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應用數學系

碩 士 論 文

三維面著色的連繫算子

Connecting operator of 3- dimensional Face Coloring

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摘　　要

這個研究主要是要去計算三維度兩個顏色的熵，但首先必須利用一個特殊的矩陣轉移以及矩陣自乘的性質所發展出來的遞迴公式去解決三維度兩個顏色下面著色的花樣生成問題。

接下來，給一個限制集則就可以定義出轉移矩陣而且它的遞迴公式也會被表現出來。最後，只需去計算連繫算子的最大特徵值即可計算出熵的問題。

Connecting operator of 3- dimensional Face Coloring

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ABSTRACT

The work investigates entropy of 3-dimensional face coloring, but we need to solve three-dimensional pattern generation problem with edge- coloring by first useing a special Matrix transfer and self-multiply matrices to establish some recursive formulas, first.

Now, given admissible set of local patterns then the transition matrix is defined and the recursive formulas are presented. Finally the entropy is obtained by computing the maximum eigenvalues of a sequence of connecting operator.

致　　謝

首先，很感謝指導教授— 林松山教授 給予我很多不管是研究學習、人生道理以及一些待人之道很多的學習，也讓我在這段時間可以順利的充實自我，對我來說這是非常寶貴的經驗，所以身為學生的我真的很感激。

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最後，要謝謝一起唸書兩年的研究室的朋友們，可以找到一起唸書一起運動一起在不同的地方留下回憶的你們，讓我這兩年在交大的生活如此精彩,真的很謝謝他(她們)。

目　　錄

1 Introduction

Here, we consider the problem of 2 symbols, then will get the set of all patterns $\sum_2^3$, first, give any admissible set $B \subseteq \sum_2^3$ and denote $\sum_2^3(B)$ be $\sum_2^3$ which is restricted in B , secondly, denote $\Gamma_{m\times n\times k}(B)$ is the quality of, $\sum_2^3(B)$,and finally, need to calculate entropy of 3-dimensional face coloring,

$$h(B) = \lim_{m,n,k\to\infty} \frac{\log \Gamma_{m\times n\times k}(B)}{mnk} \tag{1.1}$$

clearly, how to calculate $\Gamma_{m\times n\times k}(B)$ is the first problem we encountered from the equation (1.1).In order to solve the problem we face, we must study the problem of 3-dimensional pattern generation of 2 symbols in section 2and find a way to control the colors of different directions by the matrix,Y_2 .

Now, split section 2 into 3 steps as following:

Step 1 : find the recursive formula, $Y_{2\times n\times 2}$, of y-direction by $Y_{2\times 2\times 2}$, for $n \geq 3$

Step 2 : denote $Y_{2\times n\times 2} \rightarrow X_{2\times n\times 2}$ and find the recursive formula ,

Step 3 : denote $Z_{m\times n\times 2} \equiv X_{m\times n\times 2}$ and we will get $Z_{m\times n\times k}$ by $Z_{m\times n\times 2}$

In section 3, we defined $V_{2\times 2\times 2;i_y}$ as the transition matrix of $Y_{2\times 2\times 2;i_y}$, for$1 \leq i_y \leq 4$ and find that the main problem will be converted into finding $\Gamma_{m\times n\times k}(B)$ by Perron-Frrbenius theorem. Finally, using the result to calculus the entropy of (1.1), where the details will be presented in theorem 1.

2 Three-Dimensional Pattern Generation Problems

This section describes three-dimensional pattern generation problem. Here, m, n, $k \geq 2$ are fixed and indices for brevity. Let S be a set of p colors, and $Z_{m\times n\times k}$ be a fixed finite rectangular sub-lattice of Z^3, where Z^3 denotes the integer lattice on R^3 and (m, n, k) be a three-tuple of positive integer. Function $U: Z^3 \to S$ and $U_{m\times n\times k}: Z_{m\times n\times k} \to S$ are called global patterns and locally patterns respectively. The set of all patterns U is denoted by $\sum_p^3 \equiv s^{z^3}$, such that $\sum_p^3$ is the set of all patterns with p different colors in a three-dimensional lattice.

For clarity, two symbols, S = {0, 1} are considered. Let x, y and z coordinate represent 1st-, 2st- and 3st-coordinates respectively as in Fig.1. Six orderings [w] ordering are represented as the following:

$$\begin{aligned}
[x] &: [1] \succ [2] \succ [3] \\
[y] &: [2] \succ [1] \succ [3] \\
[z] &: [3] \succ [1] \succ [2] \\
[\widehat{x}] &: [1] \succ [3] \succ [2] \\
[\widehat{y}] &: [2] \succ [3] \succ [1] \\
[\widehat{z}] &: [3] \succ [2] \succ [1]
\end{aligned} \tag{2.1}$$

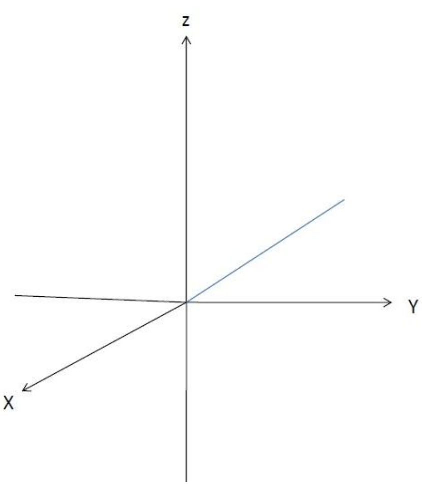


Figure 1. Three-dimension coordinate system.

On a fixed lattice $Z_{m\times n\times k}$, an ordering $[w]\succ[j]\succ[k]$ is obtained on $Z_{m\times n\times k}$, which is any one of the above ordering on $Z_{m\times n\times k}$. Therefore, the six ordering of $Z_{2\times 2\times 2}$ are presented as Fig.2, where $\alpha_i = \{0, 1\}$.

