國立交通大學

統計學研究所

碩士論文

檢定兩個自迴歸移動平均模型或兩個隨機係數自迴歸模型的相等性

Testing equality of two ARMA models or two random coefficient autoregressive models

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摘要

時間序列分析是一套動態數據處理的統計方法。基於隨機過程及數理統計理論，分析變數的產生與變數間之動態關係，進而能檢定經濟理論或對變數進行預測，以用於解決實際問題，其中，又以 Box-Jenkins 的自迴歸移動平均模型之分析方法最廣為被大家所使用。此外，理論的發展與推廣，時間序列模型至今已發展到相當複雜的程度，隨機係數自迴歸模型（Random Coefficient Autoregressive model；RCA）便是一個值得深入研究的主題，在這篇論文裡，我們提出檢定兩個自迴歸移動平均模型相等性的方法以及檢定兩個隨機係數自迴歸模型相等性的方法，並且在我們的模擬結果中顯示，我們的方法確實能使該檢定的型一錯誤達到我們設立的顯著水準。我們應用這個分析方法在實際的公司營收資料上，在我們所分析的三間公司中，顯示出有兩間公司的營收在我們所配適的模型上有顯著的差異。

關鍵詞：時間序列、自迴歸移動平均模型、隨機係數自迴歸模型

Testing equality of two ARMA models or two random coefficient autoregressive models

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Abstract

This thesis addresses the issues of testing equality of two time series models. Testing procedures for testing the equality of two ARMA models or two random coefficient autoregressive (RCA) models are proposed. For testing the equality of two ARMA models, we based on the maximum likelihood estimators to establish a testing procedure. For testing equality of two RCA models, an empirical likelihood method is developed. The proposed methods have been demonstrated to have good properties and are shown to have good performance through simulation studies. Also, the testing procedure for testing the equality of two ARMA models is illustrated through an analysis of three companies' monthly sales.

Key words: time series, ARMA model, RCA model

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Contents

List of Figures

List of Tables

1 Introduction

Time series analysis is an area of considerable activity. In the past, economists was using time series for microeconometrics, but they did not carefully explore their statistical properties. Box and Jenkins (1979) rebuild our vision of time series analysis, and then a bunch of books and articles on the subject have been published. The theories and methods have been well estabished and its influence continue to rise. For example, Shumway and Stoffer (2000) presented a balanced and comprehensive treatment of both time and frequency domain methods with accompanying theory and Brockwell and Davis (2009) provided specific techniques for handling data and at the same time to provide an understanding of the mathematical basis for the techniques. Now, time series analysis is used for many applications such as economic forecasting, sales forecasting, stock market analysis, process and quality control.

The time series data in practical problems may consist of observarions from a vector of numbers. For example, in sales forecasting, the variables include sale volume, prices and sales force, and then we can use a multivariate form of the Box-Jenkins model to analyze how is the influence of prices and sales force on sale volume. However, in multivariable time series analysis, we concentrate on input-output relationship between dependent variables and independent variables, and we rarely see the discussions about the comparion of two time series. In the above example, if there are two companys in the study, equality of two company's sales force effect on the prices may be our interests.

On the other hand, nonlinear time series models have attracted much interest during there years. Although most of the time series models discussed are linear models, it has often been found that linear models usually lead to some unexplained aspects. Many developments in nonlinear models techniques provide some alternatives to model time series, and one of examples is the random coefficient model. For this reason, we also pay attention to the comparion of two random coefficient autoregressive (RCA) time series model.

In this article, we are interented in compare two ARMA models or RCA models. Two proposed methods are introduced step by step in the following chapters for ARMA models and RCA models. In additional, we conduct simulation studies for evaluating the performance of both methods. Finally we performe our methods to real data analysis and concluding remarks are given.

2 A Test of Equality of ARMA Models

2.1 Introduction

There are many methods for modeling time series data, and the most widely recognized approach is the Box-Jenkins ARMA models. Classical Box-jenkins models describe stationary time series. A time series $\{x_t; t \in \mathbb{Z}\}$, with $\mathbb{Z} = 0, \pm 1, \pm 2, ...$ is stationary if

$(1) E|x_t|^2 < \infty$ for all $t \in \mathbb{Z}$

$(2) E(x_t)$ is constant for all $t \in \mathbb{Z}$

and

$(3) r_x(r, s) = r_x(r + t, s + t)$ for all $r, s, t \in \mathbb{Z}$,

where $r_x(r, s) = cov(x_r, x_s) = E(x_r - E(x_r), x_s - E(x_s))$ for all $r, s \in \mathbb{Z}$.

A time series $\{x_t\}$ with zero mean is an ARMA(p, q) model if it is stationary and

$$x_t = \phi_1 x_{t-1} + ... + \phi_p x_{t-p} + \omega_t - \theta_1 \omega_{t-1} - .... - \theta_q \omega_{t-q} \tag{2.1}$$

with $\phi_p \neq 0$, $\theta_q \neq 0$. Unless stated otherwise, the noise ω_t is iid $\sim N(0, \delta_\omega^2)$, where $\delta_\omega^2 > 0$. Also, the parameters p and q are called the autoregressive and the moving average orders, respectively. To express the ARMA models in an easy formula, it will be useful to write them using the AR operator and the MA operator. That is, we rewrite the formula (2.1) as

$$\phi(B)x_t = \theta(B)\omega_t \tag{2.2}$$

where $\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - ... - \phi_p B^p$, and $\theta(B) = 1 - \theta_1 B - \theta_2 B^2 - ... - \theta_q B^q$. On the other hand, since the relationships between past and future often occur at seasonal lags, it is appropriate to consider seasonal ARIMA models. The seasonal ARMA model of orders P and Q with the seasonal lags s, denoted by $ARMA(P,Q)_s$, is of the form

$$\Phi_P(B^s)x_t = \Theta_Q(B^s)\omega_t,$$

where $\phi(B^s) = 1 - \phi_1 B^s - \phi_2 B^{2s} - ... - \phi_P B^{Ps}$, and $\theta(B) = 1 - \theta_1 B - \theta_2 B^{2s} - ... - \theta_Q B^{Qs}$.

We only consider causal and invertible ARMA models in this article. An ARMA(p,q) process defined by equation (2.2) is said to be causal if there exists a sequence of constants ψ_j such that $\sum_{j=1}^{\infty} |\psi_j| < \infty$ and

$$x_t = \sum_{t=1}^{\infty} \psi_j \omega_{t-j},\ t = 0, \pm 1, \pm 2, ...$$

and said to be invertible if there exists a sequence of constants π_j such that $\sum_{j=1}^{\infty} |\pi_j| < \infty$ and

$$\omega_t = \sum_{t=1}^{\infty} \pi_j x_{t-j},\ t = 0, \pm 1, \pm 2, ...$$

Since seasonal models are special forms of the ARMA models, the description of the parameter properties is not repeated here.

We consider two time series x_t and y_t which both are ARMA(p,q) process with the forms

$$x_t = \phi_{x,1}x_{t-1} + ... + \phi_{x,p}x_{t-p} + \theta_{x,1}\omega_{x,t-1} + ... + \theta_{x,q}\omega_{x,t-q} \tag{2.3}$$

$$y_t = \phi_{y,1}y_{t-1} + ... + \phi_{y,p}y_{t-p} + \theta_{y,1}\omega_{y,t-1} + ... + \theta_{y,q}\omega_{y,t-q} \tag{2.4}$$

Denote $\underline{\beta}_x = (\phi_{x,1}, ..., \phi_{x,p}, \theta_{x,1}..., \theta_{x,q})'$ and $\underline{\beta}_y = (\phi_{y,1}, ..., \phi_{y,p}, \theta_{y,1}..., \theta_{y,q})'$, respectively. We are interested in testing the equality of two time series models, this is, $\underline{\beta}_x$ $=\underline{\beta}_y$.

