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碩士論文

概似比檢定統計量的指數加權移動平均監控一般線性資料

An Exponentially Weighted Moving Average Control Chart Based on Likelihood Ratio Test Statistics for Monitoring General Linear Profiles

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摘要

當製程品質可藉由變數間之線性關係來衡量時，本文提出統計製程管制可用於監控工業實務上。首先介紹一些概似比檢定統計量的性質。其次，我們提出概似比檢定統計量的指數加權移動平均來監控一般線性資料。最後，我們用模擬的方法來說明我們提出的這個方法之表現並討論它的優缺點。

關鍵字：概似比檢定統計量、指數加權移動平均、一般線性資料

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Abstract

When the quality of a process can be characterized by general linear profiles, a statistical process control scheme that can be used in industrial practice is proposed in the paper. First, some properties of the likelihood ratio test statistics are introduced. Next, an exponentially weighted moving average control chart based on likelihood ratio test statistics for monitoring general linear profiles is proposed. Finally, the performance of the proposed methodology is investigated through a simulation study to show its strength and weakness.

KEY WORDS: Likelihood ratio test statistics, Exponentially weighted moving average, General linear profiles

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黃偉振　謹誌于

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Contents

List of Tables

1 Introduction

1.1 Motivation

Quality improvement is a key factor for keeping competitiveness in the international market. Statistical process control (SPC) is used for applications of industrial processes by statistical methods that can help practitioner to find out the problems in processes. Roberts (1959) proposed an exponentially weighted moving average (EWMA) control chart to detect a small shift in the process mean, and it nowadays is a widely studied and accepted alternative to the traditional Shewhart $\bar{X}$ control chart (Shewhart, 1931). The likelihood ratio (LR) test is commonly used to test hypotheses for a parametric family in statistical literature because of its high power in detecting a large shift or for a large sample size. In many situations, the quality of a process may be better characterized and summarized by the relationship between the response variable and one or more explanatory variables. Thus, an EWMA control chart based on LR test statistics for monitoring general linear profiles is proposed in the paper and then some properties of the proposed methodology are investigated.

1.2 Literature Review

SPC refers to statistical methods which are extensively used to monitor and improve the quality of industrial processes and others operations. Most of the studies on SPC are focused on the charting skill which is used to monitor processes in order to distinguish special significant reasons of variation from general allocation reasons of variation in processes. In the Phase I stage, a set of process data is gathered to construct trial control limits that determine whether or not the process has been in control over the period of time, and then to model the in-control process so that reliable control limits of the control chart can be established for the later Phase II stage. In the Phase II stage, process data are compared with a pre-established standard from the previous Phase I stage to determine whether the process is in control or not.

Shewhart (1931) proposed a Shewhart $\bar{X}$ control chart which is briefly introduced as follows: At time j (≥ 1), let $X_{j1}, X_{j2}, \ldots, X_{jn}$ be *i.i.d.* $N(\mu_j, \sigma^2)$ observations, where μ_j denotes the unknown process mean at time j and σ the known positive process standard deviation. Set $\bar{X}_j \equiv \sum_{i=1}^{n} X_{ji}/n$ for $j \geq 1$. Then an out-of-control signal occurs at time j (≥ 1) if $\sqrt{n}|\bar{X}_j - \mu|/\sigma > L$, where μ denotes the known in-control process mean and L is chosen to achieve a specified in-control average run length (ARL).

Page (1954) proposed a cumulative sum (CUSUM) control chart which is briefly introduced as follows: Let $X_1, X_2, \ldots$ be independent observations such that $X_j \sim N(\mu_j, \sigma^2)$, where μ_j denotes the unknown process mean at time j and σ the known positive process standard deviation. For $j \geq 1$, two statistics C_j^+ and C_j^- are iteratively defined as

$$C_j^+ \equiv \max\{0, C_{j-1}^+ + X_j - (\mu + K)\}$$

and

$$C_j^- \equiv \max\{0, C_{j-1}^- - X_j + (\mu - K)\},$$

where $C_0^+ = C_0^- \equiv 0$, μ denotes the known in-control process mean, and K is one half of the specified shift in process mean that should be quickly detected by the scheme. Then an out-of-control signal occurs at time j (≥ 1) if $\max\{C_j^+, C_j^-\} \geq H\sigma$, where H (> 0) is chosen to achieve a specified in-control ARL.

Roberts (1959) proposed an EWMA, originally called geometric moving average, control chart which is briefly introduced as follows: At time j (≥ 1), let $X_{j1}, X_{j2}, \ldots, X_{jn}$ be *i.i.d.* $N(\mu_j, \sigma^2)$ observations, where μ_j denotes the unknown process mean at time j and σ the known positive process standard deviation. The EWMA sequence is

$$W_j \equiv (1-\lambda)W_{j-1} + \lambda(\bar{X}_j - \mu) = \lambda \sum_{k=0}^{j-1} (1-\lambda)^k (\bar{X}_{j-k} - \mu), \quad j = 1, 2, \ldots,$$

where $W_0 \equiv 0$, μ denotes the known in-control process mean, and λ is a smoothing parameter chosen in $(0, 1]$. Observe that the standard deviation of W_j is

$$\sigma_j = \sqrt{\frac{\lambda[1-(1-\lambda)^{2j}]}{n(2-\lambda)}}\,\sigma \to \sqrt{\frac{\lambda}{n(2-\lambda)}}\,\sigma$$

as $j \to \infty$. Then out-of-control signal occurs at time j (≥ 1) if $|W_j|/\sigma_j > L$, where L is chosen to achieve a specified in-control ARL.

An EWMA control chart is typically designed in the Phase II stage for a manufacturing process. Stoumbos *et al.* (2003) investigated the performance of Shewhart, EWMA, and CUSUM control charts for detecting sustained shifts and drifts in the process mean and variance. In practice, the in-control process distribution is rarely known exactly, and thus control charts are usually constructed using an approximate in-control process distribution estimated from some historical in-control process data. Jones *et al.* (2001) utilized a numerical procedure to study the run-length (RL) distribution of an EWMA control chart using estimated in-control process parameters. Jensen *et al.* (2006) provided a review of the literature that considered the effect of in-control process parameter estimation on control charts and concluded that the influence of in-control process parameter estimation on control charts should not be ignored.

In most SPC applications, it is assumed that the quality of a process can be suitably represented by the joint distribution of quality characteristics. However, in many situations, the quality of a process may be better characterized and summarized by the relationship between the response variable and one or more explanatory variables. Several methods for monitoring linear profiles have been proposed in literature, e.g., Kim *et al.* (2003) proposed a control chart for monitoring simple linear profiles in the known in-control process parameter case, and Zou *et al.* (2006) proposed a control chart based on the LR test statistics for monitoring simple linear profiles to detect a sustained shift in the unknown in-control process parameter case.

Through the modern technology that allows simultaneously monitoring all key quality characteristics during a manufacturing process, the monitored quality characteristics are generally dependent each other. The purpose of multivariate techniques is to study whether quality characteristics are simultaneously in control or not. Lowry *et al.* (1992) proposed a multivariate exponentially weighted moving average (MEWMA) control chart giving guidelines for designing this easy-to-implement multivariate procedure. The performance of their control chart is similar to that of a multivariate cumulative sum (MCUSUM) control chart (Crosier, 1988) in detecting a small shift in the process mean.

Zou *et al.* (2007) [13] proposed an MEWMA control chart for monitoring general linear profiles in the known in-control process parameter case, which is briefly introduced as follows: The process is called in control at time j (≥ 1) if

$$\boldsymbol{y}_j = X\boldsymbol{\beta} + \sigma\boldsymbol{\varepsilon}_j,$$

where $\boldsymbol{y}_j$ is the $n \times 1$ response vector at time j, X is the known $n \times p$ model matrix of full rank p ($< n$), $\boldsymbol{\beta}$ ($\equiv (\beta_0, \beta_1, ..., \beta_{p-1})^T$) is the known $p \times 1$ vector of real-valued in-control process regression parameters, σ is the known positive in-control process scale parameter, and $\boldsymbol{\varepsilon}_j$s are *i.i.d.* $N_n(\mathbf{0}_{n\times 1}, \boldsymbol{I}_n)$ standardized error vectors. Set $\boldsymbol{Z}_j \equiv (\boldsymbol{Z}_j^T(\boldsymbol{\beta}), \boldsymbol{Z}_j(\sigma))^T$, where $\boldsymbol{Z}_j(\boldsymbol{\beta}) \equiv (\hat{\boldsymbol{\beta}}_j - \boldsymbol{\beta})/\sigma$ and $\boldsymbol{Z}_j(\sigma) \equiv \Phi^{-1}(F_{\chi^2_{n-p}}((n-p)\hat{\sigma}_j^2/\sigma^2))$ with $\hat{\boldsymbol{\beta}}_j \equiv (X^TX)^{-1}X^T\boldsymbol{y}_j$, $\hat{\sigma}_j^2 \equiv (\boldsymbol{y}_j - X\hat{\boldsymbol{\beta}}_j)^T(\boldsymbol{y}_j - X\hat{\boldsymbol{\beta}}_j)/(n-p)$, $\Phi^{-1}(\cdot)$ denoting the inverse of the standard normal cumulative distribution function (c.d.f.), and $F_{\chi^2_{n-p}}(\cdot)$ the c.d.f. of the chi-squared distribution with $n-p$ degrees of freedom. The MEWMA sequence is

$$\boldsymbol{W}_j \equiv \lambda\boldsymbol{Z}_j + (1-\lambda)\boldsymbol{W}_{j-1} = \lambda\sum_{k=0}^{j-1}(1-\lambda)^k\boldsymbol{Z}_{j-k}, \quad j = 1, 2, \ldots,$$

where $\boldsymbol{W}_0 \equiv 0_{(p+1)\times 1}$ and λ is a smoothing parameter chosen in $(0, 1]$. Observe that the in-control covariance matrix of W_j is

$$\frac{\lambda[1-(1-\lambda^{2j})]}{2-\lambda}\Sigma \to \frac{\lambda}{2-\lambda}\Sigma$$

as $j \to \infty$, where

$$\mathbf{\Sigma} = \begin{pmatrix} (X^T X)^{-1} & 0_{p\times 1} \\ 0_{p\times 1}^T & 1 \end{pmatrix}.$$

Then an out-of-control signal occurs at time j (≥ 1) if $(2-\lambda)\boldsymbol{W}_j^T \mathbf{\Sigma}^{-1} \boldsymbol{W}_j/\lambda > L$, where L is chosen to achieve a specified in-control ARL.

Zou *et al.* (2007) [16] proposed a self-starting control chart based on recursive residuals for monitoring simple linear profiles to detect a sustained shift in the process intercept, slope, or standard deviation. Although current SPC methods mostly focus on detecting a sustained shift in the process mean, time-varying shifts in the process mean frequently occur in industrial applications. Thus, Zou *et al.* (2009) investigated several control charts for detecting drifts in the process mean. Recently, Zou *et al.* (2010) proposed an EWMA control chart based on the LR test statistics for monitoring the process mean and variance.

