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Asymptotic Behavior of Solutions of a Conservation Law without Convexity Conditions

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1. INTRODUCTION

In this paper we study the asymptotic behavior for the hyperbolic conservation law

$$u_t + f(u)_x = 0, \qquad t \geqslant 0, \quad -\infty < x < \infty, \tag{1.1}$$

$$u(x, 0) = u^0(x), \qquad -\infty < x < \infty, \tag{1.2}$$

with Riemann-like data for $|x|$ large. The function f is a smooth nonlinear function of u. In general, Eq. (1.1) does not have a continuous solution for all time. Shock curves appear after finite time. We will consider a piecewise continuous weak solution of (1.1) [9, 10]. It is well known that across a discontinuity line $x = x(t)$, the solution satisfies the *Rankine–Hugoniot condition* (R–H) and the *entropy condition* (E) [11],

(R–H) $\qquad x'(t) = \sigma(u_-, u_+),$

(E) $\qquad \sigma(u_-, u_+) \leqslant \sigma(u_-, u) \qquad$ for all u between u_- and u_+,

where $u_\pm = u(x(t) \pm 0, t)$ and $\sigma(u_1, u_2)$ is the shock speed defined as

$$\sigma(u_1, u_2) = \frac{f(u_1) - f(u_2)}{u_1 - u_2}.$$

We will consider the solution of (1.1), (1.2) when the initial data $u^0(x)$ are Riemann-like data for $|x|$ large, or more specifically, when $u^0(x)$ satisfies

$$\begin{aligned} u^0(x) &= u_l \qquad \text{for} \quad x \leqslant -S, \\ &= u_r \qquad \text{for} \quad x > S, \end{aligned} \tag{1.3}$$

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for some constants u_l, u_r and S, $S > 0$. For the case when f is convex (or concave) and $u_l = u_r$, the asymptotic behavior was discussed by Lax [8], Keyfitz [7], DiPerna [4] and Dafermos [3]. For the case when f is convex (or concave) and $u_l \neq u_r$, the asymptotic behavior with decay rates was recently obtained by Liu [9]. For more generally smooth f, the asymptotic behavior of solutions of (1.1)–(1.3) without decay rates was partially answered by Liu [10]. It is interesting to note that for the case when the initial data are periodic, for general f, the asymptotic behavior of solutions of (1.1), (1.2) with decay rates was also only partially answered by Greenberg and Tong [6] and Conlon [2]. In this paper we will investigate the asymptotic behavior of solutions of (1.1), (1.2) with initial conditions of the form (1.3). We assume that $f''(u)$ vanishes at a finite number of points and changes sign at these points. The main result which we obtain is that the solution approaches that of the corresponding Riemann problem at algebraic rates (we need the assumption that $f^{(n)}(u) \neq 0$ for some $n < \infty$ at points $f''(u) = 0$). In Section 2, we will consider the case when $f''(u)$ has only one zero. This case will illustrate the nature of the difficulties involved in the general one and will also be needed for the general case. In Section 3, we will consider the general case.

2. The Case When $f''(u)$ Has One Zero

Without loss of generality we assume that the smooth function f satisfies

$$f''(u) \lesseqqgtr 0 \qquad \text{for} \quad u \lesseqqgtr 0. \tag{2.1}$$

We need some definitions and notation. The readers are referred to Ballou [1] for details.

Definition 2.1. Let $\eta < 0$ be given and define $\eta^* = \eta^*(\eta)$ by

$$\eta^* = \sup\{u > \eta \colon \sigma(v, \eta) > \sigma(u, \eta)\ \forall v \in (\eta, u)\}.$$

Let $\eta > 0$ be given and define $\eta_* = \eta_*(\eta)$ by

$$\eta_* = \inf\{u < \eta \colon \sigma(v, \eta) > \sigma(u, \eta)\ \forall v \in (u, \eta)\}.$$

Let $\eta < 0$ be given and define η^{**} as the unique number that satisfies $\eta = (\eta^{**})_*$. Similarly we can define η_{**} for $\eta > 0$ as the unique number that satisfies $\eta = (\eta_{**})^*$. Note that $\eta^* = +\infty$ and $\eta_* = -\infty$ are possible.

DEFINITION 2.2. Under assumption (2.1), the solution of Eq. (1.1) with the following initial condition

$$\begin{aligned} u_\alpha(x, 0) \equiv u_\alpha^0(x) &= u_l && \text{for} \quad x \leqslant -S, \\ &= \alpha && \text{for} \quad -S < x \leqslant S, \\ &= u_r && \text{for} \quad S < x, \end{aligned}$$

where α is a constant, is called $u_\alpha(x, t)$.

The following lemmas on properties of special solutions are proved by direct constructions of these solutions.

LEMMA 2.1. *If $u_l < u_r < u_l^*$ or $u_l > u_r > u_l*$, then there exists t_α and x_α, $t_\alpha \geqslant 0$, such that for all $t \geqslant t_\alpha$,*

$$\begin{aligned} u_\alpha(x, t) &= u_l && \textit{for} \quad x - \sigma t \leqslant x_\alpha, \\ &= u_r && \textit{for} \quad x - \sigma t > x_\alpha, \end{aligned}$$

where $\sigma = \sigma(u_l, u_r)$ and

$$x_\alpha = \frac{1}{u_r - u_l} \left[\frac{u_r + u_l}{2} - \alpha \right] (2S).$$

Proof. Since the case $u_l > u_r > u_l*$ can be considered similarly, we prove the case $u_l < u_r < u_l^*$ only. We divide the proof into several cases:

A. $u_l < u_r \leqslant 0$

(i) $\alpha \leqslant u_l$; then the solution $u_\alpha(x, t)$ is

$$\left.\begin{aligned} u_\alpha(x, t) &= u_l && \text{for} \quad x \leqslant x_1(t), \\ &= h_1\left(\frac{x+S}{t}\right) && \text{for} \quad x_1(t) < x \leqslant x_2(t), \\ &= \alpha && \text{for} \quad x_2(t) < x \leqslant x_3(t), \\ &= u_r && \text{for} \quad x_3(t) < x, \end{aligned}\right\} \quad 0 < t \leqslant t_1,$$

$$\left.\begin{aligned} u_\alpha(x, t) &= u_l && \text{for} \quad x \leqslant x_1(t), \\ &= h_1\left(\frac{x+S}{t}\right) && \text{for} \quad x_1(t) < x \leqslant x_4(t), \\ &= u_r && \text{for} \quad x_4(t) < x, \end{aligned}\right\} \quad t_1 < t \leqslant t_2,$$

$$\left.\begin{aligned} u_\alpha(x, t) &= u_l && \text{for} \quad x \leqslant x_5(t), \\ &= u_r && \text{for} \quad x_5(t) < x, \end{aligned}\right\} \quad t_2 < t,$$

where h_1 is the inverse function of f' restricted in $u < 0$,

$$x_1(t) = -S + f'(u_l)t,$$
$$x_2(t) = -S + f'(\alpha)t,$$
$$x_3(t) = S + \sigma(\alpha, u_r)t,$$

$x_4(t)$ satisfies

$$x_4'(t) = \sigma\left(h_1\left(\frac{x_4(t)+S}{t}\right), u_r\right), \qquad x_4(t_1) = x_2(t_1) = x_3(t_1), t_2 > t \geqslant t_1,$$

and

$$x_4(t_2) = x_1(t_2),$$
$$x_5(t) = x_\alpha + \sigma(u_l, u_r)t, \qquad x_5(t_2) = x_1(t_2) = x_4(t_2);$$

t_1 is the time $x_3(t)$ meets $x_2(t)$ and t_2 is the time $x_4(t)$ meets $x_1(t)$. It is easy to calculate t_1; in fact,

$$t_1 = 2S/(f'(\alpha) - \sigma(\alpha, u_r)).$$

Note that $f'(\alpha) - \sigma(\alpha, u_r) > 0$ is the consequence of entropy condition (E) and assumption (2.1). To see that $x_4(t)$ will meet $x_1(t)$ at finite time t_2, we recall that

$$f'(u_r) < x_4'(t) < f'\left(h_1\left(\frac{x_4(t)+S}{t}\right)\right) = \frac{x_4(t)+S}{t},$$

which is condition (E). (The strict inequalities are due to assumption (2.1).) We can calculate $x_4''(t)$,

$$x_4''(t) = -\frac{h_1'((x_4(t)+S)/t)(x_4'(t) - (x_4(t)+S)/t)^2}{(h_1((x_4(t)+S)/t) - u_r)}$$
$$\leqslant -\frac{\min\{-h_1'((x_4(t)+S)/t)(x_4'(t) - (x_4(t)+S)/t)^2\}}{(u_r - \alpha)} = -\frac{a}{t} < 0. \qquad (2.2)$$