In this chapter, we introduce the Box-Jenkins approach for an ARMA model. The porperties and calculations of MLE are also discussed. In particular, the confidence interval for parameters of ARMA models based on MLE can be obtained in an easy way after estimating parameters. Next, we proposes two methods for constructing approximate CI based on MLE, and simulation studies which demonstrates their false positive rate are shown.

2.2 Basic Results

Maximum Likelihood Estimation (MLE) is one of the most popular parameter estimation in time series model, since it possesses a number of good asymptotic properties. However, in the general ARMA models, it is hard to express the likelihood as a function of parameters directly. For this reason, Shumway and Stoffer (2006) suggested to substitute a function of the one-step prediction errors for the explicit way to write the likelihood function. If x_t is causal ARMA(p,q) process with zero mean, the likelihood function of x_t can be written as

$$L(\underline{\beta}_x, \delta_\omega^2) = \prod_{t=1}^{n} f(x_t \mid x_{t-1}, ..., x_1),$$

The distribution of x_t given $x_{t-1}, ..., x_1$ is a Gaussian distribution with mean $x_t^{t-1} = E(x_t \mid x_{t-1}, ..., x_1)$ and variance $P_t^{t-1} = Var(x_t \mid x_{t-1}, ..., x_1)$. In addition, for ARMA models, we may write $P_t^{t-1} = \delta_\omega^2 r_t^{t-1}$ where r_t^{t-1} does not depend on δ_ω^2. In here, x_t^{t-1} and variance P_t^{t-1} are also called the one-step predictor and the mean square prediction error, respectively. They can be solved iteratively by Durbin-Levinson Algorithm (see Durbin, 1960). Now, we rewrite the likelihood function of x_t as

$$L(\underline{\beta}_x, \delta_\omega^2) = (2\pi\delta_\omega^2)^{-n/2}[r_1^0(\underline{\beta}_x) r_2^1(\underline{\beta}_x) \ldots r_n^{n-1}(\underline{\beta}_x)]^{-1/2} exp[\frac{s(\underline{\beta}_x)}{2\delta_\omega^2}], \tag{2.5}$$

where

$$s(\underline{\beta}_x) = \sum_{t=1}^{n} [\frac{(x_t - x_t^{t-1}(\underline{\beta}_x))^2}{r_t^{t-1}(\underline{\beta}_x)}]$$

Since x_t^{t-1} and P_t^{t-1} are explicitly functions of $\underline{\beta}_x$ and δ_ω^2, we can obtain maximum likelihood estimation by maximizing (2.5).

Under appropriate conditions (see Shumway and Stoffer, 2006 p.133 and Brockwell and Davis, 2006 p.258), the maximum likelihood estimation $\widehat{\underline{\beta}}_x$ for causal and invertible ARMA processes, which initialized by method of moments estimator, provide optimal estimator of $\underline{\beta}_x$ and δ_ω^2. Moreover, the asymptotic distribution of $\widehat{\underline{\beta}}_x$ is the normal distribution. It follows,

$$\sqrt{n}(\widehat{\underline{\beta}}_x - \underline{\beta}_x) \stackrel{d}{\to} N(0, V(\underline{\beta}_x)), \tag{2.6}$$

where

$$V(\underline{\beta}_x) = \begin{cases} \delta_\omega^2 \begin{pmatrix} E(\underline{U}_t\underline{U}_t') & E(\underline{U}_t\underline{V}_t') \\ E(\underline{V}_t\underline{U}_t') & E(\underline{V}_t\underline{V}_t') \end{pmatrix}, & \text{for } p \geq 1 \text{ and } q \geq 1 \\ \delta_\omega^2 E(\underline{V}_t\underline{V}_t') & \text{for p=0} \\ \delta_\omega^2 E(\underline{U}_t\underline{U}_t') & \text{for q=0} \end{cases} \tag{2.7}$$

Here, $\underline{U}_t = (U_t, ..., U_{t+1-p})'$ and $\underline{V}_t = (V_t, ..., V_{t+1-q})'$ are the autoregressive processes,

$$\phi(B)\underline{U}_t = \omega_t,$$

and

$$\theta(B)\underline{V}_t = \omega_t.$$

The asymptotic properties of maximum likelihood estimation of ARMA models can be used to construct confidence intervals of $\underline{\beta}_x$.

Although compared with estimation, confidence interval may be a second major problems, it can provide precision of the sample statistic estimation. Since the maximum likelihood estimation $\widehat{\underline{\beta}}_x$ has an asymptotic normal distribution, we can easily derive the following forms from formula (2.6):

$$\{\underline{\beta}_x \in \Re^{p+q} : (\underline{\beta}_x - \widehat{\underline{\beta}}_x)'V^{-1}(\underline{\beta}_x)(\underline{\beta}_x - \widehat{\underline{\beta}}_x) \leq n^{-1}\chi^2_{1-\alpha}(p+q)\} \tag{2.8}$$

Let v_{jj} denote the j-th diagonal element of $V(\underline{\beta}_x)$. We have the approximate $1-\alpha$ confidence region for each component of $\underline{\beta}_x$, i.e.

$$\{\underline{\beta}_{x_j} \in \Re : \mid \widehat{\underline{\beta}}_{x_j} - \underline{\beta}_{x_j} \mid \leq n^{-1/2}\Phi_{1-\alpha/2} v_{jj}^{1/2}\}, \tag{2.9}$$

where $\underline{\beta}_{x_j}$ is the j-th component of $\underline{\beta}_x$. Also, the further discussion is referred to Brockwell and Davis (2006).

2.3 Testing Methods

Let $\widehat{\underline{\beta}}_x$ and $\widehat{\underline{\beta}}_y$ be the estimations of two time series models (2.3) and (2.4). We are interested in testing where $\underline{\beta}_x$ and $\underline{\beta}_y$ are the same, i.e., testing the null hypothesis $H_0 : \underline{\beta}_x = \underline{\beta}_y$ against the alternative hypothsis $H_1 : \underline{\beta}_x \neq \underline{\beta}_y$. Basing on (2.6), we obtained two Gaussian vectors as follows:

$$\sqrt{n}(\widehat{\underline{\beta}}_x - \underline{\beta}_x) \overset{d}{\to} N(0, V(\underline{\beta}_x))$$

$$\sqrt{n}(\widehat{\underline{\beta}}_y - \underline{\beta}_y) \overset{d}{\to} N(0, V(\underline{\beta}_y)).$$

Under null hypothesis, the distribution of the difference of $\widehat{\underline{\beta}}_x$ and $\widehat{\underline{\beta}}_y$ is

$$\sqrt{n}(\widehat{\underline{\beta}}_x - \widehat{\underline{\beta}}_y) \overset{d}{\to} N(0, V(\underline{\beta}_x) + V(\underline{\beta}_y))$$

Let $V^* = (v^*_{ij})_{(p+q)\times(p+q)} = V(\underline{\beta}_x) + V(\underline{\beta}_y)$. Then a $1-\alpha$ confidence region of the difference $l = \underline{\beta}_x - \underline{\beta}_y$ could be drived in a similar way as equation (2.8) and (2.9) as follows:

$$\{l \in \Re^{p+q} : (\underline{\widehat{\beta}}_x - \underline{\widehat{\beta}}_y - l)' V^{*\ -1} (\underline{\widehat{\beta}}_x - \underline{\widehat{\beta}}_y - l) \leq n^{-1} \chi^2_{1-\alpha}(p+q)\} \tag{2.10}$$

and

$$\{l_j \in \Re : | \underline{\widehat{\beta}}_{x_j} - \underline{\widehat{\beta}}_{y_j} - l_j | \leq n^{-1/2} \Phi_{1-\alpha/2} v^{*\ -1/2}_{jj}\} \tag{2.11}$$

and $l_j = (l_1, ..., l_{p+q}) = (\phi_{x1} - \phi_{y1}, ..., \theta_{xq} - \theta_{yq})$.