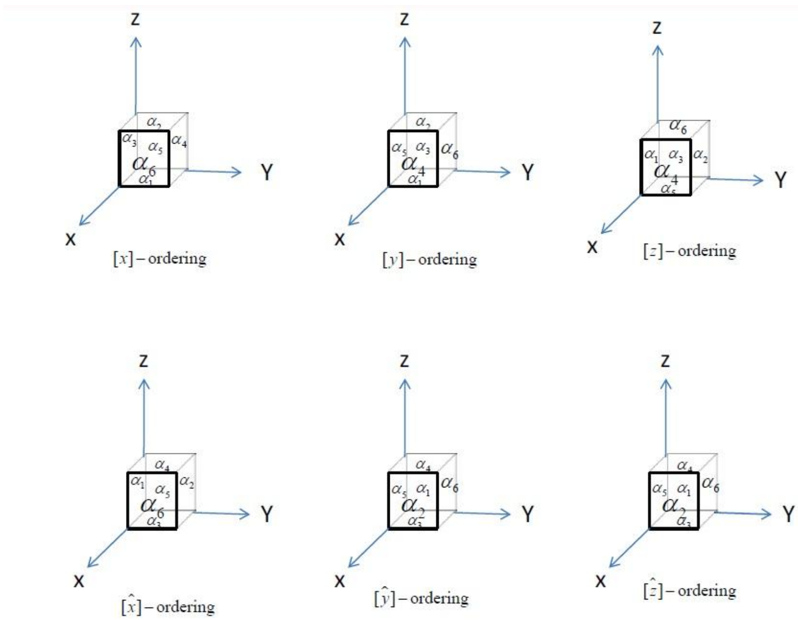


Figure 2.

2.1 Ordering Matrices

By using the six forms can get six different ordering matrices of $Z_{2\times2\times2}$ and denote the order of matrices as equation (2.2)

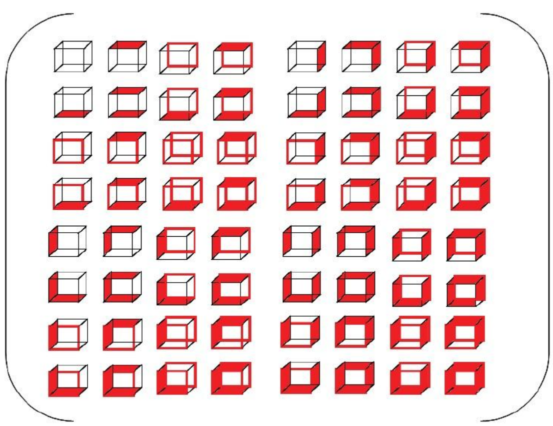
$$i_{\alpha} = 1 + \sum_{i=1}^{6} \alpha_i 2^{6-i}, where\ \alpha_i \in \{0,1\} \qquad (2.2)$$

Here, we choose [z] as the order for convenience, and we can denote the order of x, y and z-directions by i_x, i_y and i_z (2.3) respectively, where $\leqq i_x$, i_y and $i_z \leq 4$.

$$\begin{cases} i_x = 1 + \alpha_2 + \alpha_1 \times 2 \\ i_y = 1 + \alpha_4 + \alpha_3 \times 2 \\ i_x = 1 + \alpha_6 + \alpha_5 \times 2 \end{cases} \qquad (2.3)$$

For convenience again, we have to define the matrix $Y_{2\times2\times2}$ (2.4) below which present the relation between colors and each directions

$Y_{2\times2\times2} =$



$$= \left[Y_{2\times2\times2;i;j} \right]_{2^3\times2^3} = \begin{bmatrix} Y_{2\times2\times2;1} & Y_{2\times2\times2;2} \\ Y_{2\times2\times2;3} & Y_{2\times2\times2;4} \end{bmatrix} \qquad (2.4)$$

It's not different to discover that the colors of each direct of $Z_{2\times2\times2}$ be controlled in each layer of $Y_{2\times2\times2}$ respectively. It means that $Y_{2\times2\times2}$ is divided into three layers by matrix partitioning as figure 3 and the colors of y-, x-and z-direction are controlled in first, second and the third layer respectively as figure 3 below.

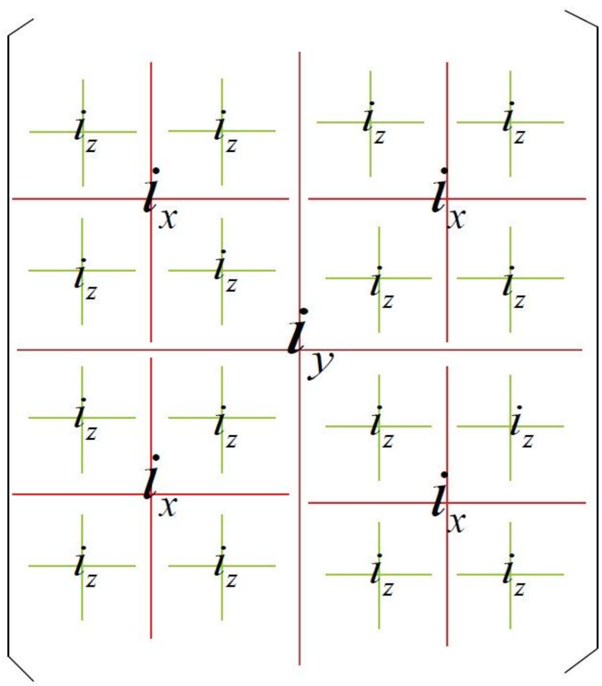


Figure 3. relation between colors and layers.

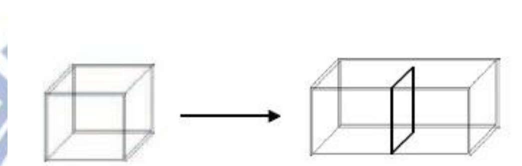
The process of investigating the pattern generation problem should be broken down in the following steps:

Step 1. find the recursive formula $Y_{2\times2\times2} \rightarrow Y_{2\times3\times2} \rightarrow \ldots \rightarrow Y_{2\times n\times2}$, that is extend on the y- direction.

Step 2. here $Y_{2\times n\times2} \rightarrow X_{2\times n\times2}$ by special Matrix transfer, then $X_{2\times n\times2}$ will have a good order of x-direction .Then we get the recursive formula like $X_{2\times n\times2} \rightarrow X_{3\times n\times2} \rightarrow \ldots \rightarrow X_{m\times n\times2}$ that is extend the x-direction by$X_{2\times n\times2}$.

Step 3. here replaces $Z_{m\times n\times2}$ with $X_{m\times n\times2}$, it means that $X_{m\times n\times2}$ is really to extend the z –direction. By using the matrix to self-multiply, we can generate $Z_{m\times n\times2} \rightarrow Z_{m\times n\times3} \rightarrow \ldots \rightarrow Z_{m\times n\times k}$, that is extend the z-direction by $Z_{m\times n\times2}$.

STEP 1.



