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碩士論文

Confidence Interval for Proportion of Conformance

良率的信賴區間估計

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Confidence Interval for Proportion of Conformance

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摘要

在工業上，良率為生產產品之品質特性落在規格界限內的比例。當我們假設良品的個數服從二項分配時，那麼建造良率的信賴區間是一個在工業應用中很重要的過程。在這項研究中，我們提出了一個方法，以改善現有良率之信賴區間估計。我們設法建立了一個程序來計算所提出的區間之覆蓋率下界。而模擬研究則提供了在不同的區間下去比較各種區間的優點與缺點。

Abstract

Proportion of conformance is defined as the proportion of products with quality characteristic inside the specification limits. The construction of confidence interval for proportion of conformance is an important problem in industrial applications, especially when the number of conforming units follows a binomial distribution. In this study, we propose an approach to improve the existing confidence intervals for proportion of conformance. We establish a procedure to calculate the lower bound for the coverage probability of proposed the intervals. A simulation study is provided to compare the performance of different intervals.

謝辭

本論文得以順利完成，衷心感謝恩師王秀瑛教授。這一年多來受業於恩師門下，在這漫長的學習過程中，老師嚴謹的治學態度與對人尊重的人格風範，一直是我學習效法的典範。回想當初請老師指導時，是既期待又怕受傷害，期待的是可以接受老師之專業指導，而怕受傷害的是擔心駑鈍的我無法完成這艱難的任務，但是老師以無比的耐心與關懷對我的指導與建議，每每讓我如沐春風，並時時督促研究上可能遇到的問題，以及灌輸正確的研究方法與知識，讓我得以突破種種學習瓶頸，終能順利完成此項研究，浩浩師恩，難以為報，謹致以最深之謝意。

Contents

1. Introduction

Proportion of conformance is defined as the proportion of products with quality characteristic inside the specification limits. The specifications are set by engineering requirements or by customers. The construction of confidence interval for proportion of conformance is an important problem in industrial applications in various sectors, including manufacturing and pharmaceuticals. Improper construction of the confidence interval may lead to serious financial losses for the manufacturers.

Wang and Lam (1996) proposed methods for constructing confidence limits for proportion of conformance when the quality characteristic follows a normal distribution. In real application, it is common that the quality characteristic follows a discrete distribution, such as a binomial distribution. In this study, we focus on exploring the coverage probability calculation for confidence intervals of proportion of conformance for the binomial distribution.

To inspect the defective rate of a product, we usually classify each unit inspected as either conforming or nonconforming to the specifications on that quality characteristic. Suppose that the production process is operating in a stable manner, such that the probability that any unit will conform to specifications is θ, and that successive units produced are independent. Then each unit produced is a realization of a Bernoulli random variable with parameter θ. If a random sample of m units of product is selected and X is the number of conforming units, then X can be assumed to follow a binomial distribution $B(m, \theta)$, that is,

$$P_\theta(X = x) = \binom{m}{x}\theta^x(1-\theta)^{m-x}, \quad x = 0,1,\dots,m$$

Suppose that we have a lower specification limit l, and an upper specification limit u. Let r denote the proportion of conformance which is defined to be the proportion that X within the specification limits l and u, that is

$$\begin{aligned} \mathrm{r} &= \mathrm{P}_\theta(l \leq X \leq u) \\ &= 1 - \mathrm{p}_l - \mathrm{p}_u, \end{aligned} \tag{1}$$

where $\mathrm{p}_l = \mathrm{P}_\theta(X \leq l)$ and $\mathrm{p}_u = \mathrm{P}_\theta(X \geq u)$.

When X is a normal distribution, estimators for p_l and p_u have been investigated in the literature (Wheeler 1970, Owen and Hua 1977, Chou and Owen 1984, Wang and Lam 1996). Related studies for conformance proportion are referred to Kotz and Johnson (1993), Kushler and Hurley (1992), Pearn, Kotz and Johnson (1992), and Wang and Lam (1996).

For the discrete distribution case, the construction of confidence interval for r is usually to construct a confidence interval $\left(\mathrm{L}_\theta(\mathrm{X}), \mathrm{U}_\theta(\mathrm{X})\right)$ of θ first, and then replace the θ in (1) by the lower confidence limit $\mathrm{L}_\theta(\mathrm{X})$ and upper confidence limit $\mathrm{U}_\theta(\mathrm{X})$ to obtain a confidence interval for r.

Although this method is an intuitive way to construct a confidence interval for r, the performance of the intervals depends on the confidence interval $\left(\mathrm{L}_\theta(\mathrm{X}), \mathrm{U}_\theta(\mathrm{X})\right)$. In this study, we evaluate the performance of confidence interval for r based on different confidence intervals for θ for the binomial distribution in terms of their coverage probability. The coverage probability of a confidence interval is defined as the probability that the confidence interval $(\mathrm{L}_\theta(\mathrm{X}), \mathrm{U}_\theta(\mathrm{X}))$ covers the true parameter θ.

In addition, it is worth noting that for the discrete distribution, the coverage probability of a confidence interval of r is a variable function of θ. In this case, the minimum coverage probability of confidence interval of r is unknown. To estimate the minimum coverage probability of confidence interval of r, we establish a lower bound for the coverage probability.