If we fit two simple one-parameter models for our analysis, we can use equation (2.11) to test equality of parameters in both models. If the number of parameter models is more then one, we can derive a simultaneous confidence interval based equation (2.11) by a Bonferroni approach . The Bonferonni approach gives

$$\alpha[PT] \approx \frac{\alpha[PF]}{C}$$

where the probability of Type I error for testing each l_j is denoted as $\alpha[PT]$, the probability that at least one occurs for the whole family of tests is denoted as $\alpha[PF]$, and C is the number of parameters in the model.

For example, if we fit two AR(2) models to obtain 95% confidence intervals for l, then $\alpha[PE] = 1 - 0.95 = 0.05$, $C = 2$, and $\alpha[PT] = 0.05/2 = 0.025$. Therefore, $\chi^2_{1-\alpha}(p+q)$ becomes $\chi^2_{0.95}(2)$ and $\Phi_{1-\alpha/2}$ becomes $\Phi_{1-0.025/2}$.

The Bonferonni approach is too conservative when the number of comparisons is large. In addition, in practical application, when the asymptotic variance-covariance matrix of the estimator is unknown, we replace $V(\underline{\beta}_x$ and $V(\underline{\beta}_y)$ by their MLEs.

Table 2.1: Testing the equality of two AR(1) models at level 0.05 (difference sample size setting)

$\phi_{x1} = \phi_{y1} = 0.3$ rep.=10000						
sample size	40	60	80	100	120	150
false positive rates	0.0556	0.0545	0.05	0.0497	0.0492	0.0508
$\phi_{x1} = \phi_{y1} = 0.7$ rep.=10000						
sample size	40	60	80	100	120	150
false positive rates	0.0603	0.0577	0.0577	0.055	0.0579	0.0536

2.4 A simulation study

In this section we conduct simulation studies to evaluate our testing results. In the first simulation study, we evaluate the performance of the confidence interval of AR(1) models basd on (2.6) in terms of their false positive rate. For our methods, the sample sizes of two time series that we want to compare may not equal, but we set the same for them in our simulation. We chose sample sizes as 40,60,80,100,120 and 150 and $\sigma_\varepsilon = 1$. For each value of sample sizes, we generated 10000 data sets from the AR(1) model with both $\phi_{x1} = \phi_{y1} = 0.3$ and $\phi_{x1} = \phi_{y1} = 0.7$. Then we computed 95% CIs for $\phi_{x1} - \phi_{y1}$. From Tables 2.1, we see that all of the false positive rates of each value of sample sizes are close to 0.05. Next, we set various paramaters of AR(1) model for the same sample size 150 in Table 2.2. Their false positive rates are also near to 0.05.

In the second simulation study, we consider the AR(2) models and their false positive rate. The false positive rates in Table 2.3 and 2.4 have no obvious difference for these two methods. It show that when the sample size is 150, the false positive rates are very close to 0.05. Therefore, if we want to compare two time series, the sample sizes of the model we fitted should not be less than 100.

Table 2.2: Testing the equality of two AR(1) models at level 0.05 (difference β setting)

$\phi_{x1} = \phi_{y1} = k$ rep.=10000 (size : 150)									
k	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45
false positive rates	0.0515	0.0523	0.0469	0.0522	0.0498	0.05	0.048	0.0553	0.0524
k	0.5	0.55	0.6	0.65	0.7	0.75	0.8	0.85	0.9
false positive rates	0.053	0.052	0.0527	0.0557	0.0542	0.0523	0.0577	0.0647	0.0603

Table 2.3: The bonferroni method for testing the equality of two AR(2) models at level 0.05

$\underline{\beta}_x = \underline{\beta}_y = (\phi_{y1}, \phi_{y2})' = (0.3, 0.3)'$ rep.=10000								
sample size	40	60	80	100	150	200	300	500
false positive rates	0.0645	0.0601	0.0586	0.0563	0.0514	0.0513	0.0495	0.0496

Table 2.4: The Chisque method for testing the equality of two AR(2) models at level 0.05

$\underline{\beta}_x = \underline{\beta}_y = (\phi_{y1}, \phi_{y2})' = (0.3, 0.3)'$ rep.=10000								
sample size	40	60	80	100	150	200	300	500
false positive rates	0.07	0.0629	0.0574	0.0523	0.0505	0.0504	0.0494	0.0515

3 A Test of Equality of RCA Models

3.1 Introduction

The first example for the random coefficient autoregressive (RCA) model was introduced and studied by Nicholls and Quinn (1982). They derived the necessary and sufficient condition for the process to be second-order stationary. In addition, they also discuss some properties and methods for the RCA model. We wrote the model RCA(1) as

$$Z_t = r_t Z_{t-1} + \varepsilon_t \tag{3.1}$$

$$r_t = \mu_r + \sigma_r u_t,$$

where ε_t's and u_t's are sequences of iid realizations from a distribution. And, ε_t and u_t are also independent. Since Wang and Ghosh (2002) defined the $\eta = \mu_r^2 + \sigma_r^2$ and called η the stationary parameter for the RCA(1) model, the necessary and sufficient condition for the process is $\eta < 1$.

A generalized form of the RCA model was introduced by Hwang and Basawa (1998). The Markovian bilinear model, the random coefficient exponential autoregressive process and the RCA model all are special cases of it. A time series Y_t is a generalized random coefficient autoregressive (GRCA) process if

$$Y_t = \phi_t' Y(t-1) + \varepsilon_t. \tag{3.2}$$

where $\phi_t = (\phi_{t1}, \cdots, \phi_{tp})'$, $Y(t-1) = (Y_{t-1}, \cdots, Y_{t-p})'$. In here,

$$E\begin{pmatrix} \phi_t \\ \varepsilon_t \end{pmatrix} = \begin{pmatrix} \phi \\ 0 \end{pmatrix} \text{ and Var} \begin{pmatrix} \phi_t \\ \varepsilon_t \end{pmatrix} = \begin{pmatrix} V_\phi & \sigma_{\phi\varepsilon} \\ \sigma'_{\phi\varepsilon} & \sigma^2_\varepsilon \end{pmatrix}$$

where $\phi = (\phi_1, \cdots, \phi_p)'$, $V_\phi = Var(\phi_t)$ is a $(p \times p)$ matrix, $\sigma_{\phi\varepsilon}=Cov(\phi_t, \varepsilon_t)$ is a $(p \times 1)$ vector, and $\sigma^2_\varepsilon= Var(\varepsilon)$. Note that a GRCA process reduces to the RCA process by setting $\sigma_{\phi\varepsilon} = \underline{0}$.

Hwang and Basawa (1998) had deriven conditional least squares and weighted least squares estimators of the mean of the random vector. Their asymptotic properties and limit distributions had also been studied. Like ARMA models, we rarely see the discussions about the comparion of two time series from RCA models. Considering two time series x_t and y_t which both satisfy formula (3.2), in this chapter, we are interested in testing the equality of ϕ_x and ϕ_y.

3.2 Basic Results

Although the conditional least-squares (LS) and weighted conditional least-squares (WLS) estimators of parameter in the general RCA model had been derived, the high order moment condition that assuming the fourth-order moment of the stationary distribution of the series exists is not easy to be verified. In particular, since the limiting distributions of these estimators also depend on other nuisance parameters, the LS or WLS procedure cannot be directly used to test the hypotheses about ϕ.

Zhao and Wang (2011) using the empirical likelihood (EL) method to the generalized RCA model. The major advantage of EL method is its performance of the confidence intervals on ϕ. In their simulation results, using EL method, the coverage probabilities of the 95% confidence intervals were maintained at around 95% throughout, but LS and WLS method can not reach level 95% when sample size n = 50, 100, 300 and 500. Moreover, they also point out the empircal likelihood method is more accurate and robust than the normal approximation-based method.

