q^{l_n}} = \infty$. Then, for all $k \geq 0$, we have*

$$m\left(\bigcup_{n=k}^{\infty} \bigcup_{\deg(Q)=n} B\left(\{Qf\}, \frac{1}{q^{n+l_n}}\right)\right) > \frac{1}{q^{c+1}}. \tag{3.1}$$

Proof. We first exclude the case $q = 2$.

Let $l'_n = \max\{l_n, c\}$, $\forall n \in \mathbb{N}$. Then, we have $\sum q^{-l'_n} = \infty$. Assume that (3.1) is false. Hence, there exists $k_0 \in \mathbb{N}$ such that

$$m\left(\bigcup_{n=k_0}^{N} \bigcup_{\deg(Q)=n} B\left(\{Qf\}, \frac{1}{q^{n+l'_n}}\right)\right) \leq \frac{1}{q^{c+1}} \text{ , for all } N \geq k_0. \tag{3.2}$$

We define a set

$$L_N := \left\{\deg(Q) = N : \{Qf\} \in \bigcup_{n=k_0}^{N} \bigcup_{\deg(Q')=n} B\left(\{Q'f\}, \frac{1}{q^{n+l'_n}}\right) \setminus \bigcup_{n=k_0}^{N-1} \bigcup_{\deg(Q')=n} B\left(\{Q'f\}, \frac{1}{q^{n+l'_n}}\right)\right\}.$$

We first estimate the number of elements of L_N. Let

$$\bigcup_{n=k_0}^{N-1} \bigcup_{\deg(Q')=n} B\left(\{Q'f\}, \frac{1}{q^{n+l'_n}}\right) = \bigcup_i B\left(\{Q_i f\}, \frac{1}{q^{d_i}}\right),$$

where $B\left(\{Q_i f\}, \frac{1}{q^{d_i}}\right)$ are disjoint, $\forall i$. By (3.2), we get

$$\begin{aligned}
\frac{1}{q^{c+1}} &\geq m\left(\bigcup_{n=k_0}^{N-1} \bigcup_{\deg(Q')=n} B\left(\{Q'f\}, \frac{1}{q^{n+l'_n}}\right)\right) \\
&= m\left(\bigcup_i B\left(\{Q_i f\}, \frac{1}{q^{d_i}}\right)\right) \\
&= \sum_i m\left(B\left(\{Q_i f\}, \frac{1}{q^{d_i}}\right)\right) \\
&= \sum_i \frac{1}{q^{d_i}}.
\end{aligned}$$

Using Lemma 3.4, the number of Q with $\deg(Q) \leq N$ such that $\{Qf\}$ belong to $\bigcup_i B\left(\{Q_i f\}, \frac{1}{q^{d_i}}\right)$ is at most $\sum_i \max\left\{q^{N+c-d_i}, 1\right\} = \max\left\{q^{N+c}\sum_i q^{-d_i}, q^N\right\} =$

q^N. Thus, the number of L_N is at least $q^{N+1} - q^N - q^N = q^N(q-2)$. Next, we claim that

$$\bigcup_{Q \in L_N} B\left(\{Qf\}, \frac{1}{q^{N+l'_N}}\right)$$
$$\subset \bigcup_{n=k_0}^{N} \bigcup_{\deg(Q')=n} B\left(\{Q'f\}, \frac{1}{q^{n+l'_n}}\right) \setminus \bigcup_{n=k_0}^{N-1} \bigcup_{\deg(Q')=n} B\left(\{Q'f\}, \frac{1}{q^{n+l'_n}}\right). \tag{3.3}$$

In order to show this, fix $Q_1 \in L_N$. Suppose there exists a polynomial Q_2 with $\deg(Q_2) = u < N$ and $B\left(\{Q_1 f\}, q^{-N-l'_N}\right) \cap B\left(\{Q_2 f\}, q^{-u-l'_u}\right) \neq \emptyset$. We know that $\{Q_1 f\}$ does not belong to $B\left(\{Q_2 f\}, q^{-u-l'_u}\right)$. Hence,

$$B\left(\{Q_2 f\}, \frac{1}{q^{u+l'_u}}\right) \subset B\left(\{Q_1 f\}, \frac{1}{q^{N+l'_N}}\right).$$

Then, we get

$$|\{Q_1 f\} - \{Q_2 f\}| < \frac{1}{q^{N+l'_N}}.$$

By Lemma 3.4, the number of $\{Qf\}$ belonging to $B\left(\{Q_1 f\}, q^{-N-l'_N}\right)$ is at most $\max\{q^{N-N-l'_N+c}, 1\} = \max\{q^{-l'_N+c}, 1\} = 1$. Thus, we get $\{Q_1 f\} = \{Q_2 f\}$, a contradiction. Consequently, (3.3) holds. Now, we show that any two balls appearing on the left side of (3.3) are disjoint. We again use proof by contradiction. Therefore, suppose there are two different polynomials Q_1, $Q_2 \in L_N$ such that $B\left(\{Q_1 f\}, q^{-N-l'_N}\right)$ and $B\left(\{Q_2 f\}, q^{-N-l'_N}\right)$ are not disjoint. Thus, we know that these two balls are equal. This implies that

$$|\{Q_1 f\} - \{Q_2 f\}| = |\{(Q_1 - Q_2)f\}| < \frac{1}{q^{N+l'_N}}.$$

Hence,

$$\{(Q_1 - Q_2)f\} \in B\left(0, q^{-N-l'_N}\right).$$

By Lemma 3.4 again, the number of $\{Qf\}$ in $B\left(\{Q_1 f\}, q^{-N-l'_N}\right)$ is at most $\max\{q^{c-l'_N}, 1\} = 1$. Consequently, $\{Q_1 f\} = \{Q_2 f\}$, a contradiction. By the latter claim and (3.3), we now obtain