The rest of the thesis is organized as follows. Section 2 introduces the existing methods for constructing the upper limits or the lower limit of the proportion of conformance. Section 3 describes the main result. In Section 4, simulation studies are conducted to show the comparison results. In Section 5, an example of deriving the coverage probabilities of confidence intervals for proportion of conformance is provided. Finally, in Section 6, we give a conclusion about the confidence interval for proportion of conformance.

2. Preliminary

Many applications require inferences concerning the probability that the number of conforming units in a future sample is less or equal to (or greater than) some specified number J. Those intervals were described previously in Chandra and Hahn (1981). If the proportion of conforming units in the population are known to equal to θ, the probability p_{LE} that X, the number of conforming units in a sample of size m, will be less than or equal to a pre-specified number J is computed from the binomial cumulative distribution function as

$$p_{LE} = P_{\theta}(X \leq J) = \sum_{i=0}^{J} \binom{m}{i} \theta^{i} (1-\theta)^{m-i} \tag{2}$$

Usually, θ is unknown and only sample data on the number of conforming units in the previous sample are available. Since p_{LE} is a decreasing function of θ, the following two-steps procedure described previously in Chandra and Hahn (1981) is used to find a two-sided confidence interval for p_{LE}:

(i) Obtain a two-sided confidence interval $(L_{\theta}(X), U_{\theta}(X))$ for θ.

(ii) Substitute these values for θ into (2) to obtain the desired two-sided confidence interval on p_{LE}.

This two-steps method are mainly based on a confidence interval of θ, and the performance of confidence interval of the conformance proportion depends on the confidence interval of θ. We introduce several confidence intervals for the binomial proportion θ in the literature (Brown, Cai and DasGupta 2002, Wang 2007).

Four intervals for θ are introduced below. Let k be the upper $\alpha/2$ cutoff point of the standard normal distribution.

1. The exact binomial interval. The $1-\alpha$ exact interval has the form

$$CI_E(X) = \left((1 + \frac{(m - X + 1)F(1 - \frac{\alpha}{2}, 2m - 2X + 2, 2X)}{X})^{-1}, \right.$$

$$\left. (1 + \frac{m - X}{(X + 1)F(1 - \frac{\alpha}{2}, 2X + 2, 2m - 2X)})^{-1} \right).$$

   where $F(r, v, w)$ denotes the 100rth percentile of the F distribution with v and w degrees of freedom.

2. The Wald interval. Let $\hat{p} = \frac{X}{m}$ and $\hat{q} = 1 - \hat{p}$. The approximate interval is

$$CI_{WL}(X) = \left(\hat{p} - k(\hat{p}\hat{q})^{\frac{1}{2}} m^{-\frac{1}{2}}, \quad \hat{p} + k(\hat{p}\hat{q})^{\frac{1}{2}} m^{-\frac{1}{2}} \right).$$

3. The Wilson interval. Let $\tilde{X} = X + \frac{k^2}{2}$ and $\tilde{m} = m + k^2$. Let $\tilde{p} = \frac{\tilde{X}}{\tilde{m}}$ and $\tilde{q} = 1 - \tilde{p}$. The 1-$\alpha$ Wilson interval has the form

$$CI_{WS}(X) = \left(\tilde{p} - \frac{km^{\frac{1}{2}}}{m + k^2} (\hat{p}\hat{q} + \frac{k^2}{4m})^{\frac{1}{2}}, \quad \tilde{p} + \frac{km^{\frac{1}{2}}}{m + k^2} (\hat{p}\hat{q} + \frac{k^2}{4m})^{\frac{1}{2}} \right).$$

4. The Agresti-Coull interval. The 1-α Agresti-Coull interval is

$$CI_{AC}(X) = \left(\tilde{p} - k(\tilde{p}\tilde{q})^{\frac{1}{2}} \tilde{m}^{-\frac{1}{2}}, \quad \tilde{p} + k(\tilde{p}\tilde{q})^{\frac{1}{2}} \tilde{m}^{-\frac{1}{2}} \right).$$

Based on the above confidence intervals for θ and the two-steps method, a lower limit or an upper limit for r can be constructed. Since the existing two-steps method for estimating r is basically to construct a lower limit or an upper limit for r , there is not much investigation established to construct a confidence interval for r.

Although we can simply apply a method similar to the two-steps method to construct a confidence interval $\left(L_r(X), U_r(X)\right)$ of r. However, the coverage probability of the confidence interval constructed by the two-steps method is much lower than the nominal level.

In the next section, we propose a modified confidence interval for r based on the two-step method. The modified confidence intervals lead to a more satisfactory result.

3. The main results

In this section, we propose a procedure to construct a confidence interval $(\mathrm{L_r}(\mathrm{X}), \mathrm{U_r}(\mathrm{X}))$ for r, which mainly uses the result from Wang (2007) and the two-steps procedure from Chandra and Hahn (1981).

Procedure 1: constructing a confidence interval for r

Step 1. Select a level $1-\alpha$ confidence interval $(\mathrm{L_\theta}(\mathrm{X}), \mathrm{U_\theta}(\mathrm{X}))$ for θ.

Step 2. Calculate 1/(1+M), where M=$\left[\binom{\mathrm{m}}{u}(\mathrm{m}-u)/\binom{\mathrm{m}}{l}l\right]^{1/(u-l+1)}$.