Let ϕ_0 denote the true parameter value for ϕ and $G_t(\phi) = Y_t Y(t-1) - Y(t-1)Y'(t-1)\phi$. Then the log-empirical likelihood ratio is

$$l(\phi) = 2\sum_{t=1}^{n} log(1+\lambda^t G_t(\phi)), \tag{3.3}$$

where $\lambda \in \mathbb{R}^p$ satisfies

$$\frac{1}{n}\sum_{t=1}^{n}\frac{G_t(\phi)}{1+\lambda^t G_t(\phi)} = 0.$$

Under appropriate conditions (see Zhao and Wang, 2011), $l(\phi_0)$ converges to the chi-square distribution with degrees of freedom p, i.e.

$$l(\phi_0) \xrightarrow{d} \chi^2(p) \text{ as } n \to \infty$$

Then, for $0 < \alpha < 1$, an asymptotic $100(1-\alpha)\%$ confidence region of ϕ is given by

$$\{\phi \in \Re^p : l(\phi) \leq \chi^2_\alpha(p)\}$$

where $\chi^2_\alpha(p)$ is the upper α-quantile of the chi-square with degrees of freedom p.

3.3 Testing Methods

We consider two time series x_t and y_t which both are from RCA(p) models:

$$x_t = \phi_{xt} X(t-1) + \varepsilon_{xt}$$

$$y_t = \phi_{yt} Y(t-1) + \varepsilon_{yt}$$

We are interested in testing if ϕ_x and ϕ_y are equivalent. Let $l_x(\phi_x)$ and $l_y(\phi_y)$ be the log empirical likelihood ratio of x_t and y_t, respectively. Note that the asymptotic distributions of $l_x(\phi_x)$ and $l_y(\phi_y)$ both are chi-square distributions with degrees of freedom p. This means that

$$l_x(\phi_x) \xrightarrow{d} \chi^2(p) \text{ as } n \to \infty$$

$$l_y(\phi_y) \xrightarrow{d} \chi^2(p) \text{ as } n \to \infty$$

We recall a random variate of the F-distribution arises as the ratio of two appropriately scaled chi-square variates. Therefore, for testing the null hypothesis $H_0 : \phi_x = \phi_y$ against the alternative hypothsis $H_1 : \phi_x \neq \phi_x$, the test statistic F and its asymptotic distribution is

$$F = \frac{l_x(\phi_x)}{l_y(\phi_y)} \xrightarrow{d} F(p,p) \text{ as } n \to \infty \tag{3.4}$$

We rejects $H_0 : l(\phi_x) = l(\phi_y)$ if $F < F_\alpha(p,p)$, where $F_\alpha(p,p)$ is the upper α-quantile of the F distribution with parameters (p,p). Since the above foumla includes the ratio of functions l_x and l_y, we could not obtain the confidence region using this method directly.

3.4 A simulation study

In the simulation study, the sample size is selected to be 150 through this section. Since the beta density function can have different shapes depending on the parameter values, we consider that r_t are iid from the beta(a, b) distribution and (3.1) can be written as

$$Z_t = r_t Z_{t-1} + \varepsilon_t, \tag{3.5}$$

where $r_t \overset{iid}{\sim} Beta(a, b)$, $\varepsilon_t \overset{iid}{\sim} N(0, \sigma_\varepsilon)$, $\mu_r = E(r_t)$ and $\sigma_r = Var(r_t)$. Then, for any a, $b > 0$, the stationary parameter is

$$\begin{aligned}
\eta &= \mu_r^2 + \sigma_r^2 \\
&= (\frac{a}{a+b})^2 + \frac{ab}{(a+b+1)(a+b)^2} \\
&= (\frac{a}{a+b})^2 (\frac{a+b+1}{a+b+1}) + \frac{ab}{(a+b+1)(a+b)^2} \\
&= \frac{a^3 + a^2 b + a^2 + ab}{(a+b+1)(a+b)^2} \\
&= \frac{a^3 + a^2 b + a^2 + ab}{a^3 + 3a^2 b + 3ab^2 + b^3 + a^2 + 2ab + b^2} \\
&= \frac{(a^3 + a^2 b + a^2 + ab)}{(a^3 + a^2 b + a^2 + ab) + 2a^2 b + 3ab^2 + b^3 + ab + b^2}.
\end{aligned}$$

Since $2a^2b + 3ab^2 + b^3 + ab + b^2 > 0$, the stationary condition ($\eta < 1$) is always satisfied.

In this simulation, we set the pairs (a, b) as (0.5,2), (2,2) and (2,0.5). The corresponding μ_r is 0.2, 0.5 and 0.8, respectively. Table 3.1 shows the rejection rates such that the type I error in our testing result is at level 0.05. The simulation replicants is 1000, and the rejection rates are very close 0.05 in difference parmaters setting.

Table 3.1: The EL method for testing the equality of two RCA(1) models at level 0.05

μ_r	0.2	0.5	0.8
(a, b)	$(0.5, 2)$	$(2, 2)$	$(2, 0.5)$
$\sigma_\varepsilon^2 = 1$	0.049	0.051	0.057
$\sigma_\varepsilon^2 = 2$	0.052	0.045	0.061
$\sigma_\varepsilon^2 = 5$	0.046	0.067	0.068

4 Application

In this chapter, we illustrate our testing method by a real data example. The data sets we use are the monthly sales of FamilyMart, President Chain Store and Poya in Taiwan. These data were obtained from Taiwan Economic Journal (TEJ), http://www.finasia.biz/ensite/.

First, we consider the data consists of 145 records for the monthly sales of FamilyMart and President Chain Store ranging from March 2000 to March 2012 presented in Figure 4.1, which show that the two time series are nonstationary. We take first differneces for both series in natural log scale. The sample autocorrelation and partial autocorrelation functions are also ploted in Figure 4.2 . Since the output in Figure 4.2 shows that the first differnece of the natural logarithms dies down very slowly at the seasonal level, we also take seasonal differneces with lag 12 for both series and denote them as Z_{1t} and Z_{2t} , $t = 1, ..., n$, respectively.

The ACF and PACF of the two series Z_{1t} and Z_{2t} are shown in Figure 4.3 and 4.4 which are used to identify a suitable model for the two time series. We determine the ARMA order of Z_{1t} first. At seasonal level, the ACF and PACF of Z_{1t} suggest that we may consider first-order seasonal MA model with the yearly seasonal period $MA(1)_{12}$ or second-order seasonal AR model with the yearly seasonal period $AR(2)_{12}$ to fit seasonal part. Since the coefficient of $MA(1)_{12}$ we estimated is very close to 1, we prefer to ues $AR(2)_{12}$ model tentatively. At nonseasonal level the PACF cuts off at lag 3 and the ACF dies down. We may fit an $AR(3)$ model. Although the partial autocorelation at lag 9 is significant, it is hard to explain why the sales depend on the past ninth month.

We combine the seasonal model and nonseasonal model above. This gives the overall model ARMA$(3,0)(2,0)_{12}$ for Z_{1t}. Since the ACF and PACF of Z_{2t} have similar pattern for Z_{1t}, we directly use the same model to fit Z_{2t}. We can see that both residuls of fited models look like white noise and their ACF and PACF in Figure 4.5 and 4.6 have no spikes in any lag. Hence, we conclude that our models is adequate and the coefficients we estimated are given in Table 4.5.