1.3 Outline

The paper is organized as follows. In Section 2, an EWMA control chart based on LR test statistics for monitoring general linear profiles is proposed and then some properties of the proposed monitoring scheme are investigated. In Section 3, a simulation study is presented to illustrate the proposed methodology. In Section 4, performance comparisons and conclusions are given. Possible future work is discussed in Section 5.

2 An EWMA Control Chart Based on LR Test Statistics for Monitoring General Linear Profiles

In this section, an EWMA control chart based on LR test statistics for monitoring general linear profiles is proposed and then some properties of the proposed methodology are investigated.

2.1 Model

In this subsection, consider the following model for the purpose of monitoring general linear profiles: Suppose that

$$\boldsymbol{y}_j = X_j\boldsymbol{\beta}_j + \sigma_j\boldsymbol{\varepsilon}_j, \quad j = 1, 2, ..., \tag{2.1}$$

where $\boldsymbol{y}_j$ is the $n_j \times 1$ response vector at time j, X_j is the known $n_j \times p$ model matrix of full rank p $(< n_j)$ at time j, $\boldsymbol{\beta}_j$ $(\equiv (\beta_{j0}, \beta_{j1}, ..., \beta_{j,p-1})^T)$ is the unknown $p \times 1$ vector of real-valued process regression parameters at time j, σ_j is the unknown positive process scale parameter at time j, and $\boldsymbol{\varepsilon}_j$s are independent $N_{n_j}(\mathbf{0}_{n_j\times 1}, \boldsymbol{I}_{n_j})$ standardized error vectors. Set $\boldsymbol{\theta}_j \equiv (\boldsymbol{\beta}_j^T, \sigma_j)^T$ $(\in \mathcal{R}^p \times (0, \infty) \equiv \Theta)$ for $j \geq 1$. Then $\boldsymbol{\theta}_j$ is the process parameter vector at time j for $j \geq 1$. For $j \geq 1$, the process is called in control at time j if $\boldsymbol{\theta}_j = \boldsymbol{\theta}$, where $\boldsymbol{\theta}$ $(\equiv (\boldsymbol{\beta}^T, \sigma)^T \equiv (\beta_0, \beta_1, \ldots, \beta_{p-1}, \sigma)^T \in \Theta)$ is the in-control process parameter vector.

When $\boldsymbol{\theta}$ is unknown, it is assumed that the historical in-control process data $\{X_0, \boldsymbol{y}_0\}$ are available such that

$$\boldsymbol{y}_0 = X_0\boldsymbol{\beta} + \sigma\boldsymbol{\varepsilon}_0, \tag{2.2}$$

where $\boldsymbol{y}_0$ is the $n_0 \times 1$ response vector, X_0 is the known $n_0 \times p$ model matrix of full rank p $(< n_0)$, and $\boldsymbol{\varepsilon}_0$ is an $N_{n_0}(\mathbf{0}_{n_0\times 1}, \boldsymbol{I}_{n_0})$ standardized error vector independent of $\{\boldsymbol{\varepsilon}_j : j \geq 1\}$.

2.2 Known and Unconstrained Case

In this subsection, consider the case where the process parameter vector at time j (≥ 1) may be different from the known in-control process parameter vector. For $j \geq 1$, the joint probability density function (p.d.f.) of $\boldsymbol{y}_j$ is

$$\begin{aligned} f(\boldsymbol{y}_j; \boldsymbol{\theta}_j) &= \frac{1}{(2\pi)^{n_j/2}\sigma_j^{n_j}} \exp\left\{-\frac{\|\boldsymbol{y}_j - X_j\boldsymbol{\beta}_j\|^2}{2\sigma_j^2}\right\} \\ &= \frac{1}{(2\pi)^{n_j/2}\sigma_j^{n_j}} \exp\left\{-\frac{(\hat{\boldsymbol{\beta}}_j - \boldsymbol{\beta}_j)^T X_j^T X_j(\hat{\boldsymbol{\beta}}_j - \boldsymbol{\beta}_j) + n_j\hat{\sigma}_j^2}{2\sigma_j^2}\right\}, \end{aligned}$$

where $\hat{\boldsymbol{\beta}}_j \equiv (X_j^T X_j)^{-1} X_j^T \boldsymbol{y}_j$ and $\hat{\sigma}_j \equiv \|[\boldsymbol{I}_{n_j} - X_j(X_j^T X_j)^{-1} X_j^T]\boldsymbol{y}_j\| / \sqrt{n_j}$. Then, for $j \geq 1$, the log-likelihood function for $\boldsymbol{\theta}_j$ is

$$\begin{aligned} \ell_j(\boldsymbol{\theta}_j) &\equiv \log[f(\boldsymbol{y}_j; \boldsymbol{\theta}_j)] \\ &= -\frac{n_j}{2}\log(2\pi) - \frac{n_j}{2}\log(\sigma_j^2) - \frac{(\hat{\boldsymbol{\beta}}_j - \boldsymbol{\beta}_j)^T X_j^T X_j(\hat{\boldsymbol{\beta}}_j - \boldsymbol{\beta}_j) + n_j\hat{\sigma}_j^2}{2\sigma_j^2}. \end{aligned}$$

and thus the maximum log-likelihood is

$$\ell_j(\hat{\boldsymbol{\theta}}_j) = -\frac{n_j}{2} - \frac{n_j}{2}\log(2\pi) - \frac{n_j}{2}\log(\hat{\sigma}_j^2),$$

where $\hat{\boldsymbol{\theta}}_j$ $(\equiv (\hat{\boldsymbol{\beta}}_j^T, \hat{\sigma}_j)^T)$ is the maximum likelihood estimator (MLE) of $\boldsymbol{\theta}_j$. Hence the LR test statistic at time j (≥ 1) is

$$W_j \equiv 2[\ell_j(\hat{\boldsymbol{\theta}}_j) - \ell_j(\boldsymbol{\theta})] = \frac{(\hat{\boldsymbol{\beta}}_j - \boldsymbol{\beta})^T X_j^T X_j (\hat{\boldsymbol{\beta}}_j - \boldsymbol{\beta})}{\sigma^2} + n_j\left[-1 + \frac{\hat{\sigma}_j^2}{\sigma^2} - \log\left(\frac{\hat{\sigma}_j^2}{\sigma^2}\right)\right], \quad (2.3)$$

where

$$\frac{(\hat{\boldsymbol{\beta}}_j - \boldsymbol{\beta})^T X_j^T X_j (\hat{\boldsymbol{\beta}}_j - \boldsymbol{\beta})}{\sigma^2} \sim \frac{\sigma_j^2}{\sigma^2}\chi_p^2\left(\frac{(\boldsymbol{\beta}_j - \boldsymbol{\beta})^T X_j^T X_j (\boldsymbol{\beta}_j - \boldsymbol{\beta})}{\sigma_j^2}\right),$$

$$\frac{\hat{\sigma}_j^2}{\sigma^2} \sim \frac{\sigma_j^2}{\sigma^2}\frac{\chi_{n_j-p}^2}{n_j},$$

and $(\hat{\boldsymbol{\beta}}_j - \boldsymbol{\beta})^T X_j^T X_j (\hat{\boldsymbol{\beta}}_j - \boldsymbol{\beta})/\sigma^2$ is independent of $n_j[\hat{\sigma}_j^2/\sigma^2 - 1 - \log(\hat{\sigma}_j^2/\sigma^2)]$.

When the process is out of control at time j (≥ 1), the distribution of W_j only depends on p, $n_j - p$, σ_j/σ, and $(\boldsymbol{\beta}_j - \boldsymbol{\beta})^T X_j^T X_j (\boldsymbol{\beta}_j - \boldsymbol{\beta})/\sigma_j^2$ $(\equiv \tau_j^2)$, and $W_j \xrightarrow{d} \infty$ as $\min\{n_j$, the minimum eigenvalue of $X_j^T X_j\} \to \infty$, where

$$E_{\boldsymbol{\theta}_j}(W_j) = \tau_j^2\frac{\sigma_j^2}{\sigma^2} + n_j\left[\log\left(\frac{n_j}{2}\right) - \psi\left(\frac{n_j - p}{2}\right) - 1 + \frac{\sigma_j^2}{\sigma^2} - \log\left(\frac{\sigma_j^2}{\sigma^2}\right)\right],$$

and

$$\mathrm{Var}_{\boldsymbol{\theta}_j}(W_j) = E_{\boldsymbol{\theta}_j}(W_j^2) - [E_{\boldsymbol{\theta}_j}(W_j)]^2$$

with $\psi(x) \equiv d\log[\Gamma(x)]/dx$ and $\psi'(x) \equiv d\psi(x)/dx$ for $x > 0$. See Appendix A.1 for some relevant formulas about both $\psi(x)$ and $\psi'(x)$. See Appendix A.2 for $E_{\boldsymbol{\theta}_j}(W_j^2)$ for $j \geq 1$.

When the process is in control at time j (≥ 1), the distribution of W_j only depends on p and $n_j - p$, and

$$n_j \left[-1 + \frac{\hat{\sigma}_j^2}{\sigma^2} - \log\left(\frac{\hat{\sigma}_j^2}{\sigma^2}\right) \right] = \frac{n_j(\hat{\sigma_j^2} - \sigma^2)^2}{2\sigma^4} + O_p\left(\frac{1}{\sqrt{n_j}}\right) \xrightarrow{d} Z^2$$

as $n_j \to \infty$, where $Z \sim N(0,1)$. Then $W_j \xrightarrow{d} \chi^2_{p+1}$ as $n_j \to \infty$, where

$$E_{\boldsymbol{\theta}}(W_j) \equiv E_{\boldsymbol{\theta}_j}(W_j)|_{\boldsymbol{\theta}_j = \boldsymbol{\theta}} = n_j \left[\log\left(\frac{n_j}{2}\right) - \psi\left(\frac{n_j - p}{2}\right) \right] = p + 1 + O\left(\frac{1}{n_j}\right),$$

and

$$\mathrm{Var}_{\boldsymbol{\theta}}(W_j) \equiv \mathrm{Var}_{\boldsymbol{\theta}_j}(W_j)|_{\boldsymbol{\theta}_j = \boldsymbol{\theta}} = n_j \left[n_j \psi'\left(\frac{n_j - p}{2}\right) - 2 \right] = 2(p+1) + O\left(\frac{1}{n_j}\right).$$

See Appendix A.3 for both $E_{\boldsymbol{\theta}}(W_j)$ and $\mathrm{Var}_{\boldsymbol{\theta}}(W_j)$ as $n_j \to \infty$.