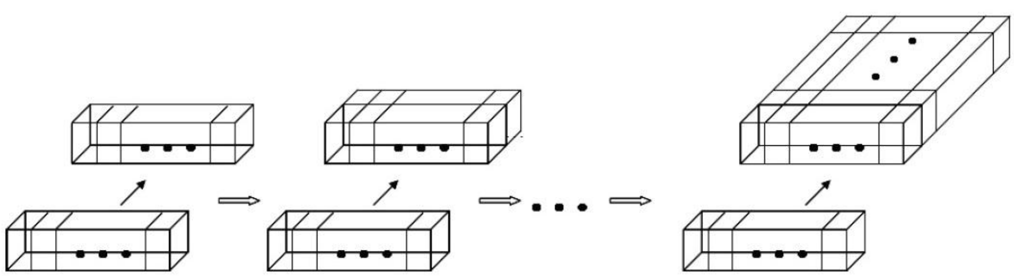
$$Y_{2\times2\times2} \rightarrow Y_{2\times3\times2}$$

STEP 2.

Change the order of matrix $Y_{2\times3\times2}$ by a special transfer, then the new matrix $X_{2\times3\times2}$ will have a good nature to make the calculation of $X_{2\times n\times2} \rightarrow X_{3\times n\times2} \rightarrow \ldots \rightarrow X_{m\times n\times2}$ to be simple.

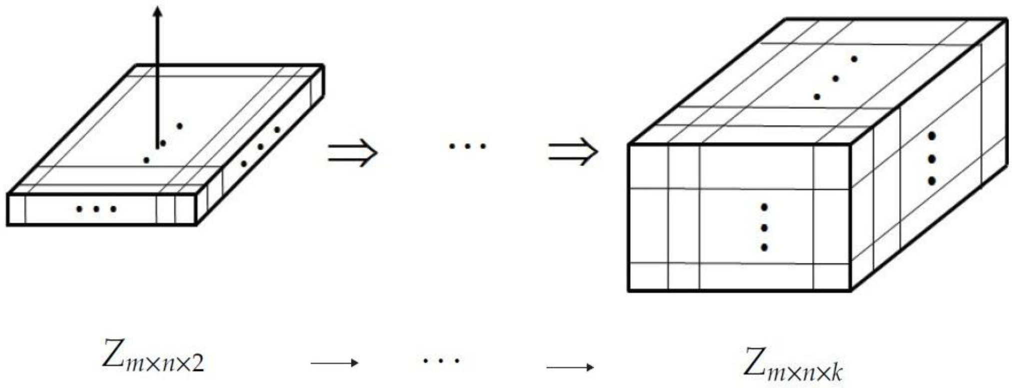
And also have $Y_{2\times3\times2} \rightarrow Y_{2\times4\times2} \rightarrow \ldots \rightarrow Y_{2\times n\times2}$, that is extend on the y- direction.

STEP 3.

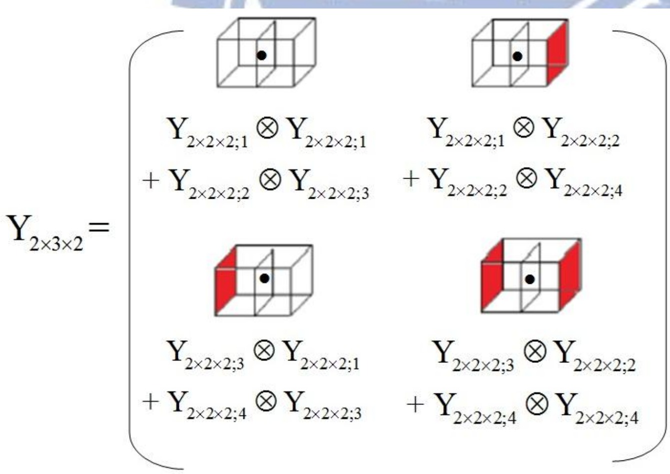


$$X_{2\times n\times2} \otimes X_{2\times n\times2} = X_{3\times n\times2} \Rightarrow X_{2\times n\times2} \otimes X_{3\times n\times2} = X_{4\times n\times2} \Rightarrow \cdots \Rightarrow X_{2\times n\times2} \otimes X_{m-1\times n\times2} = X_{m\times n\times2}$$

STEP 4.



Now, we talk about the details of those steps. First, we will make $Y_{2\times3\times2}$ by using the properties of $Y_{2\times2\times2}$ as the following:



$$=\begin{bmatrix} Y_{2\times3\times2;1} & Y_{2\times3\times2;2} \\ Y_{2\times3\times2;3} & Y_{2\times3\times2;4} \end{bmatrix}$$

Secondly, we have to define a # transfer, then we will have $Y_{2\times n\times 2} \rightarrow X_{2\times n\times 2}$ by $M_n^{\#}$ transfer

Definition 2.1.

A is a 4×4 matrix, then the # transfer is mean if

$$A=\begin{bmatrix} A_{1;1} & A_{1;2} & A_{2;1} & A_{2;2} \\ A_{1;3} & A_{1;4} & A_{2;3} & A_{2;4} \\ A_{3;1} & A_{3;2} & A_{4;1} & A_{4;2} \\ A_{3;3} & A_{3;4} & A_{4;3} & A_{4;4} \end{bmatrix}, then\ A^{\#}=\begin{bmatrix} A_{1;1} & A_{2;1} & A_{1;2} & A_{2;2} \\ A_{3;1} & A_{4;1} & A_{3;2} & A_{4;2} \\ A_{1;3} & A_{2;3} & A_{1;4} & A_{2;4} \\ A_{3;3} & A_{4;3} & A_{3;4} & A_{4;4} \end{bmatrix} \quad (2.5)$$

Which $A^{\#}_{i;j}=A_{j;i}\ \forall i;j=1,2,3,4$

Definition 2.2.

