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C_{11} CONTRACTIONS ARE REFLEXIVE. II

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ABSTRACT. It has been shown previously by the author that any completely nonunitary C_{11} contraction with finite defect indices is reflexive. In this note we show that this is true even without the completely nonunitary assumption.

Recall that a bounded linear operator T on a complex, separable Hilbert space is *reflexive* if Alg Lat T = Alg T, where Alg Lat T and Alg T denote, respectively, the weakly closed algebra of operators which leave invariant every invariant subspace of T and the weakly closed algebra generated by T and I. It was shown in **[9]** that every completely nonunitary (c.n.u.) C_{11} contraction with finite defect indices is reflexive and it was conjectured that the same is true for arbitrary C_{11} contractions. In this note we move one step closer to establish this conjecture by dropping the completely nonunitary assumption, i.e. we prove that any C_{11} contraction with finite defect indices (a direct sum of a unitary operator and a c.n.u. C_{11} contraction) is reflexive. Note that this is not entirely trivial since in general we do not know whether the direct sum of two reflexive operators is reflexive (cf. **[3**, Question 2]).

In the discussion below we will follow the notations established in **[9]**. We also need some more facts from **[10]**. Let T be a c.n.u. C_{11} contraction with defect indices $d_T = d_{T^*} \equiv n < \infty$. Then T can be considered as defined on $H \equiv [H_n^2 \oplus \overline{\Delta L_n^2}] \ominus \{\Theta_T w \oplus \Delta w: w \in H_n^2\}$ by $T(f \oplus g) = P(e^{it}f \oplus e^{it}g)$ for $f \oplus g \in H$, where Θ_T denotes the characteristic function of T, $\Delta = (I - \Theta_T^*\Theta_T)^{1/2}$ and P denotes the (orthogonal) projection onto H. Since Θ_T is outer from both sides, there exists an outer scalar multiple δ of Θ_T (cf. **[7**, p. 217]). Let Ω be a contractive analytic function such that $\Omega\Theta_T = \Theta_T\Omega = \delta I$. Let U denote the operator of multiplication by e^{it} on $\overline{\Delta_* L_n^2}$, where $\Delta_* = (I - \Theta_T\Theta_T^*)^{1/2}$, and let $X: H \to \overline{\Delta_* L_n^2}$, $Y: \overline{\Delta_* L_n^2} \to H$ be the operators defined by $X(f \oplus g) = -\Delta_* f + \Theta_T g$ for $f \oplus g \in H$ and $Yu = P(0 \oplus \Omega u)$ for $u \in \overline{\Delta_* L_n^2}$. Then X and Y are quasi-affinities which intertwine T and U and satisfy $YX = \delta(T)$ and $XY = \delta(U)$ (cf. **[10**, Lemma 2.1]).

Any absolutely continuous unitary operator U_a on K is, by the spectral theorem, unitarily equivalent to the operator of multiplication by e^{it} on $L^2(E_1) \oplus \cdots \oplus L^2(E_k)$, where k may be infinite and $E_1, \ldots, E_k$ are Borel subsets of the

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unit circle C with $E_1 \supseteq E_2 \supseteq \cdots \supseteq E_k$. In particular, U is unitarily equivalent to the operator of multiplication by e^{it} on $L^2(F_1) \oplus \cdots \oplus L^2(F_n)$, where $C \supseteq F_1 \supseteq F_2 \supseteq \cdots \supseteq F_n$. Let Z_1: $K \to L^2(E_1) \oplus \cdots \oplus L^2(E_k)$ and Z_2: $\overline{\Delta_* L_n^2} \to L^2(F_1) \oplus \cdots \oplus L^2(F_n)$ be the implementing unitary transformations.

Now we are ready to start. In the following lemmas we consider a C_{11} contraction with finite defect indices whose unitary part is absolutely continuous. We first find operators in its double commutant. Lemma 2 deals with the reflexivity and the double commutant property.