For $j \geq 1$, set

$$\bar{W}_j \equiv \frac{W_j - E_{\boldsymbol{\theta}}(W_j)}{\sqrt{\mathrm{Var}_{\boldsymbol{\theta}}(W_j)}}. \tag{2.4}$$

2.3 Known and Constrained Case

In this subsection, consider the case where the process parameter vector at time j (≥ 1) may be different from the known in-control process parameter vector with the constraint $\sigma_j \geq \sigma$. Then the maximum log-likelihood is

$$\ell_j(\boldsymbol{\theta}_j^*) = -\frac{n_j}{2}\log(2\pi) - \frac{n_j}{2}\log(\sigma_j^{*2}) - \frac{n_j\hat{\sigma}_j^2}{2\sigma_j^{*2}},$$

where $\boldsymbol{\theta}_j^*$ $(\equiv (\boldsymbol{\beta}_j^{*T}, \sigma_j^*)^T = (\hat{\boldsymbol{\beta}}_j^T, \max\{\hat{\sigma}_j, \sigma\})^T)$ is the MLE of $\boldsymbol{\theta}_j$. Hence the LR test statistic at time j (≥ 1) is

$$\begin{aligned} W_j^* &\equiv 2[\ell_j(\boldsymbol{\theta}_j^*) - \ell_j(\boldsymbol{\theta})] \\ &= \frac{(\hat{\boldsymbol{\beta}}_j - \boldsymbol{\beta})^T X_j^T X_j (\hat{\boldsymbol{\beta}}_j - \boldsymbol{\beta})}{\sigma^2} + n_j\left[-1 + \frac{\hat{\sigma}_j^2}{\sigma^2} - \log\left(\frac{\hat{\sigma}_j^2}{\sigma^2}\right)\right] \cdot 1_{\{\hat{\sigma}_j > \sigma_j\}}, \end{aligned} \tag{2.5}$$

where $(\hat{\boldsymbol{\beta}}_j - \boldsymbol{\beta})^T X_j^T X_j (\hat{\boldsymbol{\beta}}_j - \boldsymbol{\beta})/\sigma^2$ is independent of $n_j[-1+\hat{\sigma}_j^2/\sigma^2 - \log(\hat{\sigma}_j^2/\sigma^2)] \cdot 1_{\{\hat{\sigma}_j > \sigma_j\}}$.

When the process is out of control at time j (≥ 1), the distribution of W_j^* only depends on p, $n_j - p$, σ_j/σ and τ_j^2, and $W_j^* \xrightarrow{d} \infty$ as $\min\{n_j$, the minimum eigenvalue of $X_j^T X_j\} \to \infty$, where

$$\begin{aligned} E_{\boldsymbol{\theta}_j}(W_j^*) &= (p + \tau_j^2)\frac{\sigma_j^2}{\sigma^2} - n_j\left[1 + \log\left(\frac{\sigma_j^2}{n_j\sigma^2}\right)\right] E(1_{\{H_j \geq a_j\}}) \\ &\quad + \frac{\sigma_j^2}{\sigma^2} E(H_j \cdot 1_{\{H_j \geq a_j\}}) - n_j E(\log(H_j) \cdot 1_{\{H_j \geq a_j\}}), \end{aligned}$$

and

$$\mathrm{Var}_{\boldsymbol{\theta}_j}(W_j^*) = E_{\boldsymbol{\theta}_j}(W_j^{*2}) - [E_{\boldsymbol{\theta}_j}(W_j^*)]^2$$

with $H_j \equiv n_j\hat{\sigma}_j^2/\sigma_j^2 \sim \chi^2_{n_j - p}$, $a_j \equiv n_j\sigma^2/\sigma_j^2$, and $1_{\{H_j > n_j\}}$ denoting the indicator function for $\{H_j > n_j\}$. See Appendix A.4 for both $E_{\boldsymbol{\theta}_j}(W_j^*)$ and $E_{\boldsymbol{\theta}_j}(W_j^{*2})$ in detail.

When the process is in control at time j (≥ 1), the distribution of W_j^* only depends on p and $n_j - p$, and

$$
\begin{aligned}
& n_j \left[-1 + \frac{\hat{\sigma}_j^2}{\sigma^2} - \log\left(\frac{\hat{\sigma}_j^2}{\sigma^2} \right) \right] \cdot 1_{\{\hat{\sigma}_j > \sigma\}} \\
= \; & \frac{n_j(\hat{\sigma}_j^2 - \sigma^2)^2}{2\sigma^4} \cdot 1_{\{\sqrt{n_j}(\hat{\sigma}_j^2 - \sigma^2)/(\sqrt{2}\sigma^2) > 0\}} + O_p\left(\frac{1}{\sqrt{n_j}} \right) \\
\xrightarrow{d} \; & Z^2 \cdot 1_{\{Z > 0\}}
\end{aligned}
$$

as $n_j \to \infty$, where $Z \sim N(0,1)$. Then $W_j^* \xrightarrow{d} \chi_p^2/2 + \chi_{p+1}^2/2$ as $n_j \to \infty$, where

$$
\begin{aligned}
E_{\boldsymbol{\theta}}(W_j^*) \; & \equiv \; E_{\boldsymbol{\theta}_j}(W_j^*)|_{\boldsymbol{\theta}_j = \boldsymbol{\theta}} \\
= \; & p + [n_j \log(n_j) - n_j] E(1_{\{H_j > n_j\}}) + E[(H_j \cdot 1_{\{H_j > n_j\}}) - n_j E(\log(H_j) \cdot 1_{\{H_j > n_j\}}),
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{Var}_{\boldsymbol{\theta}}(W_j^*) \; & \equiv \; \mathrm{Var}_{\boldsymbol{\theta}_j}(W_j^*)|_{\boldsymbol{\theta}_j = \boldsymbol{\theta}} \; = \; E_{\boldsymbol{\theta}}(W_j^{*2}) - (E_{\boldsymbol{\theta}}(W_j^*))^2 \\
= \; & p(2+p) + [n_j \log(n_j) - n_j]^2 E(1_{\{H_j > n_j\}}) + 2[n_j \log(n_j) - n_j] E(H_j \cdot 1_{\{H_j > n_j\}}) \\
& -2n_j [n_j \log(n_j) - n_j] E(\log(H_j) \cdot 1_{\{H_j > n_j\}}) + E(H_j^2 \cdot 1_{\{H_j > n_j\}}) \\
& + n_j^2 E([\log(H_j)]^2 \cdot 1_{\{H_j > n_j\}}) \\
& -2n_j E(H_j \log(H_j) \cdot 1_{\{H_j > n_j\}}) - [E_{\boldsymbol{\theta}}(W_j^*)]^2.
\end{aligned}
$$

See Appendix A.4 for both $E_{\boldsymbol{\theta}}(W_j^*)$ and $\mathrm{Var}_{\boldsymbol{\theta}}(W_j^*)$.

For $j \geq 1$, set

$$
\bar{W}_j^* \equiv \frac{W_j^* - E_{\boldsymbol{\theta}}(W_j^*)}{\sqrt{\mathrm{Var}_{\boldsymbol{\theta}}(W_j^*)}}. \tag{2.6}
$$

2.4 Unknown and Unconstrained Case

In this subsection, consider the case where the process parameter vector at time j (≥ 1) may be different from the unknown in-control process parameter vector. For j (≥ 1), the joint p.d.f. of $(\boldsymbol{y}_0^T, \boldsymbol{y}_j^T)^T$ is

$$\begin{aligned} f(\boldsymbol{y}_0, \boldsymbol{y}_j; \boldsymbol{\theta}, \boldsymbol{\theta}_j) &= f(\boldsymbol{y}_0; \boldsymbol{\theta}) \cdot f(\boldsymbol{y}_j; \boldsymbol{\theta}_j) \\ = \frac{1}{(2\pi)^{n_0/2}\sigma^{n_0}} &\exp\left\{-\frac{\|\boldsymbol{y}_0 - X_0\boldsymbol{\beta}\|^2}{2\sigma^2}\right\} \cdot f(\boldsymbol{y}_j; \boldsymbol{\theta}_j) \\ = \frac{1}{(2\pi)^{n_0/2}\sigma_0^{n_0}} &\exp\left\{-\frac{(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})^T X_0^T X_0(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) + n_0\hat{\sigma}^2}{2\sigma^2}\right\} \cdot f(\boldsymbol{y}_j; \boldsymbol{\theta}_j), \end{aligned}$$

where $\hat{\boldsymbol{\beta}} \equiv (X_0^T X_0)^{-1} X_0^T \boldsymbol{y}_0$ and $\hat{\sigma} \equiv \|[\boldsymbol{I}_{n_0} - X_0(X_0^T X_0)^{-1} X_0^T]\boldsymbol{y}_0\|/\sqrt{n_0}$.

Then, for $j \geq 1$, the log-likelihood function for $(\boldsymbol{\theta}^T, \boldsymbol{\theta}_j^T)^T$ is

$$\begin{aligned} \ell_{0,j}(\boldsymbol{\theta}, \boldsymbol{\theta}_j) &\equiv \log[f(\boldsymbol{y}_0; \boldsymbol{\theta})] + \ell_j(\boldsymbol{\theta}_j) \\ = & -\frac{n_0}{2}\log(2\pi) - \frac{n_0}{2}\log(\sigma^2) - \frac{(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})^T X_0^T X_0(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) + n_0\hat{\sigma}^2}{2\sigma^2} + \ell_j(\boldsymbol{\theta}_j) \\ \equiv & \ \ell_0(\boldsymbol{\theta}) + \ell_j(\boldsymbol{\theta}_j) \end{aligned}$$

and thus the maximum log-likelihood is

$$\begin{aligned} \ell_{0,j}(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\theta}}_j) &= \ell_0(\hat{\boldsymbol{\theta}}) + \ell_j(\hat{\boldsymbol{\theta}}_j) = -\frac{n_0}{2} - \frac{n_0}{2}\log(2\pi) - \frac{n_0}{2}\log(\hat{\sigma}^2) + \ell_j(\hat{\boldsymbol{\theta}}_j) \\ &= -\frac{n_0 + n_j}{2} - \frac{n_0 + n_j}{2}\log(2\pi) - \frac{n_0}{2}\log(\hat{\sigma}^2) - \frac{n_j}{2}\log(\hat{\sigma}_j^2), \end{aligned}$$

where $(\hat{\boldsymbol{\theta}}^T, \hat{\boldsymbol{\theta}}_j^T)^T$ ($\equiv (\hat{\boldsymbol{\beta}}^T, \hat{\sigma}, \hat{\boldsymbol{\theta}}_j^T)^T$) is the MLE of $(\boldsymbol{\theta}^T, \boldsymbol{\theta}_j^T)^T$. Hence the LR test statistic at time j (≥ 1) is

$$W_{0,j} \equiv 2[\ell_{0,j}(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\theta}}_j) - \ell_{0,j}(\tilde{\boldsymbol{\theta}}, \tilde{\boldsymbol{\theta}}_j)] = (n_0 + n_j) \log(\tilde{\sigma}^2) - n_0 \log(\hat{\sigma}^2) - n_j \log(\hat{\sigma}_j^2), \quad (2.7)$$

where $\tilde{\boldsymbol{\theta}} \equiv (\tilde{\boldsymbol{\beta}}^T, \tilde{\sigma})^T$ and $\tilde{\boldsymbol{\theta}}_j \equiv (\tilde{\boldsymbol{\beta}}_j^T, \tilde{\sigma}_j)^T = \tilde{\boldsymbol{\theta}}$ with

$$\tilde{\boldsymbol{\beta}} \equiv (X_0^T X_0 + X_j^T X_j)^{-1}(X_0^T \boldsymbol{y}_0 + X_j^T \boldsymbol{y}_j)$$

and

$$\begin{aligned}
\tilde{\sigma} &\equiv \frac{\|[\boldsymbol{I}_{n_0+n_j} - (X_0^T, X_j^T)^T (X_0^T X_0 + X_j^T X_j)^{-1}(X_0^T, X_j^T)](\boldsymbol{y}_0^T, \boldsymbol{y}_j^T)^T\|}{\sqrt{n_0 + n_j}} \\
&= \sqrt{\frac{n_0\hat{\sigma}^2 + n_j\hat{\sigma}_j^2 + (\hat{\boldsymbol{\beta}}_j - \hat{\boldsymbol{\beta}})^T[(X_0^T X_0)^{-1} + (X_j^T X_j)^{-1}]^{-1}(\hat{\boldsymbol{\beta}}_j - \hat{\boldsymbol{\beta}})}{n_0 + n_j}}.
\end{aligned}$$