Step 3. Let $\mathrm{g}(l,u)$=1/(1+M).

Set the upper bound $\mathrm{U_r}(\mathrm{X})$

$$\equiv \begin{cases} \mathrm{P}_{\mathrm{g(l,u)}}(l \le X \le u), & \mathrm{g}(l,u) \in (\mathrm{L_\theta}(\mathrm{X}), \mathrm{U_\theta}(\mathrm{X})) \\ \max\left[\mathrm{P}_{\mathrm{L_\theta(X)}}(l \le X \le u), \mathrm{P}_{\mathrm{U_\theta(X)}}(l \le X \le u)\right], & \mathrm{g}(l,u) \notin (\mathrm{L_\theta}(\mathrm{X}), \mathrm{U_\theta}(\mathrm{X})) \end{cases}$$

and the lower bound

$$\mathrm{L_r}(\mathrm{X}) \equiv \min\left[\mathrm{P}_{\mathrm{L_\theta(X)}}(l \le X \le u), \mathrm{P}_{\mathrm{U_\theta(X)}}(l \le X \le u)\right].$$

The interval $(\mathrm{L_r}(\mathrm{X}), \mathrm{U_r}(\mathrm{X}))$ is the level $1-\alpha$ confidence interval for r.

The performance of the confidence interval $(\mathrm{L_r}(\mathrm{X}), \mathrm{U_r}(\mathrm{X}))$ for r depends on the confidence interval $(\mathrm{L_\theta}(\mathrm{X}), \mathrm{U_\theta}(\mathrm{X}))$. The following theorem shows that the coverage probability of the proposed interval $(\mathrm{L_r}(\mathrm{X}), \mathrm{U_r}(\mathrm{X}))$ for a θ has a lower bound, which is the coverage probability of $(\mathrm{L_\theta}(\mathrm{X}), \mathrm{U_\theta}(\mathrm{X}))$ for a θ. The performance of the two intervals of their coverage probabilities is discussed in simulation study, which is consistent to the result of the following theorem.

Theorem 1.

Let X follow a binomial distribution $B(m, \theta)$, *and assume that a confidence interval* $(L_\theta(X), U_\theta(X))$ *for* θ *has a coverage probability* w *at* $\theta = \theta_0$. *Then the coverage probability of* $\left(L_\gamma(X), U_\gamma(X)\right)$ *obtained by the procedure 1 based on the confidence interval* $\left(L_\theta(X), U_\theta(X)\right)$ *has a lower bound* w *at* $\theta = \theta_0$.

Proof. Assume that a confidence interval for θ based on X is $(L_\theta(X), U_\theta(X))$.

For a θ, the proportion of conformance $P_\theta(l \le X \le u)$ is

$$\sum_{i=l}^{u} \binom{m}{i} \theta^i (1-\theta)^{m-i} \tag{3}$$

and the lower and upper specification limits are l and u, respectively.

There are three cases for l and u.

(i) l=0, u>0

(ii) l<m, u=m

(iii) l>0, u<m

We will show that this theorem is valid for these three cases.

(i) l=0, u>0

In this case, by Lemma 1 in Wang (2007), (3) is a decreasing function of θ, see Figure 1. Let θ^* denote the true value of θ. If $\theta^* \in (L_\theta(X), U_\theta(X))$, then we intend to show that $P_{\theta^*}(X \le u) \in (L_r(X), U_r(X))$.

Since $P_\theta(X \le u)$ is a decreasing function of θ, we have $P_{U_\theta(X)}(X \le u) < P_{\theta^*}(X \le u) < P_{L_\theta(X)}(X \le u)$,resulting $L_r(X) < P_{\theta^*}(X \le u) < U_r(X)$. Therefore we obtain

$$P_{\theta^*}(X \le u) \in (L_r(X), U_r(X)).$$

Thus we may have over w probability of covering the unknown r with our interval estimator.

(ii) l<m, u=m

In this case, by Lemma 1 in Wang (2007), (3) is an increasing function of θ, see Figure 2. Let θ^* denote the true value of θ. If $\theta^* \in (L_\theta(X), U_\theta(X))$, then we intend to show that $P_{\theta^*}(X \ge l) \in (L_r(X), U_r(X))$. Since $P_\theta(X \ge l)$ is an increasing function of θ,we have

$$P_{L_\theta(X)}(X \ge l) < P_{\theta^*}(X \ge l) < P_{U_\theta(X)}(X \ge l),$$

resulting $L_r(X) < P_{\theta^*}(X \ge l) < U_r(X)$.Therefore we obtain

$$P_{\theta^*}(X \ge l) \in (L_r(X), U_r(X)).$$

Thus we may have over w probability of covering the unknown r with our interval estimator.

(iii)l>0, u<m

In this case, by Lemma 1 in Wang (2007), (3) is an unimodal function of θ. We consider two situations.

The first one is the situation that $g(l,u) \in (L_\theta(X), U_\theta(X))$, see Figure 3.

And the other one is the situation that $g(l,u) \notin (L_\theta(X), U_\theta(X))$. First, assume that $g(l,u)$ lies in $(L_\theta(X), U_\theta(X))$. Since the function (3) reach its maximum value at $\theta = g(l,u)$, we use $g(l,u)$ to construct the upper bound for the confidence interval.