LEMMA 1. *Let $S = U_a \oplus T$, where U_a is an absolutely continuous unitary operator on K and T is a c.n.u. C_{11} contraction with finite defect indices on H. Then $\{S\}'' = \{\psi(U_a) \oplus P[\begin{smallmatrix} A & 0 \\ B & \psi \end{smallmatrix}]$: $\psi \in L^\infty$, $A\Theta_T = \Theta_T A_0$ and $B\Theta_T + \psi\Delta = \Delta A_0$ for some bounded analytic function $A_0\}$.*

PROOF. For any $V \in \{S\}''$, $V = V_1 \oplus V_2$ where $V_1 \in \{U_a\}''$ and $V_2 \in \{T\}''$. Hence

$$V_1 = \psi_1(U_a) \quad \text{and} \quad V_2 = P\begin{bmatrix} A & 0 \\ B & \psi_2 \end{bmatrix},$$

where $\psi_1, \psi_2 \in L^\infty$ and A, B satisfy $A\Theta_T = \Theta_T A_0$ and $B\Theta_T + \psi_2\Delta = \Delta A_0$ for some bounded analytic function A_0 (cf. [9, Lemma 2]). Let $W = \delta(U_a)V_1 \oplus XV_2Y \equiv W_1 \oplus W_2$. For any $u \in \overline{\Delta_* L_n^2}$, we have

$$W_2 u = XV_2Yu = XP\begin{bmatrix} A & 0 \\ B & \psi_2 \end{bmatrix} P\begin{bmatrix} 0 \\ \Omega u \end{bmatrix}$$

$$= XP\begin{bmatrix} 0 \\ \psi_2\Omega u \end{bmatrix} = -\Delta_* 0 + \Theta_T\psi_2\Omega u = \delta\psi_2 u.$$

This shows that $W_2 = (\delta\psi_2)(U)$. Hence $W = (\delta\psi_1)(U_a) \oplus (\delta\psi_2)(U)$. Next we show that $W \in \{U_a \oplus U\}''$. Since $W_1 \in \{U_a\}''$ and $W_2 \in \{U\}''$, we have only to check that (i) any operator Q: $K \to \overline{\Delta_* L_n^2}$ intertwining U_a and U intertwines W_1 and W_2 and (ii) any operator R: $\overline{\Delta_* L_n^2} \to K$ intertwining U and U_a intertwines W_2 and W_1. To prove (i), note that YQ: $K \to H$ intertwines U_a and T. Since $V = V_1 \oplus V_2 \in \{S\}''$, we have $YQV_1 = V_2YQ$. Applying X from the left on both sides, we obtain $XYQV_1 = XV_2YQ$ or $\delta(U)QV_1 = W_2Q$. But $\delta(U)QV_1 = Q\delta(U_a)V_1 = QW_1$. Hence Q intertwines W_1 and W_2, proving (i). (ii) can be proved in a similar fashion. Thus $W \in \{U_a \oplus U\}''$ as asserted and therefore $W = \xi(U_a \oplus U)$ for some $\xi \in L^\infty$. But we already have $W = (\delta\psi_1)(U_a) \oplus (\delta\psi_2)(U)$. It follows that $\xi = \delta\psi_1$ a.e. on E_1 and $\xi = \delta\psi_2$ a.e. on F_1, whence $\psi_1 = \psi_2$ a.e. on $E_1 \cap F_1$. Let ψ in L^∞ be such that $\psi = \psi_1$ a.e. on E_1 and $\psi = \psi_2$ a.e. on F_1. Then $V = \psi(U_a) \oplus P[\begin{smallmatrix} A & 0 \\ B & \psi \end{smallmatrix}]$ as asserted.

For the converse, let $V = V_1 \oplus V_2 = \psi(U_a) \oplus P[\begin{smallmatrix} A & 0 \\ B & \psi \end{smallmatrix}]$ for some $\psi \in L^\infty$. Again, we consider $W = \delta(U_a)V_1 \oplus XV_2Y$. As before, it can be shown that $W = (\delta\psi)(U_a \oplus U) \in \{U_a \oplus U\}''$. Since $V_1 \in \{U_a\}''$ and $V_2 \in \{T\}''$ (cf. [9, Lemma 2]), to show that $V \in \{S\}''$ we have to check (i) any operator Q: $K \to H$ intertwining U_a and T intertwines V_1 and V_2 and (ii) any operator R: $H \to K$

intertwining T and U_a intertwines V_2 and V_1. Here we only prove (i). Since $XQ: K \to \overline{\Delta_* L^2_n}$ intertwines U_a and U and $W \in \{U_a \oplus U\}''$, we have $XQ\delta(U_a)V_1 = XV_2YXQ$. It follows from the injectivity of X that $Q\delta(U_a)V_1 = V_2YXQ$. But $V_2YXQ = V_2\delta(T)Q = V_2Q\delta(U_a)$ and hence we have $QV_1\delta(U_a) = V_2Q\delta(U_a)$. Since $\delta(U_a)$ has dense range, we conclude that $QV_1 = V_2Q$ as asserted. Similarly for (ii). Hence $V \in \{S\}''$, completing the proof.