$M_1^{\#}$ is a transfer to the matrix ,it contains the following two-step

1. Cut the matrix to be a $2^1\times 2^1$ matrix
2. do #-transfer on each block

i.e. $M_1^{\#}=\begin{bmatrix} [M_{1;1}]^{\#} & [M_{1;2}]^{\#} \\ [M_{2;1}]^{\#} & [M_{2;2}]^{\#} \end{bmatrix}$

Similarly, define $M_n^{\#}$ is a transfer to the matrix ,it contains the following two-step

1.Cut the matrix to be a $2^n\times 2^n$ matrix

2.do #-transfer on each block

i.e. $M_n^{\#}=\left[[M_{i;j}]^{\#}\right]_{2^n\times 2^n}$

when we $Y_{2\times3\times2}\rightarrow X_{2\times3\times2}$ by $M_1^{\#}$, we find the recursive formula

$$Y_{2\times2\times2}\rightarrow Y_{2\times3\times2}\rightarrow \ldots \rightarrow Y_{2\times n\times2},$$

by $Y_{2\times2\times2}\otimes X_{2\times n\text{-}1\times2}=Y_{2\times n\times2}\ \forall n\in N$ which $Y_{2\times n\times2}\rightarrow X_{2\times n\times2}$ by $M_1^{\#}\rightarrow M_2^{\#}\rightarrow\cdots\rightarrow M_{n\text{-}2}^{\#}$

Thirdly, we cut $X_{2\times n\times2;k}$ to be a $2^{n\text{-}1}\times2^{n\text{-}1}$ matrix, which k is mean the y-direction's color. $\forall k=1,2,3,4$, then we have

$$\left[X_{3\times n\times2;(k_1,k_2)}\right]^a_b=\sum_{i=1}^{2^{n\text{-}1}}\left[X_{2\times n\times2;k_1}\right]^a_i\otimes\left[X_{2\times n\times2;k_2}\right]^i_b \quad (2.6)$$

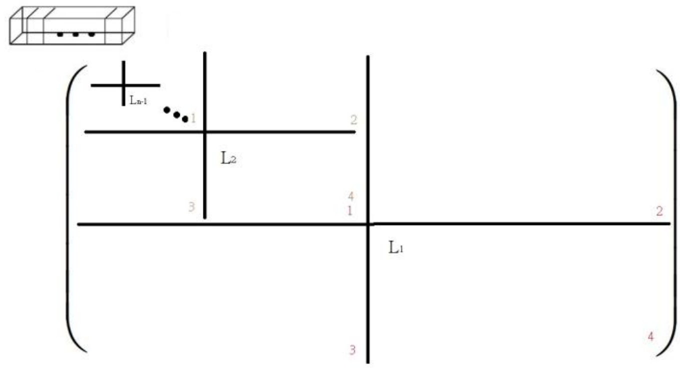
which $k_i=\{1,2,3,4\}\forall i\in N$, which k_i is mean the y-direction's color of the i-th box from the front. And $[A]^a_b$ is mean the a-th row and b-th column $\forall a,b=\{1,2,\cdots,2^{n\text{-}1}\}$

Similarly, the recursive formula can be developed by using the properties of $X_{2\times n\times 2;k}$ as the following:

cut $X_{m\text{-}1\times n\times 2;(k_1,\cdots,k_{m-2})}$ to be a $2^{n-1}\times 2^{n-1}$ matrix, which ki is mean the y-direction's color. $\forall k_i = 1,2,3,4$, then we have

$$\left[X_{m\times n\times 2;(k_1,k_2,\cdots,k_{m-1})}\right]_b^a = \sum_{i=1}^{2^{n-1}}\left[X_{m\text{-}1\times n\times 2;(k_1,\cdots,k_{m-2})}\right]_i^a \otimes \left[X_{2\times n\times 2;k_{m-1}}\right]_b^i \qquad (2.7)$$

We denoted $X_{m\times n\times 2;(k_1,k_2,\cdots,k_{m-1})} = \left[X_{m\times n\times 2;(k_1,k_2,\cdots,k_{m-1});(l_1,\cdots,l_{n-1})}\right], \forall l_i \in \{1,2,3,4\}$, which l_i is be taken by Z-shaped like the following figure.



Finally, denote $Z_{m\times n\times 2;(k_1,k_2,\cdots,k_{m-1});(l_1,\cdots,l_{n-1})} = X_{m\times n\times 2;(k_1,k_2,\cdots,k_{m-1});(l_1,\cdots,l_{n-1})}$, and we have $Z_{m\times n\times 2} \rightarrow Z_{m\times n\times 3}$ by Matrix multiplication

$$Z_{m\times n\times 2;(k_1^{(1)},\cdots,k_{m-1}^{(1)});(l_1^{(1)},\cdots,l_{n-1}^{(1)})}\; Z_{m\times n\times 2;(k_1^{(2)},\cdots,k_{m-1}^{(2)});(l_1^{(2)},\cdots,l_{n-1}^{(2)})} \qquad (2.8)$$

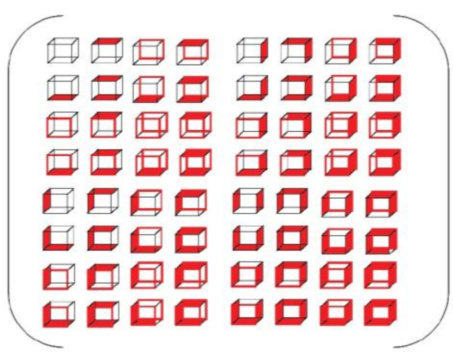
In the same way,

denote $Z_{m\times n\times 3;(k_1,k_2,\cdots,k_{m-1});(l_1,\cdots,l_{n-1})} = \left[Z_{m\times n\times 2;(k_1,k_2,\cdots,k_{m-1});(l_1,\cdots,l_{n-1});(t_1,t_2)}\right], \forall t_i \in \{1,2,3,4\}$ which t_i is be taken by Z-shaped, which t_i is mean the Z-direction's color.

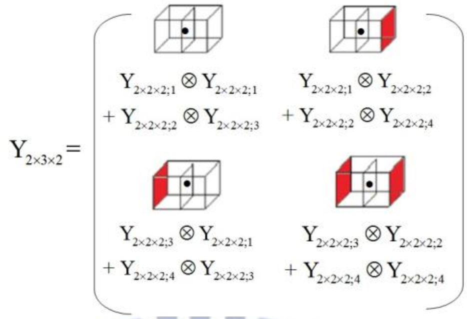
2.2 Show $Y_{2\times 2\times 2} \rightarrow Z_{3\times 3\times 3}$

Now , we show the case $Y_{2\times 2\times 2} \rightarrow Z_{3\times 3\times 3}$

First, we will make $Y_{2\times 3\times 2}$ by using the properties of $Y_{2\times 2\times 2}$ as the following:

$Y_{2\times 2\times 2} =$ 

then



Now, We only consider the case $Y_{2\times3\times2;1} = Y_{2\times2\times2;1} \otimes Y_{2\times2\times2;1} + Y_{2\times2\times2;2} \otimes Y_{2\times2\times2;3}$, then the other cases will be the same

Step 2. Change the order of matrix $Y_{2\times3\times2;1}$ by M_1^* transfer, then we will have a new matrix $X_{2\times3\times2;1}$