Let θ^* denote the true value of θ. If $\theta^* \in (L_\theta(X), U_\theta(X))$, then we intend to show that $P_{\theta^*}(l \leq X \leq u) \in (L_r(X), U_r(X))$. Since

$$P_{\theta^*}(l \leq X \leq u) < P_{g(l,u)}(l \leq X \leq u) \text{ and}$$

$$\min\left[P_{L_\theta(X)}(l \leq X \leq u), P_{U_\theta(X)}(l \leq X \leq u)\right] < P_{\theta^*}(l \leq X \leq u)$$

we have $L_r(X) < P_{\theta^*}(l \leq X \leq u) < U_r(X)$. Therefore we obtain

$$P_{\theta^*}(l \leq X \leq u) \in (L_r(X), U_r(X)).$$

The other situation is $g(l,u) \notin (L_\theta(X), U_\theta(X))$, so we do not use $g(l,u)$ to construct the upper bound for the confidence interval. It is similar to the first two cases.

Thus we may have over w chance of covering the unknown r with our interval estimator. That is, the coverage probability of above confidence interval for r has a lower bound w. The proof is complete.

4. Simulation results

In this section, we calculate the coverage probability of $(L_\theta(X), U_\theta(X))$ and $(L_r(X), U_r(X))$ for a θ, respectively. Then we compare the coverage probability of $(L_r(X), U_r(X))$ and its corresponding interval $(L_\theta(X), U_\theta(X))$ of θ. The coverage probabilities of the level 0.95 intervals with respect to different parameters when the true parameter θ is 0.5 are shown in Tables 1 and 2. Table 1 lists the coverage probability of several intervals with $l = 0$ and $u = 10$ corresponding to different n when the true θ is 0.5, and m is 30. From Wang (2007), the minimum coverage probability of $(L_\theta(X), U_\theta(X))$ can be exactly calculated. The coverage probability of $(L_r(X), r(X))$ is approximated by simulation.

Table 1. Coverage probabilities of proportion intervals and conformance intervals

n	Exact θ	Exact r	Wald θ	Wald r	Wilson θ	Wilson r	AC θ	AC r
5	1.000	1.000	0.930	0.930	0.930	0.930	0.930	0.930
20	0.961	0.961	0.941	0.941	0.961	0.961	0.961	0.961
30	0.966	0.966	0.922	0.922	0.947	0.947	0.947	0.947
50	0.963	0.963	0.953	0.953	0.953	0.953	0.953	0.953
70	0.974	0.974	0.946	0.946	0.946	0.946	0.946	0.946
90	0.967	0.967	0.937	0.937	0.937	0.937	0.937	0.937
100	0.959	0.959	0.945	0.945	0.959	0.959	0.959	0.959

Table 2. Coverage probabilities of proportion intervals and conformance intervals

n	Exact θ	Exact r	Wald θ	Wald r	Wilson θ	Wilson r	AC θ	AC r
5	1.000	1.000	0.933	0.933	0.933	0.971	0.933	0.971
20	0.964	0.980	0.931	0.946	0.964	0.980	0.964	0.980
30	0.963	0.986	0.937	0.960	0.963	0.986	0.963	0.986
50	0.964	0.981	0.933	0.967	0.950	0.967	0.950	0.967
70	0.957	0.980	0.936	0.959	0.936	0.959	0.936	0.959
90	0.970	0.982	0.941	0.966	0.957	0.982	0.957	0.982
100	0.951	0.974	0.935	0.974	0.951	0.974	0.951	0.974

Table 1 lists the coverage probabilities of the confidence intervals for r are equal to the confidence intervals for θ because we only consider an upper confidence bound or a lower confidence bound. Table 2 lists the coverage probabilities of several intervals with $l = 5$ and $u = 20$ corresponding to different n when the true θ is 0.5, and m is 30. From Tables 1 and 2, the coverage probabilities of the exact intervals for the proportion of conformance are higher than or equal to the other intervals because the exact intervals are substantially longer. The Wald intervals are derived by the large-sample approximation theory. And the coverage probabilities of the Wald intervals for binomial proportion are lower than the other intervals, which may be due to the fluctuation of the simulation and small sample size. It is well known that the Wilson interval and the Agresti-Coull interval are better than the Wald interval (see Agresti and Coull 1988; Brown, Cai, and DasGupta 2001). But even if we use the Wilson interval and the Agresti-Coull interval to construct the confidence intervals for proportion of conformance, their coverage probabilities still cannot be very close to the nominal level. Because all the simulation results show that the coverage probabilities of the confidence intervals for proportion of conformance are higher than or equal to the confidence intervals for a binomial proportion, which are consistent to the result of Theorem 1. Therefore, the coverage probabilities of corresponding intervals for proportion of conformance can approximate the level 0.95. In addition, The simulation results show that the coverage probability of $\left(L_r(X), U_r(X)\right)$ is very close to that of $(L_\theta(X), U_\theta(X))$ when the sample size is large, resulting that the lower bound provided in Theorem 1 can be used to approximate the coverage probability of $\left(L_r(X), U_r(X)\right)$.

The simulation for the cases when θ is not equal to 0.5 and m is 30 is given in Tables 4-19.