LEMMA 2. *Let* $S = U_a \oplus T$ *be as in Lemma* 1.

(1) *If* $E_1 \cup F_1 \neq C$ *a.e., then* Alg Lat S = Alg $S = \{S\}''$.

(2) *If* $E_1 \cup F_1 = C$ *a.e., then* Alg Lat S = Alg $S = \{\varphi(S): \varphi \in H^\infty\}$.

In particular, S *is reflexive and* $\{S\}''$ = Alg S *if and only if* $E_1 \cup F_1 \neq C$ *a.e.*

PROOF. (1) In this case, it suffices to show that Alg Lat $S \subseteq \{S\}''$ and $\{S\}'' \subseteq$ Alg S. To prove the former, let $V \in$ Alg Lat S. Then $V = V_1 \oplus V_2$, where $V_1 \in$ Alg Lat U_a = Alg U_a and $V_2 \in$ Alg Lat T = Alg T since U_a and T are both reflexive (cf. [6] and [9]). Hence

$$V_1 = \psi_1(U_a) \quad \text{and} \quad V_2 = P\begin{bmatrix} A & 0 \\ B & \psi_2 \end{bmatrix},$$

where $\psi_1, \psi_2 \in L^\infty$ and A, B satisfy $A\Theta_T = \Theta_T A_0$ and $B\Theta_T + \psi_2\Delta = \Delta A_0$ for some A_0.

Consider the subspace

$$\mathfrak{M} = \left\{Z_1^{-1}(\chi_{E_1} f \oplus \cdots \oplus \chi_{E_k} f) \oplus Z_2^{-1}(\chi_{F_1} f \oplus \cdots \oplus \chi_{F_n} f): f \in L^2\right\}$$

of $K \oplus \overline{\Delta_* L^2_n}$. Note that $\mathfrak{M}$ is a (closed) invariant subspace for $U_a \oplus U$. Let $\mathfrak{N} = \overline{(\delta(U_a) \oplus Y)\mathfrak{M}}$. Then $\mathfrak{N}$ is invariant for S and hence $\overline{V\mathfrak{N}} \subseteq \mathfrak{N}$. Applying $I \oplus X$ on both sides, we obtain $\overline{(I \oplus X)V\mathfrak{N}} \subseteq \overline{(I \oplus X)\mathfrak{N}}$. But

$$\begin{aligned}\overline{(I \oplus X)V\mathfrak{N}} &= \overline{(I \oplus X)(V_1 \oplus V_2)(\delta(U_a) \oplus Y)\mathfrak{M}} \\ &= \overline{(V_1\delta(U_a) \oplus XV_2Y)\mathfrak{M}} = \overline{(\psi_1(U_a) \oplus \psi_2(U))\delta(U_a \oplus U)\mathfrak{M}},\end{aligned}$$

where the last equality was proved in Lemma 1, and

$$\overline{(I \oplus X)\mathfrak{N}} = \overline{(I \oplus X)(\delta(U_a) \oplus Y)\mathfrak{M}} = \overline{\delta(U_a \oplus U)\mathfrak{M}}.$$

Since δ is an outer function, $\delta(U_a \oplus U)|\mathfrak{M}$ is a quasi-affinity on $\mathfrak{M}$ (cf. [10, Lemma 2.3]). We conclude from above that $\overline{(\psi_1(U_a) \oplus \psi_2(U))\mathfrak{M}} \subseteq \mathfrak{M}$. Hence for any $f \in L^2$, there exists $\psi \in L^2$ such that $\chi_{E_1}\psi_1 f = \chi_{E_1}\psi$ a.e. and $\chi_{F_1}\psi_2 f = \chi_{F_1}\psi$ a.e. In particular, for $f \equiv 1$ this implies that $\psi_1 = \psi$ a.e. on E_1 and $\psi_2 = \psi$ a.e. on F_1. Therefore, $V = \psi(U_a) \oplus P[\begin{smallmatrix} A & 0 \\ B & \psi \end{smallmatrix}] \in \{S\}''$ by Lemma 1.