Thus for sufficiently large t, $x_4'(t) < f'(u_l)$. This would ensure that $x_4(t)$ meets $x_1(t)$ at finite time t_2. To find x_α, we proceed as in Liu [9]. We take A sufficiently large so that $u_\alpha(x, t) = u_r$ for $x = A + \sigma t$. It is easy to see from (1.1) and (R–H) that

$$\eta(t) = \int_{-\infty}^{A+\sigma t} (u_\alpha(x, t) - u_l)\, dx$$

is time invariant. For $t \geqslant t_\alpha = t_2$, $\eta(t) = (u_r - u_l)(A - x_\alpha)$. So

$$\eta(t) = \eta(0) = \int_{-\infty}^{A} (u_\alpha(x, 0) - u_l)\, dx$$

$$= \int_{-S}^{S} (\alpha - u_l)\, dx + \int_{S}^{A} (u_r - u_l)\, dx.$$

Thus

$$x_\alpha = \frac{1}{u_r - u_l}\left[\frac{u_r + u_l}{2} - \alpha\right](2S).$$

We prove the lemma in this subcase.

(ii) $u_l < \alpha \leqslant u_r$, then the solution $u_\alpha(x, t)$ is

$$\begin{aligned} u_\alpha(x, t) &= u_l && \text{for } x \leqslant x_1(t), \\ &= \alpha && \text{for } x_1(t) < x \leqslant x_2(t), \qquad 0 < t \leqslant t_1, \\ &= u_r && \text{for } x_2(t) < x, \\ u_\alpha(x, t) &= u_l && \text{for } x \leqslant x_3(t), \\ &= u_r && \text{for } x_3(t) < x, \qquad t_1 < t, \end{aligned}$$

where

$$\begin{aligned} x_1(t) &= -S + \sigma(u_l, \alpha)t, \\ x_2(t) &= S + \sigma(\alpha, u_r)t, \\ x_3(t) &= x_\alpha + \sigma(u_l, u_r)t, \qquad t > t_1, \\ x_1(t_1) &= x_2(t_1) = x_3(t_1). \end{aligned}$$

It is easy to see that $t_\alpha = t_1$ exists and x_α can be obtained as in A(i). This proves the lemma for this subcase.

(iii) $u_r < \alpha \leqslant 0$, then the solution $u_\alpha(x, t)$ is

$$\begin{aligned} u_\alpha(x, t) &= u_l && \text{for } x \leqslant x_1(t), \\ &= \alpha && \text{for } x_1(t) < x \leqslant x_2(t), \\ &= h_1\left(\frac{x - S}{t}\right) && \text{for } x_2(t) < x \leqslant x_3(t), \qquad 0 < t \leqslant t_1, \\ &= u_r && \text{for } x_3(t) < x, \end{aligned}$$

$$
\begin{aligned}
u_\alpha(x,t) &= u_l && \text{for } x \leqslant x_4(t), && \\
&= h_1\left(\frac{x-S}{t}\right) && \text{for } x_4(t) < x \leqslant x_3(t), && t_1 < t \leqslant t_2, \\
&= u_r && \text{for } x_3(t) < x, && \\
u_\alpha(x,t) &= u_l && \text{for } x \leqslant x_5(t), && t_2 < t, \\
&= u_r && \text{for } x_5(t) < x, &&
\end{aligned}
$$

where

$$
\begin{aligned}
x_1(t) &= -S + \sigma(u_l, \alpha)t, \\
x_2(t) &= S + f'(\alpha)t, \\
x_3(t) &= S + f'(u_r)t,
\end{aligned}
$$

$x_4(t)$ satisfies

$$
\begin{aligned}
x_4'(t) &= \sigma\left(u_l, h_1\left(\frac{x_4(t)-S}{t}\right)\right), \\
x_4(t_1) &= x_1(t_1) = x_2(t_1), \qquad t \geqslant t_1,
\end{aligned}
$$

and

$$x_5(t) = x_\alpha + \sigma(u_l, u_r)t = x_\alpha + \sigma t, \qquad x_5(t_2) = x_4(t_2) = x_3(t_2).$$

Using arguments similar to those in case A(i), we can prove that t_2 is finite. Thus we can choose $t_\alpha = t_2$ in this case. x_α can be similarly determined.

(iv) $u_r^{**} \geqslant u_l^*$, $0 < \alpha \leqslant u_l^*$, then the solution $u_\alpha(x,t)$ is

$$
\begin{aligned}
u_\alpha(x,t) &= u_l && \text{for } x \leqslant x_1(t) && \\
&= \alpha && \text{for } x_1(t) < x \leqslant x_2(t), && \\
&= h_1\left(\frac{x-S}{t}\right) && \text{for } x_2(t) < x \leqslant x_3(t), && 0 < t \leqslant t_1, \\
&= u_r && \text{for } x_3(t) < x, && \\
u_\alpha(x,t) &= u_l && \text{for } x \leqslant x_4(t), && \\
&= h_1\left(\frac{x-S}{t}\right) && \text{for } x_4(t) < x \leqslant x_3(t), && t_1 < t \leqslant t_2, \\
&= u_r && \text{for } x_3(t) < x, && \\
u_\alpha(x,t) &= u_l && \text{for } x \leqslant x_5(t), && t_2 < t, \\
&= u_r && \text{for } x_5(t) < x, &&
\end{aligned}
$$

where

$$x_1(t) = -S + \sigma(u_l, \alpha)t,$$
$$x_2(t) = S + \sigma(\alpha, \alpha_*)t = S + f'(\alpha_*)t,$$
$$x_3(t) = S + f'(u_r)t;$$

$x_4(t)$ satisfies

$$x_4'(t) = \sigma\left(u_l, h_1\left(\frac{x_4(t) - S}{t}\right)\right),$$

$$x_4(t_1) = x_1(t_1) = x_2(t_1), \qquad t \geqslant t_1,$$

$$x_5(t) = x_\alpha + \sigma t, \qquad x_5(t_2) = x_4(t_2) = x_3(t_2).$$

Similar arguments as in A(i,iii) can be used to complete the proof of the lemma in this subcase.

(v) $u_r^{**} \geqslant u_l^*$, $u_l^* < \alpha \leqslant u_r^{**}$, then the solution $u_\alpha(x, t)$ is

$$\begin{aligned} u_\alpha(x,t) &= u_l && \text{for } x \leqslant x_1(t), \\ &= h_2\left(\frac{x+S}{t}\right) && \text{for } x_1(t) < x \leqslant x_2(t), \\ &= \alpha && \text{for } x_2(t) < x \leqslant x_3(t), \qquad 0 < t \leqslant t_1, \\ &= h_1\left(\frac{x-S}{t}\right) && \text{for } x_3(t) < x \leqslant x_4(t), \\ &= u_r && \text{for } x_4(t) < x, \end{aligned}$$

$$\begin{aligned} u_\alpha(x,t) &= u_l && \text{for } x \leqslant x_1(t), \\ &= h_2\left(\frac{x+S}{t}\right) && \text{for } x_1(t) < x \leqslant x_5(t), \\ &= u_F(x,t) && \text{for } x_5(t) < x \leqslant x_3(t), \qquad t_1 < t \leqslant t_2, \\ & h_1\left(\frac{x-S}{t}\right) && \text{for } x_3(t) < x \leqslant x_4(t), \\ &= u_r && \text{for } x_4(t) < x, \end{aligned}$$

$$\begin{aligned} u_\alpha(x,t) &= u_l && \text{for } x \leqslant x_6(t), \\ &= u_F(x,t) && \text{for } x_6(t) < x \leqslant x_3(t), \\ &= h_1\left(\frac{x-S}{t}\right) && \text{for } x_3(t) < x \leqslant x_4(t), \qquad t_2 < t \leqslant t_3, \\ &= u_r && \text{for } x_4(t) < x, \end{aligned}$$

$$\begin{aligned} u_\alpha(x,t) &= u_l && \text{for} \quad x \leqslant x_7(t), \\ &= h_1\left(\frac{x-S}{t}\right) && \text{for} \quad x_7(t) < x \leqslant x_4(t), \qquad t_3 < t \leqslant t_4, \\ &= u_r && \text{for} \quad x_4(t) < x, \\ u_\alpha(x,t) &= u_l && \text{for} \quad x \leqslant x_8(t), \\ &= u_r && \text{for} \quad x_8(t) < x, \qquad t_4 < t, \end{aligned}$$

where

$$\begin{aligned} x_1(t) &= -S + \sigma(u_l, u_l^*)t = -S + f'(u_l^*)t, \\ x_2(t) &= -S + f'(\alpha)t, \\ x_3(t) &= S + \sigma(\alpha, \alpha_*)t = S + f'(\alpha_*)t, \\ x_4(t) &= S + f'(u_r)t, \end{aligned}$$