We performe our methods to test equality of two models. The testing statistic is

$$\begin{aligned}
\chi^2 &= (\hat{\beta}_x - \hat{\beta}_y)' V^{*\,-1} (\hat{\beta}_x - \hat{\beta}_y) \\
&= \left(\begin{pmatrix} -0.2714 \\ -0.1231 \\ -0.3302 \\ -0.8729 \\ -0.4028 \end{pmatrix} - \begin{pmatrix} -0.3972 \\ -0.1788 \\ -0.2086 \\ -0.8623 \\ -0.4187 \end{pmatrix} \right)' V^{*\,-1} \left(\begin{pmatrix} -0.2714 \\ -0.1231 \\ -0.3302 \\ -0.8729 \\ -0.4028 \end{pmatrix} - \begin{pmatrix} -0.3972 \\ -0.1788 \\ -0.2086 \\ -0.8623 \\ -0.4187 \end{pmatrix} \right) \\
&= \begin{pmatrix} 0.1258 \\ 0.0557 \\ -0.1216 \\ -0.0106 \\ 0.0159 \end{pmatrix}' \begin{pmatrix} 0.0149 & 0.0050 & 0.0011 & -0.0014 & -0.0024 \\ 0.0050 & 0.0160 & 0.0044 & -0.0017 & -0.0015 \\ 0.0011 & 0.0044 & 0.0143 & 0.0000 & -0.0003 \\ -0.0014 & -0.0017 & 0.0000 & 0.0140 & 0.0085 \\ -0.0024 & -0.0015 & -0.0003 & 0.0085 & 0.0134 \end{pmatrix}^{-1} \begin{pmatrix} 0.1258 \\ 0.0557 \\ -0.1216 \\ -0.0106 \\ 0.0159 \end{pmatrix} \\
&= 2.602746
\end{aligned}$$

where $V^* =$

$$\begin{pmatrix} 0.0071 & 0.0019 & 0.0003 & -0.0005 & -0.0010 \\ 0.0019 & 0.0073 & 0.0016 & -0.0004 & -0.0005 \\ 0.0003 & 0.0016 & 0.0069 & 0.0002 & 0.0000 \\ -0.0005 & -0.0004 & 0.0002 & 0.0074 & 0.0044 \\ -0.0010 & -0.0005 & 0.0000 & 0.0044 & 0.0070 \end{pmatrix} + \begin{pmatrix} 0.0078 & 0.0031 & 0.0008 & -0.0009 & -0.0014 \\ 0.0031 & 0.0087 & 0.0028 & -0.0013 & -0.0010 \\ 0.0008 & 0.0028 & 0.0074 & -0.0002 & -0.0003 \\ -0.0009 & -0.0013 & -0.0002 & 0.0066 & 0.0041 \\ -0.0014 & -0.0010 & -0.0003 & 0.0041 & 0.0064 \end{pmatrix}$$

Since $\chi^2 = 2.602746 < 11.0705 = \chi^2_{0.05}(5)$, we did not reject the equality of two series under our model assumption.

On the other hand, we are also interested in the variation of FamilyMart's and Poya's monthly sales in the same periods that we analyzed above. We directly take log and differneces in seasonal and nonseasonal lag of Poya's monthly sales and denote it by Z_{3t}. The model $ARMA(3,0)(2,0)_{12}$ is considered as well. We performe bonferroni method and chi-square method for testing the equality of paramaters which we estimated to FamilyMart's and Poya's monthly sales. The chi-square statistic 11.60805 is larger than $\chi^2_{0.05}(5)$. This means that there are significantly difference between FamilyMart's and Poya's relationships between past sales and future sales. However, the critical value by bonferroni approach is $z_{1-0.05/(2*5)} = 2.575829$ and z-value for the five coefficirnt are 2.16, 0.97, 1.49, 1.63, 0.64. There are not any significantly difference between the parameters in two $ARMA(3,0)(2,0)_{12}$ models by bonferroni approach.

Table 4.1: The 145 records for the monthly sales of FamilyMart

1176297	1203511	1298980	1341173	1448275	1412358	1377681	1420655	1324594	1370169
1424020	1257277	1393381	1401237	1493782	1561164	1700699	1742855	1557325	1606533
1496431	1630714	1603934	1625136	1699289	1743039	1897031	1958004	2022666	2000199
1857083	1883538	1736939	1817184	1823917	1757139	1842472	1889945	2034507	2115684
2290404	2231388	2170777	2152775	2047819	2052770	2192003	1972375	2126223	2143391
2357776	2346279	2525170	2551546	2356080	2331998	2213939	2248976	2186831	2172723
2263059	2364689	2434188	2273040	2627003	2703091	2646670	2635123	2736831	2640543
2738463	2338039	2270752	2493470	2694421	2783042	2740841	2692549	2542157	2670539
2408413	2414213	2416360	2502406	2555435	2551918	2728505	2698956	3029872	3344389
3081076	2937270	2753587	2818224	3006540	3026450	2921727	3081081	3274186	3159104
3456286	3470185	3228930	3221119	2952750	3033530	3228163	2899779	3036564	2956038
3258665	3212950	3632916	3679114	3434640	3541790	3171044	3154109	3184423	3260491
3325341	3268256	3717275	3632261	4003762	3960021	3704590	3783590	3547716	3566106
3495592	3308275	3507977	3796799	3927056	3892256	4186706	4203710	4444292	4411963
4145609	4107737	4523114	3898680	4269153					

Table 4.2: The 145 records for the monthly sales of President Chain Store

4491115	4757277	4917705	5317334	5151417	4944261	5114518	4688220	4851881	5245460
4551829	5006136	4955858	5402798	5796654	6071025	6134253	5310996	5542568	5071628
5662084	5440804	5478789	5664451	5795443	6204563	6358800	6711169	6613039	6059449
6141493	5703570	5831072	5902463	5909544	5948880	6123298	6549613	6634948	7244164
7106254	6868259	6737832	6369917	6270463	7005126	6193627	6329060	6378422	6926771
6892067	7314727	7363984	6762306	6739042	6445986	6585437	6427478	6693389	6636050
6874298	8426333	9389209	10189078	8048829	7797608	7920930	7751074	7486858	7957398
6906009	7772104	8182944	8531847	8088966	9657812	9241675	8694701	8637923	8097559
8187449	7775909	8573222	8272678	8087873	9447797	9192908	9296334	8561123	8253002
8471859	7997619	8403231	8286532	8240892	7955810	7937409	8451530	8620256	9401185
9279031	8521024	8904654	8407819	8185085	8678734	7479726	7982511	8173621	9085255
8521446	8795788	8813036	8322949	8602572	8440369	8860415	8761825	8861694	9267065
8895036	9686463	9495227	10289154	10189891	10040908	10105227	9341217	9730864	9336890
9540881	9406303	9534967	10243258	10393956	10806588	10657507	10811993	10941694	10379517
10659172	11675088	9969214	10604789						

Table 4.3: The 145 records for the monthly sales of Poya

134572	139882	151795	142373	169890	158217	156978	144326	131484	146331
159908	154044	167596	164856	159300	160481	175428	174350	178467	171466
159049	170817	153637	154914	158808	159861	169925	159645	166328	190791
228079	203493	191163	203540	217475	209811	227167	202533	226091	213989
241005	239018	245058	239038	217248	260849	266234	237624	220658	225923
245033	231372	239919	280470	289460	279621	240378	284378	271904	270610
255527	256112	265460	262181	278458	275864	272014	281022	270523	319695
295752	269613	263292	264167	276487	272641	288919	298156	331839	296026
307825	352557	309917	338401	319048	312168	338692	317888	359587	366206
402671	363113	347163	395244	380332	416281	374001	376488	382949	382992
391630	477309	436935	443974	423617	448455	524183	415991	405746	401773
451212	412836	474039	511685	508552	494973	460700	556776	523646	508097
490038	471239	499398	468760	534932	550774	525182	516074	465039	538118
546607	513171	457284	465630	469685	513494	564373	563342	566643	538218
486194	571026	619879	511171	486322					