Next we show that $\{S\}'' \subseteq$ Alg S. Let $V \in \{S\}''$. By [5, Theorem 7.1], it suffices to show that Lat $S^{(n)} \subseteq$ Lat $V^{(n)}$ for any $n > 1$, where

$$S^{(n)} = \underbrace{S \oplus \cdots \oplus S}_{n} \quad \text{and} \quad V^{(n)} = \underbrace{V \oplus \cdots \oplus V}_{n}.$$

Since $S^{(n)}$ is an operator of the same type as S and $V^{(n)} \in \{S^{(n)}\}''$, it is clear that we have only to check for $n = 1$, i.e. Lat $S \subseteq$ Lat V. To prove this, let $\mathfrak{M} \in$ Lat S. By Lemma 1, $V = V_1 \oplus V_2 = \psi(U_a) \oplus P[\begin{smallmatrix} A & 0 \\ B & \psi \end{smallmatrix}]$ for some $\psi \in L^\infty$ and A, B.

Let $W = \delta(U_a)V_1 \oplus XV_2Y$. As proved in Lemma 1, $W = (\delta\psi)(U_a \oplus U) \in \{U_a \oplus U\}''$. Since by our assumption $E_1 \cup F_1 \neq C$ a.e., every invariant subspace for $U_a \oplus U$ is bi-invariant, i.e. invariant for any operator in $\{U_a \oplus U\}''$. In particular, $\mathfrak{N} \equiv \overline{(I \oplus X)\mathfrak{M}}$ is invariant for W, i.e. $\overline{W\mathfrak{N}} \subseteq \mathfrak{N}$. Applying $\delta(U_a) \oplus Y$ on both sides, we obtain $\overline{(\delta(U_a) \oplus Y)W\mathfrak{N}} \subseteq \overline{(\delta(U_a) \oplus Y)\mathfrak{N}}$. But

$$\begin{aligned}\overline{(\delta(U_a) \oplus Y)W\mathfrak{N}} &= \overline{(\delta(U_a)V_1\delta(U_a) \oplus YXV_2YX)\mathfrak{M}} \\ &= \overline{(V_1 \oplus V_2)\delta(U_a \oplus T)^2\mathfrak{M}} = \overline{(V_1 \oplus V_2)\mathfrak{M}},\end{aligned}$$

where the last equality follows from the fact that $\delta(U_a \oplus T)|\mathfrak{M}$ is a quasi-affinity on $\mathfrak{M}$. (This can be proved in the same fashion as [10, Lemma 2.3].) On the other hand, $\overline{(\delta(U_a) \oplus Y)\mathfrak{N}} = \overline{\delta(U_a \oplus T)\mathfrak{M}} = \mathfrak{M}$. We conclude that $\overline{(V_1 \oplus V_2)\mathfrak{M}} \subseteq \mathfrak{M}$ whence $\mathfrak{M} \in \operatorname{Lat} V$. This completes the proof of (1).

(2) As in (1), let

$$V = \psi_1(U_a) \oplus P\begin{bmatrix} A & 0 \\ B & \psi_2 \end{bmatrix}$$

be an operator in Alg Lat S. This time we consider the subspace

$$\mathfrak{N} = \{Z_1^{-1}(\chi_{E_1}f \oplus \cdots \oplus \chi_{E_k}f) \oplus Z_2^{-1}(\chi_{F_1}f \oplus \cdots \oplus \chi_{F_n}f) : f \in H^2\}$$

of $K \oplus \overline{\Delta_* L_n^2}$. Since $E_1 \cup F_1 = C$ a.e., it is easy to check that $\mathfrak{N}$ is closed and invariant for $U_a \oplus U$. As in the first part of (1), we derive that for any $f \in H^2$ there exists $\varphi \in H^2$ such that $\chi_{E_1}\psi_1 f = \chi_{E_1}\varphi$ a.e. and $\chi_{F_1}\psi_2 f = \chi_{F_1}\varphi$ a.e. Hence for $f \equiv 1$, we have $\psi_1 = \varphi$ a.e. on E_1 and $\psi_2 = \varphi$ a.e. on F_1. Therefore $V = \varphi(U_a) \oplus P[\begin{smallmatrix} A & 0 \\ B & \varphi \end{smallmatrix}]$. Using the fact that $\{P(0 \oplus g) : g \in \overline{\Delta L_n^2}\}$ is dense in H (cf. [9, proof of Lemma 2]), we can easily show that $P[\begin{smallmatrix} A & 0 \\ B & \varphi \end{smallmatrix}] = \varphi(T)$. Hence $V = \varphi(U_a \oplus T) = \varphi(S)$, completing the proof.