When the process is in control at time j (≥ 1), the distribution of $W_{0,j}$ only depends on p, $n_0 - p$, and $n_j - p$, and $W_{0,j} \stackrel{a.s.}{\to} W_j$ as min{n_0, the minimum eigenvalue of $X_0^T X_0$} $\to \infty$, see Appendix A.5 in detail. Then $W_{0,j} \stackrel{d}{\to} \chi^2_{p+1}$ as $\min\{n_0/n_j, n_j\} \to \infty$. For $j \geq 1$, set

$$\bar{W}_{0,j} \equiv \frac{W_{0,j} - E_{\boldsymbol{\theta}}(W_{0,j})}{\sqrt{\mathrm{Var}_{\boldsymbol{\theta}}(W_{0,j})}}, \quad (2.8)$$

where

$$
\begin{aligned}
E_{\boldsymbol{\theta}}(W_{0,j}) &\equiv E_{\boldsymbol{\theta},\boldsymbol{\theta}_j}(W_{0,j})|_{\boldsymbol{\theta}_j=\boldsymbol{\theta}} \\
&= n_0\left[\log\left(\frac{n_0}{2}\right)-\psi\left(\frac{n_0-p}{2}\right)\right]+n_j\left[\log\left(\frac{n_j}{2}\right)-\psi\left(\frac{n_j-p}{2}\right)\right] \\
&\quad -(n_0+n_j)\left[\log\left(\frac{n_0+n_j}{2}\right)-\psi\left(\frac{n_0+n_j-p}{2}\right)\right] \\
&= p+1+O\left(\frac{1}{\min\{n_0,n_j\}}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{Var}_{\boldsymbol{\theta}}(W_{0,j}) &\equiv \mathrm{Var}_{\boldsymbol{\theta},\boldsymbol{\theta}_j}(W_{0,j})|_{\boldsymbol{\theta}_j=\boldsymbol{\theta}} \\
&= n_0\left[n_0\psi'\left(\frac{n_0-p}{2}\right)-2\right]+n_j\left[n_j\psi'\left(\frac{n_j-p}{2}\right)-2\right] \\
&\quad -(n_0+n_j)\left[(n_0+n_j)\psi'\left(\frac{n_0+n_j-p}{2}\right)-2\right] \\
&= 2(p+1)+O\left(\frac{1}{\min\{n_0,n_j\}}\right).
\end{aligned}
$$

2.5 Unknown and Constrained Case

In this subsection, consider the case where the process parameter vector at time j (≥ 1) may be different from the unknown in-control process parameter vector with the constraint $\sigma_j \geq \sigma$. Then the maximum log-likelihood is

$$
\ell_{0,j}(\boldsymbol{\theta}^*,\boldsymbol{\theta}_j^*) = -\frac{n_0+n_j}{2}-\frac{n_0+n_j}{2}\log(2\pi)-\frac{n_0}{2}\log(\sigma^{*2})-\frac{n_j}{2}\log(\sigma_j^{*2}),
$$

where $(\boldsymbol{\theta}^*, \boldsymbol{\theta}_j^*)^T$ $(\equiv (\boldsymbol{\beta}^{*T}, \sigma^*, \boldsymbol{\beta}_j^{*T}, \sigma_j^*)^T = (\hat{\boldsymbol{\beta}}^T, \sigma^*, \hat{\boldsymbol{\beta}}_j^T, \sigma_j^*)^T)$ is the MLE of $(\boldsymbol{\theta}^T, \boldsymbol{\theta}_j^T)^T$ with

$$\sigma^{*2} = \hat{\sigma}^2 \cdot 1_{\{\hat{\sigma}_j \geq \hat{\sigma}\}} + \frac{n_0\hat{\sigma}^2 + n_j\hat{\sigma}_j^2}{n_0 + n_j} \cdot 1_{\{\hat{\sigma}_j < \hat{\sigma}\}}$$

and

$$\sigma_j^{*2} = \hat{\sigma}_j^2 \cdot 1_{\{\hat{\sigma}_j \geq \hat{\sigma}\}} + \frac{n_0\hat{\sigma}^2 + n_j\hat{\sigma}_j^2}{n_0 + n_j} \cdot 1_{\{\hat{\sigma}_j < \hat{\sigma}\}}.$$

Hence the LR test statistic at time j (≥ 1) is

$$W_{0,j}^* \equiv 2[\ell_{0,j}(\boldsymbol{\theta}^*, \boldsymbol{\theta}_j^*) - \ell_{0,j}(\tilde{\boldsymbol{\theta}}, \tilde{\boldsymbol{\theta}}_j)] = (n_0 + n_j)\log(\tilde{\sigma}^2) - n_0\log(\sigma^{*2}) - n_j\log(\sigma_j^{*2}). \quad (2.9)$$

When the process is in control at time j (≥ 1), the distribution of $W_{0,j}^*$ only depends on p, $n_0 - p$, and $n_j - p$, and $W_{0,j}^* \xrightarrow{a.s.} W_j^*$ as $\min\{n_0,$ the minimum eigenvalue of $X_0^T X_0\} \to \infty$. Then $W_{0,j}^* \xrightarrow{d} \chi_p^2/2 + \chi_{p+1}^2/2$ as $\min\{n_0/n_j, n_j\} \to \infty$. For $j \geq 1$, set

$$\tilde{W}_{0,j}^* \equiv \frac{W_{0,j}^* - E_{\boldsymbol{\theta}}(W_{0,j}^*)}{\sqrt{\mathrm{Var}_{\boldsymbol{\theta}}(W_{0,j}^*)}}, \quad (2.10)$$

where both $E_{\boldsymbol{\theta}}(W_{0,j}^*) \equiv E_{\boldsymbol{\theta},\boldsymbol{\theta}_j}(W_{0,j}^*)|_{\boldsymbol{\theta}_j=\boldsymbol{\theta}}$ and $\mathrm{Var}_{\boldsymbol{\theta}}(W_{0,j}^*) \equiv \mathrm{Var}_{\boldsymbol{\theta},\boldsymbol{\theta}_j}(W_{0,j}^*)|_{\boldsymbol{\theta}_j=\boldsymbol{\theta}}$ are given in Appendix A.6.

2.6 Proposed Monitoring Scheme

Based on the standardized LR test statistics, the EWMA sequence is defined as

$$U_j \equiv \lambda\tilde{W}_j + (1-\lambda)U_{j-1}, \quad j = 1, 2, ..., \quad (2.11)$$

where $U_0 \equiv 0$, λ is a smoothing parameter in $(0, 1]$, and $\tilde{W}_j = \bar{W}_j$ in Section 2.2, or $\bar{W}_j^*$ in Section 2.3, or $\bar{W}_{0,j}$ in Section 2.4, or $\bar{W}_{0,j}^*$ in Section 2.2. Then

$$U_j = \lambda \sum_{k=0}^{j-1} (1-\lambda)^k \tilde{W}_{j-k}, \quad j = 1, 2, \ldots. \tag{2.12}$$

Due to we use the LR test statistics, then the larger value is more likely become out-of-control. An out-of-control signal occurs at time j (≥ 1) if

$$U_j^* \equiv \frac{U_j - E_{\boldsymbol{\theta}}(U_j)}{\sqrt{\mathrm{Var}_{\boldsymbol{\theta}}(U_j)}} > C, \tag{2.13}$$

where $E_{\boldsymbol{\theta}}(U_j) \equiv E_{\boldsymbol{\theta}, \boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_j}(U_j)|_{\boldsymbol{\theta}_1 = \ldots = \boldsymbol{\theta}_j = \boldsymbol{\theta}} = 0$,

$$\mathrm{Var}_{\boldsymbol{\theta}}(U_j) \equiv \mathrm{Var}_{\boldsymbol{\theta}, \boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_j}(U_j)|_{\boldsymbol{\theta}_1 = \ldots = \boldsymbol{\theta}_j = \boldsymbol{\theta}} = \frac{\lambda[1-(1-\lambda)^{2j}]}{2-\lambda} \tag{2.14}$$

for known $\boldsymbol{\theta}$,

$$\begin{aligned} \mathrm{Var}_{\boldsymbol{\theta}}(U_j) &\equiv \mathrm{Var}_{\boldsymbol{\theta}, \boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_j}(U_j)|_{\boldsymbol{\theta}_1 = \ldots = \boldsymbol{\theta}_j = \boldsymbol{\theta}} \\ = \frac{\lambda[1-(1-\lambda)^{2j}]}{2-\lambda} &+ 2\lambda^2 \sum_{0 \leq k_1 < k_2 \leq j-1} (1-\lambda)^{k_1+k_2} \mathrm{Cov}_{\boldsymbol{\theta}}(\tilde{W}_{j-k_1}, \tilde{W}_{j-k_2}) \end{aligned} \tag{2.15}$$

for unknown $\boldsymbol{\theta}$ with $\mathrm{Cov}_{\boldsymbol{\theta}}(\tilde{W}_{j-k_1}, \tilde{W}_{j-k_2}) \equiv \mathrm{Cov}_{\boldsymbol{\theta}, \boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_{j-k_1}}(\tilde{W}_{j-k_1}, \tilde{W}_{j-k_2})|_{\boldsymbol{\theta}_1 = \ldots = \boldsymbol{\theta}_{j-k_1} = \boldsymbol{\theta}}$, and C is chosen to achieve a specified in-control ARL. When θ is unknown, $\mathrm{Cov}_{\boldsymbol{\theta}}(\tilde{W}_{j-k_1}, \tilde{W}_{j-k_2})$ depends on X_0, X_{j-k_1}, and X_{j-k_2} for $0 \leq k_1 < k_2 \leq j-1$. In the paper, each $\mathrm{Cov}_{\boldsymbol{\theta}}(\tilde{W}_{j-k_1}, \tilde{W}_{j-k_2})$ is approximated by a Monte Carlo estimate.

3 A Simulation Study

In this section, in order to study the performance of the proposed EWMA monitoring scheme based on the LR test statistics, the out-of-control ARLs are compared with those in Kim *et al.* (2003) and Zou *et al.* (2007) [13] through a simulation study.

First, consider the case where the in-control parameter vector θ is known. Then C in equation (2.13) can be evaluated as follows:

Step 1 : Choose the desired in-control ARL and the smoothing constant λ. Here the in-control ARL and λ is chosen as, respectively, 200 and 0.2. Then we can start to construct the EWMA control chart, and in order to conveniently introduce the steps, we consider the case where the in-control parameter vector θ is constrained, and unconstrained case is also suitable for use.