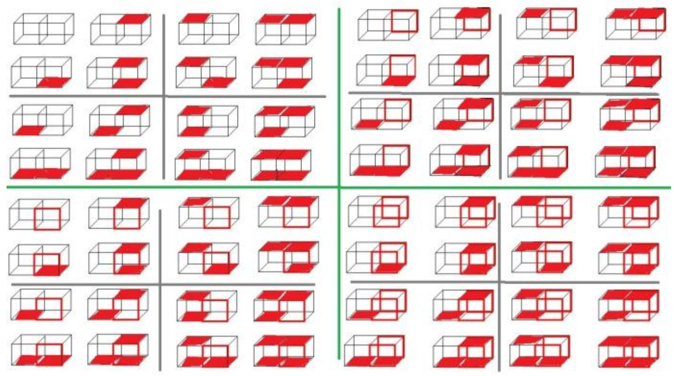
i.e. $Y_{2\times3\times2;1} \xrightarrow{M_1^\#} X_{2\times3\times2;1}$,by the definition of $M_1^\#$ transfer, it contains the following two-step.

1. Cut $Y_{2\times3\times2;1}$ to be a $2^1 \times 2^1$ matrix, i.e. $Y_{2\times3\times2;1} = \begin{bmatrix} Y_{2\times3\times2;1;1} & Y_{2\times3\times2;1;2} \\ Y_{2\times3\times2;1;3} & Y_{2\times3\times2;1;4} \end{bmatrix}$
2. Do # -transfer on every $Y_{2\times3\times2;1;I}$,i.e. $X_{2\times3\times2;1;i} = \left[Y_{2\times3\times2;1;i}\right]^\#,\ \forall i = \{1,2,3,4\}$

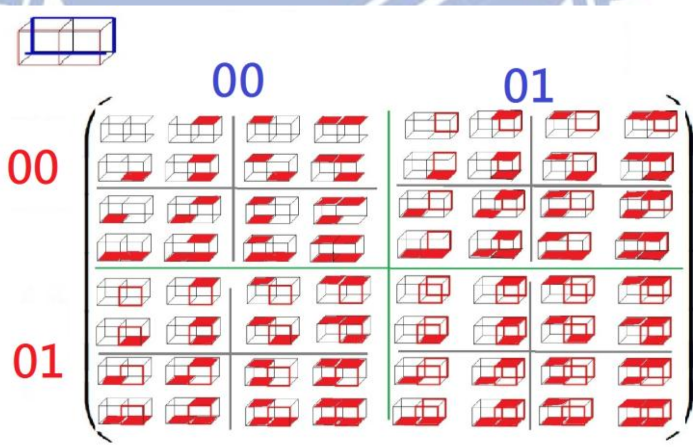
Now, we only consider $X_{2\times3\times2;1;1} = \left[Y_{2\times3\times2;1;1}\right]^\#$,

$Y_{2\times3\times2;1;1} =$

Then $X_{2\times3\times2;1;1} = \left[Y_{2\times3\times2;1;1}\right]^{\#} =$



We observe this figure, then we will find the rows and columns of $X_{2\times3\times2;1;1}$ is sequential. When boxes in the same block which is cut by green line, then the X-direction's color of boxes is the same. As shown below



Similarly, repeat the same steps to get $X_{2\times3\times2;1;i}, \forall i = \{2,3,4\}$,

then we have $X_{2\times3\times2;1} = \begin{bmatrix} X_{2\times3\times2;1;1} & X_{2\times3\times2;1;2} \\ X_{2\times3\times2;1;3} & X_{2\times3\times2;1;4} \end{bmatrix}$

Similarly, we find the rows and columns of $Y_{2\times3\times2;1}$ is sequential like fig4.

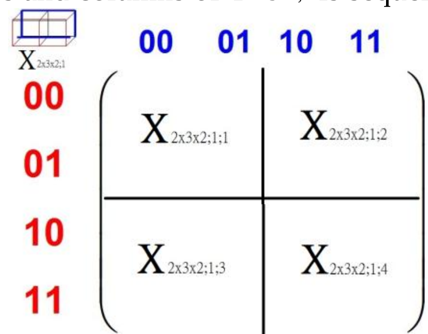


Figure 4

In the same way, repeat the same steps to get $X_{2\times3\times2}=\begin{bmatrix} X_{2\times3\times2;1} & X_{2\times3\times2;2} \\ X_{2\times3\times2;3} & X_{2\times3\times2;4} \end{bmatrix}$

Step 3. $X_{2\times3\times2} \rightarrow X_{3\times3\times2}$

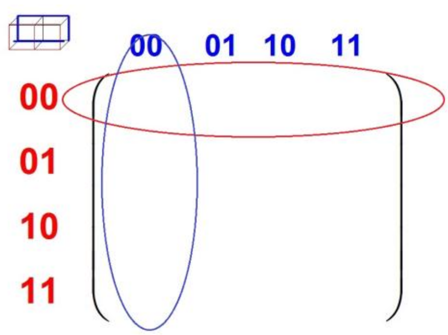
cut $X_{2\times3\times2;k_i}$ to be a $2^{n-1}\times2^{n-1}$ matrix,

$$\left[X_{3\times3\times2;(k_1,k_2)}\right]_b^a = \sum_{i=1}^{2^2}\left[X_{2\times3\times2;k_1}\right]_i^a \otimes \left[X_{2\times3\times2;k_2}\right]_b^i ,$$

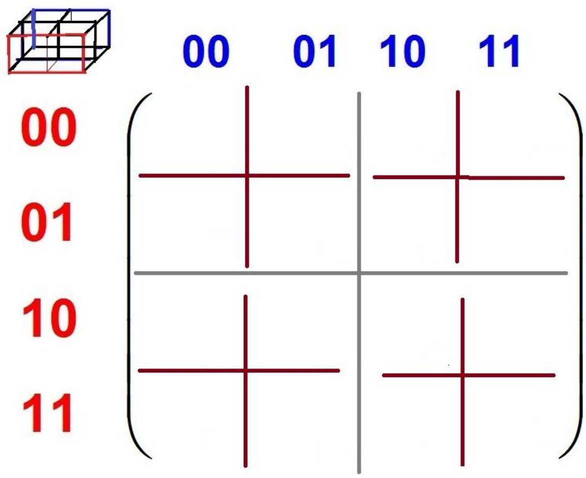
$k_i=\{1,2,3,4\}\ \forall i=\{1,2\}$, which k_i is mean the y-direction's color of the i-th box from the front. And when $k_1,k_2=\{1,4\}$ is mean the y-direction's color of the box was same color.