$x_5(t)$ satisfies

$$x_5'(t) = \sigma\left(h_2\left(\frac{x_5(t)+S}{t}\right), h_2\left(\frac{x_5(t)+S}{t}\right)_*\right)$$

with

$$x_5(t_1) = x_2(t_1) = x_3(t_1), \qquad t \geqslant t_1,$$

$x_6(t)$ satisfies

$$x_6'(t) = \sigma(u_l, u_F(x_6(t), t))$$

with

$$x_6(t_2) = x_1(t_2) = x_5(t_2), \qquad t \geqslant t_2,$$

$x_7(t)$ satisfies

$$x_7'(t) = \sigma\left(u_l, h_1\left(\frac{x_7(t)-S}{t}\right)\right)$$

with

$$x_7(t_3) = x_6(t_3) = x_3(t_3), \qquad t \geqslant t_3,$$

and finally

$$x_8(t) = x_\alpha + \sigma t, \qquad x_8(t_4) = x_7(t_4) = x_4(t_4).$$

The curve $x_5(t)$ has the property that $x_5''(t) < 0$. The function $u_F(x, t)$ can be constructed as follows: First, we draw a line through (x, t) which is tangent to curve $x_5(t)$. This line is unique and contacts with $x_5(t)$ at one point, let it be $(\bar{x}, \bar{t})$. Then we let

$$u_F(x, t) = h_2 \left(\frac{\bar{x} + S}{\bar{t}}\right)_* .$$

The existence of the curve $x_5(t)$ and the solution $u_F(x, t)$ can be found in Ballou [1]. The fact that times t_1, t_2, t_3 and t_4 exist and are finite can be established similarly. This completes the proof in this subcase.

(vi) $u_r^{**} \geqslant u_l^*$, $u_r^{**} < \alpha$, then the solution $u_\alpha(x, t)$ is

$$u_\alpha(x, t) = \begin{cases} u_l & \text{for } x \leqslant x_1(t), \\ h_2\left(\dfrac{x+S}{t}\right) & \text{for } x_1(t) < x \leqslant x_3(t), \\ \alpha & \text{for } x_3(t) < x \leqslant x_4(t), \\ u_r & \text{for } x_4(t) < x, \end{cases} \qquad 0 < t \leqslant t_1,$$

$$u_\alpha(x, t) = \begin{cases} u_l & \text{for } x \leqslant x_1(t), \\ h_2\left(\dfrac{x+S}{t}\right) & \text{for } x_1(t) < x < x_5(t), \\ u_r & \text{for } x_5(t) < x, \end{cases} \qquad t_1 < t \leqslant t_2,$$

$$u_\alpha(x, t) = \begin{cases} u_l & \text{for } x \leqslant x_1(t), \\ h_2\left(\dfrac{x+S}{t}\right) & \text{for } x_1(t) < x \leqslant x_6(t), \\ u_F(x, t) & \text{for } x_6(t) < x < x_2(t), \\ u_r & \text{for } x_2(t) < x, \end{cases} \qquad t_2 < t \leqslant t_3,$$

$$u_\alpha(x, t) = \begin{cases} u_l & \text{for } x \leqslant x_7(t), \\ u_F(x, t) & \text{for } x_7(t) < x \leqslant x_2(t), \\ u_r & \text{for } x_2(t) < x, \end{cases} \qquad t_3 < t \leqslant t_4,$$

$$u_\alpha(x, t) = \begin{cases} u_l & \text{for } x \leqslant x_8(t), \\ u_r & \text{for } x_8(t) < x. \end{cases} \qquad t_4 < t,$$

where

$$x_1(t) = -S + \sigma(u_l, u_l^*)t = -S + f'(u_l^*)t$$

$$x_2(t) = -S + f'(u_r^{**})t,$$

$$x_3(t) = -S + f'(\alpha)t,$$

$$x_4(t) = S + \sigma(\alpha, u_r)t,$$

$x_5(t)$ satisfies

$$x_5'(t) = \sigma\left(h_2\left(\frac{x_5(t)+S}{t}\right), u_r\right)$$

with

$$x_5(t_1) = x_3(t_1) = x_4(t_1), \qquad t \geqslant t_1,$$

$x_6(t)$ satisfies

$$x_6'(t) = \sigma\left(h_2\left(\frac{x_6(t)+S}{t}\right), h_2\left(\frac{x_6(t)+S}{t}\right)_*\right)$$

with

$$x_6(t_2) = x_2(t_2) = x_5(t_2), \qquad t \geqslant t_2,$$

$x_7(t)$ satisfies

$$x_7'(t) = \sigma(u_l, u_F(x_7(t), t))$$

with

$$x_7(t_3) = x_1(t_3) = x_6(t_3), \qquad t \geqslant t_3,$$

and

$$x_8(t) = x_\alpha + \sigma t, \qquad x_8(t_4) = x_7(t_4) = x_2(t_4).$$

The function $u_F(x, t)$ is connected to the curve $x_6(t)$ as in A(v). It is straightforward to verify that t_4 is finite; however, we do not go through this here.

(vii) $u_r^{**} < u_l^*$, $0 < \alpha \leqslant u_r^{**}$, then the solution $u_\alpha(x, t)$ is

$$\begin{aligned} u_\alpha(x,t) &= u_l && \text{for } x \leqslant x_1(t), \\ &= \alpha && \text{for } x_1(t) < x \leqslant x_2(t), \\ &= h_1\left(\frac{x-S}{t}\right) && \text{for } x_2(t) < x \leqslant x_3(t), \\ &= u_r && \text{for } x_3(t) < x, \end{aligned} \qquad 0 < t \leqslant t_1,$$

$$\begin{aligned} u_\alpha(x,t) &= u_l && \text{for} \quad x \leqslant x_4(t), \\ &= h_1\left(\frac{x-S}{t}\right) && \text{for} \quad x_4(t) < x \leqslant x_3(t), \qquad t_1 < t \leqslant t_2, \\ &= u_r && \text{for} \quad x_3(t) < x, \\ u_\alpha(x,t) &= u_l && \text{for} \quad x \leqslant x_5(t), \\ &= u_r && \text{for} \quad x_5(t) < x, \qquad t_2 < t, \end{aligned}$$

where

$$\begin{aligned} x_1(t) &= -S + \sigma(u_l, \alpha)t, \\ x_2(t) &= S + f'(\alpha_*)t, \\ x_3(t) &= S + f'(u_r)t, \end{aligned}$$

$x_4(t)$ satisfies

$$x_4'(t) = \left(u_l, h_1\left(\frac{x_4(t)-S}{t}\right)\right)$$

with

$$x_4(t_1) = x_1(t_1) = x_2(t_1), \qquad t \geqslant t_1,$$

and

$$x_5(t) = x_\alpha + \sigma t,$$

with $x_5(t_2) = x_4(t_2) = x_3(t_2)$. This completes the proof for this subcase.

(viii) $u_r^{**} < u_l^*$, $u_r^{**} < \alpha \leqslant u_l^*$, then the solution $u_\alpha(x,t)$ is

$$\begin{aligned} u_\alpha(x,t) &= u_l && \text{for} \quad x \leqslant x_1(t), \\ &= \alpha && \text{for} \quad x_1(t) < x \leqslant x_2(t), \qquad 0 < t \leqslant t_1, \\ &= u_r && \text{for} \quad x_2(t) < x, \\ u_\alpha(x,t) &= u_l && \text{for} \quad x \leqslant x_3(t), \\ &= u_r && \text{for} \quad x_3(t) < x, \qquad t_1 < t, \end{aligned}$$

where

$$\begin{aligned} x_1(t) &= -S + \sigma(u_l, \alpha)t, \\ x_2(t) &= S + \sigma(\alpha, u_r)t, \\ x_3(t) &= x_\alpha + \sigma t \end{aligned}$$

with $x_1(t_1) = x_2(t_1) = x_3(t_1)$. This complete the proof for this subcase.