Table 4.5: The estimation of the parimaters of $ARMA(3,0)(2,0)_{12}$ model

FamilyMart

parameter	ϕ_{x1}	ϕ_{x2}	ϕ_{x3}	ϕ_{x12}	ϕ_{x24}
estimation	-0.2714	-0.1231	-0.3302	-0.8729	-0.4028
S.E.	0.0842	0.0854	0.0829	0.0858	0.0837
z-value	3.223278	1.4415	3.9831	10.1737	4.8124

variance of residuals estimated as 0.001545

President Chain stores

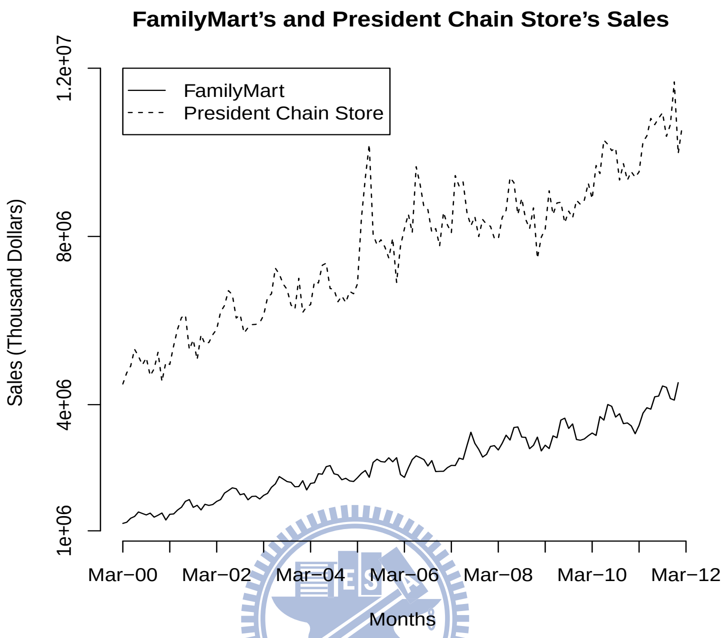
parameter	ϕ_{y1}	ϕ_{y2}	ϕ_{y3}	ϕ_{y12}	ϕ_{y24}
estimation	-0.3972	-0.1788	-0.2086	-0.8623	-0.4187
S.E.	0.0882	0.0931	0.0857	0.0814	0.0800
z-value	4.5034	1.9205	2.4341	10.5934	5.2338

varance of residuals estimated as 0.002339

Poya

parameter	ϕ_{z1}	ϕ_{z2}	ϕ_{z3}	ϕ_{z12}	ϕ_{z24}
estimation	-0.5362	-0.2480	-0.1504	-0.6702	-0.3224
S.E.	0.0889	0.0969	0.0874	0.0904	0.0927
z-value	6.0315	2.5593	1.7208	7.4137	3.4779

variance of residuals estimated as 0.004062



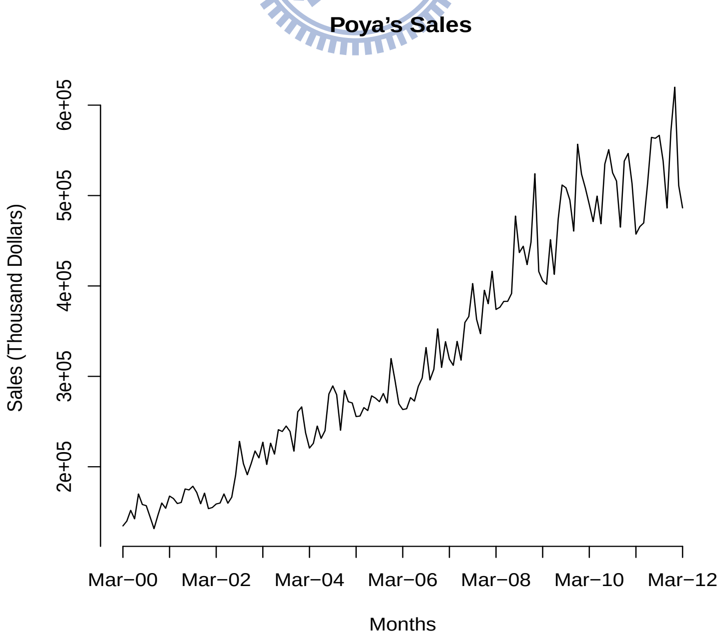


Figure 4.1: Monthly Sales of FamilyMart, President Chain stores and Poya

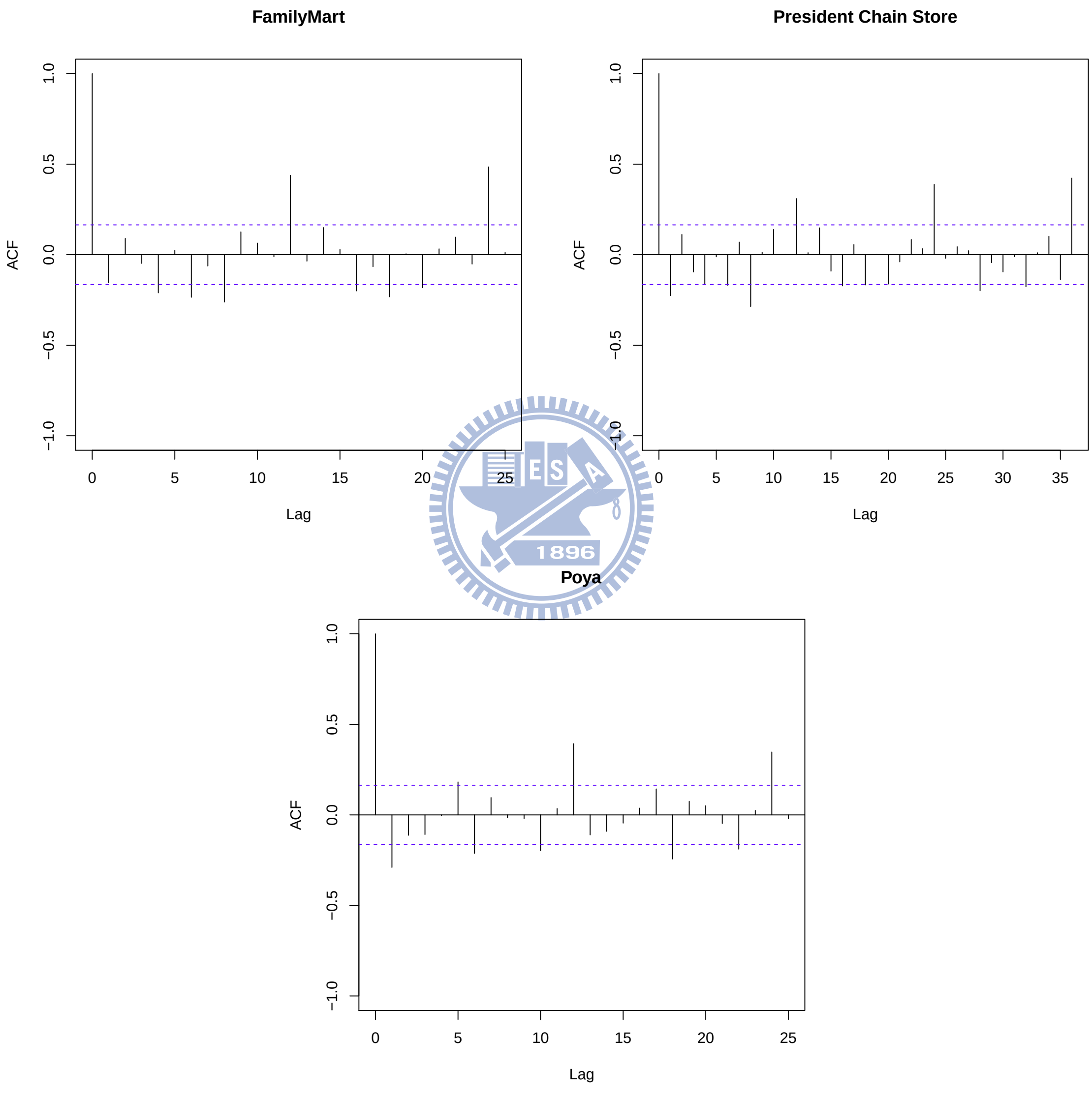


Figure 4.2: the ACF of the first differneces of the natural logarithms.