$$\begin{aligned}
& m\left(\bigcup_{n=k_0}^{N}\bigcup_{\deg(Q)=n} B\left(\{Qf\},\frac{1}{q^{n+l'_n}}\right)\right)\\
&\geq m\left(\bigcup_{n=k_0}^{N-1}\bigcup_{\deg(Q)=n} B\left(\{Qf\},\frac{1}{q^{n+l'_n}}\right)\right)+m\left(\bigcup_{Q\in L_N} B\left(\{Qf\},\frac{1}{q^{N+l'_N}}\right)\right)\\
&\geq m\left(\bigcup_{n=k_0}^{N-1}\bigcup_{\deg(Q)=n} B\left(\{Qf\},\frac{1}{q^{n+l'_n}}\right)\right)+(q-2)q^N\frac{1}{q^{N+l'_N}}\\
&\geq m\left(\bigcup_{n=k_0}^{N-2}\bigcup_{\deg(Q)=n} B\left(\{Qf\},\frac{1}{q^{n+l'_n}}\right)\right)+\frac{q-2}{q^{l'_{N-1}}}+\frac{q-2}{q^{l'_N}}\\
&\geq\cdots\geq(q-2)\sum_{n=k_0}^{N}\frac{1}{q^{l'_n}}.
\end{aligned}$$

As the series $\sum_{n=1}^{\infty} q^{-l'_n}$ diverges, we have a contradiction for N large enough.

Now, we consider the case $q=2$. Since $\sum_{n\geq 0} q^{-l'_n}=\infty$, we have either $\sum_{n\geq 0} q^{-l'_{2n}}=\infty$ or $\sum_{n\geq 0} q^{-l'_{2n+1}}=\infty$. Without loss of generality, assume that the first case holds. Then, the same proof as above can be used with the one difference that is instead of L_N, we consider

$$\begin{aligned}
L_{2N}:=\Bigg\{\deg(Q)=2N:\{Qf\}\in &\bigcup_{n=k_0}^{2N}\bigcup_{\deg(Q')=n} B\left(\{Q'f\},\frac{1}{q^{n+l'_n}}\right)\\
&\setminus\bigcup_{n=k_0}^{2N-2}\bigcup_{\deg(Q')=n} B\left(\{Q'f\},\frac{1}{q^{n+l'_n}}\right)\Bigg\}.
\end{aligned}$$

Hence, we obtain

$$m\left(\bigcup_{n=k_0}^{2N}\bigcup_{\deg(Q)=n} B\left(\{Qf\},\frac{1}{q^{n+l'_n}}\right)\right)\geq d\sum_{n=\lceil\frac{k_0}{2}\rceil}^{N} q^{-l'_{2n}}$$

for some $d>0$. As the series $\sum_{n=1}^{\infty} q^{-l'_{2n}}=\infty$, we have a contradiction again for N large enough. ∎

Proposition 3.1.

$$S\supseteq\{f\in\mathbb{L}: f\ \textit{is badly approximable}\}$$

Proof. Let f be badly approximable. We have to show that

$$m\left(\bigcap_{k=0}^{\infty}\bigcup_{n=k}^{\infty}\bigcup_{\deg(Q)=n} B\left(\{Qf\}, \frac{1}{q^{n+l_n}}\right)\right) = 1.$$

By Lemma 3.5, we obtain

$$m\left(\bigcup_{n=k}^{\infty}\bigcup_{\deg(Q)=n} B\left(\{Qf\}, \frac{1}{q^{n+l_n}}\right)\right) > \frac{1}{q^{c+1}} > 0, \forall k.$$

Consequently,

$$m\left(\bigcap_{k=0}^{\infty}\bigcup_{n=k}^{\infty}\bigcup_{\deg(Q)=n} B\left(\{Qf\}, \frac{1}{q^{n+l_n}}\right)\right) > 0$$

and Lemma 3.3 implies the claim. ∎

Proposition 3.2.

$$S \subseteq \{f \in \mathbb{L} : f \textit{ is badly approximable}\}$$

Proof. Assume that f is not badly approximable. We will show that a sequence l_n we can choose such that $\sum_{n=1}^{\infty} \frac{1}{q^{l_n}} = \infty$ but for almost every $g \in \mathbb{L}$, there are at most finitely many Q with

$$|\{Qf\} - g| < \frac{1}{q^{n+l_n}},\ \ Q \in \mathbb{F}_q[X],\ \ \deg(Q) = n.$$

Let us choose (R_i, S_i) such that

$$\left|f - \frac{R_i}{S_i}\right| \leq \frac{1}{q^{2n_i+2i}}, \forall i \in \mathbb{N},$$

where $\deg(S_i) = n_i$, and define

$$\begin{cases} t_0 = 0 \\ t_i = n_i + i, i \geq 1 \end{cases},\ \text{ and for } t_{i-1} \leq n < t_i,\ \ l_n = t_i - n\ .$$

Then, we have

$$\sum_{n=1}^{\infty} \frac{1}{q^{l_n}} \geq \sum_{i=1}^{\infty} \frac{1}{q^{l_{t_i-1}}} = \sum_{i=1}^{\infty} \frac{1}{q^{t_i-(t_i-1)}} = \sum_{i=1}^{\infty} \frac{1}{q} = \infty.$$

On the other hand, let Q be a polynomial such that $\deg(Q) < t_i$, $\forall i$. Then,

$$\left| Qf - \frac{QR_i}{S_i} \right| \le \frac{q^{t_i}}{q^{2n_i+2i}}.$$

This implies that

$$\left| \{Qf\} - \frac{R_i'}{S_i} \right| \le \frac{1}{q^{n_i+i}} < 1.$$

Note that $\deg(R_i') < \deg(S_i)$. Therefore,

$$\begin{aligned} \bigcup_{t_{i-1} \le n < t_i} \bigcup_{\deg(Q)=n} B\left(\{Qf\}, \frac{1}{q^{n+l_n}}\right) &= \bigcup_{t_{i-1} \le n < t_i} \bigcup_{\deg(Q)=n} B\left(\{Qf\}, \frac{1}{q^{t_i}}\right) \\ &\subset \bigcup_{\deg(R_i') < \deg(S_i)} B\left(\frac{R_i'}{S_i}, \frac{1}{q^{t_i}}\right). \end{aligned}$$