(ix) $u_r^{**} < u_l^*$, $u_l^* < \alpha$, then the solution $u(x, t)$ is

$$
\begin{aligned}
u_\alpha(x,t) &= u_l && \text{for } x \leqslant x_1(t), && \\
&= h_2\left(\frac{x+S}{t}\right) && \text{for } x_1(t) < x \leqslant x_2(t), && 0 < t \leqslant t_1, \\
&= \alpha && \text{for } x_2(t) < x \leqslant x_3(t), && \\
&= u_r && \text{for } x_3(t) < x, && \\
u_\alpha(x,t) &= u_l && \text{for } x \leqslant x_1(t), && \\
&= h_2\left(\frac{x+S}{t}\right) && \text{for } x_1(t) < x \leqslant x_4(t), && t_1 < t \leqslant t_2, \\
&= u_r && \text{for } x_4(t) < x, && \\
u_\alpha(x,t) &= u_l && \text{for } x \leqslant x_5(t), && t_2 < t, \\
&= u_r && \text{for } x_5(t) < x, &&
\end{aligned}
$$

where

$$
\begin{aligned}
x_1(t) &= -S + \sigma(u_l, u_l^*)t = -S + f'(u_l^*)t, \\
x_2(t) &= -S + f'(\alpha)t, \\
x_3(t) &= S + \sigma(\alpha, u_r)t,
\end{aligned}
$$

$x_4(t)$ satisfies

$$x_4'(t) = \sigma\left(h_2\left(\frac{x_4(t)+S}{t}\right), u_r\right)$$

with

$$x_4(t_1) = x_2(t_1) = x_3(t_1), \qquad t \geqslant t_1,$$

and finally

$$x_5(t) = x_\alpha + \sigma t$$

with $x_5(t_2) = x_1(t_2) = x_4(t_2)$. This complete the proof for subcase A.

B. $u_l < 0 \leqslant u_r < u_l^*$

(i) $\alpha \leqslant u_l$, this case is similar to A(i).

(ii) $u_l < \alpha \leqslant u_r$, this case is similar to A(ii,iv,v).

(iii) $u_r < \alpha \leqslant u_l^*$, this case is also similar to A(ii) (two shocks).

(iv) $u_l^* < \alpha$, then the solution $u(x, t)$ is

$$
\begin{aligned}
u_\alpha(x,t) &= u_l && \text{for} \quad x \leqslant x_1(t), \\
&= h_2\left(\frac{x+S}{t}\right) && \text{for} \quad x_2(t) < x \leqslant x_2(t), \qquad 0 < t \leqslant t_1, \\
&= \alpha && \text{for} \quad x_2(t) < x \leqslant x_3(t), \\
&= u_r && \text{for} \quad x_3(t) < x, \\
u_\alpha(x,t) &= u_l && \text{for} \quad x \leqslant x_1(t), \\
&= h_2\left(\frac{x+S}{t}\right) && \text{for} \quad x_1(t) < x \leqslant x_4(t), \qquad t_1 < t \leqslant t_2, \\
&= u_r && \text{for} \quad x_4(t) < x, \\
u_\alpha(x,t) &= u_l && \text{for} \quad x \leqslant x_5(t), \qquad t_2 < t, \\
&= u_r && \text{for} \quad x_5(t) < x,
\end{aligned}
$$

where

$$
\begin{aligned}
x_1(t) &= -S + \sigma(u_l, u_l^*)t, \\
x_2(t) &= -S + f'(\alpha)t, \\
x_3(t) &= S + \sigma(\alpha, u_r)t,
\end{aligned}
$$

$x_4(t)$ satisfies

$$x_4'(t) = \sigma\left(h_2\left(\frac{x_4(t)+S}{t}\right), u_r\right)$$

with

$$x_4(t_1) = x_3(t_1) = x_2(t_1),$$

and

$$x_5(t) = x_\alpha + \sigma t$$

with $x_5(t_2) = x_4(t_2) = x_1(t_2)$. This completes the proof for this subcase B.

Q.E.D.

LEMMA 2.2. *If* $0 < u_l \leqslant u_r$, *then there exists* t_α, $X_l(\alpha)$ *and* $X_r(\alpha)$, $t_\alpha \geqslant 0$, *such that*

$$
\begin{aligned}
u_\alpha(x, t_\alpha) &= u_l && \textit{for} \quad x \leqslant X_l(\alpha), \\
&= u_\alpha(x, t_\alpha) && \textit{for} \quad X_l(\alpha) < x \leqslant X_r(\alpha), \\
&= u_r && \textit{for} \quad X_r(\alpha) < x,
\end{aligned}
$$

with

$$u_\alpha(x, t_\alpha) \geqslant (u_l*)^* > 0.$$

Remark. We have a similar theorem for $u_r \leqslant u_l < 0$.

Proof. We divide the range of α into several cases:

(i) $0 \leqslant \alpha$, this case is trivial.

(ii) $u_l* \leqslant \alpha < 0$, then the solution $u_\alpha(x, t)$ is

$$\begin{aligned} u_\alpha(x,t) &= u_l && \text{for } x \leqslant x_1(t), \\ &= \alpha && \text{for } x_1(t) < x \leqslant x_2(t), \\ &= h_2\left(\frac{x-S}{t}\right) && \text{for } x_2(t) < x < x_4(t), \\ &= u_r && \text{for } x_4(t) < x, \end{aligned} \qquad 0 < t \leqslant t_1,$$

$$\begin{aligned} u_\alpha(x,t) &= u_l && \text{for } x \leqslant x_5(t), \\ &= h_2\left(\frac{x-S}{t}\right) && \text{for } x_5(t) < x \leqslant x_4(t), \\ &= u_r && \text{for } x_4(t) < x, \end{aligned} \qquad t_1 < t \leqslant t_2,$$

where

$$\begin{aligned} x_1(t) &= -S + \sigma(u_l, \alpha)t, \\ x_2(t) &= S + \sigma(\alpha, \alpha^*)t = S + f'(\alpha^*)t, \\ x_3(t) &= S + f'((u_{l^*})^*)t, \\ x_4(t) &= S + f'(u_r)t, \end{aligned}$$

$x_5(t)$ satisfies

$$x_5'(t) = \sigma\left(u_l, h_2\left(\frac{x_5(t)-S}{t}\right)\right)$$

with

$$x_5(t_1) = x_1(t_1) = x_2(t_1), \qquad t \geqslant t_1,$$

and finally $x_5(t_2) = x_3(t_2)$. It is obvious that we can choose $t = t_2$, $X_l(\alpha) = x_5(t_2)$, $X_r(\alpha) = x_4(t_2)$ to complete the proof for this subcase.

(iii) $(u_r)_{**} \leqslant \alpha < u_l{*}$, then the solution $u_\alpha(x, t)$ is

$$\begin{aligned} u_\alpha(x, t) &= u_l && \text{for } x \leqslant x_1(t), \\ &= h_1\left(\frac{x+S}{t}\right) && \text{for } x_1(t) < x \leqslant x_2(t), \\ &= \alpha && \text{for } x_2(t) < x \leqslant x_3(t), \qquad 0 < t \leqslant t_1, \\ &= h_2\left(\frac{x-S}{t}\right) && \text{for } x_3(t) < x \leqslant x_4(t), \\ &= u_r && \text{for } x_4(t) < x, \end{aligned}$$

$$\begin{aligned} u_\alpha(x, t) &= u_l && \text{for } x \leqslant x_1(t), \\ &= h_1\left(\frac{x+S}{t}\right) && \text{for } x_1(t) < x \leqslant x_5(t), \\ &= u_F(x, t) && \text{for } x_5(t) < x \leqslant x_3(t), \qquad t_1 < t \leqslant t_2, \\ &= h_2\left(\frac{x-S}{t}\right) && \text{for } x_3(t) < x \leqslant x_4(t), \\ &= u_r && \text{for } x_4(t) < x, \end{aligned}$$

where

$$\begin{aligned} x_1(t) &= -S + \sigma(u, u_{l^*})t = -S + f'(u_{l^*})t, \\ x_2(t) &= -S + f'(\alpha)t, \\ x_3(t) &= S + \sigma(\alpha, \alpha^*)t = S + f'(\alpha^*)t, \\ x_4(t) &= S + f'(u_r)t, \end{aligned}$$

$x_5(t)$ satisfies

$$x_5'(t) = \sigma\left(h_1\left(\frac{x_5(t)+S}{t}\right), h_1\left(\frac{x_5(t)+S}{t}\right)_*\right)$$

with $x_5(t_1) = x_2(t_1) = x_3(t_1)$, $t \geqslant t_1$, and t_2 is the time when $x_5(t_2) = x_1(t_2)$. The curve $x_5(t)$ and function $u_F(x, t)$ is similar to the case A(v). Choose $t_\alpha = t_2$, $X_l(\alpha) = x_5(t_2)$, and $X_r(\alpha) = x_4(t_2)$ to complete the proof.