Now comes our main result.

Theorem 3. *Any C_{11} contraction S with finite defect indices is reflexive. Moreover, $\{S\}'' = \operatorname{Alg} S$ if and only if $E_1 \cup F_1 \neq C$ a.e.*

Proof. Let $S = U_s \oplus U_a \oplus T$ on $L \oplus K \oplus H$ be such that U_s and U_a are singular and absolutely continuous unitary operators, respectively, and T is a c.n.u. C_{11} contraction (cf. [7, p. 9] and [4]). We first show that $\operatorname{Alg} U_s \oplus \operatorname{Alg}(U_a \oplus T) = \operatorname{Alg} S$. By [5, Theorem 7.1], this is equivalent to $\operatorname{Lat} U_s^{(n)} \oplus \operatorname{Lat}(U_a \oplus T)^{(n)} = \operatorname{Lat} S^{(n)}$ for all $n \geqslant 1$. Since $S^{(n)} = U_s^{(n)} \oplus (U_a \oplus T)^{(n)}$ is of the same type as $S = U_s \oplus (U_a \oplus T)$, it suffices to check for $n = 1$, i.e. $\operatorname{Lat} U_s \oplus \operatorname{Lat}(U_a \oplus T) = \operatorname{Lat} S$. Let $\mathfrak{M} \in \operatorname{Lat} S$. We can decompose the $C_{1\cdot}$ contraction $S|\mathfrak{M}$ as $S|\mathfrak{M} = S_1 \oplus S_2 \oplus S_3$ on $\mathfrak{M} = \mathfrak{M}_1 \oplus \mathfrak{M}_2 \oplus \mathfrak{M}_3$, where S_1 and S_2 are singular and absolutely continuous unitary operators and S_3 is a c.n.u. $C_{1\cdot}$ contraction. Note that $\mathfrak{M}_1$ and $\mathfrak{M}_2 \oplus \mathfrak{M}_3$ are invariant for S. To complete the proof, we have to show that $\mathfrak{M}_1 \subseteq L$ and $\mathfrak{M}_2 \oplus \mathfrak{M}_3 \subseteq K \oplus H$.

Let W be the operator of multiplication by e^{it} on $L_n^2 \oplus \overline{\Delta L_n^2}$. Then $Z \equiv U_s \oplus U_a \oplus W$ is the minimal unitary dilation of S. It follows that Z is a unitary dilation of

$S_2 \oplus S_3$. There exists a reducing subspace $\mathcal{L}$ for Z such that $Z|\mathcal{L}$ is the minimal unitary dilation of $S_2 \oplus S_3$ (cf. [7, p. 13]). Since S_2 is absolutely continuous and S_3 is c.n.u., $Z|\mathcal{L}$ must be absolutely continuous (cf. [7, p. 84]). On the other hand, we have $\operatorname{Lat} U_s \oplus \operatorname{Lat}(U_a \oplus W) = \operatorname{Lat} Z$ (cf. [2, Lemma 1]). Hence we infer that $\mathcal{L} \subseteq K \oplus (L_n^2 \oplus \overline{\Delta L_n^2})$. Therefore $\mathfrak{M}_2 \oplus \mathfrak{M}_3 \subseteq \mathcal{L} \cap (L \oplus K \oplus H) \subseteq K \oplus H$. Along the same line, an even simpler argument can be applied to $\mathfrak{M}_1$ and shows that $\mathfrak{M}_1 \subseteq L$. Thus we have $\operatorname{Alg} U_s \oplus \operatorname{Alg}(U_a \oplus T) = \operatorname{Alg} S$.

If $V \in \operatorname{Alg} \operatorname{Lat} S$, then $V = V_1 \oplus V_2$ where $V_1 \in \operatorname{Alg} \operatorname{Lat} U_s = \operatorname{Alg} U_s$ and $V_2 \in \operatorname{Alg} \operatorname{Lat}(U_a \oplus T) = \operatorname{Alg}(U_a \oplus T)$ by Lemma 2. From above we conclude that $V \in \operatorname{Alg} S$ whence S is reflexive. Since $\{U_s\}'' = \operatorname{Alg} U_s$ (cf. [8]) and $\{U_s\}'' \oplus \{U_a \oplus T\}'' = \{S\}''$ (cf. [1, Proposition 1.3]), Lemma 2 implies that $\{S\}'' = \operatorname{Alg} S$ if and only if $E_1 \cup F_1 \neq C$ a.e.

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