5. Real Data Example

We illustrate the proposed methods by a real data example about Department Required Test. The data from the ROC College Entrance Examination Center in 2011 are available on the website **http://www.ceec.edu.tw/**. We intend to analyze the scores of the subject, Scientific Mathematics (abbreviated Sci-Math), which is the most important subject in the Department Required Test. We obtain a random sample of 1000 scores from about 37,000 high school students who attained the Department Required Test. The examination time is 80 minutes and the range of test score is from 0 to 100 for this subject.

If a student has score greater than or equal to 60, we regard this student passing the test. Therefore, we are interested in estimating the probability that the count of students passing the test is between 500 and 700 among 1000 students. In this case, we assume that the count of students passing the test is a random variable, which follows a binomial distribution $\mathrm{B}(1000, \theta)$, where the true θ is 0.3891. We are interested in investigating the probability $\mathrm{r} = \mathrm{P}_\theta(500 \leq X \leq 700)$. By following the procedure provided in Sections 2 and 3, we first calculate the level 0.95 confidence intervals for θ based on the four intervals**,** the exact binomial interval, Wald interval, Wilson interval, and Agresti-Coull interval. Then we follow Procedure 1 to construct the confidence interval for r. The performances of different confidence intervals are presented in Table 3.

Table 3. Coverage probabilities for the proportion of conformance intervals

n	Exact	Wald	Wilson	AC
20	0.967	0.932	0.967	0.967
30	0.975	0.915	0.949	0.949
50	0.958	0.936	0.958	0.958
70	0.966	0.954	0.954	0.954
90	0.966	0.949	0.957	0.957
100	0.960	0.947	0.947	0.947
300	0.953	0.947	0.947	0.947
600	0.952	0.948	0.944	0.944

It shows that the confidence interval constructed by Procedure 1 can lead to a satisfactory result and the coverage probabilities approximate the nominal level when the sample size is large.

6. Conclusions

We purpose a method to improve the existing confidence intervals for the proportion of conformance. Four confidence intervals, the exact binomial interval, Wald interval, Wilson interval, and Agresti-Coull interval, are discussed.

In the simulation studies, all the simulation results show that the coverage probabilities of the proposed intervals for the proportion of conformance are higher than or equal to the corresponding intervals for the binomial proportion, which is consistent to the result of Theorem 1. The coverage probabilities of the Wald intervals for the binomial proportion are lower than the other intervals, but the coverage probabilities of corresponding intervals for proportion of conformance can approximate the nominal level. With the results in this paper, the procedure can be directly used to construct the confidence interval for proportion of conformance.

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Figures

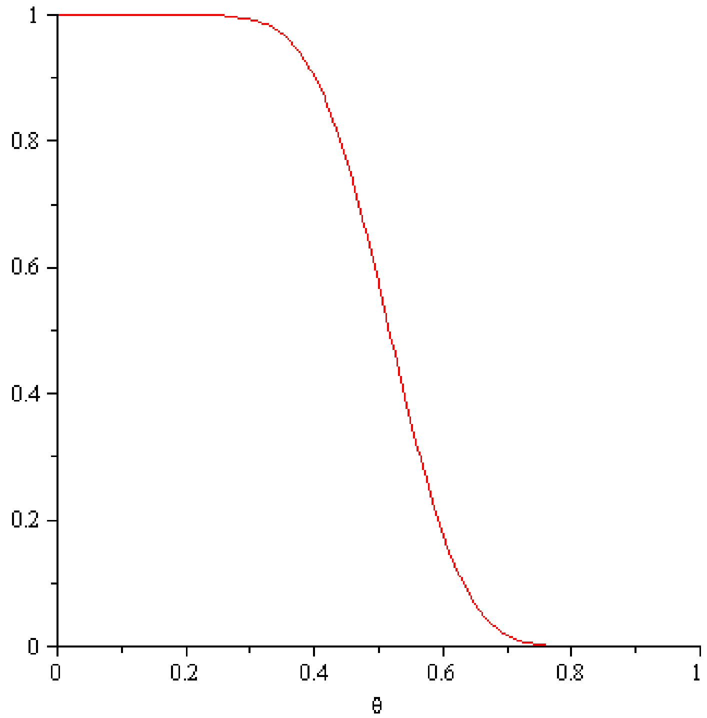

Figure 1. For a fixed u with $0<u<m$, $P_{\theta}(X \leq u)$ is a decreasing function of θ.