Then, we can estimate the measure of union of these balls

$$\begin{aligned} m\left(\bigcup_{t_{i-1} \le n < t_i} \bigcup_{\deg(Q)=n} B\left(\{Qf\}, \frac{1}{q^{n+l_n}}\right) \right) &\le m\left(\bigcup_{\deg(R_i) < \deg(S_i)} B\left(\frac{R_i'}{S_i}, \frac{1}{q^{t_i}}\right) \right) \\ &\le \frac{|S_i|}{q^{t_i}} = \frac{q^{n_i}}{q^{t_i}} = \frac{1}{q^i}. \end{aligned}$$

So we get

$$\sum_{i=1}^{\infty} m\left(\bigcup_{t_{i-1} \le n < t_i} \bigcup_{\deg(Q)=n} B\left(\{Qf\}, \frac{1}{q^{n+l_n}}\right) \right) \le \sum_{i=1}^{\infty} \frac{1}{q^i} < \infty.$$

Hence, for almost every $g \in \mathbb{L}$, there are at most finitely many $Q's$ such that $|\{Qf\} - g| < q^{-n-l_n}$ with $\deg(Q) = n$. ∎

Finally, Proposition 3.1 and Proposition 3.2 imply Theorem 1.13.

Chapter 4

Kurzweil's Theorem in Higher Dimension

Here, we are going to prove Theorem 1.14 from the introduction. Therefore, fix a $r \times s$ matrix A. We first need a technical lemma.

Lemma 4.1. *If $A\mathbf{u}^\intercal \in \mathbb{F}_q[X]^r$ for some $\mathbf{u}^\intercal \neq \mathbf{0}$, then A is not badly approximable.*

Proof. Assume that A is badly approximable. Let us fix some notation. First, set

$$A = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1s} \\ f_{21} & f_{22} & \cdots & f_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ f_{r1} & f_{r2} & \cdots & f_{rs} \end{bmatrix}, \quad \mathbf{u}^\intercal = \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_s \end{bmatrix}$$

and

$$A\mathbf{u}^\intercal = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1s} \\ f_{21} & f_{22} & \cdots & f_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ f_{r1} & f_{r2} & \cdots & f_{rs} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_s \end{bmatrix} = \begin{bmatrix} U_1 f_{11} + \cdots + U_s f_{1s} \\ U_1 f_{21} + \cdots + U_s f_{2s} \\ \vdots \\ U_1 f_{r1} + \cdots + U_s f_{rs} \end{bmatrix} \equiv \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_r \end{bmatrix}$$

which is in $\mathbb{F}_q[X]^r$. Then, by Dirichlet's theorem,

$$
\begin{aligned}
|Q_1 f_{11} + Q_2 f_{21} + \cdots + Q_r f_{r1} - P_1| &< q^{-\lfloor \frac{Nr}{s-1} \rfloor} \\
|Q_1 f_{12} + Q_2 f_{22} + \cdots + Q_r f_{r2} - P_2| &< q^{-\lfloor \frac{Nr}{s-1} \rfloor} \\
&\vdots \\
|Q_1 f_{1s-1} + Q_2 f_{2s-1} + \cdots + Q_r f_{rs-1} - P_{s-1}| &< q^{-\lfloor \frac{Nr}{s-1} \rfloor}
\end{aligned}
$$

has infinitely many solutions in $Q_1, Q_2, \ldots, Q_r$ and $P_1, P_2, \ldots, P_{s-1}$ with $N = \max_{1 \le i \le r} \deg(Q_i)$. Next, multiply both sides of the above inequality by $|U_s|$ and set $Q'_i := U_s Q_i$ and $P'_j := U_s P_j$ for all i, j. Then, we obtain

$$
\begin{aligned}
|Q'_1 f_{11} + Q'_2 f_{21} + \cdots + Q'_r f_{r1} - P'_1| &< |U_s| q^{-\lfloor \frac{Nr}{s-1} \rfloor} \\
|Q'_1 f_{12} + Q'_2 f_{22} + \cdots + Q'_r f_{r2} - P'_2| &< |U_s| q^{-\lfloor \frac{Nr}{s-1} \rfloor} \\
&\vdots \\
|Q'_1 f_{1s-1} + Q'_2 f_{2s-1} + \cdots + Q'_r f_{rs-1} - P'_{s-1}| &< |U_s| q^{-\lfloor \frac{Nr}{s-1} \rfloor}.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
|Q'_1 f_{11} + Q'_2 f_{21} + \cdots + Q'_r f_{r1} - P'_1| &< q^{-\lfloor \frac{N'r}{s-1} \rfloor - c_1} \\
|Q'_1 f_{12} + Q'_2 f_{22} + \cdots + Q'_r f_{r2} - P'_2| &< q^{-\lfloor \frac{N'r}{s-1} \rfloor - c_1} \\
&\vdots \\
|Q'_1 f_{1s-1} + Q'_2 f_{2s-1} + \cdots + Q'_r f_{rs-1} - P'_{s-1}| &< q^{-\lfloor \frac{N'r}{s-1} \rfloor - c_1}
\end{aligned}
\tag{4.1}
$$

has infinitely many solutions in $Q'_1, \ldots, Q'_r$ and $P'_1, \ldots, P'_r$, where $N' = \max_{i=1,\ldots,r} \deg(Q'_i)$ and c_1 is a suitable constant. Now, consider

$$
\begin{aligned}
& U_s f_{1s} Q'_1 + \cdots + U_s f_{rs} Q'_r \\
&\quad = \sum_{i=1}^{r} \left(R_i - U_1 f_{i1} - \cdots - U_{s-1} f_{is-1}\right) Q'_i \\
&\quad = \sum_{i=1}^{r} Q'_i R_i - \sum_{j=1}^{s-1} U_j \left(Q'_1 f_{1j} + \cdots + Q'_r f_{rj} - P'_j\right) - \sum_{j=1}^{s-1} U_j P'_j
\end{aligned}
$$

This implies that

$$
\sum_{i=1}^{r} U_s f_{is} Q'_i + \sum_{j=1}^{s-1} U_j P'_j - \sum_{i=1}^{r} Q'_i R_i = -\sum_{j=1}^{s-1} U_j \left(Q'_1 f_{1j} + \cdots + Q'_r f_{rj} - P'_j\right)
$$