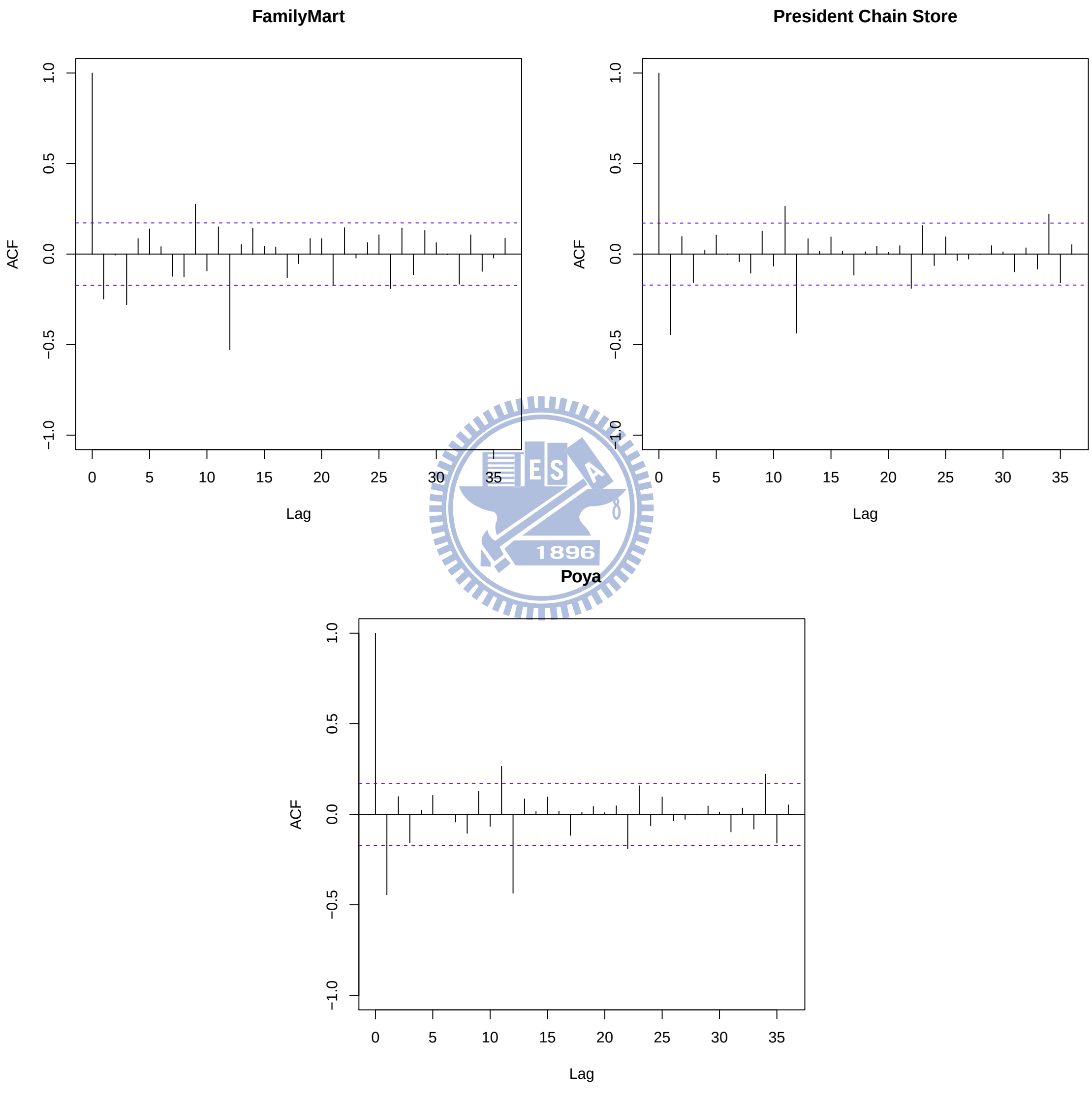


Figure 4.3: the ACF of both series took log and two difference (lag 1 and lag 12)

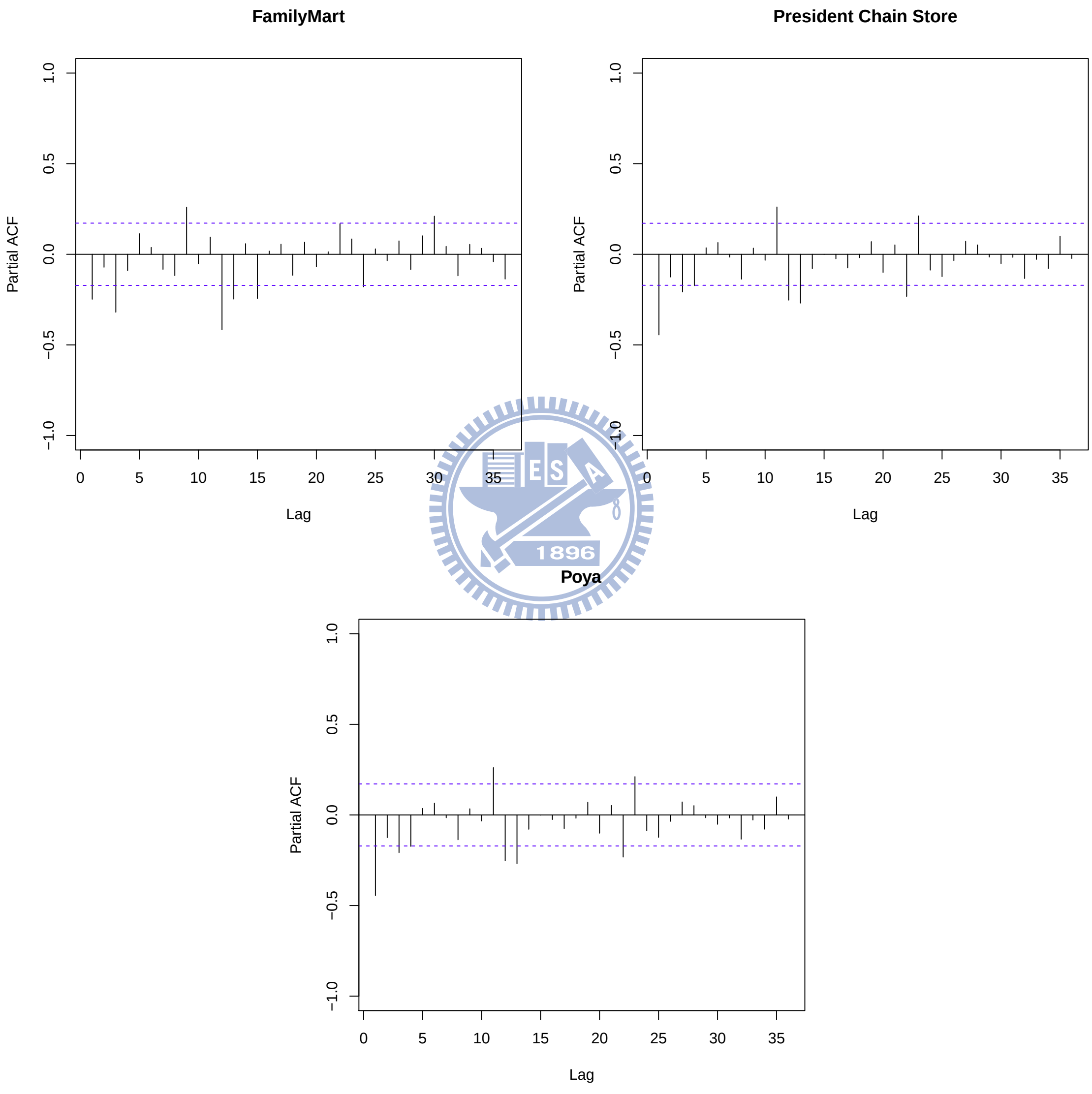


Figure 4.4: the PACF of both series took log and two difference (lag 1 and lag 12)