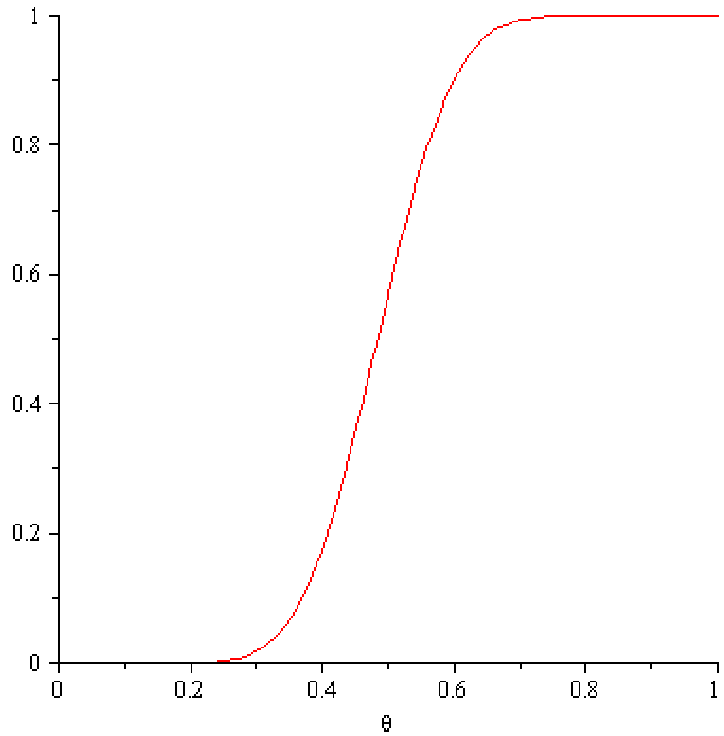

Figure 2. For a fixed *l* with 0<*l*<m, $P_\theta(l \le X)$ is an increasing function of θ.

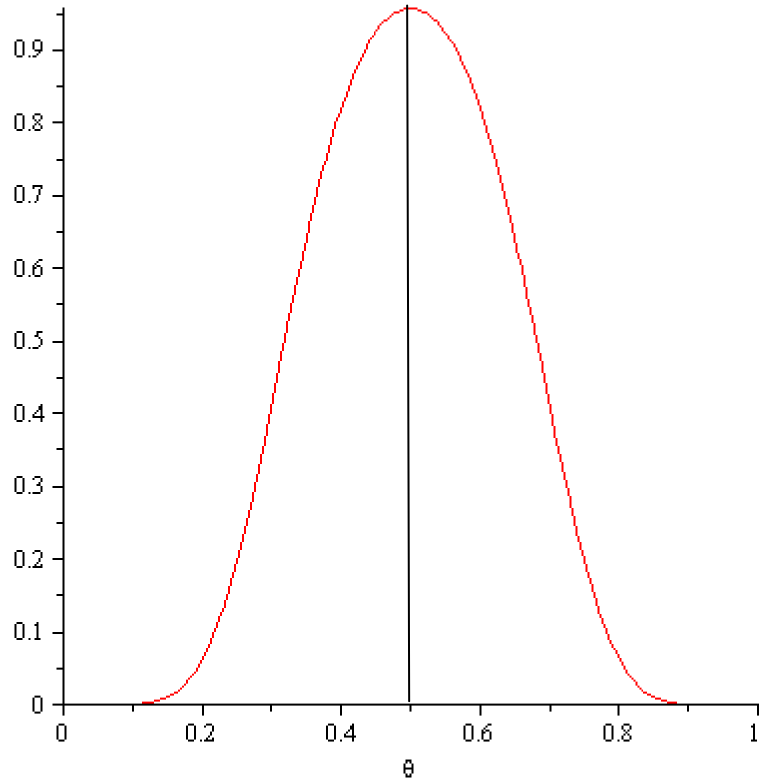

Figure 3. For fixed *l* and *u* with 0<*l*<*u*<m, $P_\theta(l \le X \le u)$ is a unimodal function of θ.

Tables

Table 4. Coverage probabilities with l=0, u=10 when the true θ is 0.4

n	Exact θ	Exact r	Wald θ	Wald r	Wilson θ	Wilson r	AC θ	AC r
5	0.987	0.987	0.825	0.825	0.987	0.987	0.987	0.987
20	0.977	0.977	0.942	0.942	0.963	0.963	0.963	0.963
30	0.972	0.972	0.921	0.921	0.921	0.921	0.921	0.921
50	0.977	0.977	0.940	0.940	0.959	0.959	0.959	0.959
70	0.956	0.956	0.934	0.934	0.956	0.956	0.956	0.956
90	0.974	0.974	0.951	0.951	0.951	0.951	0.951	0.951
100	0.945	0.945	0.932	0.932	0.945	0.945	0.945	0.945

Table 5. Coverage probabilities with l=0, u=10 when the true θ is 0.3

n	Exact θ	Exact r	Wald θ	Wald r	Wilson θ	Wilson r	AC θ	AC r
5	0.998	0.998	0.787	0.787	0.966	0.966	0.966	0.966
20	0.978	0.978	0.953	0.953	0.978	0.978	0.978	0.978
30	0.970	0.970	0.910	0.910	0.935	0.935	0.956	0.956
50	0.978	0.978	0.943	0.943	0.965	0.965	0.965	0.965
70	0.957	0.957	0.936	0.936	0.945	0.945	0.945	0.945
90	0.960	0.960	0.945	0.945	0.948	0.948	0.948	0.948
100	0.961	0.961	0.950	0.950	0.950	0.950	0.950	0.950

Table 6. Coverage probabilities with l=0, u=10 when the true θ is 0.2

n	Exact θ	Exact r	Wald θ	Wald r	Wilson θ	Wilson r	AC θ	AC r
5	0.992	0.992	0.688	0.688	0.949	0.949	0.949	0.949
20	0.975	0.975	0.928	0.928	0.948	0.948	0.948	0.948
30	0.978	0.978	0.952	0.952	0.956	0.956	0.956	0.956
50	0.975	0.975	0.946	0.946	0.955	0.955	0.955	0.955
70	0.965	0.965	0.950	0.950	0.948	0.948	0.948	0.948
90	0.963	0.963	0.931	0.931	0.957	0.957	0.957	0.957
100	0.961	0.961	0.946	0.946	0.948	0.948	0.948	0.948