Hence,

$$\left|\sum_{i=1}^{r} U_s f_{is} Q_i' + \sum_{j=1}^{s-1} U_j P_j' - \sum_{i=1}^{r} Q_i' R_i\right|$$
$$\leq \max_{j=1,\ldots,s-1} \left\{|U_j| \left|Q_1' f_{1j} + \cdots + Q_r' f_{rj} - P_j'\right|\right\} < q^{-\lfloor \frac{N' r}{s-1} \rfloor - c_2},$$

where c_2 is a suitable constant. Dividing both sides by $|U_s|$ gives

$$\left|\sum_{i=1}^{r} f_{is} Q_i' + \frac{\sum_{j=1}^{s} U_j P_j' - \sum_{i=1}^{r} Q_i' R_i}{U_s}\right| < q^{-\lfloor \frac{N' r}{s-1} \rfloor - c_3},$$

where c_3 is a suitable constant. Since U_s divides Q_i' and P_j' for all i, j, we obtain

$$T = \frac{\sum_{j=1}^{s} U_j P_j' - \sum_{i=1}^{r} Q_i' R_i}{U_s}$$

is a polynomial. Thus, we have proved that

$$|Q_1' f_{1s} + \cdots + Q_r' f_{rs} + T| < q^{-\lfloor \frac{Nr}{s-1} \rfloor - c_3}.$$

Now, set $\mathbf{q}' = [Q_1', Q_2', \ldots, Q_r']$ and $c = \min\{c_1, c_3\}$. By the above inequality and (4.1),

$$\|\{\mathbf{q}' A\}\| < q^{-\lfloor \frac{N' r}{s-1} \rfloor - c}$$

has infinitely many solutions. Consequently, we obtain that A is not badly approximable, a contradiction. Hence, the proof is finished. ∎

For the next five lemmas, we assume that $A = [f_{ij}]_{r \times s}$ is badly approximable. Then, there exists a constant $c > 0$ such that for all $\mathbf{q} \in \mathbb{F}_q[X]^r, \mathbf{q} \neq \mathbf{0}$ with $\deg(\mathbf{q}) = n$,

$$\|\{\mathbf{q} A\}\| > \frac{1}{q^{\lfloor \frac{nr}{s} \rfloor + c}}.$$

Lemma 4.2. *$\{\{\mathbf{q} A\} : \mathbf{q} \in \mathbb{F}_q[X]^r\}$ is dense in $\mathbb{L}^s$.*

Proof. Let us fix $n \in \mathbb{N}$ and $\mathbf{g} = [g_1, g_2, \ldots, g_s] \in \mathbb{L}^s$, where

$$g_j = g_j^{(1)} X^{-1} + g_j^{(2)} X^{-2} + \cdots, \ \forall j = 1, 2, \ldots, s.$$

We have to show that there exists $\mathbf{q} = [Q_1, Q_2, \ldots, Q_r]$ with $\deg(Q_i) = N_i$ such that

$$\|\{\mathbf{q} A\} - \mathbf{g}\| < \frac{1}{q^n}. \tag{4.2}$$

First, for $i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, s$, set

$$f_{ij} = f_{ij}^{(1)} X^{-1} + f_{ij}^{(2)} X^{-2} + \cdots ,$$

and let

$$\mathbf{a}_i = \begin{bmatrix} a_i^{(0)} \\ a_i^{(1)} \\ \vdots \\ a_i^{(N)} \end{bmatrix}^{\mathsf{T}}, \; A_{ij} = \begin{bmatrix} f_{ij}^{(1)} & f_{ij}^{(2)} & \cdots & f_{ij}^{(n)} \\ f_{ij}^{(2)} & f_{ij}^{(3)} & \cdots & f_{ij}^{(n+1)} \\ \vdots & \vdots & \ddots & \vdots \\ f_{ij}^{(N+1)} & f_{ij}^{(N+2)} & \cdots & f_{ij}^{(N+n)} \end{bmatrix}, \; \mathbf{b}_j = \begin{bmatrix} g_j^{(1)} \\ g_j^{(2)} \\ \vdots \\ g_j^{(n)} \end{bmatrix}^{\mathsf{T}}.$$

Finally, set

$$\mathbf{a} = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_r \end{bmatrix}^{\mathsf{T}}, \; A' = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1s} \\ A_{21} & A_{22} & \cdots & A_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ A_{r1} & A_{r2} & \cdots & A_{rs} \end{bmatrix}, \; \mathbf{b} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \vdots \\ \mathbf{b}_s \end{bmatrix}^{\mathsf{T}}.$$

Then, the inequality (4.2) has a solution if and only if $\mathbf{a}A' = \mathbf{b}$ has a solution $\mathbf{a}$. In order to prove that this system is solvable, we have to show that $\operatorname{rank}(A') = sn$ as N is large enough. Assume that there exist $\alpha_1, \ldots, \alpha_{sn}$ not all zero such that

$$\begin{aligned} &\alpha_1(f_{11}^{(1)}, \ldots, f_{11}^{(N+1)}, \ldots, f_{r1}^{(1)}, \ldots, f_{r1}^{(N+1)}) \\ &\quad + \ldots + \alpha_{sn}(f_{1s}^{(n)}, \ldots, f_{1s}^{(N+n)}, \ldots, f_{rs}^{(n)}, \ldots, f_{rs}^{(N+n)}) = \mathbf{0}, \end{aligned} \tag{4.3}$$

Now, we set $\mathbf{p} = [P_1, P_2, \ldots, P_s] \in \mathbb{F}_q[X]^s$, $\mathbf{p} \neq \mathbf{0}$ with

$$\begin{aligned} P_1(X) &= \alpha_1 + \alpha_2 X + \cdots + \alpha_n X^{n-1} \\ P_2(X) &= \alpha_{n+1} + \alpha_{n+2} X + \cdots + \alpha_{2n} X^{n-1} \\ &\vdots \\ P_s(X) &= \alpha_{(s-1)n+1} + \alpha_{(s-1)n+2} X + \cdots + \alpha_{sn} X^{n-1}. \end{aligned}$$