(iv) $\alpha < (u_r)_{**}$, then the solution $u_\alpha(x, t)$ is

$$\begin{aligned} u_\alpha(x, t) &= u_l && \text{for } x \leqslant x_1(t), \\ &= h_2\left(\frac{x+S}{t}\right) && \text{for } x_1(t) < x \leqslant x_3(t), \\ &= \alpha && \text{for } x_3(t) < x \leqslant x_4(t), \qquad 0 < t \leqslant t_1, \\ &= u_r && \text{for } x_4(t) < x, \end{aligned}$$

$$
\begin{aligned}
u_\alpha(x, t) &= u_l && \text{for} \quad x \leqslant x_1(t), \\
&= h_2\left(\frac{x+S}{t}\right) && \text{for} \quad x_1(t) < x \leqslant x_5(t), \qquad t_1 < t \leqslant t_2, \\
&= u_r && \text{for} \quad x_5(t) < x, \\
u_\alpha(x, t) &= u_l && \text{for} \quad x \leqslant x_1(t), \\
&= h_2\left(\frac{x+S}{t}\right) && \text{for} \quad x_1(t) < x \leqslant x_6(t), \\
&= u_F(x, t) && \text{for} \quad x_6(t) < x \leqslant x_2(t), \qquad t_2 < t \leqslant t_3, \\
&= u_r && \text{for} \quad x_2(t) < x,
\end{aligned}
$$

where

$$
\begin{aligned}
x_1(t) &= -S + \sigma(u_l, u_{l^*})t, \\
x_2(t) &= -S + f'(u_r)t = -S + \sigma(u_r)_{**}, u_r)t, \\
x_3(t) &= -S + f'(\alpha)t, \\
x_4(t) &= S + \sigma(\alpha, u_r)t,
\end{aligned}
$$

$x_5(t)$ satisfies

$$x_5'(t) = \sigma\left(h_2\left(\frac{x_5(t)+S}{t}\right), u_r\right)$$

with

$$x_5(t_1) = x_3(t_1) = x_4(t_1), \qquad t \geqslant t_1,$$

$x_6(t)$ satisfies

$$x_6'(t) = \sigma\left(h_2\left(\frac{x_6(t)+S}{t}\right), h_2\left(\frac{x_6(t)+S}{t}\right)^*\right)$$

with

$$x_6(t_2) = x_2(t_2) = x_5(t_2), \qquad t \geqslant t_2,$$

and finally $x_6(t_3) = x_1(t_3)$. Choose $t_\alpha = t_3$, $X_l(\alpha) = x_1(t_3)$, and $X_r(\alpha) = x_2(t_3$ to complete the proof. Q.E.D

LEMMA 2.3. *If* $u_l < u_l^* \leqslant u_r$, *let*

$$x_l(t) = \sup\{x : u_\alpha(x', t) = u_l \ \forall x' < x\},$$

then there exists x_α, $t_\alpha \geqslant 0$ *such that*

(i) $x_l'(t) \geqslant \sigma(u_l, u_l^*)$ *for all* $t > 0$ *except at finite points of* t,

(ii) $x_l(t) \leqslant x_\alpha + \sigma(u_l, u_l^*)t$,

(iii) $\lim_{t\to\infty}[x_\alpha + \sigma(u_l, u_l^*)t - x_l(t)] = 0$,

(iv) $u_l(x, t) \geqslant u_l^*$ *for* $x > x_\alpha + \sigma(u_l, u_l^*)t$, $t \geqslant t_\alpha$.

Remark. We have a similar lemma for the case $u_l > u_l* \geqslant u_r$.

Proof. (i) $\alpha \leqslant (u_r)_{**}$, then the solution $u_\alpha(x, t)$ is

$$
\begin{aligned}
u_\alpha(x, t) &= u_l && \text{for} \quad x \leqslant x_1(t), \\
&= h_1\left(\frac{x+S}{t}\right) && \text{for} \quad x_1(t) < x \leqslant x_3(t), && 0 < t \leqslant t_1, \\
&= \alpha && \text{for} \quad x_3(t) < x \leqslant x_4(t), \\
&= u_r && \text{for} \quad x_4(t) < x, \\
u_\alpha(x, t) &= u_l && \text{for} \quad x \leqslant x_1(t), \\
&= h_1\left(\frac{x+S}{t}\right) && \text{for} \quad x_1(t) < x \leqslant x_5(t), && t_1 < t \leqslant t_2, \\
&= u_r && \text{for} \quad x_5(t) < x, \\
u_\alpha(x, t) &= u_l && \text{for} \quad x \leqslant x_1(t), \\
&= h_1\left(\frac{x+S}{t}\right) && \text{for} \quad x_1(t) < x \leqslant x_6(t), && t_2 < t \leqslant t_3, \\
&= u_F(x, t) && \text{for} \quad x_6(t) < x \leqslant x_9(t), \\
&= u_r && \text{for} \quad x_9(t) < x, \\
u_\alpha(x, t) &= u_l && \text{for} \quad x \leqslant x_7(t), \\
&= u_F(x, t) && \text{for} \quad x_7(t) < x \leqslant x_9(t), && t_3 < t, \\
&= u_r && \text{for} \quad x_9(t) < x,
\end{aligned}
$$

where

$$
\begin{aligned}
x_1(t) &= -S + f'(u_l)t, \\
x_2(t) &= -S + f'(u_r**)t, \\
x_3(t) &= -S + f'(\alpha)t, \\
x_4(t) &= S + \sigma(\alpha, u_r)t,
\end{aligned}
$$

$x_5(t)$ satisfies

$$x_5'(t) = \sigma\left(h_1\left(\frac{x_5(t)+S}{t}\right), u_r\right)$$

with

$$x_5(t_1) = x_3(t_1) = x_4(t_1), \qquad t \geqslant t_1,$$

$x_6(t)$ satisfies

$$x_6'(t) = \sigma\left(h_1\left(\frac{x_5(t)+S}{t}\right), h_1\left(\frac{x_5(t)+S}{t}\right)^*\right)$$

with

$$x_6(t_2) = x_2(t_2) = x_5(t_2), \qquad t \geqslant t_2,$$

and

$$\begin{aligned} x_7(t) = x_1(t_3) + \sigma(u_l, u_l^*)(t - t_3) = x_6(t_3) + f'(u_l^*)(t - t_3), \qquad t \geqslant t_3, \\ x_9(t) = x_2(t_2) + \sigma(u_r**, u_r)(t - t_2) \\ = x_2(t_2) + f'(u_r)(t - t_2). \end{aligned}$$

Obviously,

$$\begin{aligned} x_l(t) &= x_1(t), \qquad 0 < t \leqslant t_3, \\ &= x_7(t), \qquad x_3 < t; \end{aligned}$$

choose $x_\alpha = x_1(t_3) - f'(u^*)t_3$, $t_\alpha = t_3$ to complete the proof for this subcase.