Step 2 : Due to the in-control ARL = 200, so we generate *i.i.d.* in-control W_1^*, ..., W_{200}^*, and we want to standardized the W_j^*, so we evaluate the $E_{\boldsymbol{\theta}}(W_j^*)$ and $\mathrm{Var}_{\boldsymbol{\theta}}(W_j^*)$, which are used to evaluate $\tilde{W}_j^*$. And the data X_j can be random when the p and $n_j - p$ is known.

Step 3 : Evaluate the EWMA sequences $\{U_1, \ldots, U_{200}\}$ by equation (2.12), and we want to standardized the U_j, so we evaluate the $\mathrm{Var}_{\boldsymbol{\theta}}(U_1), \ldots, \mathrm{Var}_{\boldsymbol{\theta}}(U_{200})$ by equation (2.14), which use to evaluate U_j^* by equation (2.13).

Step 4 : We want to know the limit value every time, which use to estimate the C. So we find the maximum of $\{U_1^*, \ldots, U_{200}^*\}$ ($\equiv Q^{(1)}$), repeat above steps for 50,000 times to obtain $Q^{(1)}, \ldots, Q^{(50,000)}$.

Step 5 : Due to the in-control ARL is the mean of run length, so we think that the C is between the first sample quartile and third sample quartile of $Q^{(r)}, r = 1, \ldots, 50,000$. Therefore we sort the $Q^{(r)}, r = 1, \ldots, 50,000$, and evaluate its first sample quartile $(\equiv q_1)$, second sample quartile $(\equiv q_2)$ and third sample quartile $(\equiv q_3)$.

Step 6 : Choose C as each of q_1, q_2 and q_3 to evaluate the corresponding in-control ARL for 50,000 times. Use the interpolation method to find the value C with the specified in-control ARL$= 200$.

To evaluate the out-of-control ARL, consider the same shifts in θ in Zou *et al.* (2007) [13]. The out-of-control ARL can be evaluated as follows:

Step 1 : Due to we want to evaluate the out-of-control ARL, we simulate 50,000 times to average the RL. Then we generate *i.i.d.* W_1^* and W_2^* and standardized them, which use to evaluate U_1 and U_2.

Step 2 : Standardized both U_1 and U_2 to U_1^* and U_2^*, then to judge U_1^* whether larger than C. If it is true, then the RL$= j = 1$, else continue the process: generate *i.i.d.* $W_j^*, j = 1, \ldots$, until the value $U_j^* > C, j \geq 1$. Then the stopping time ℓ is the RL of this time.

Step 3 : Repeat Steps 1 and 2 50,000 times to average the RLs, which can obtain the out-of-control ARL.

Next, consider the case where the in-control parameter vector θ is unknown. Then C in equation (2.13) can be evaluated as follows:

Step 1 : Choose the desired in-control ARL and the smoothing constant λ. Here the in-control ARL and λ is chosen as, respectively, 200 and 0.2. Then we can start to construct the EWMA control chart, and in order to conveniently introduce the steps, we consider the case where the in-control parameter vector θ is constrained, and unconstrained case is also suitable for use.

Step 2 : Due to the in-control ARL = 200, so we generate *i.i.d.* in-control $W_{0,1}^*$, $\ldots$, $W_{0,200}^*$, and we want to standardized the $W_{0,j}^*$, so we evaluate the $E_{\boldsymbol{\theta}}(W_{0,j}^*)$ and $\mathrm{Var}_{\boldsymbol{\theta}}(W_{0,j}^*)$, which are used to evaluate $\tilde{W}_j^*$. The $W_{0,j}^*$ and $W_{0,j'}^*$ depends on all of p, $n_0 - p$, $n_1 - p$, $n_2 - p, \ldots$ $(j \neq j')$, so the X_0 and X_j must be a design matrix. e.x.,

$$X : n \times p \quad X_j = \begin{pmatrix} X \\ \vdots \\ X \end{pmatrix}_{m_j n \times p}, X_0 = \begin{pmatrix} X \\ \vdots \\ X \end{pmatrix}_{m_0 n \times p} \quad m_0, m_j \geq 1 \quad \text{for all} \quad j \geq 1$$

Step 3 : Evaluate the EWMA sequences $\{U_1, \ldots, U_{200}\}$ by equation (2.12), and we want to standardized the U_j, so we evaluate the $\mathrm{Var}_{\boldsymbol{\theta}}(U_1), \ldots, \mathrm{Var}_{\boldsymbol{\theta}}(U_{200})$ by equation (2.15), which use to evaluate U_j^* by equation (2.13).

Step 4 : We want to know the limit value every time, which use to estimate the C. So we find the maximum of $\{U_1^*, \ldots, U_{200}^*\}$ $(\equiv Q^{(1)})$, repeat above steps for 50,000 times to obtain $Q^{(1)}, \ldots, Q^{(50{,}000)}$.

Step 5 : Due to the in-control ARL is the mean of run length, so we think that the C is between the first sample quartile and third sample quartile of $Q^{(r)}, r = 1, \ldots, 50,000$. Therefore we sort the $Q^{(r)}, r = 1, \ldots, 50,000$, and evaluate its first sample quartile $(\equiv q_1)$, second sample quartile $(\equiv q_2)$ and third sample quartile $(\equiv q_3)$.

Step 6 : Choose C as each of q_1, q_2 and q_3 to evaluate the corresponding in-control ARL for 50,000 times. Use the interpolation method to find the value C with the specified in-control ARL$= 200$.

To evaluate the out-of-control ARL, consider the same shifts in θ in Zou *et al.* (2007) [13]. The out-of-control ARL can be evaluated as follows:

Step 1 : Due to we want to evaluate the out-of-control ARL, we simulate 50,000 times to average the RL. Then we generate *i.i.d.* $W^*_{0,1}$ and $W^*_{0,2}$ and standardized them, which use to evaluate U_1 and U_2.

Step 2 : Standardized both U_1 and U_2 to U^*_1 and U^*_2, then to judge U^*_1 whether larger than C. If it is true, then the RL$= j = 1$, else continue the process: generate *i.i.d.* $W^*_{0,j}, j = 1, \ldots,$ until the value $U^*_j > C, j \geq 1$. Then the stopping time ℓ is the RL of this time.

Step 3 : Repeat Steps 1 and 2 50,000 times to average the RLs, which can obtain the out-of-control ARL.

Then we consider the simplest case of model (2.1) to compared with Kim *et al.* (2003) and Zou *et al.* (2007) [13] through a simulation study, in which the parameters in the in-control model are $\boldsymbol{\beta} = (3, 2)^T$, $\sigma = 1$, and

$$X_0 = \begin{pmatrix} X \\ \vdots \\ X \end{pmatrix}_{mn \times p}$$

$X = (1_{4\times1}, \boldsymbol{x})$ with $\boldsymbol{x} = (2, 4, 6, 8)^T$.

4 Comparisons and Conclusions

In this section, first, we compared our propose EWMA control chart with the ZTW and KMW control charts from Kim *et al.* (2003) and Zou *et al.* (2007) [13]. We follow the table by Zou *et al.* (2007) [13], which in term of out-of-control ARL in the simplest case of model (2.1) for shifts in β_0, β_1 and σ , respectively. Because of our performance under unconstrained case is not better, therefore we don't present in this. Then the performance under known and constrained case are given in Table 5-1, 5-2 and 5-3. On Table 5-1 and 5-2, those table show that for β_0 and β_1, our charts' performance is not better than others with small shifts, but it is better than others with large shifts. We think that, the LR test statistic is more powerful with large shift by nature. But on Table 5-3 shows that change for σ, our performance is better than others with any shifts. Above result show that, our propose EWMA chart do not have better performance for shifts in intercept and slope, but have better performance for shifts in σ. We think that it maybe because of the method of ZTW is using MEWMA, the proportion of $\boldsymbol{\beta}$ and σ are the same, however the proportion of σ is larger than $\boldsymbol{\beta}$ by our proposed.

In practice, the in-control process parameters are rarely known, and thus control charts usually are constructed using estimates from the historical in-control process data in place of the unknown in-control parameters. Therefore we present the performance under unknown and constrained case are given in Table 5-1, 5-2 and 5-3 left side, those tables show that the performance is similar to known parameter about 125 historical in-control process data.

From the section 2, we can know that the out-of-control ARL is depends on all of p, $n_j - p$, σ_j/σ and τ_j^2, where the noncentrality parameter

$$\tau_j^2 = (\boldsymbol{\beta}_j - \boldsymbol{\beta})^T X_j^T X_j (\boldsymbol{\beta}_j - \boldsymbol{\beta})/\sigma_j^2,$$

$$X_j = X \quad \text{for all} \quad j \geq 1,$$

and both p and $n_j - p$ are constant. Therefore we present the performance under combinations of σ_j/σ and $\tau^2\sigma_j^2/\sigma^2$, are given in Table 5-4 and 5-5. Those tables show that the performance reduce fast at σ_j/σ.

In this paper, our propose EWMA chart based on LR test statistics have better performance for shift in σ. And we provide the mean and variance for likelihood ratio test statistic that not presented in any papers. This can be taken as a reference.

5 Future Work

In this section, we provide some idea to extend or improve study. The future research could substitute: Consider the case where only β may change; or only σ may change to observe the performance whether more better for β_0 and β_1 with small shift. Another the research could consider possible transformation for w_j^* obtain better performance for β_0 and β_1 with small shifts. e.g., $T = \Phi^{-1}(F_{\boldsymbol{\theta}}(W_j^*))$, then in the in-control $T \sim N(0, 1)$, then C can be easily found in some literatures, where $F(\cdot)$ is $c.d.f.$, $\Phi^{-1}(\cdot)$ is inverse of the standard normal cumulative distribution faction. Zou *et al.* (2010) proposed for jointly monitoring the process mean and variance, consider an MEWMA control chart based on LR test statistics for monitoring general linear profiles.