Hence, (4.3) can be rewritten to

$$|\{P_1 f_{i1} + \cdots + P_s f_{is}\}| < q^{-N-1}$$

for all $i = 1, 2, \ldots, r$. This implies that

$$\|\{A\mathbf{p}^{\mathsf{T}}\}\| < q^{-N-1}.$$

On the other hand, since A is badly approximable, Lemma 4.1 implies that $A\mathbf{p}^{\intercal}$ does not belong to $\mathbb{F}_q[X]^r$, $\forall \mathbf{p} \neq \mathbf{0}$. Consequently, since the number of $\mathbf{p}$ is finite (since n is fixed),

$$\min_{\deg(\mathbf{p})<n,\, \mathbf{p}\neq \mathbf{0}} \|\{A\mathbf{p}^{\intercal}\}\| \geq q^{-N-1},$$

for N large enough, a contradiction. Hence, we obtain that $\mathbf{p} = \mathbf{0}$ which implies $\alpha_1 = \alpha_2 = \cdots = \alpha_{sn} = 0$. Thus, our claimed result is proved. Therefore, there exists a solution of $\mathbf{a}A' = \mathbf{b}$. This implies that for all $n \in \mathbb{N}$, (4.2) has a solution. Finally, we have proved that $\{\{\mathbf{q}A\} : \mathbf{q} \in \mathbb{F}_q[X]^r\}$ is dense in $\mathbb{L}$. ∎

The next two lemmas are proved as in the last chapter. Consequently, we will omit the proofs.

Lemma 4.3 (0-1 law)**.** *Let a measurable set E in $\mathbb{L}^s$ be invariant under the action $\cdot + \{\mathbf{q}A\}$ for all $\mathbf{q} \in \mathbb{F}_q[X]^r$. Then, we have $m(E) = 0$ or 1.*

Lemma 4.4. *Let*

$$E := \left\{ \mathbf{g} : \|\{\mathbf{q}A\} - \mathbf{g}\| < \frac{1}{q^{\lfloor \frac{nr}{s} \rfloor + l_n}} \text{ with } n = \deg(\mathbf{q}) \text{ has infinitely many solutions} \right\}.$$

Then, E is invariant under the action $\cdot + \{\mathbf{q}A\}$ for all $\mathbf{q} \in \mathbb{F}_q[X]^r$ and hence $m(E) = 0$ or 1.

Next, we need the following result which is similar to Lemma 3.4 from the last chapter.

Lemma 4.5. *Let $\mathbf{g} \in \mathbb{L}^s$ and $d > 0$. Then, the number of $\{\mathbf{q}A\}$ with $\deg(\mathbf{q}) \leq N$ belonging to $B\left(\mathbf{g}, q^{-d}\right)$ is at most $\max\{q^{Nr+cs-ds}, 1\}$.*

Proof. For the proof, we use the second method of proof of Lemma 3.4. Therefore, fix $\mathbf{q}, \mathbf{q}' \in \mathbb{F}_q[X]^r$ with $\deg(\mathbf{q}), \deg(\mathbf{q}') \leq N$. Since A is badly approximable, we have

$$\|\{\mathbf{q}A\} - \{\mathbf{q}'A\}\| = \|\{(\mathbf{q} - \mathbf{q}')A\}\| > \frac{1}{q^{\lfloor \frac{\deg(\mathbf{q}-\mathbf{q}')r}{s} \rfloor + c}} \geq \frac{1}{q^{\lfloor \frac{Nr}{s} \rfloor + c}}.$$

This means that the distance between any two points $\{\mathbf{q}A\}$ and $\{\mathbf{q}'A\}$ is more than $q^{-\lfloor \frac{Nr}{s} \rfloor - c}$. Then, we consider the following two cases.

1. If $q^{-\lfloor \frac{Nr}{s} \rfloor - c} \geq q^{-d}$, then there is at most one point in $B(\mathbf{g}, q^{-d})$.

2. If $q^{-\lfloor \frac{Nr}{s} \rfloor - c} < q^{-d}$, then the number of points in $B(\mathbf{g}, q^{-d})$ is at most

$$\frac{\left(q^{-d}\right)^s}{\left(q^{-\lfloor \frac{Nr}{s} \rfloor - c}\right)^s} \leq q^{Nr+cs-ds}.$$

Hence, our claim is proved. ∎

Lemma 4.6. *Let l_n be a sequence with $\sum q^{-sl_n} = \infty$. Then, for all $k \geq 0$, we have*

$$m\left(\bigcup_{n=k}^{\infty} \bigcup_{\deg(\mathbf{q})=n} B\left(\{\mathbf{q}A\}, \frac{1}{q^{\lfloor \frac{nr}{s} \rfloor + l_n}}\right)\right) > \frac{1}{q^{cs+1}}. \tag{4.4}$$

Proof. We first exclude the case $q = 2$ and $r = 1$.

Let $l'_n = \max\{l_n, c\}$, $\forall n \in \mathbb{N}$. Then, we have $\sum q^{-sl'_n} = \infty$. Assume that (4.4) is incorrect. Hence, there exists $k_0 \in \mathbb{N}$ such that

$$m\left(\bigcup_{n=k_0}^{N} \bigcup_{\deg(\mathbf{q})=n} B\left(\{\mathbf{q}A\}, \frac{1}{q^{\lfloor \frac{nr}{s} \rfloor + l'_n}}\right)\right) \leq \frac{1}{q^{cs+1}} \text{ , for all } N \geq k_0. \tag{4.5}$$

We define a set

$$L_N = \left\{\deg(\mathbf{q}) = N : \{\mathbf{q}A\} \in \bigcup_{n=k_0}^{N} \bigcup_{\deg(\mathbf{q}')=n} B\left(\{\mathbf{q}'A\}, \frac{1}{q^{\lfloor \frac{nr}{s} \rfloor + l'_n}}\right) \right.$$
$$\left. \setminus \bigcup_{n=k_0}^{N-1} \bigcup_{\deg(\mathbf{q}')=n} B\left(\{\mathbf{q}'A\}, \frac{1}{q^{\lfloor \frac{nr}{s} \rfloor + l'_n}}\right)\right\}.$$