Table 7. Coverage probabilities with l=0, u=10 when the true θ is 0.1

n	Exact θ	Exact r	Wald θ	Wald r	Wilson θ	Wilson r	AC θ	AC r
5	0.991	0.991	0.408	0.408	0.920	0.920	0.920	0.920
20	0.994	0.994	0.868	0.868	0.965	0.965	0.965	0.965
30	0.995	0.995	0.811	0.811	0.981	0.981	0.981	0.981
50	0.986	0.986	0.899	0.899	0.968	0.968	0.968	0.968
70	0.957	0.957	0.923	0.923	0.940	0.940	0.973	0.973
90	0.972	0.972	0.938	0.938	0.960	0.960	0.960	0.960
100	0.958	0.958	0.930	0.930	0.937	0.937	0.977	0.977

Table 8. Coverage probabilities with l=0, u=10 when the true θ is 0.05

n	Exact θ	Exact r	Wald θ	Wald r	Wilson θ	Wilson r	AC θ	AC r
5	0.974	0.974	0.230	0.230	0.974	0.974	0.974	0.974
20	0.976	0.976	0.622	0.622	0.916	0.916	0.976	0.976
30	0.992	0.992	0.783	0.783	0.948	0.948	0.992	0.992
50	0.990	0.990	0.910	0.910	0.956	0.956	0.956	0.956
70	0.974	0.974	0.873	0.873	0.974	0.974	0.974	0.974
90	0.975	0.975	0.934	0.934	0.949	0.949	0.949	0.949
100	0.987	0.987	0.887	0.887	0.968	0.968	0.968	0.968

Table 9. Coverage probabilities with l=5, u=20 when the true θ is 0.4

n	Exact θ	Exact r	Wald θ	Wald r	Wilson θ	Wilson r	AC θ	AC r
5	0.988	0.988	0.823	0.823	0.988	0.988	0.988	0.988
20	0.964	0.973	0.933	0.933	0.964	0.964	0.964	0.964
30	0.966	0.974	0.940	0.940	0.966	0.966	0.966	0.966
50	0.965	0.980	0.949	0.960	0.965	0.976	0.965	0.976
70	0.973	0.982	0.947	0.966	0.960	0.979	0.960	0.979
90	0.973	0.978	0.953	0.965	0.953	0.965	0.953	0.965
100	0.957	0.977	0.939	0.954	0.951	0.972	0.951	0.972

Table 10. Coverage probabilities with l=5, u=20 when the true θ is 0.3

n	Exact θ	Exact r	Wald θ	Wald r	Wilson θ	Wilson r	AC θ	AC r
5	0.999	1.000	0.804	0.829	0.974	0.999	0.974	1.000
20	0.971	0.989	0.950	0.968	0.944	0.989	0.944	0.989
30	0.983	0.987	0.927	0.947	0.967	0.987	0.967	0.987
50	0.981	0.992	0.923	0.934	0.960	0.979	0.973	0.992
70	0.974	0.984	0.960	0.970	0.958	0.968	0.958	0.968
90	0.959	0.961	0.947	0.949	0.959	0.961	0.959	0.961
100	0.969	0.971	0.960	0.962	0.960	0.962	0.960	0.962

Table 11. Coverage probabilities with l=5, u=20 when the true θ is 0.2

n	Exact θ	Exact r	Wald θ	Wald r	Wilson θ	Wilson r	AC θ	AC r
5	0.997	1.000	0.693	0.696	0.936	1.000	0.936	1.000
20	0.967	0.983	0.913	0.929	0.943	0.983	0.943	0.983
30	0.975	0.979	0.882	0.885	0.958	0.961	0.958	0.961
50	0.971	0.971	0.960	0.960	0.971	0.971	0.971	0.971
70	0.980	0.980	0.940	0.940	0.966	0.966	0.966	0.966
90	0.961	0.961	0.941	0.941	0.955	0.955	0.955	0.955
100	0.963	0.963	0.926	0.926	0.943	0.943	0.943	0.943

Table 12. Coverage probabilities with l=5, u=20 when the true θ is 0.1

n	Exact θ	Exact r	Wald θ	Wald r	Wilson θ	Wilson r	AC θ	AC r
5	0.989	1.000	0.408	0.419	0.921	1.000	0.921	1.000
20	0.983	0.983	0.863	0.863	0.955	0.955	0.955	0.955
30	0.976	0.976	0.791	0.791	0.976	0.976	0.976	0.976
50	0.976	0.976	0.875	0.875	0.976	0.976	0.976	0.976
70	0.976	0.976	0.916	0.916	0.944	0.944	0.944	0.944
90	0.968	0.968	0.934	0.934	0.955	0.955	0.955	0.955
100	0.978	0.978	0.935	0.935	0.964	0.964	0.964	0.964