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Table 5.1: ARL comparisons among EWMA, ZTW, and KMW control charts for shifts in β_0 with $\boldsymbol{\theta}_j = (\beta_0 + \delta_0\sigma, \beta_1, \sigma)^T$ and $\tau_j^2 = 4\,\delta_0^2$ for $j \geq 1$

β_0							
δ_0	τ_j^2	EWMA $m_0 = 5$	EWMA $m_0 = 25$	EWMA $m_0 = 125$	EWMA $m_0 = \infty$	ZTW	KMW
.1000	0.0400	178.2	174.5	172.9	171.1	131.5	133.7
.2000	0.1600	134.6	120.3	116.3	113.1	59.9	59.1
.3000	0.3600	88.7	72.7	68.6	67.2	29.6	28.3
.4000	0.6400	55.0	41.2	38.9	38.1	17.2	16.2
.5000	1.0000	33.6	24.6	22.7	22.1	11.5	10.7
.6000	1.4400	21.5	15.2	13.9	13.8	8.5	7.9
.8000	2.5600	9.9	6.9	6.5	6.1	5.8	5.1
1.0000	4.0000	5.5	3.9	3.6	3.4	4.1	3.8
1.5000	9.0000	2.2	1.6	1.6	1.5	2.6	2.4
2.0000	16.0000	1.4	1.1	1.1	1.1	2.0	1.9

Table 5.2: ARL comparisons among EWMA, ZTW, and KMW control charts for shifts in β_1 with $\boldsymbol{\theta}_j = (\beta_0, \beta_1 + \delta_1\sigma, \sigma)^T$ and $\tau_j^2 = 120\,\delta_1^2$ for $j \geq 1$

β_1							
δ_1	τ_j^2	EWMA $m_0 = 5$	EWMA $m_0 = 25$	EWMA $m_0 = 125$	EWMA $m_0 = \infty$	ZTW	KMW
.0250	0.0750	168.5	157.6	155.5	153.0	99.0	101.6
.0375	0.1688	133.6	117.8	115.3	112.3	57.4	61.0
.0500	0.3000	102.9	83.5	79.4	77.4	35.0	36.5
.0625	0.4688	74.2	57.8	53.7	53.2	23.1	24.6
.0750	0.6750	53.5	39.2	36.4	35.4	16.4	17.0
.1000	1.2000	27.7	19.2	17.6	17.4	9.8	10.3
.1250	1.8750	15.3	10.6	9.6	9.5	6.9	7.2
.1500	2.7000	9.4	6.4	5.9	5.8	5.3	5.5
.2000	4.8000	4.4	3.1	2.9	2.8	3.7	3.8
.2500	7.5000	2.6	2.0	1.8	1.8	2.9	2.9

Table 5.3: ARL comparisons among EWMA, ZTW, and KMW control charts for shifts in σ with $\boldsymbol{\theta}_j = (\beta_0, \beta_1, \delta\sigma)^T$ for $j \geq 1$

	σ					
δ	EWMA $m_0 = 5$	EWMA $m_0 = 25$	EWMA $m_0 = 125$	EWMA $m_0 = \infty$	ZTW	KMW
1.1000	72.6	56.8	51.1	51.1	76.2	72.8
1.1500	48.9	35.1	30.9	30.2	48.7	48.1
1.2000	34.4	23.2	20.5	20.0	33.2	33.5
1.2500	25.1	16.7	14.7	14.6	24.1	24.9
1.3000	19.3	12.6	11.1	11.0	18.4	19.4
1.4000	12.3	7.9	7.0	7.0	12.1	12.7
1.6000	6.5	4.3	4.0	4.0	7.0	7.2
1.8000	4.2	2.9	2.8	2.8	4.9	5.1
2.2000	2.5	1.9	1.8	1.8	3.1	3.2
2.6000	1.8	1.4	1.4	1.4	2.3	2.5
3.0000	1.5	1.3	1.3	1.3	1.9	2.1

Table 5.4: ARLs for shifts in $\boldsymbol{\theta}$ with $\boldsymbol{\theta}_j = (\boldsymbol{\beta}_j^T, \delta\sigma)^T$ and $\tau_j^2 = \tau^2/\delta^2$ for $j \geq 1$

	$m_0 = 5$ $m_0 = 25$ $m_0 = 125$ $m_0 = \infty$	δ: 1.1^0	1.1^1	1.1^2	1.1^3	1.1^4	1.1^5	1.1^6	1.1^7	1.1^8	1.1^9	1.1^{10}	1.1^{11}	1.1^{12}
τ^2	0.0^2	200.0	72.6	32.5	16.6	9.8	6.3	4.5	3.3	2.6	2.1	1.8	1.6	1.4
		200.0	56.8	20.9	10.4	6.2	4.1	3.0	2.3	1.9	1.6	1.4	1.3	1.2
		200.0	51.2	18.9	9.5	5.6	3.8	2.8	2.2	1.8	1.6	1.4	1.3	1.2
		200.0	51.1	18.2	9.0	5.4	3.7	2.7	2.1	1.8	1.5	1.4	1.3	1.2
	0.2^2	178.2	70.1	31.4	16.3	9.6	6.3	4.4	3.3	2.6	2.1	1.8	1.6	1.4
		174.5	51.3	20.3	10.3	6.1	4.1	3.0	2.3	1.9	1.6	1.4	1.3	1.2
		172.9	47.2	18.5	9.3	5.6	3.8	2.8	2.2	1.8	1.6	1.4	1.3	1.2
		171.1	43.2	17.1	8.7	5.3	3.6	2.7	2.1	1.8	1.5	1.4	1.3	1.2
	0.4^2	132.9	57.5	27.6	14.9	9.1	6.1	4.3	3.3	2.6	2.1	1.8	1.6	1.4
		120.3	41.3	17.9	9.5	5.9	4.0	3.0	2.3	1.9	1.6	1.4	1.3	1.2
		116.3	37.7	16.2	8.6	5.4	3.7	2.7	2.2	1.8	1.6	1.4	1.3	1.2
		113.1	34.4	15.0	8.2	5.1	3.5	2.6	2.1	1.8	1.5	1.4	1.3	1.2
	0.6^2	88.7	43.1	22.5	13.3	8.4	5.7	4.2	3.2	2.5	2.1	1.8	1.6	1.4
		72.7	30.0	14.6	8.5	5.5	3.8	2.9	2.3	1.9	1.6	1.4	1.3	1.2
		68.6	27.5	13.3	7.7	5.0	3.5	2.7	2.1	1.8	1.5	1.4	1.3	1.2
		67.2	25.3	12.5	7.2	4.7	3.4	2.6	2.1	1.7	1.5	1.4	1.3	1.2
	0.8^2	55.0	30.4	17.9	11.3	7.5	5.4	4.0	3.1	2.5	2.1	1.8	1.6	1.4
		41.2	20.8	11.7	7.3	5.0	3.6	2.7	2.2	1.8	1.6	1.4	1.3	1.2
		39.0	18.9	10.7	6.7	4.5	3.3	2.6	2.1	1.7	1.5	1.4	1.3	1.2
		38.1	17.4	9.9	6.3	4.3	3.2	2.5	2.0	1.7	1.5	1.4	1.3	1.2
	1.0^2	33.6	21.6	14.0	9.5	6.7	4.9	3.7	3.0	2.4	2.0	1.8	1.6	1.4
		24.6	14.4	9.1	6.2	4.4	3.4	2.6	2.1	1.8	1.6	1.4	1.3	1.2
		22.7	13.1	8.4	5.7	4.1	3.1	2.5	2.0	1.7	1.5	1.4	1.3	1.2
		22.1	12.3	7.8	5.4	3.9	3.0	2.4	2.0	1.7	1.5	1.3	1.2	1.2
	1.2^2	21.5	15.2	10.8	7.8	5.8	4.5	3.5	2.9	2.3	2.0	1.7	1.5	1.4
		15.2	10.1	7.2	5.2	3.9	3.1	2.5	2.1	1.8	1.6	1.4	1.3	1.2
		13.9	9.4	6.5	4.8	3.6	2.9	2.3	1.9	1.7	1.5	1.4	1.3	1.2
		13.8	8.7	6.2	4.5	3.5	2.8	2.2	1.9	1.6	1.5	1.3	1.2	1.2
	1.4^2	14.4	11.0	8.4	6.5	5.1	4.1	3.3	2.7	2.3	1.9	1.7	1.5	1.4
		10.0	7.5	5.7	4.4	3.5	2.8	2.3	2.0	1.7	1.5	1.4	1.3	1.2
		9.2	6.9	5.2	4.1	3.2	2.6	2.2	1.9	1.6	1.5	1.3	1.3	1.2
		8.6	6.5	5.0	3.9	3.1	2.5	2.1	1.8	1.6	1.4	1.3	1.2	1.2
	1.6^2	9.9	8.2	6.7	5.5	4.5	3.7	3.0	2.5	2.2	1.9	1.7	1.5	1.4
		6.9	5.6	4.6	3.7	3.1	2.6	2.2	1.9	1.7	1.5	1.4	1.3	1.2
		6.5	5.2	4.2	3.5	2.9	2.4	2.1	1.8	1.6	1.4	1.3	1.2	1.2
		6.1	4.9	4.0	3.3	2.8	2.4	2.0	1.8	1.6	1.4	1.3	1.2	1.2
	1.8^2	7.2	6.3	5.4	4.6	3.9	3.3	2.8	2.4	2.1	1.8	1.6	1.5	1.4
		5.1	4.4	3.8	3.2	2.8	2.4	2.1	1.8	1.6	1.5	1.4	1.3	1.2
		4.7	4.1	3.5	3.0	2.6	2.2	2.0	1.7	1.6	1.4	1.3	1.2	1.2
		4.5	3.9	3.3	2.9	2.5	2.2	1.9	1.7	1.5	1.4	1.3	1.2	1.2
	2.0^2	5.5	5.0	4.4	3.9	3.4	3.0	2.6	2.3	2.0	1.8	1.6	1.5	1.3
		3.9	3.5	3.1	2.8	2.5	2.2	1.9	1.7	1.6	1.4	1.3	1.2	1.2
		3.6	3.2	2.9	2.6	2.3	2.1	1.8	1.7	1.5	1.4	1.3	1.2	1.2
		3.4	3.1	2.8	2.5	2.2	2.0	1.8	1.6	1.5	1.4	1.3	1.2	1.2

Table 5.5: ARLs for shifts in $\boldsymbol{\theta}$ with $\boldsymbol{\theta}_j = (\boldsymbol{\beta}_j^T, \delta\sigma)^T$ and $\tau_j^2 = \tau^2/\delta^2$ for $j \geq 1$