We first estimate the number of elements of L_N. Let

$$\bigcup_{n=k_0}^{N-1} \bigcup_{\deg(\mathbf{q}')=n} B\left(\{\mathbf{q}'A\}, \frac{1}{q^{\lfloor \frac{nr}{s} \rfloor + l'_n}}\right) = \bigcup_i B\left(\{\mathbf{q}_i A\}, \frac{1}{q^{d_i}}\right),$$

where $B\left(\{\mathbf{q}_i A\}, q^{-d_i}\right)$ are disjoint $\forall i$. By (4.5), we get

$$\begin{aligned}
\frac{1}{q^{cs+1}} &\geq m\left(\bigcup_{n=k_0}^{N-1} \bigcup_{\deg(\mathbf{q}')=n} B\left(\{\mathbf{q}'A\}, \frac{1}{q^{\lfloor \frac{nr}{s} \rfloor + l'_n}}\right)\right) \\
&= m\left(\bigcup_i B\left(\{\mathbf{q}_i A\}, \frac{1}{q^{d_i}}\right)\right) \\
&= \sum_i m\left(B\left(\{\mathbf{q}_i A\}, \frac{1}{q^{d_i}}\right)\right) \\
&= \sum_i \frac{1}{q^{sd_i}}.
\end{aligned}$$

Using Lemma 4.5, the number of $\mathbf{q}$ with $\deg(\mathbf{q}) \le N$ such that $\{\mathbf{q}A\}$ belongs to $\bigcup_i B\left(\{\mathbf{q}_i A\}, q^{-d_i}\right)$ is at most $\sum_i \max\left\{q^{Nr+cs-sd_i}, 1\right\} = \max\left\{q^{Nr+cs}\sum_i q^{-sd_i}, q^{Nr}\right\}$ $= q^{Nr}$. Then, the number of elements in L_N is at least $q^{(N+1)r} - q^{Nr} - q^{Nr} = q^{Nr}(q^r - 2)$. Next, we claim that

$$\bigcup_{\mathbf{q}\in L_N} B\left(\{\mathbf{q}A\}, \frac{1}{q^{\lfloor \frac{Nr}{s}\rfloor + l'_N}}\right)$$

$$\subset \bigcup_{n=k_0}^{N}\bigcup_{\deg(\mathbf{q})=n} B\left(\{\mathbf{q}'A\}, \frac{1}{q^{\lfloor \frac{nr}{s}\rfloor + l'_n}}\right) \setminus \bigcup_{n=k_0}^{N-1}\bigcup_{\deg(\mathbf{q})=n} B\left(\{\mathbf{q}'A\}, \frac{1}{q^{\lfloor \frac{nr}{s}\rfloor + l'_n}}\right). \tag{4.6}$$

In order to show this, fix $\mathbf{q}_1 \in L_N$. Suppose there exists $\mathbf{q}_2 \in \mathbb{F}_q[X]^r$ with $\deg(\mathbf{q}_2) = u < N$ and $B\left(\{\mathbf{q}_1 A\}, q^{-\lfloor \frac{Nr}{s}\rfloor - l'_N}\right) \cap B\left(\{\mathbf{q}_2 A\}, q^{-\lfloor \frac{ur}{s}\rfloor - l'_u}\right) \neq \emptyset$. We know that $\{\mathbf{q}_1 A\}$ does not belong to $B\left(\{\mathbf{q}_2 A\}, q^{-\lfloor \frac{ur}{s}\rfloor - l'_u}\right)$. Hence,

$$B\left(\{\mathbf{q}_2 A\}, \frac{1}{q^{\lfloor \frac{ur}{s}\rfloor + l'_u}}\right) \subset B\left(\{\mathbf{q}_1 A\}, \frac{1}{q^{\lfloor \frac{Nr}{s}\rfloor + l'_N}}\right).$$

Then, we obtain

$$\|\{\mathbf{q}_1 A\} - \{\mathbf{q}_2 A\}\| < \frac{1}{q^{\lfloor \frac{Nr}{s}\rfloor + l'_N}}.$$

By Lemma 4.5, the number of $\{\mathbf{q}A\}$ belonging to $B\left(\{\mathbf{q}_1 A\}, q^{-\lfloor \frac{Nr}{s}\rfloor - l'_N}\right)$ is at most $\max\{q^{Nr-Nr-sl'_N+cs}, 1\} = \max\{q^{-sl'_N+cs}, 1\} = 1$. Thus, we get $\{\mathbf{q}_1 A\} = \{\mathbf{q}_2 A\}$, a contradiction. Consequently, (4.6) holds. Now, we show that any two balls appearing on the left side of (4.6) are disjoint. We again use proof by contradiction. Suppose there are $\mathbf{q}_1$, $\mathbf{q}_2 \in L_N$ such that $B\left(\{\mathbf{q}_1 A\}, q^{-\lfloor \frac{Nr}{s}\rfloor - l'_N}\right)$ and $B\left(\{\mathbf{q}_2 A\}, q^{-\lfloor \frac{Nr}{s}\rfloor - l'_N}\right)$ are not disjoint. Thus, we know that these two balls are equal. This implies that

$$\|\{\mathbf{q}_1 A\} - \{\mathbf{q}_2 A\}\| = \|\{(\mathbf{q}_1 - \mathbf{q}_2)A\}\| < \frac{1}{q^{\lfloor \frac{Nr}{s}\rfloor + l'_N}}.$$

Hence,

$$\{(\mathbf{q}_1 - \mathbf{q}_2)A\} \in B\left(\mathbf{0}, \frac{1}{q^{\lfloor \frac{Nr}{s}\rfloor + l'_N}}\right).$$