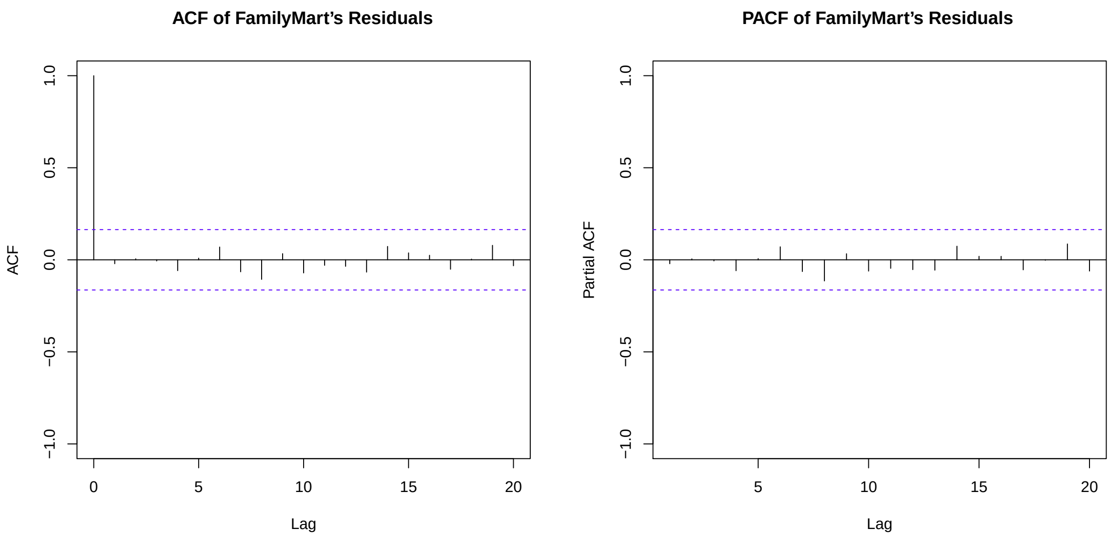


Figure 4.5: the ACF and PACF of residuals of the $ARMA(3,0)(2,0)_{12}$ model of Z_{1t}

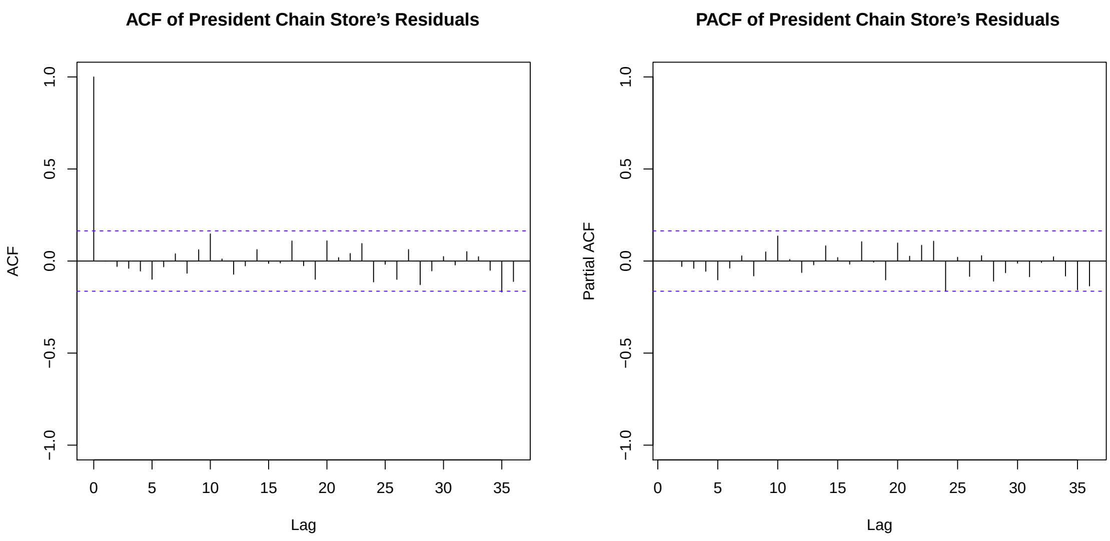


Figure 4.6: the ACF and PACF of residuals of the $ARMA(3,0)(2,0)_{12}$ model of Z_{2t}.

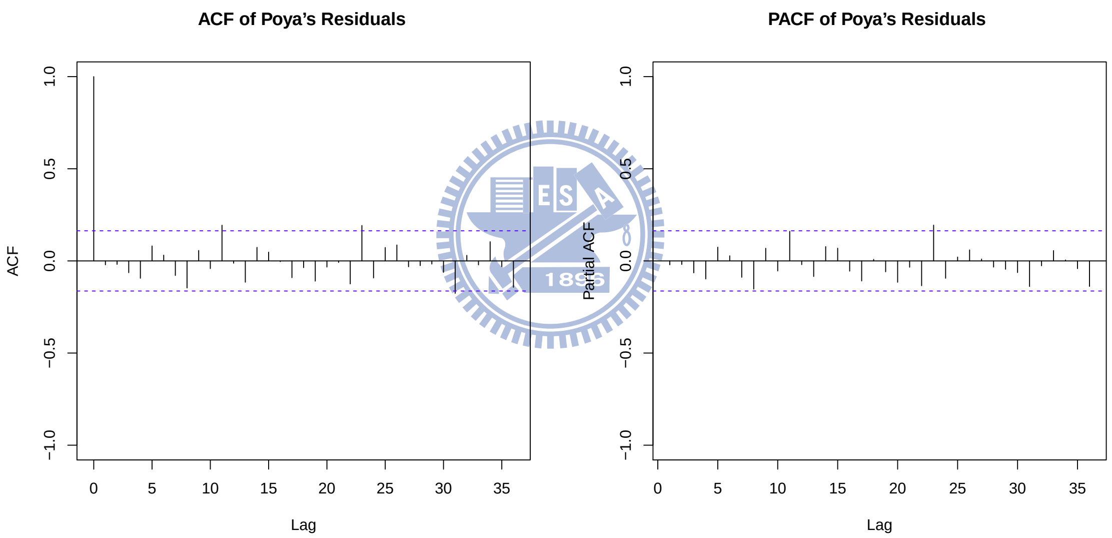


Figure 4.7: the ACF and PACF of residuals of the $ARMA(3,0)(2,0)_{12}$ model fitting Z_{3t}.

5 Conclusions

In this thesis, we study and review the literature on estimation and inference for ARMA models based on MLE method. For comparing time series, we proposed an approach to test the equality of the parameters estimated from two time series. We also presente the Bonferroni approach for multiple testing. In addition to the classical ARMA based methods to compare two time series, we considered the RCA models as well. We performe the empirical likelihood estimation for both RCA models, and then test equality of their means of random coefficients by the F distribution. We also considere beta distribution for the random coefficient of the RCA(1) model and show that the stationary condition is always satisfied. For testing for ARMA models or RCA models, our simulations verify the testing results can attain a desired level mostly. Finally, we practice our methods for real data. The data consists of three companies' monthly sales, namely, FamilyMart, President Chain Store and Poya. In our analysis, we conclude that there are significantly difference between FamilyMart's and Poya's sales behavior.

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