Now, we consider a,b=1

Ex: $\left[X_{3\times3\times2;(k_1,k_2)}\right]_1^1 = \sum_{i=1}^{2^2}\left[X_{2\times3\times2;k_1}\right]_i^1 \otimes \left[X_{2\times3\times2;k_2}\right]_1^i$,as shown below



Then we have $\mathrm{X}_{3\times3\times2;(k_1,k_2)} =$



We denoted $\mathrm{X}_{3\times3\times2;(k_1,k_2)}$ by take Z-shaped twice, which is like fig5.

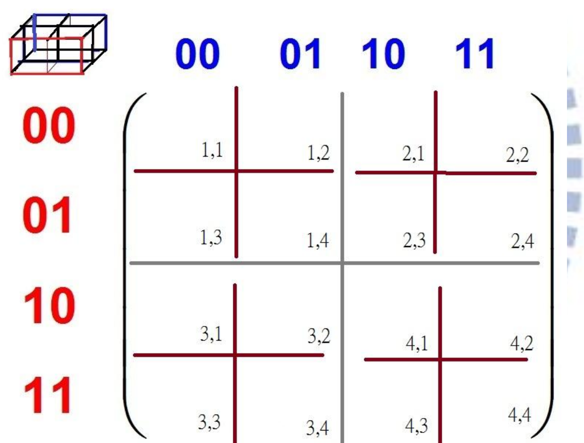


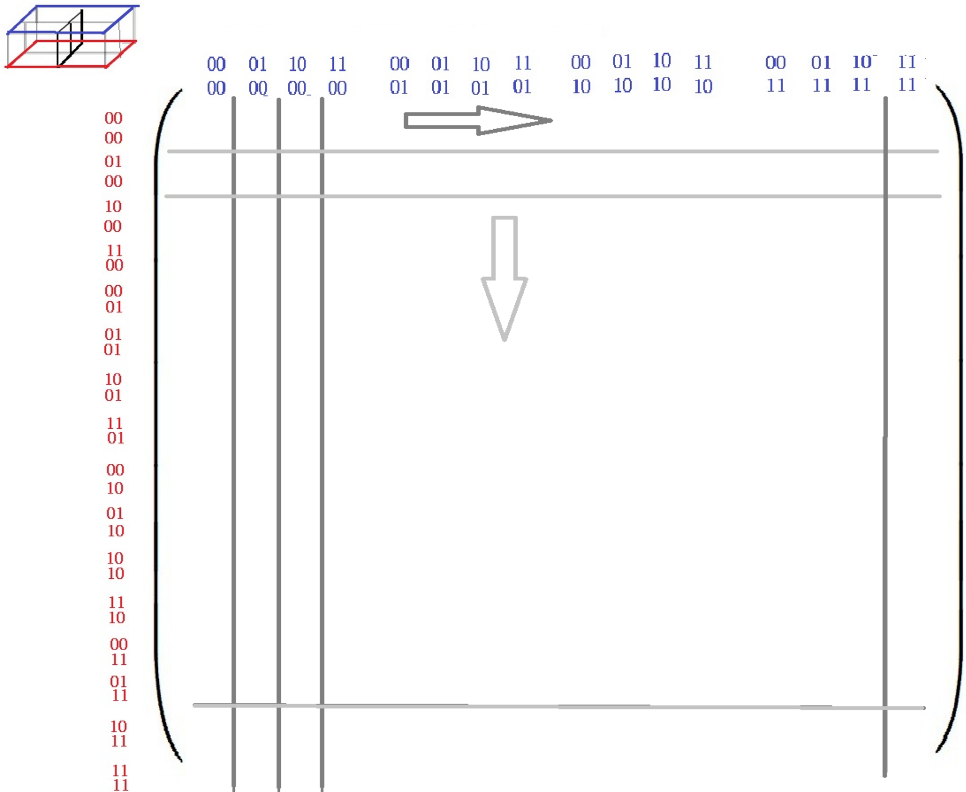
Figure 5

i.e. $\mathrm{X}_{3\times3\times2;(k_1,k_2)} = \lfloor \mathrm{X}_{3\times3\times2;(k_1,k_2);(l_1,l_2)} \rfloor$,which l_i is be taken by Z-shaped and is also mean the x-direction's color of the i-th box from the right.

And when $l_1, l_2 = \{1,4\}$ is mean the x-direction's color of the box was same color.

Step 4.

Finally, we also find the rows and columns of $X_{3\times3\times2;(k_1,k_2);(l_1,l_2)}$ is sequential



So we denote $Z_{3\times3\times2;(k_1,k_2);(l_1,l_2)} = X_{3\times3\times2;(k_1,k_2);(l_1,l_2)}$,and we have $Z_{3\times3\times2} \rightarrow Z_{3\times3\times3}$

By $Z_{3\times3\times2;(k_1^{(1)},k_2^{(1)});(l_1^{(1)},l_2^{(1)})}$ $Z_{3\times3\times2;(k_1^{(2)},k_2^{(2)});(l_1^{(2)},l_2^{(2)})}$

3 Transition Matrices and Connecting Operator

3.1 Transition Matrices

Based on the process of the ordering matrix, we have to define transition matrix as the following :

1. Given an admissible set $\mathrm{B} \subseteq \sum_{2\times 2\times 2}^{Z_3}$

2.Define,

$$\begin{cases} \mathrm{V}_{2\times 2\times 2;i,j} = 1, \; if \;\; \mathrm{Y}_{2\times 2\times 2;i,j} \in \mathrm{B} \\ \mathrm{V}_{2\times 2\times 2;i,j} = 0, \; if \;\; \mathrm{Y}_{2\times 2\times 2;i,j} \notin \mathrm{B} \end{cases}$$