	$m_0 = 5$ $m_0 = 25$ $m_0 = 125$ $m_0 = \infty$	δ 1.1^0	1.1^1	1.1^2	1.1^3	1.1^4	1.1^5	1.1^6	1.1^7	1.1^8	1.1^9	1.1^{10}	1.1^{11}	1.1^{12}
	2.2^2	4.3	4.0	3.7	3.4	3.0	2.7	2.4	2.2	1.9	1.7	1.6	1.4	1.3
		3.1	2.9	2.7	2.4	2.2	2.0	1.8	1.7	1.5	1.4	1.3	1.2	1.2
		2.9	2.7	2.5	2.3	2.1	1.9	1.7	1.6	1.5	1.4	1.3	1.2	1.2
		2.8	2.6	2.4	2.2	2.0	1.9	1.7	1.6	1.4	1.3	1.3	1.2	1.1
	2.4^2	3.5	3.4	3.1	2.9	2.7	2.5	2.2	2.0	1.8	1.7	1.5	1.4	1.3
		2.6	2.5	2.3	2.1	2.0	1.9	1.7	1.6	1.5	1.4	1.3	1.2	1.2
		2.4	2.3	2.2	2.0	1.9	1.8	1.6	1.5	1.4	1.3	1.3	1.2	1.2
		2.3	2.2	2.1	2.0	1.8	1.7	1.6	1.5	1.4	1.3	1.2	1.2	1.1
	2.6^2	2.9	2.8	2.7	2.6	2.4	2.3	2.1	1.9	1.8	1.6	1.5	1.4	1.3
		2.2	2.1	2.0	1.9	1.8	1.7	1.6	1.5	1.4	1.3	1.3	1.2	1.2
		2.0	2.0	1.9	1.8	1.7	1.7	1.6	1.5	1.4	1.3	1.2	1.2	1.1
		2.0	1.9	1.9	1.8	1.7	1.6	1.5	1.4	1.4	1.3	1.2	1.2	1.1
	2.8^2	2.5	2.4	2.4	2.3	2.2	2.1	2.0	1.8	1.7	1.6	1.5	1.4	1.3
		1.9	1.9	1.8	1.7	1.7	1.6	1.5	1.5	1.4	1.3	1.3	1.2	1.2
		1.8	1.7	1.7	1.7	1.6	1.5	1.5	1.4	1.3	1.3	1.2	1.2	1.1
		1.7	1.7	1.7	1.6	1.6	1.5	1.5	1.4	1.3	1.3	1.2	1.2	1.1
	3.0^2	2.2	2.2	2.1	2.1	2.0	1.9	1.8	1.7	1.6	1.5	1.4	1.4	1.3
		1.6	1.7	1.6	1.6	1.6	1.5	1.5	1.4	1.3	1.3	1.2	1.2	1.1
		1.6	1.6	1.6	1.5	1.5	1.5	1.4	1.4	1.3	1.3	1.2	1.2	1.1
		1.5	1.5	1.5	1.5	1.5	1.4	1.4	1.3	1.3	1.2	1.2	1.2	1.1
τ^2	3.2^2	1.9	1.9	1.9	1.9	1.8	1.8	1.7	1.7	1.6	1.5	1.4	1.3	1.3
		1.5	1.5	1.5	1.5	1.5	1.4	1.4	1.4	1.3	1.3	1.2	1.2	1.1
		1.4	1.4	1.4	1.4	1.4	1.4	1.4	1.3	1.3	1.2	1.2	1.2	1.1
		1.4	1.4	1.4	1.4	1.4	1.4	1.3	1.3	1.3	1.2	1.2	1.1	1.1
	3.4^2	1.7	1.7	1.7	1.7	1.7	1.7	1.6	1.6	1.5	1.4	1.4	1.3	1.2
		1.4	1.4	1.4	1.4	1.4	1.4	1.3	1.3	1.3	1.2	1.2	1.2	1.1
		1.3	1.3	1.3	1.3	1.3	1.3	1.3	1.3	1.2	1.2	1.2	1.1	1.1
		1.3	1.3	1.3	1.3	1.3	1.3	1.3	1.3	1.2	1.2	1.2	1.1	1.1
	3.6^2	1.6	1.6	1.6	1.6	1.6	1.6	1.5	1.5	1.4	1.4	1.3	1.3	1.2
		1.3	1.3	1.3	1.3	1.3	1.3	1.3	1.3	1.2	1.2	1.2	1.1	1.1
		1.2	1.2	1.3	1.3	1.3	1.3	1.3	1.2	1.2	1.2	1.2	1.1	1.1
		1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.1	1.1
	3.8^2	1.5	1.5	1.5	1.5	1.5	1.5	1.5	1.4	1.4	1.4	1.3	1.3	1.2
		1.2	1.2	1.2	1.2	1.3	1.2	1.2	1.2	1.2	1.2	1.2	1.1	1.1
		1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.1	1.1	1.1
		1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.1	1.1	1.1
	4.0^2	1.4	1.4	1.4	1.4	1.4	1.4	1.4	1.4	1.4	1.3	1.3	1.2	1.2
		1.1	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.1	1.1	1.1
		1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1
		1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1

Appendix

A.1

$$
\begin{aligned}
\psi(1) &= -\gamma, \\
\psi(\frac{1}{2}) &= -\gamma - 2\log(2), \\
\psi(x+1) &= \psi(x) + \frac{1}{x}, \\
\psi(x) &= -\gamma + \sum_{n=1}^{\infty} \frac{x-1}{n(n+x-1)}, \\
\psi(x) &= \ln x - \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{12x^4} - \frac{1}{252x^6} + \ldots, \\
\psi'(1) &= \frac{\pi^2}{6}, \\
\psi'(\frac{1}{2}) &= \frac{\pi^2}{2}, \\
\psi'(x+1) &= \psi'(x) - \frac{1}{x^2}, \\
\psi'(x) &= \sum_{k=0}^{\infty} \frac{1}{(x+k)^2}, \\
\psi'(x) &\sim \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{12x^7} - \frac{1}{30x^9} \ldots,
\end{aligned}
$$

where $\gamma = 0.577215664\ldots$ is the Euler constant and $x \in (0, \infty)$.

A.2

$$
\begin{aligned}
& E_{\boldsymbol{\theta}_j}(W_j^2) \\
= & \left(\frac{\sigma_j^2}{\sigma^2}\right)^2 \left\{ 2\left[p + 2\frac{(\boldsymbol{\beta}_j - \boldsymbol{\beta})^T X_j^T X_j (\boldsymbol{\beta}_j - \boldsymbol{\beta})}{\sigma_j^2}\right] + \left[p + \frac{(\boldsymbol{\beta}_j - \boldsymbol{\beta})^T X_j^T X_j (\boldsymbol{\beta}_j - \boldsymbol{\beta})}{\sigma_j^2}\right]^2 \right\} \\
& + n_j^2 \left[\log\left(\frac{\sigma_j^2}{n_j \sigma^2}\right) + 1\right]^2 - 2\frac{n_j \sigma_j^2}{\sigma^2}\left[\log\left(\frac{\sigma_j^2}{n_j \sigma^2}\right) + 1\right](n_j - p) \\
& + n_j^2 \left[\log\left(\frac{\sigma_j^2}{n_j \sigma^2}\right) + 1\right]\left[\psi\left(\frac{n_j - p}{2}\right) + \log(2)\right] \\
& - 2\frac{n_j \sigma_j^2}{\sigma^2}(n_j - p)\left[\psi\left(\frac{n_j - p + 2}{2}\right) + \log(2)\right] \\
& + \left(\frac{\sigma_j^2}{\sigma^2}\right)^2 (n_j - p)(2 + n_j - p)^2 + n_j^2 \left\{\psi'\left(\frac{n_j - p}{2}\right) + \left[\psi\left(\frac{n_j - p}{2}\right) + log(2)\right]^2\right\}
\end{aligned}
$$

A.3

$$
\begin{aligned}
& n_j \left[\log\left(\frac{n_j}{2}\right) - \psi\left(\frac{n_j - p}{2}\right)\right] \\
= \; & n_j \left[\log\left(\frac{n_j}{2}\right) - \log\left(\frac{n_j - p}{2}\right) + \frac{1}{n_j - p} + O\left(\frac{1}{n_j^2}\right)\right] \\
= \; & n_j \left[\log\left(\frac{n_j}{n_j - p}\right) + \frac{1}{n_j} + O\left(\frac{1}{n_j^2}\right)\right] \\
= \; & n_j \left[\log\left(1 + \frac{p}{n_j - p}\right) + \frac{1}{n_j} + O\left(\frac{1}{n_j^2}\right)\right] \\
= \; & n_j \left[\frac{p}{n_j - p} + \frac{1}{n_j} + O\left(\frac{1}{n_j^2}\right)\right] \\
= \; & p + 1 + O\left(\frac{1}{n_j}\right),
\end{aligned}
$$

$$
\begin{aligned}
& n_j\left[n_j\psi'\left(\frac{n_j-p}{2}\right)-2\right] \\
= \; & n_j\left[n_j\frac{2}{n_j-p}+n_j\frac{4}{2(n_j-p)^2}-2+O\left(\frac{1}{n_j^2}\right)\right] \\
= \; & n_j\left[\frac{2}{1-p/n_j}+\frac{2}{n_j(1-p/n_j)^2}-2+O\left(\frac{1}{n_j^2}\right)\right] \\
= \; & n_j\left[2+\frac{2p}{n_j}+\frac{2}{n_j}-2+O\left(\frac{1}{n_j^2}\right)\right] \\
= \; & 2(p+1)+O\left(\frac{1}{n_j}\right).
\end{aligned}
$$

A.4

$$
\begin{aligned}
& E_{\boldsymbol{\theta}_j}(W_j^{*2}) \\
= \; & \left(\frac{\sigma_j^2}{\sigma^2}\right)^2\left\{2\left[p+2\frac{(\boldsymbol{\beta}_j-\boldsymbol{\beta})^T X_j^T X_j(\boldsymbol{\beta}_j-\boldsymbol{\beta})}{\sigma_j^2}\right]+\left[p+\frac{(\boldsymbol{\beta}_j-\boldsymbol{\beta})^T X_j^T X_j(\boldsymbol{\beta}_j-\boldsymbol{\beta})}{\sigma_j^2}\right]^2\right\} \\
& +n_j^2\left[\log\left(\frac{n_j\sigma_0^2}{\sigma_j^2}\right)-1\right]^2 E(1_{\{H_j\geq a_j\}})+2n_j\frac{\sigma_j^2}{\sigma_0^2}\left[\log\left(\frac{n_j\sigma_0^2}{\sigma_j^2}\right)-1\right]E(H_j\cdot 1_{\{H_j\geq a_j\}}) \\
& -2n_j^2\left[\log\left(\frac{n_j\sigma_0^2}{\sigma_j^2}\right)-1\right]E(\log(H_j)\cdot 1_{\{H_j\geq a_j\}})+\left(\frac{\sigma_j^2}{\sigma^2}\right)^2 E(H_j^2\cdot 1_{\{H_j\geq a_j\}}) \\
& +n_j^2 E([\log(H_j)]^2\cdot 1_{\{H_j\geq a_j\}})-2n_j\frac{\sigma_j^2}{\sigma^2}E(H_j\log(H_j)\cdot 1_{\{H_j\geq a_j\}}),
\end{aligned}
$$

where

$$
\begin{aligned}
E(1_{\{H_j \geq a_j\}}) &= \int_{a_j}^{\infty} \frac{x^{v-1} e^{-\frac{x}{2}}}{\Gamma(v) 2^v} dx \\
&= 1 - \frac{1}{\Gamma(v) 2^v} \sum_{m=0}^{\infty} \frac{(-1)^m a_j^{v+m}}{2^m \cdot m!(v+m)}, \\
&\quad \text{with} \quad v \equiv \frac{n_j - p}{2} \\
E(H_j \cdot 1_{\{H_j \geq a_j\}}) &= \int_{a_j}^{\infty} \frac{x^{v} e^{-\frac{x}{2}}}{\Gamma(v) 2^v} dx \\
&= (n_j - p) - \frac{1}{\Gamma(v) 2^v} \sum_{m=0}^{\infty} \frac{(-1)^m a_j^{v+m+1}}{2^m \cdot m!(v+m+1)}, \\
E(\log(H_j) \cdot 1_{\{H_j \geq a_j\}}) &= \int_{a_j}^{\infty} \log(x) \frac{x^{v-1} e^{-\frac{x}{2}}}{\Gamma(v) 2^v} dx \\
&= [\psi(v + \log(2))] - \frac{1}{\Gamma(v) 2^v} \sum_{m=0}^{\infty} \frac{(-1)^m a_j^m}{2^m \cdot m!} \left(\frac{\log(a_j) a_j^v}{v+m} - \frac{a_j^v}{(v+m)^2} \right), \\
E(H_j \log(H_j) \cdot 1_{\{H_j \geq a_j\}}) &= \int_{a_j}^{\infty} \log(x) \frac{x^{v} e^{-\frac{x}{2}}}{\Gamma(v) 2^v} dx \\
&= (n_j - p) \left[\psi\left(\frac{n_j - p + 2}{2} + \log(2) \right) \right] - \frac{1}{\Gamma(v) 2^v} \sum_{m=0}^{\infty} \frac{(-1)^m a_j^m}{2^m \cdot m!} G_1(a_j, v, m)
\end{aligned}
$$