(ii) $u_r** < \alpha \leqslant u_l$, then the solution $u_\alpha(x, t)$ is

$$\begin{aligned} u_\alpha(x, t) &= u_l && \text{for } x \leqslant x_1(t), \\ &= h_1\left(\frac{x+S}{t}\right) && \text{for } x_1(t) < x \leqslant x_2(t), \\ &= \alpha && \text{for } x_2(t) < x \leqslant x_3(t), \qquad 0 < t \leqslant t_1, \\ &= h_2\left(\frac{x-S}{t}\right) && \text{for } x_3(t) < x \leqslant x_4(t), \\ &= u_r && \text{for } x_4(t) < x, \end{aligned}$$

$$
\begin{aligned}
u_\alpha(x,t) &= u_l && \text{for } x \leqslant x_1(t), \\
&= h_1\left(\frac{x+S}{t}\right) && \text{for } x_1(t) < x \leqslant x_5(t), \\
&= u_F(x,t) && \text{for } x_5(t) < x \leqslant x_3(t), \qquad t_1 < t \leqslant t_2, \\
&= h_2\left(\frac{x-S}{t}\right) && \text{for } x_3(t) < x \leqslant x_4(t), \\
&= u_r && \text{for } x_4(t) < x, \\
u_\alpha(x,t) &= u_l && \text{for } x \leqslant x_6(t), \\
&= u_F(x,t) && \text{for } x_6(t) < x \leqslant x_3(t), \\
&= h_2\left(\frac{x-S}{t}\right) && \text{for } x_3(t) < x \leqslant x_4(t), \qquad t_2 < t, \\
&= u_r && \text{for } x_4(t) < x,
\end{aligned}
$$

where

$$
\begin{aligned}
x_1(t) &= -S + f'(u_l)t, \\
x_2(t) &= -S + f'(\alpha)t, \\
x_3(t) &= S + f'(\alpha^*)t, \\
x_4(t) &= S + f'(u_r)t,
\end{aligned}
$$

$x_5(t)$ satisfies

$$x_5'(t) = \sigma\left(h_1\left(\frac{x_5(t)+S}{t}\right), h_1\left(\frac{x_5(t)+S}{t}\right)_*\right)$$

with

$$x_5(t_1) = x_2(t_1) = x_3(t_1), \qquad t \geqslant t_1$$

and

$$x_6(t) = x_1(t_2) + f'(u_l^*)(t-t_2) = x_5(t_2) + f'(u_l^*)(t-t_2).$$

It is obvious that we have

$$
\begin{aligned}
x_l(t) &= x_1(t), && 0 \leqslant t \leqslant t_2, \\
&= x_6(t), && t_2 < t.
\end{aligned}
$$

Choose $x_\alpha = x_1(t_2) - f'(u_l^*)\,t_2$, $t_\alpha = t_2$ to complete the proof for this subcase.

(iii) $u_l < \alpha < 0$, then the solution $u_\alpha(x, t)$ is

$$\left.\begin{aligned} u_\alpha(x,t) &= u_l && \text{for } x \leqslant x_1(t), \\ &= \alpha && \text{for } x_1(t) < x \leqslant x_2(t), \\ &= h_2\left(\frac{x-S}{t}\right) && \text{for } x_2(t) < x \leqslant x_4(t), \\ &= u_r && \text{for } x_4(t) < x, \end{aligned}\right\} \quad 0 < t \leqslant t_1,$$

$$\left.\begin{aligned} u_\alpha(x,t) &= u_l && \text{for } x \leqslant x_5(t), \\ &= h_2\left(\frac{x-S}{t}\right) && \text{for } x_5(t) < x \leqslant x_4(t), \\ &= u_r && \text{for } x_4(t) < x, \end{aligned}\right\} \quad t_1 < t,$$

where

$$\begin{aligned} x_1(t) &= -S + \sigma(u_l, \alpha)t, \\ x_2(t) &= S + \sigma(\alpha, \alpha^*) = S + f'(\alpha^*)t, \\ x_3(t) &= S + f'(u_l^*)t, \\ x_4(t) &= S + f'(u_r)t, \end{aligned}$$

$x_5(t)$ satisfies

$$x_5'(t) = \sigma\left(u_l, h_2\left(\frac{x_5(t) - S}{t}\right)\right)$$

with

$$x_5(t_1) = x_1(t_1) = x_2(t_1), \qquad t \geqslant t_1.$$

It is obvious that

$$\begin{aligned} x_l(t) &= x_1(t) && \text{for } 0 \leqslant t \leqslant t_1, \\ &= x_5(t) && \text{for } t_1 < t. \end{aligned}$$

Choose $x_\alpha = S$, $t_\alpha = 0$ to complete the proof.

(iv) $0 \leqslant \alpha \leqslant u_l^*$, this case is similar to the above case.

(v) $u_l^* < \alpha$, this case is trivial.

Thus we complete the proof for the lemma. Q.E.D.

Before we present our main theorems, we state an ordering principle due originally to Douglis [5]. See also Wu [13], Ballou [1] and Keyfitz [7].

ORDERING PRINCIPLE. Let the function f be smooth function, and let $u(x, t)$ and $v(x, t)$ be piecewise smooth weak solutions satisfying condition (E) to the Cauchy problem (1.1), (1.2), where the initial data are $u(x, 0)$ and $v(x, 0)$, respectively. Then $u(x, 0) \leqslant v(x, 0)$ $\forall x \in (-\infty, \infty)$ implies $u(x, t) \leqslant v(x, t)$ $\forall t \geqslant 0$, $\forall x \in (-\infty, \infty)$.

DEFINITION 2.3. For solutions of (1.1), (1.2), (1.3), $u(x, t)$, let

$$x_l(t) = \sup\{x: u(x', t) = u_l \ \forall x' \leqslant x\},$$
$$x_r(t) = \inf\{x: u(x', t) = u_r \ \forall x' \geqslant x\}.$$

Now we state our main theorems.

THEOREM 2.4. *If $u_l < u_r < u_l^*$, then there exists t_0 and x_0, $t_0 \geqslant 0$, such that for all $t \geqslant t_0$,*

$$\begin{aligned} u(x, t) &= u_l \quad \text{for} \quad x \leqslant x_0 + \sigma t, \\ &= u_r \quad \text{for} \quad x > x_0 + \sigma t, \end{aligned}$$

where $\sigma = \sigma(u_l, u_r)$ and

$$x_0 = \frac{1}{u_r - u_l} \int_{-N}^{N} \left(\frac{u_r + u_l}{2} - u^0(x) \right) dx, \qquad N \geqslant S.$$

Remark. We have a similar theorem for $u_l > u_r > u_l *$.

Proof. Let

$$M = \sup\{u^0(x): -S \leqslant x \leqslant S\}, \tag{2.3}$$
$$m = \inf\{u^0(x): -S \leqslant x \leqslant S\}. \tag{2.4}$$

From Lemma 2.1, let the solutions corresponding to $\alpha = M$ and $\alpha = m$ be respectively $u_M(x, t)$ and $u_m(x, t)$ with corresponding x_M, x_m, t_M, t_m. It is easy to see that $x_M \leqslant x_m$. If $x_M = x_m$, which is the case $M = m$, then we are done. So assume $x_M < x_m$. Using the Ordering Principle, we have

$$u_m(x, t) \leqslant u(x, t) \leqslant u_M(x, t) \qquad \forall x, \ \forall t \geqslant 0.$$

Thus for all $t \geqslant \bar{t} = \max\{t_M, t_m\}$, we have

$$\begin{aligned} u(x, t) &= u_l \quad \text{for} \quad x \leqslant x_M + \sigma t, \\ &= u_r \quad \text{for} \quad x > x_m + \sigma t \end{aligned}$$

and

$$u_l = u_m(x, t) \leqslant u(x, t) \leqslant u_M(x, t) = u_r \qquad \text{for} \quad x_M + \sigma t < x \leqslant x_m + \sigma t.$$