3. The recursive formula for y-direction is as following:

$$\mathrm{V}_{2\times 3\times 2} = \begin{bmatrix} \mathrm{V}_{2\times 2\times 2;1} \otimes \mathrm{V}_{2\times 2\times 2;1} & \mathrm{V}_{2\times 2\times 2;1} \otimes \mathrm{V}_{2\times 2\times 2;2} \\ + \mathrm{V}_{2\times 2\times 2;2} \otimes \mathrm{V}_{2\times 2\times 2;3} & + \mathrm{V}_{2\times 2\times 2;2} \otimes \mathrm{V}_{2\times 2\times 2;4} \\ \mathrm{V}_{2\times 2\times 2;3} \otimes \mathrm{V}_{2\times 2\times 2;1} & \mathrm{V}_{2\times 2\times 2;3} \otimes \mathrm{V}_{2\times 2\times 2;2} \\ + \mathrm{V}_{2\times 2\times 2;4} \otimes \mathrm{V}_{2\times 2\times 2;3} & + \mathrm{V}_{2\times 2\times 2;4} \otimes \mathrm{V}_{2\times 2\times 2;4} \end{bmatrix}$$

we get $\mathrm{V}_{2\times 3\times 2} \rightarrow \mathrm{H}_{2\times 3\times 2}$ by $M_1^{\#}$, we find the recursive formula

$\mathrm{V}_{2\times 2\times 2} \rightarrow \mathrm{V}_{2\times 3\times 2} \rightarrow \ldots \rightarrow \mathrm{V}_{2\times n\times 2}$, by $\mathrm{V}_{2\times 2\times 2} \otimes \mathrm{H}_{2\times n\text{-}1\times 2} = \mathrm{V}_{2\times n\times 2} \quad \forall n \in \mathrm{N}$

which $\mathrm{V}_{2\times n\times 2} \rightarrow \mathrm{H}_{2\times n\times 2}$ by $\mathrm{M}_1^{\#} \rightarrow \mathrm{M}_2^{\#} \rightarrow \cdots \rightarrow \mathrm{M}_{n\text{-}2}^{\#}$

5. we cut $\mathrm{H}_{2\times n\times 2;k}$ to be a $2^{n\text{-}1} \times 2^{n\text{-}1}$ matrix, which k is mean the y-direction's color.

Then we have $\left[\mathrm{H}_{3\times n\times 2;(k_1,k_2)}\right]_b^a = \sum_{i=1}^{2^{n-1}} \left[\mathrm{H}_{2\times n\times 2;k_1}\right]_i^a \otimes \left[\mathrm{H}_{2\times n\times 2;k_2}\right]_b^i$

Similarly, the recursive formula can be developed by using the properties of $\mathrm{V}'_{2\times n\times 2;k}$ as the following:

cut $\mathrm{H}_{m\text{-}1\times n\times 2;(k_1,\cdots,k_{m-2})}$ to be a $2^{n\text{-}1} \times 2^{n\text{-}1}$ matrix, which ki is mean the y-direction's color. $\forall k_i = 1,2,3,4$

Then we have $\left[\mathrm{H}_{m\times n\times 2;(k_1,k_2,\cdots,k_{m-1})}\right]_b^a = \sum_{i=1}^{2^{n-1}} \left[\mathrm{H}_{m\text{-}1\times n\times 2;(k_1,\cdots,k_{m-2})}\right]_i^a \otimes \left[\mathrm{H}_{2\times n\times 2;k_{m\text{-}1}}\right]_b^i$

6. We denoted $\mathrm{H}_{m\times n\times 2;(k_1,k_2,\cdots,k_{m-1})} = \left[\mathrm{H}_{m\times n\times 2;(k_1,k_2,\cdots,k_{m-1});(l_1,\cdots,l_{n-1})}\right], \forall l_i \in \{1,2,3,4\}$

which l_i is be taken by Z-shaped

Finally, denote $\mathrm{S}_{m\times n\times 2;(k_1,\cdots,k_{m-1});(l_1,\cdots,l_{n-1})} = \mathrm{H}_{m\times n\times 2;(k_1,\cdots,k_{m-1});(l_1,\cdots,l_{n-1})}$

Theorem 3.1.

Given $B \subseteq \sum_{2\times 2\times 2}^{Z_3}$,Let $\forall k_i = \{1,4\}$ and $\forall l_i \in \{1,4\}$,Then, for any m,n,k≧1,

$$h(B) \geq \frac{\log \rho\left(S_{m\times n\times 2;(k_1^{(1)},\cdots,k_{m-1}^{(1)});(l_1^{(1)},\cdots,l_{n-1}^{(1)})} S_{m\times n\times 2;(k_1^{(2)},\cdots,k_{m-1}^{(2)});(l_1^{(2)},\cdots,l_{n-1}^{(2)})} \cdots S_{m\times n\times 2;(k_1^{(k)},\cdots,k_{m-1}^{(k)});(l_1^{(k)},\cdots,l_{n-1}^{(k)})}\right)}{(m-1)\cdot(n-1)\cdot k}$$

Proof:

$\Gamma_{m\times n\times tk} \geq \left|(S_{m\times n\times 2;(k_1,\cdots,k_{m-1});(l_1,\cdots,l_{n-1})})^{t-1}\right|$ is implies to $\Gamma_{pm\times qn\times tk} \geq \left|(S_{m\times n\times 2;(k_1,\cdots,k_{m-1});(l_1,\cdots,l_{n-1})})^{t-1}\right|^{p\cdot q}$

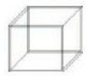
Then

$$\begin{aligned} h(B) &= \lim_{p,q,s\to\infty} \frac{\log \Gamma_{pm\times qn\times tk}}{pm\times qn\times sk} \\ &\geq \lim_{p,q,s\to\infty} \frac{\log\left|(S_{m\times n\times 2;(k_1,\cdots,k_{m-1});(l_1,\cdots,l_{n-1})})^{t-1}\right|^{p\cdot q}}{pm\times qn\times sk} \\ &= \lim_{s\to\infty} \frac{\log\left|(S_{m\times n\times 2;(k_1,\cdots,k_{m-1});(l_1,\cdots,l_{n-1})})^{t-1}\right|}{m\cdot n\cdot sk} \end{aligned}$$

By Perron-Forbenius Theorem,

$$h(B) \geq \frac{\log \rho\left(S_{m\times n\times 2;(k_1^{(1)},\cdots,k_{m-1}^{(1)});(l_1^{(1)},\cdots,l_{n-1}^{(1)})} S_{m\times n\times 2;(k_1^{(2)},\cdots,k_{m-1}^{(2)});(l_1^{(2)},\cdots,l_{n-1}^{(2)})} \cdots S_{m\times n\times 2;(k_1^{(k)},\cdots,k_{m-1}^{(k)});(l_1^{(k)},\cdots,l_{n-1}^{(k)})}\right)}{(m-1)\cdot(n-1)\cdot k},$$