with

$$
G_1(a_j, v, m) \equiv \left(\frac{\log(a_j) a_j^{v+1}}{v+m+1} - \frac{a_j^{v+1}}{(v+m+1)^2} \right)
$$

$$E(H_j^2 \cdot 1_{\{H_j \geq a_j\}}) = \int_{a_j}^{\infty} \frac{x^{v+1} e^{-\frac{x}{2}}}{\Gamma(v) 2^v} dx$$

$$= (n_j - p)(n_j - p - 2) - \frac{1}{\Gamma(v) 2^v} \sum_{m=0}^{\infty} \frac{(-1)^m a_j^{v+m+2}}{2^m \cdot m!(v+m+2)}$$

$$E([\log(H_j)]^2 \cdot 1_{\{H_j \geq a_j\}}) = \int_{a_j}^{\infty} [\log(x)]^2 \frac{x^{v-1} e^{-\frac{x}{2}}}{\Gamma(v) 2^v} dx$$

$$= \psi'(v) + [\psi(v) + \log(2)]^2 - \frac{1}{\Gamma(v) 2^v} \sum_{m=0}^{\infty} \frac{(-1)^m a_j^m}{2^m \cdot m!} G_2(a_j, v, m)$$

with

$$G_2(a_j, v, m) \equiv \left[\frac{[\log(a_j)]^2 a_j^v (v+m) - \log(a_j) a_j^v}{(v+m)^2} - \frac{\log(a_j) a_j^v (v+m)^2 - a_j^v 2(v+m)}{(v+m)^4} \right]$$

A.5

$$
\begin{aligned}
& W_{0,j} \\
= \; & (n_0+n_j)\log\left(\frac{n_0\hat{\sigma}^2 + n_j\hat{\sigma}_j^2 + (\hat{\boldsymbol{\beta}}_j-\hat{\boldsymbol{\beta}})^T[(X_0^TX_0)^{-1}+(X_j^TX_j)^{-1}](\hat{\boldsymbol{\beta}}_j-\hat{\boldsymbol{\beta}})}{(n_0+n_j)\sigma^2}\right) \\
& -n_0\log\left(\frac{\hat{\sigma}^2}{\sigma^2}\right) - n_j\log\left(\frac{\hat{\sigma}_j^2}{\sigma^2}\right) \\
\stackrel{a.s.}{\to} \; & \left[\frac{n_0(\hat{\sigma}^2-\sigma^2)}{\sigma^2} + \frac{n_j(\hat{\sigma}_j^2-\sigma^2)}{\sigma^2} + \frac{(\hat{\boldsymbol{\beta}}_j-\hat{\boldsymbol{\beta}})^T[(X_0^TX_0)^{-1}+(X_j^TX_j)^{-1}](\hat{\boldsymbol{\beta}}_j-\hat{\boldsymbol{\beta}})}{\sigma^2}\right] \\
& -n_0\left(\frac{\hat{\sigma}^2-\sigma^2}{\sigma^2}\right) - n_j\log\left(\frac{\hat{\sigma}_j^2}{\sigma^2}\right) + O_p\left(\frac{1}{\sqrt{n_0}}\right) \\
\stackrel{a.s.}{\to} \; & W_j + O_p\left(\frac{1}{\sqrt{n_0}}\right)
\end{aligned}
$$

as $\min\{n_0, \text{the minimum eigenvalue of } X_0^TX_0\} \to \infty$.

A.6

$$
W_{0,j}^* \equiv 2[\ell_{0,j}(\boldsymbol{\theta}^*, \boldsymbol{\theta}_j^*) - \ell_{0,j}(\tilde{\boldsymbol{\theta}}, \tilde{\boldsymbol{\theta}}_j)] = W_{0,j} - 2[\ell_{0,j}(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\theta}}_j) - \ell_{0,j}(\boldsymbol{\theta}^*, \boldsymbol{\theta}_j^*)].
$$

If the process is in control at time j (≥ 1), then

$$
\begin{aligned}
g(B_j) &\equiv 2[\ell_{0,j}(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\theta}}_j) - \ell_{0,j}(\boldsymbol{\theta}^*, \boldsymbol{\theta}_j^*)] \\
&= 1_{\{\hat{\sigma}_j < \hat{\sigma}_0\}}\left\{-n_0\log(\hat{\sigma}) - n_j\log(\hat{\sigma}_j) + (n_0+n_j)\log\left(\frac{n_0\hat{\sigma}_0^2 + n_j\hat{\sigma}_j^2}{n_0+n_j}\right)\right\} \\
&= 1_{\{B_j<b_j\}}\{-n_0\log(1-B_j) - n_j\log(B_j) + [n_0\log(n_0) + n_j\log(n_j) \\
&\quad -(n_0+n_j)\log((n_0+n_j))]\},
\end{aligned}
$$

where $B_j \equiv H_j/(H_0 + H_j) \sim \text{beta}\left(\frac{n_j-p}{2}, \frac{n_0-p}{2}\right)$, $H_0 \equiv n_0\hat{\sigma}^2/\sigma^2 \sim \chi^2_{n_0-p}$, $g(B_j)$ is a function of B_j, $b_j \equiv n_j/(n_0 + n_j)$, and $1_{\{B_j<b_j\}}$ denotes the indicator function for $\{B_j < b_j\}$. $E_{\boldsymbol{\theta}_j}(W^*_{0,j}) = E_{\boldsymbol{\theta}_j}(W_{0,j}) - E_{\boldsymbol{\theta}_j}(g(B_j))$, where

$$\begin{aligned} E_{\boldsymbol{\theta}_j}(g(B_j)) &= [n_0\log(n_0) + n_j\log(n_j) - (n_0+n_j)\log(n_0+n_j)]E(1_{\{B_j<b_j\}}) \\ &\quad -n_0E(\log(1-B_j)\cdot 1_{\{B_j<b_j\}}) - n_jE(\log(B_j)\cdot 1_{\{B_j<b_j\}}) \end{aligned}$$

$$\text{Var}_{\boldsymbol{\theta}_j}(W^*_{0,j}) = \text{Var}_{\boldsymbol{\theta}_j}(W_{0,j} - g(B_j)) = \text{Var}_{\boldsymbol{\theta}_j}(W_{0,j}) + \text{Var}_{\boldsymbol{\theta}_j}(g(B_j)) - 2\text{Cov}_{\boldsymbol{\theta}_j}(W_{0,j}, g(B_j)),$$

where

$$\text{Var}_{\boldsymbol{\theta}_j}(g(B_j)) = E_{\boldsymbol{\theta}_j}(g(B_j)^2) - [E_{\boldsymbol{\theta}_j}(g(B_j))]^2$$

and

$$\begin{aligned} &\text{Cov}_{\boldsymbol{\theta}_j}(W_{0,j}, g(B_j)) \\ =\ & n_0^2\text{Cov}(\log(1-B_j), \log(1-B_j)\cdot 1_{\{B_j<b_j\}}) + n_0n_j\text{Cov}(\log(1-B_j), \log(B_j)\cdot 1_{\{B_j<b_j\}}) \\ & -n_0[n_0\log(n_0) + n_j\log(n_j) - (n_0+n_j)\log(n_0+n_j)]\text{Cov}(\log(1-B_j), 1_{\{B_j<b_j\}}) \\ & +n_0n_j\text{Cov}(\log(B_j), \log(1-B_j)\cdot 1_{\{B_j<b_j\}}) + n_j^2\text{Cov}(\log(B_j), \log(B_j)\cdot 1_{\{B_j<b_j\}}) \\ & -n_j[n_0\log(n_0) + n_j\log(n_j) - (n_0+n_j)\log(n_0+n_j)]\text{Cov}(\log(B_j), 1_{\{B_j<b_j\}}) \end{aligned}$$

with

$$\begin{aligned} E_{\boldsymbol{\theta}_j}(g(B_j)^2) &= E(\{n_0^2[\log(1-B_j)]^2 + n_j^2[\log(B_j)]^2 + 2n_0n_j\log(1-B_j)\log(B_j) \\ &\quad +[n_0\log(n_0) + n_j\log(n_j) - (n_0+n_j)\log(n_0+n_j)]^2 \\ &\quad -2n_0[n_0\log(n_0) + n_j\log(n_j) - (n_0+n_j)\log(n_0+n_j)]\log(1-B_j) \\ &\quad -2n_j[n_0\log(n_0) + n_j\log(n_j) - (n_0+n_j)\log(n_0+n_j)]\log(B_j)\}\cdot 1_{\{B_j<b_j\}}). \end{aligned}$$

Let $\alpha \equiv (n_j - p)/2$ and $\beta \equiv (n_0 - p)/2$

$$\begin{aligned}
& E(1_{\{B_j<b_j\}}) \\
= & \int_0^{b_j} \frac{\Gamma(\alpha+\beta)x^{\alpha-1}(1-x)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}dx \quad = \quad \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\sum_{m=0}^{\infty}(-1)^m C_m^{\beta-1}\frac{b_j^{\alpha+m}}{\alpha+m} \\
& E(\log(B_j)\cdot 1_{\{B_j<b_j\}}) \\
= & \int_0^{b_j} \log(x)\frac{\Gamma(\alpha+\beta)x^{\alpha-1}(1-x)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}dx \quad = \quad \frac{\partial}{\partial\alpha}E(1_{\{B_j<b_j\}}) \\
& E(\log(1-B_j)\cdot 1_{\{B_j<b_j\}}) \\
= & \int_0^{b_j} \log(1-x)\frac{\Gamma(\alpha+\beta)x^{\alpha-1}(1-x)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}dx \quad = \quad \frac{\partial}{\partial\beta}E(1_{\{B_j<b_j\}}) \\
& E(\log(B_j)\log(1-B_j)\cdot 1_{\{B_j<b_j\}}) \\
= & \int_0^{b_j} \log(x)\log(1-x)\frac{\Gamma(\alpha+\beta)x^{\alpha-1}(1-x)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}dx \quad = \quad \frac{\partial^2}{\partial\alpha\partial\beta}E(1_{\{B_j<b_j\}}) \\
& E([\log(B_j)]^2\cdot 1_{\{B_j<b_j\}}) \\
= & \int_0^{b_j} [\log(x)]^2\frac{\Gamma(\alpha+\beta)x^{\alpha-1}(1-x)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}dx \quad = \quad \frac{\partial^2}{\partial\alpha^2}E(1_{\{B_j<b_j\}}) \\
& E([\log(1-B_j)]^2\cdot 1_{\{B_j<b_j\}}) \\
= & \int_0^{b_j} [\log(1-x)]^2\frac{\Gamma(\alpha+\beta)x^{\alpha-1}(1-x)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}dx \quad = \quad \frac{\partial^2}{\partial\beta^2}E(1_{\{B_j<b_j\}})
\end{aligned}$$