By Lemma 4.5 again, the number of $\{\mathbf{q}A\}$ belonging to $B\left(\mathbf{0}, q^{-\lfloor \frac{Nr}{s}\rfloor - l'_N}\right)$ is at most $\max\{q^{cs-sl'_N}, 1\} = 1$. Consequently, $\{\mathbf{q}_1 A\} = \{\mathbf{q}_2 A\}$, a contradiction. By the latter claim and (4.6), we obtain

$$\begin{aligned}
&m\left(\bigcup_{n=k_0}^{N}\bigcup_{\deg(\mathbf{q}')=n}B\left(\{\mathbf{q}'A\},\frac{1}{q^{\lfloor\frac{nr}{s}\rfloor+l'_n}}\right)\right)\\
&\quad\geq m\left(\bigcup_{n=k_0}^{N-1}\bigcup_{\deg(\mathbf{q}')=n}B\left(\{\mathbf{q}'A\},\frac{1}{q^{\lfloor\frac{nr}{s}\rfloor+l'_n}}\right)\right)+m\left(\bigcup_{\mathbf{q}\in L_N}B\left(\{\mathbf{q}A\},\frac{1}{q^{\lfloor\frac{Nr}{s}\rfloor+l'_N}}\right)\right)\\
&\quad\geq m\left(\bigcup_{n=k_0}^{N-1}\bigcup_{\deg(\mathbf{q}')=n}B\left(\{\mathbf{q}'A\},\frac{1}{q^{\lfloor\frac{nr}{s}\rfloor+l'_n}}\right)\right)+\left(q^r-2\right)q^{Nr}\left(\frac{1}{q^{\lfloor\frac{Nr}{s}\rfloor+l'_N}}\right)^s\\
&\quad\geq m\left(\bigcup_{n=k_0}^{N-2}\bigcup_{\deg(\mathbf{q}')=n}B\left(\{\mathbf{q}'A\},\frac{1}{q^{\lfloor\frac{nr}{s}\rfloor+l'_n}}\right)\right)+\frac{(q^r-2)}{q^{sl'_{N-1}}}+\frac{(q^r-2)}{q^{sl'_N}}\\
&\quad\geq\cdots\geq\left(q^r-2\right)\sum_{n=k_0}^{N}\frac{1}{q^{sl'_n}}.
\end{aligned}$$

As the series $\sum q^{-sl'_n}$ diverges, we have a contradiction for N large enough.

Now, we consider the case $q=2$ and $r=1$. Since $\sum_{n\geq 0}q^{-sl'_n}=\infty$, we have either $\sum_{n\geq 0}q^{-sl'_{2n}}=\infty$ or $\sum_{n\geq 0}q^{-sl'_{2n+1}}=\infty$. Without loss of generality, assume that the first case holds. Then, the same proof as above can be used with the only difference that instead of L_N, we consider

$$L_{2N}:=\left\{\deg(\mathbf{q})=2N:\{\mathbf{q}A\}\in\bigcup_{n=k_0}^{2N}\bigcup_{\deg(\mathbf{q}')=n}B\left(\{\mathbf{q}'A\},\frac{1}{q^{\lfloor\frac{nr}{s}\rfloor+l'_n}}\right)\right.$$
$$\left.\setminus\bigcup_{n=k_0}^{2N-2}\bigcup_{\deg(\mathbf{q}')=n}B\left(\{\mathbf{q}'A\},\frac{1}{q^{\lfloor\frac{nr}{s}\rfloor+l'_n}}\right)\right\}.$$

Hence, we obtain

$$m\left(\bigcup_{n=k_0}^{2N}\bigcup_{\deg(\mathbf{q})=n}B\left(\{\mathbf{q}A\},\frac{1}{q^{\lfloor\frac{nr}{s}\rfloor+l'_n}}\right)\right)\geq d\sum_{n=\lceil\frac{k_0}{2}\rceil}^{N}q^{-sl'_{2n}}$$

for some $d>0$. As the series $\sum_{n=1}^{\infty}q^{-sl'_{2n}}=\infty$, we have a contradiction again for N large enough. ∎

Proposition 4.1.

$$S_{r,s}\supseteq\left\{A\in\mathbb{L}^{r\times s}: A \text{ is badly approximable}\right\}$$

Proof. Let A be badly approximable. Then, we have to show that

$$m\left(\bigcap_{k=0}^{\infty}\bigcup_{n=k}^{\infty}\bigcup_{\deg(\mathbf{q})=n} B\left(\{\mathbf{q}A\}, \frac{1}{q^{\lfloor \frac{nr}{s}\rfloor + l_n}}\right)\right) = 1.$$

By Lemma 4.6, we obtain that for all $k \geq 0$,

$$m\left(\bigcup_{n=k}^{\infty}\bigcup_{\deg(\mathbf{q})=n} B\left(\{\mathbf{q}A\}, \frac{1}{q^{\lfloor \frac{nr}{s}\rfloor + l_n}}\right)\right) > \frac{1}{q^{cs+1}} > 0.$$

Consequently,

$$m\left(\bigcap_{k=0}^{\infty}\bigcup_{n=k}^{\infty}\bigcup_{\deg(\mathbf{q})=n} B\left(\{\mathbf{q}A\}, \frac{1}{q^{\lfloor \frac{nr}{s}\rfloor + l_n}}\right)\right) > 0$$

and Lemma 4.4 implies our claim. ∎

Proposition 4.2.

$$S_{r,s} \subseteq \left\{A \in \mathbb{L}^{r\times s} : A \textit{ is badly approximable}\right\}$$

Proof. Assume that A is not badly approximable. We will show that we can choose a sequence l_n such that $\sum_{n=1}^{\infty} q^{-sl_n} = \infty$ but for almost every $\mathbf{g} \in \mathbb{L}^s$, there are finitely many $\mathbf{q}$ with

$$\|\{\mathbf{q}A\} - \mathbf{g}\| < \frac{1}{q^{\lfloor \frac{nr}{s}\rfloor + l_n}},\ \mathbf{q} \in \mathbb{F}_q[X]^r,\ \deg(\mathbf{q}) = n.$$

Since A is not badly approximable, there exists a sequence $\mathbf{q}^{(i)} = \left[Q_1^{(i)}, Q_2^{(i)}, \ldots, Q_r^{(i)}\right]$ belonging to $\mathbb{F}_q[X]^r$ with $\deg(\mathbf{q}^{(i)}) = n_i$ and n_i increasing such that

$$\|\{\mathbf{q}^{(i)}A\}\| < \frac{1}{q^{\lfloor \frac{r(n_i+i)}{s}\rfloor + i}}, \forall i \in \mathbb{N}.$$

Define

$$\begin{cases} t_0 = 0 \\ t_i = n_i + i, i \geq 1 \end{cases},\ \text{and for } t_{i-1} \leq n < t_i,\ l_n = \lfloor \frac{r(t_i - n)}{s} \rfloor.$$