3.2 Computation of $\lambda_{Z_{;m\times n\times 2}}$ and entropy

Let a forbidden set $B^c =$ 

$$\text{Then } V_{2\times2\times2} = \begin{bmatrix} 0&1&0&1&0&1&0&1\\ 1&1&1&1&1&1&1&1\\ 0&1&0&1&0&1&0&1\\ 1&1&1&1&1&1&1&1\\ 0&1&0&1&0&1&0&1\\ 1&1&1&1&1&1&1&1\\ 0&1&0&1&0&1&0&1\\ 1&1&1&1&1&1&1&1 \end{bmatrix}$$

$$\Rightarrow V_{2\times3\times2;k} = \begin{bmatrix} 0&0&0&0&0&2&0&2&0&0&0&0&0&2&0&2\\ 0&0&0&0&2&2&2&2&0&0&0&0&2&2&2&2\\ 0&0&0&0&0&2&0&2&0&0&0&0&0&2&0&2\\ 0&0&0&0&2&2&2&2&0&0&0&0&2&2&2&2\\ 0&2&0&2&0&2&0&2&0&2&0&2&0&2&0&2\\ 2&2&2&2&2&2&2&2&2&2&2&2&2&2&2&2\\ 0&2&0&2&0&2&0&2&0&2&0&2&0&2&0&2\\ 2&2&2&2&2&2&2&2&2&2&2&2&2&2&2&2\\ 0&0&0&0&0&2&0&2&0&0&0&0&0&2&0&2\\ 0&0&0&0&2&2&2&2&0&0&0&0&2&2&2&2\\ 0&0&0&0&0&2&0&2&0&0&0&0&0&2&0&2\\ 0&0&0&0&2&2&2&2&0&0&0&0&2&2&2&2\\ 0&2&0&2&0&2&0&2&0&2&0&2&0&2&0&2\\ 2&2&2&2&2&2&2&2&2&2&2&2&2&2&2&2\\ 0&2&0&2&0&2&0&2&0&2&0&2&0&2&0&2\\ 2&2&2&2&2&2&2&2&2&2&2&2&2&2&2&2 \end{bmatrix}$$

$\forall k = \{1,4\}$

$$
H_{2\times3\times2;k}=\begin{bmatrix}
0&0&0&2&0&0&0&2&0&0&0&2&0&0&0&2\\
0&0&2&2&0&0&2&2&0&0&2&2&0&0&2&2\\
0&2&0&2&0&2&0&2&0&2&0&2&0&2&0&2\\
2&2&2&2&2&2&2&2&2&2&2&2&2&2&2&2\\
0&0&0&2&0&0&0&2&0&0&0&2&0&0&0&2\\
0&0&2&2&0&0&2&2&0&0&2&2&0&0&2&2\\
0&2&0&2&0&2&0&2&0&2&0&2&0&2&0&2\\
2&2&2&2&2&2&2&2&2&2&2&2&2&2&2&2\\
0&0&0&2&0&0&0&2&0&0&0&2&0&0&0&2\\
0&0&2&2&0&0&2&2&0&0&2&2&0&0&2&2\\
0&2&0&2&0&2&0&2&0&2&0&2&0&2&0&2\\
2&2&2&2&2&2&2&2&2&2&2&2&2&2&2&2\\
0&0&0&2&0&0&0&2&0&0&0&2&0&0&0&2\\
0&0&2&2&0&0&2&2&0&0&2&2&0&0&2&2\\
0&2&0&2&0&2&0&2&0&2&0&2&0&2&0&2\\
2&2&2&2&2&2&2&2&2&2&2&2&2&2&2&2
\end{bmatrix}
$$

$\forall k=\{1,4\}$

when $H_{2\times3\times2;k}$ to be a $2^2\times2^2$ matrix,then

$$\left[H_{3\times3\times2;(k_1,k_2)}\right]_b^a=\sum_{i=1}^{2^2}\left[H_{2\times3\times2;k_1}\right]_i^a\otimes\left[H_{2\times3\times2;k_2}\right]_b^i$$

We find the matrix $\left[H_{3\times3\times2;k_1}\right]_b^a,\forall a,b\in\{1,2,3,4\}$ is the same for any $k_1,k_2\in\{1,4\}$, since $\left[H_{2\times3\times2;k}\right]_b^a,\forall a,b\in\{1,2,3,4\}$ is the same for any $k\in\{1,4\}$

Now, We denoted $H_{3\times3\times2;(k_1,k_2)}=\left[H_{3\times3\times2;(k_1,k_2);(l_1,l_2)}\right],\forall l_i\in\{1,4\}$ which l_i is be taken by Z-shaped so

$$H_{3\times 3\times 2;(k_1,k_2);(l_1,l_2)} = \begin{bmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&16\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&16&16\\
0&0&0&0&0&0&0&0&0&0&0&0&0&16&0&16\\
0&0&0&0&0&0&0&0&0&0&0&0&16&16&16&16\\
0&0&0&0&0&0&0&0&0&0&0&16&0&0&0&16\\
0&0&0&0&0&0&0&0&0&0&16&16&0&0&16&16\\
0&0&0&0&0&0&0&0&0&16&0&16&0&16&0&16\\
0&0&0&0&0&0&0&0&16&16&16&16&16&16&16&16\\
0&0&0&0&0&0&0&16&0&0&0&0&0&0&0&16\\
0&0&0&0&0&0&16&16&0&0&0&0&0&0&16&16\\
0&0&0&0&0&16&0&16&0&0&0&0&0&16&0&16\\
0&0&0&0&16&16&16&16&0&0&0&0&16&16&16&16\\
0&0&0&16&0&0&0&16&0&0&0&16&0&0&0&16\\
0&0&16&16&0&0&16&16&0&0&16&16&0&0&16&16\\
0&16&0&16&0&16&0&16&0&16&0&16&0&16&0&16\\
16&16&16&16&16&16&16&16&16&16&16&16&16&16&16&16
\end{bmatrix},$$

$\forall k_1,k_2,l_1,l_2 = \{1,,4\}$

Finally, denote $S_{m\times n\times 2;(k_1,\cdots,k_{m-1});(l_1,\cdots,l_{n-1})} = H_{m\times n\times 2;(k_1,\cdots,k_{m-1});(l_1,\cdots,l_{n-1})}$

So we can calculate the maximum eigenvalue of $S_{3\times 3\times 2;(k_1,k_2);(l_1,l_2)}$ $\forall k_1,k_2,l_1,l_2 = \{1,4\}$ by using matlab, then the answer is 109.6656 and log109.6656=2.04

Hence, $h(B) \geq \dfrac{2.04}{2\times 2\times 1} = 0.51$

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