Table 13. Coverage probabilities with l=5, u=20 when the true θ is 0.05

n	Exact θ	Exact r	Wald θ	Wald r	Wilson θ	Wilson r	AC θ	AC r
5	0.999	1.000	0.220	0.221	0.982	0.983	0.982	0.983
20	0.982	0.982	0.660	0.660	0.982	0.982	0.982	0.982
30	0.980	0.980	0.818	0.818	0.920	0.920	0.980	0.980
50	0.983	0.983	0.936	0.936	0.941	0.941	0.983	0.983
70	0.963	0.963	0.885	0.885	0.945	0.945	0.971	0.971
90	0.971	0.971	0.828	0.828	0.940	0.940	0.971	0.971
100	0.979	0.979	0.904	0.904	0.950	0.950	0.950	0.950

Table 14. Coverage probabilities with l=15, u=25 when the true θ is 0.5

n	Exact θ	Exact r	Wald θ	Wald r	Wilson θ	Wilson r	AC θ	AC r
5	1.000	1.000	0.948	0.948	0.948	0.971	0.948	0.971
20	0.955	0.973	0.925	0.973	0.955	0.973	0.955	0.973
30	0.940	0.967	0.914	0.941	0.940	0.953	0.940	0.953
50	0.957	0.958	0.923	0.924	0.942	0.942	0.942	0.942
70	0.966	0.966	0.952	0.952	0.952	0.952	0.952	0.952
90	0.960	0.960	0.934	0.934	0.949	0.949	0.949	0.949
100	0.962	0.962	0.944	0.944	0.962	0.962	0.962	0.962

Table 15. Coverage probabilities with l=15, u=25 when the true θ is 0.4

n	Exact θ	Exact r	Wald θ	Wald r	Wilson θ	Wilson r	AC θ	AC r
5	0.989	1.000	0.845	0.918	0.989	1.000	0.989	1.000
20	0.960	0.961	0.937	0.944	0.960	0.960	0.960	0.960
30	0.960	0.960	0.941	0.941	0.960	0.960	0.960	0.960
50	0.956	0.956	0.956	0.956	0.956	0.956	0.956	0.956
70	0.961	0.961	0.947	0.947	0.947	0.947	0.947	0.947
90	0.973	0.973	0.950	0.950	0.950	0.950	0.950	0.950
100	0.963	0.963	0.954	0.954	0.963	0.963	0.963	0.963

Table 16. Coverage probabilities with l=15, u=25 when the true θ is 0.3

n	Exact θ	Exact r	Wald θ	Wald r	Wilson θ	Wilson r	AC θ	AC r
5	0.999	1.000	0.797	0.830	0.966	1.000	0.966	1.000
20	0.978	0.978	0.955	0.955	0.955	0.955	0.978	0.978
30	0.976	0.976	0.912	0.912	0.954	0.954	0.954	0.954
50	0.967	0.967	0.944	0.944	0.951	0.951	0.951	0.951
70	0.967	0.967	0.948	0.948	0.951	0.951	0.951	0.951
90	0.965	0.965	0.952	0.952	0.949	0.949	0.949	0.949
100	0.963	0.963	0.946	0.946	0.952	0.952	0.952	0.952

Table 17. Coverage probabilities with l=15, u=25 when the true θ is 0.2

n	Exact θ	Exact r	Wald θ	Wald r	Wilson θ	Wilson r	AC θ	AC r
5	0.991	1.000	0.662	0.671	0.940	0.940	0.940	0.949
20	0.979	0.979	0.929	0.929	0.957	0.957	0.957	0.957
30	0.983	0.983	0.888	0.888	0.953	0.953	0.953	0.953
50	0.972	0.972	0.944	0.944	0.972	0.972	0.972	0.972
70	0.962	0.962	0.933	0.933	0.962	0.962	0.962	0.962
90	0.982	0.982	0.945	0.945	0.968	0.968	0.968	0.968
100	0.972	0.972	0.948	0.948	0.959	0.959	0.959	0.959

Table 18. Coverage probabilities with l=15, u=25 when the true θ is 0.1

n	Exact θ	Exact r	Wald θ	Wald r	Wilson θ	Wilson r	AC θ	AC r
5	0.996	0.996	0.426	0.430	0.914	0.914	0.914	0.914
20	0.986	0.986	0.887	0.887	0.953	0.953	0.953	0.953
30	0.992	0.992	0.789	0.789	0.979	0.979	0.979	0.979
50	0.985	0.985	0.888	0.888	0.968	0.968	0.968	0.968
70	0.956	0.956	0.932	0.932	0.956	0.956	0.967	0.967
90	0.975	0.975	0.952	0.952	0.949	0.949	0.975	0.975
100	0.979	0.979	0.953	0.953	0.969	0.969	0.969	0.969

Table 19. Coverage probabilities with l=15, u=25 when the true θ is 0.05

n	Exact θ	Exact r	Wald θ	Wald r	Wilson θ	Wilson r	AC θ	AC r
5	0.978	0.978	0.226	0.228	0.978	0.978	0.978	0.978
20	0.981	0.981	0.637	0.637	0.927	0.927	0.981	0.981
30	0.980	0.980	0.765	0.765	0.936	0.936	0.980	0.980
50	0.988	0.988	0.920	0.920	0.967	0.967	0.967	0.967
70	0.982	0.982	0.877	0.877	0.982	0.982	0.982	0.982
90	0.970	0.970	0.950	0.950	0.949	0.949	0.949	0.949
100	0.986	0.986	0.886	0.886	0.970	0.970	0.970	0.970