From Definition 2.3, for $t \geqslant \bar{t}$, it is easy to see that

$$x_M + \sigma t \leqslant x_l(t) \leqslant x_r(t) \leqslant x_m + \sigma t.$$

Furthermore $x_l(t)$ and $x_r(t)$ are Lipschitz continuous curves with slopes

$$x_l'(t) = \sigma(u_l, u(x_l(t) + 0, t)),$$

and

$$x_r'(t) = \sigma(u_r, u(x_r(t) - 0, t)),$$

respectively, and with bounded second derivatives. Since $u(x_l(t) + 0, t)$ and $u(x_r(t) - 0, t)$ are both between u_l and u_r for $t \geqslant \bar{t}$, we have from (E)

$$x_l'(t) \geqslant \sigma(u_l, u_r) = \sigma \geqslant x_r'(t).$$

Thus $x_l'(t) - \sigma \geqslant 0$ and $x_r'(t) - \sigma \leqslant 0$ or $(x_l(t) - \sigma t)$ is nondecreasing and $x_r(t) - \sigma t$ is nonincreasing for $t \geqslant \bar{t}$. But $x_l(t) - \sigma t \leqslant x_r(t) - \sigma t$ for $t \geqslant \bar{t}$. Thus if there exists $t_0 \geqslant \bar{t}$ such that $x_l(t_0) = x_r(t_0)$, then we have $x_l(t) = x_r(t)$ and $x_l'(t) = \sigma = x_r'(t)$ for all $t \geqslant t_0$. If this is the case, then we are done. Now suppose the opposite, that is, $x_l(t) < x_r(t)$ for all $t \geqslant \bar{t}$; then $x_l(t) - \sigma t$ is nondecreasing and bounded and $x_r(t) - \sigma t$ is nonincreasing and bounded for all $t \geqslant \bar{t}$. Hence

$$\lim_{t \to \infty} (x_l(t) - \sigma t) = X_l, \qquad \lim_{t \to \infty} x_l'(t) = \sigma, \tag{2.5}$$

$$\lim_{t \to \infty} (x_r(t) - \sigma t) = X_r, \qquad \lim_{t \to \infty} x_r'(t) = \sigma, \tag{2.6}$$

with $X_l \leqslant X_r$. From the entropy condition (E),

$$f'(u_l) > \sigma(u_l, u_r) > f'(u_r),$$

we can choose sufficiently small δ such that

$$f'(u) > \sigma > f'(v) \qquad \text{for all } u \in (u_l, u_l + \delta),\ v \in (u_r - \delta, u_r). \tag{2.7}$$

From (2.5) and (2.6), we can choose sufficiently large t_δ, such that $u(x_l(t) + 0, t) \in (u_r - \delta, u_r)$ and $u(x_r(t) - 0, t) \in (u_l, u_l + \delta)$ for all $t \geqslant t_\delta$. Now for $t \geqslant t_\delta$, through $(x_l(t) + 0, t)$ and $(x_r(t) - 0, t)$ we draw characteristics backward in time. They would intersect along a discontinuity line whose slope is approximately σ due to (2.7) and (R–H). (Note that they cannot terminate to a contact discontinuity before they meet.) But it is obvious that this discontinuity line violates (E). Q.E.D.

THEOREM 2.5. *If $0 < u_l \leqslant u_r$, let*

$$p_1(t) = \min_x \int_{-\infty}^{x} (u(y, t) - u_l)\, dy,$$

$$q_1(t) = \max_x \int_{x}^{\infty} (u(y, t) - u_r)\, dy,$$

then

(i) $p_1'(t) \geqslant 0$, $q_1'(t) \leqslant 0$,

(ii) *there exists t_0, $t_0 \geqslant 0$, such that for all $t \geqslant t_0$, $p_1(t) = p_1(t_0) = p_1(\infty)$, $q_1(t) = q_1(t_0) = q_1(\infty)$.*

Proof. (i) follows from Liu [9, Theorem 1(i)]. Let M and m be as defined in (2.3), (2.4). From Lemma 2.2 and the Ordering Principle, we know that

$$u_M(x, t) \geqslant u(x, t) \geqslant u_m(x, t), \qquad t \geqslant 0, \quad -\infty < x < \infty,$$

and for $t \geqslant t_0 = \max\{t_M, t_m\}$,

$$u_M(x, t) \geqslant (u_l*)^*, \qquad u_m(x, t) \geqslant (u_l*)^*.$$

Hence $u(x, t) \geqslant (u_l*)^*$ for all $t \geqslant t_0$ and

$$\begin{aligned} u(x, t_0) &= u_l \qquad \text{for} \quad x \leqslant X_l(M), \\ &= u_r \qquad \text{for} \quad x > X_r(m). \end{aligned}$$

Thus for $t \geqslant t_0$, $u(x, t)$ are restricted in the region $f''(u(x, t) > 0$. Hence the theorem follows from Liu [9, Theorem 1(ii)]. Q.E.D.

Remark. We have a similar theorem for the case $u_r \leqslant u_l < 0$.

THEOREM 2.6. *If $0 < u_l \leqslant u_r$, let $p_1(t)$ and $q_1(t)$ be as defined in Theorem 2.5. Define the generalized N-waves as*

$$\begin{aligned} N_2(x, t) &= u_l \qquad \textit{for} \quad x - f'(u_l)t \leqslant -\sqrt{-2p_1(\infty) f''(u_l) t}, \\ &= u_r \qquad \textit{for} \quad x - f'(u_r)t \geqslant \sqrt{2q_1(\infty) f''(u_r) t}, \qquad t > 0. \\ &= h_2\left(\frac{x}{t}\right) \qquad \textit{otherwise}; \end{aligned}$$

then we have

(i) *the edges of* N_2 *and* u *have finite distance for all time, i.e.,*

$$|x_l(t) - f'(u_l)t + \sqrt{-2p_1(\infty)f''(u_l)t}| + |x_r(t) - f'(u_r)t - \sqrt{2q_1(\infty)f''(u_r)t}| = O(S),$$

(ii) $|u(x, t) - N_2(x, t)| \leqslant \Lambda_1^{-1}O(S)\,t^{-1}$ *for any* x *that lies between* $\max(x_l(t), f'(u_l)t - \sqrt{-2p_1(\infty)f''(u_l)t})$ *and* $\min(x_r(t), f'(u_r)t + \sqrt{2q_1(\infty)f''(u_r)t})$,

(iii) $|u(x, t) - N_2(x, t)| = O(S)\,t^{-1/2}$ *for* x *between* $x_l(t)$ *and* $(f'(u_l)t - \sqrt{-2p_1(\infty)f''(u_l)t})$ *or between* $x_r(t)$ *and* $(f'(u_r)t + \sqrt{2q_1(\infty)f''(u_r)t})$,

(iv) $u(x, t) = N_2(x, t)$ *if* x *lies outside the regions of* (ii) *and* (iii), *where* $x_l(t)$ *and* $x_r(t)$ *are defined in Definition* 2.3, $\Lambda_1 = \min_{B \geqslant u \geqslant (u_l,\cdot)} f''(u)$, *and* B *is a bound for* $u^0(x)$.

Remark. We have a similar theorem for the case $u_r \leqslant u_l < 0$.

Proof. This theorem is an easy consequence of Lemma 2.2, Theorem 2.5, and Theorem 4 of Liu [9]. We omit the proof.

THEOREM 2.7. *If* $u_l < u_l^* \leqslant u_r$, *then there exist* x_0, $t_0 \geqslant 0$, *such that*

(i) $x_l'(t) \geqslant \sigma(u_l, u_l^*)$,

(ii) $x_l(t) \leqslant x_0 + \sigma(u_l, u_l^*)t$,

(iii) $\lim_{t\to\infty}[x_0 + \sigma(u_l, u_l^*)t - x_l(t)] = 0$,

(iv) $u(x, t) \geqslant ((u_l^*)_*)^*$ *for* $x > x_l(t)$, $t \geqslant t_0$.

Remark. We have a similar theorem for the case $u_l > u_l{*} \geqslant u_r$.

Proof. (i) is obvious. From (i), $(x_l(t) - \sigma(u_l, u_l^*)t)$ is nondecreasing. But $x_l(t) \leqslant x_m + \sigma(u_l, u_l^*)t$, where x_m is the x_α when $\alpha = m$ in Lemma 2.3 and m is the number defined in (2.4). Thus $(x_l(t) - \sigma(u_l, u_l^*)t)$ is nondecreasing and bounded. Hence, $\lim_{t\to\infty}(x_l(t) - \sigma(u_l, u_l^*)t)$ exists; let it be x_0. We already proved (ii) and (iii). Now it is easy to see that we can find a time t_1 sufficiently large, such that $u(x, t_1) \geqslant ((u_l^*)_*)^*$ for $x_l(t_1) < x < x_0 + \sigma(u_l, u_l^*)\,t_1$ and $u(x_0 + \sigma(u_l, u_l^*)\,t_1 + 0, t_1) = u_l^*$. Furthermore, the line segment $x_0 + \sigma(u_l, u_l^*)t$, $t \geqslant t_1$, is the characteristic line passing through the point $(x_0 + \sigma(u_l, u_l^*)\,t_1 + 0, t_1)$ and $t_1 \geqslant t_m$. Thus we have $u(x_0 + \sigma(u_l, u_l^*)t + 0, t) = u_l^*$ and $u(x, t) \geqslant u_l^*$ for all $t \geqslant t_1$, $x > x_m + \sigma(u_l, u_l^*)t$. Hence we can choose t_1 sufficiently large, such that $u(x, t) \geqslant u_l^*$ for all $t \geqslant t_1$, $x > x_0 + \sigma(u_l, u_l^*)t$. Choose $t_0 = t_1$ to complete the proof for (iv).