Then, we have

$$\sum_{n=1}^{\infty} \frac{1}{q^{sl_n}} \geq \sum_{i=1}^{\infty} \frac{1}{q^{sl_{t_i-1}}} = \sum_{i=1}^{\infty} \frac{1}{q^{s\lfloor \frac{r(t_i-(t_i-1))}{s}\rfloor}} \geq \sum_{i=1}^{\infty} \frac{1}{q^r} = \infty.$$

On the other hand, assume without loss of generality that $q^{n_i} = \|\mathbf{q}^{(i)}\| = |Q_1^{(i)}|$. We have to show that

$$\bigcup_{t_{i-1}\le n<t_i}\bigcup_{\deg(\mathbf{q})=n} B\left(\{\mathbf{q}A\}, q^{-\lfloor\frac{nr}{s}\rfloor-l_n}\right) \subset \bigcup B\left(\{\mathbf{q}'A\}, q^{-\lfloor\frac{rt_i}{s}\rfloor+2}\right),$$

where the right union runs over all $\mathbf{q}' = [Q_1', Q_2', \ldots, Q_r']$ which fulfil the conditions

$$|Q_1'| \le q^{n_i-1},\ |Q_2'| \le q^{t_i-1},\ \ldots,\ |Q_r'| \le q^{t_i-1}.$$

Fix $\{\mathbf{q}A\}$ with $t_{i-1} \le \deg(\mathbf{q}) = n < t_i$. Then, there exists a polynomial h such that $|Q_1 + hQ_1^{(i)}| \le q^{n_i-1}$. Note that $|h| \le q^{t_i-1-n_i}$. Now set

$$\mathbf{q}' = [Q_1 + hQ_1^{(i)}, Q_2 + hQ_2^{(i)}, \ldots, Q_r + hQ_r^{(i)}].$$

Then, we obtain

$$\|\{\mathbf{q}A\} - \{\mathbf{q}'A\}\| \le |h|\|\{\mathbf{q}^{(i)}A\}\| < q^{t_i-1-n_i}\, q^{-\lfloor\frac{rt_i}{s}\rfloor-i} = q^{-\lfloor\frac{rt_i}{s}\rfloor-1}.$$

Note that

$$q^{-\lfloor\frac{nr}{s}\rfloor-l_n} = q^{-\lfloor\frac{nr}{s}\rfloor-\lfloor\frac{r(t_i-n)}{s}\rfloor} < q^{-\frac{nr}{s}-\frac{r(t_i-n)}{s}+2} \le q^{-\lfloor\frac{rt_i}{s}\rfloor+2}.$$

Hence, we have

$$B\left(\{\mathbf{q}A\}, q^{-\lfloor\frac{nr}{s}\rfloor-l_n}\right) \subset B\left(\{\mathbf{q}'A\}, q^{-\lfloor\frac{rt_i}{s}\rfloor+2}\right).$$

Therefore, our claim is proved. Now, we estimate the measure of union of these balls

$$m\left(\bigcup_{t_{i-1}\le n<t_i}\bigcup_{\deg(\mathbf{q})=n} B\left(\{\mathbf{q}A\}, \frac{1}{q^{\lfloor\frac{nr}{s}\rfloor+l_n}}\right)\right) \le q^{s\left(-\lfloor\frac{rt_i}{s}\rfloor+2\right)} q^{n_i+t_i(r-1)} \le q^{3s-i}.$$

Consequently,

$$\sum_{i=1}^{\infty} m\left(\bigcup_{t_{i-1}\le n<t_i}\bigcup_{\deg(\mathbf{q})=n} B\left(\{\mathbf{q}A\}, \frac{1}{q^{\lfloor\frac{nr}{s}\rfloor+l_n}}\right)\right) \le \sum_{i=1}^{\infty} q^{3s-i} < \infty.$$

Hence, for almost every $\mathbf{g} \in \mathbb{L}^s$, there are finitely many $\mathbf{q}$'s with $\deg(\mathbf{q}) = n$ satisfying $\|\{\mathbf{q}A\} - \mathbf{g}\| < q^{-\lfloor\frac{nr}{s}\rfloor-l_n}$. ∎

Finally, Proposition 4.1 and Proposition 4.2 imply Theorem 1.14.

Chapter 5

Conclusion

We conclude this thesis with some remarks.

First, note that in the real number case, the approximation function in Kurzweil's theorem is assumed to be monotonic (compare with the theorems of Kristensen in Section 1.2 were also some monotonicity assumptions are used). In this work, on the other hand, the approximation function is of the form q^{-n-l_n} with no monotonicity assumptions on l_n. However, note that our approximation function tends to 0 as n tends to infinity (this was not assumed by Kurzweil). In fact, we guess that if one replaces our approximation function by q^{-l_n}, then in order for the result to hold, a monotonicity condition on l_n similar to the one used by Kurzweil is needed.

Next, we briefly discuss possible almost sure results for the number of solutions of (1.6) (similar considerations can be made for the higher dimensional case). Let us denote a sequence of random variables counting solutions of (1.6) by

$$X_N := \#\{\text{solutions in } Q \text{ with } Q \text{ monic of (1.6) with } n \leq N\}.$$

Then, we have that

$$X_N = \sum_{n \leq N} \sum_{\deg(Q)=n} \chi_{B\left(\{Qf\}, q^{-n-l_n}\right)},$$

where χ_A denotes the indicator function of the set A. Hence, we obtain for the expected value

$$\mathbb{E}\left[X_N\right] = \sum_{n\le N} \sum_{\deg(Q)=n} q^{-n-l_n} = \sum_{n\le N} q^{-l_n}.$$

An interesting question is whether or not one can prove a strong law of large numbers for the number of solutions? More precisely, is it true that for any badly approximable f, we have

$$X_N \sim \sum_{n\le N} q^{-l_n} \qquad a.s.\ ?$$

If yes, what can be said about the error term (which should then depend on Diophantine approximation properties of f)?

Overall, there are still interesting questions left concerning inhomogeneous Diophantine approximation in the field of formal Laurent series.

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