Q.E.D.

THEOREM 2.8. *Under the assumptions of Theorem* 2.7, *let*

$$q(t) = \max_x \int_x^{\infty} (u(y, t) - u_r)\, dy;$$

then

(i) $q'(t) \leqslant 0$,

(ii) *there exists* t_0, *such that* $q(t) = q(t_0)$ *for all* $t \geqslant t_0$.

Proof. (i) follows from Liu [9, Theorem 1(i)]. To prove (ii), take the t_0 of Theorem 2.7 as the t_0 we want. Assume that the maximum point in the definition of $q(t)$ is taken place at $x^*(t)$. We want to prove (1) $u(x^*(t), t) = u_r$ and $u(x, t)$ is continuous at $x^*(t)$, and (2) $q'(t) = 0$ for all $t \geqslant t_0$. If $x^*(t)$ is a discontinuity, then since $x^*(t) > x_l(t)$, we must have $u(x^*(t) - 0, t) > u(x^*(t) + 0, t)$. But in this case, $x^*(t)$ is not the maximum point. Hence $u(x, t)$ must be continuous at $x^*(t)$. Now if $u(x^*(t), t) \neq u_r$, then $x^*(t)$ cannot be the maximum point too. This proves (1). To prove (2), we know that from (1), $dx^*(t)/dt$ exists and is equal to $f'(u_r)$. From the definition of $q(t)$, we have

$$q(t) = \int_{x^*(t)}^{x_r(t) - 0} (u(y, t) - u_r)\, dy.$$

Hence

$$\begin{aligned} q'(t) &= [u(x_r(t) - 0, t) - u_r]\, x_r'(t) + \int_{x^*(t)}^{x_r(t) - 0} u_t\, dy \\ &= [u(x_r(t) - 0, t) - u_r]\, x_r'(t) - f(u(x_r(t) - 0)) - f(u_r) \\ &= 0 \qquad \text{(R–H)}. \end{aligned}$$

This proves (ii). Q.E.D.

THEOREM 2.9. *Under the assumptions of Theorems* 2.7 *and* 2.8, *let* x_0 *and* $q(\infty) = q(t_0)$ *be the respective constants in Theorems* 2.7 *and* 2.8. *Define the following one-sided generalized N-wave*

$$\begin{aligned} N(x, t) &= u_l && \textit{for} \quad x \leqslant x_0 + \sigma(u_l, u_l^*)t, \\ &= u_r && \textit{for} \quad x > f'(u_r)t + \sqrt{2q(\infty) f''(u_r)t} + x_0, \\ &= h_2\left(\frac{x - x_0}{t}\right) && \textit{otherwise}. \end{aligned}$$

Then we have

(i) *there exists* $t_0 \geqslant 0$, *such that* $x_l(t) = x_0 + \sigma(u_l, u_l^*)t$ *for all* $t \geqslant t_0$,

(ii) $|u(x,t) - N(x,t)| \leqslant A^{-1}O(S)\,t^{-1}$ *for* x *between* $x_0 + \sigma(u_l, u_l^*)t$ *and* $\min(x_r(t), f'(u_r)t + \sqrt{2q(\infty)f''(u_r)t} + x_0)$,

(iii) $|x_r(t) - f'(u_r)t - \sqrt{2q(\infty)f''(u_r)t}| = O(S)$,

(iv) $|u(x,t) - N(x,t)| \leqslant O(S)\,t^{-1/2}$ *for* x *between* $x_r(t)$ *and* $f'(u_r)t + \sqrt{2q(\infty)f''(u_r)t} + x_0$, *where* $A = \min_{((u_l^*)_*)^* \leqslant u \leqslant B} f''(u)$.

Remark. We have a similar theorem for the case $u_r \leqslant u_l{*} < u_l$.

Proof. Let $-X(t) = x_l(t) - x_0 - f'(u_l^*)t$, then

$$-X'(t) = x_l'(t) - f'(u_l^*) = \sigma(u_l, u(x_l(t)+0,t)) - \sigma(u_l, u_l^*).$$

From Theorem 2.7, $X(t) \to 0$, $X'(t) \to 0$ as $t \to \infty$. Hence we can expand $\sigma(u_l, u(x_l(t)+0,t))$ to obtain

$$\sigma(u_l, u(x_l(t)+0,t)) \cong \sigma(u_l, u_l^*) - \frac{f''(u_l^*)}{2(u_l - u_l^*)}(u - u_l^*)^2.$$

But

$$u_l^* = h_2(f'(u_l^*)) \approx h_2\left(\frac{x_0 + f'(u_l^*)t - x_0}{t}\right),$$

hence for t large

$$u(x_l(t)+0,t) \cong h_2\left(\frac{x_l(t) - x_0}{t}\right)$$

$$\cong h_2\left(\frac{-X(t) + f'(u_l^*)t}{t}\right) = h_2\left(f'(u_l^*) - \frac{X(t)}{t}\right)$$

$$= u_l^* - h_2'(f'(u_l^*))\frac{X(t)}{t}.$$

Hence

$$X'(t) \cong -A\frac{X^2(t)}{t^2} \qquad \text{with} \quad A = \frac{f''(u_l^*)}{2(u_l^* - u_l)}(h_2'(f'(u_l^*)))^2,$$

or

$$X(t) \cong \frac{t}{ct - A} \qquad \text{and} \qquad X(t) \to \frac{1}{c} \qquad \text{as} \quad t \to \infty.$$

But as we know $X(t) \to 0$ as $t \to \infty$. Hence $c = \infty$, which means that for some finite t_0, $X(t_0) = 0$. This proves (i). Parts (ii), (iii) and (iv) are an easy consequence of Theorem 2.7, Theorem 2.8 and Liu [9, Theorem 4].

LEMMA 2.10. *If $u_l = u_r = 0$ and $f(u) \cong f'(0)u + Au^{2n+1}$, $n \geqslant 1$, $A > 0$, for $|u|$ small, then we have*

$$|u_\alpha(x, t)| = O(S)\, t^{-1/(2n+1)}.$$

Proof. If $\alpha < 0$, then the solution $u_\alpha(x, t)$ is

$$\begin{aligned} u_\alpha(x,t) &= 0 && \text{for} \quad x \leqslant x_1(t), && \\ &= h_1\left(\frac{x+S}{t}\right) && \text{for} \quad x_1(t) < x \leqslant x_2(t), && \\ &= \alpha && \text{for} \quad x_2(t) < x \leqslant x_3(t), && 0 < t \leqslant t_1, \\ &= 0 && \text{for} \quad x_3(t) < x, && \\ u_\alpha(x,t) &= 0 && \text{for} \quad x \leqslant x_1(t), && \\ &= h_1\left(\frac{x+S}{t}\right) && \text{for} \quad x_1(t) < x \leqslant x_4(t), && t_1 < t, \\ &= 0 && \text{for} \quad x_4(t) < x, && \end{aligned}$$

where

$$\begin{aligned} x_1(t) &= -S + f'(0)t, \\ x_2(t) &= -S + f'(\alpha)t, \\ x_3(t) &= S + \sigma(\alpha, 0)t, \end{aligned}$$

$x_4(t)$ satisfies

$$x_4'(t) = \sigma\left(h_1\left(\frac{x_4(t)+S}{t}\right), 0\right) = f\left(h_1\left(\frac{x_4(t)+S}{t}\right)\right) \Big/ h_1\left(\frac{x_4(t)+S}{t}\right),$$

with

$$x_4(t_1) = x_2(t_1) = x_3(t_1), \qquad t \geqslant t_1.$$

From the entropy condition,

$$x_4'(t) < f'\left(h_1\left(\frac{x_4(t)+S}{t}\right)\right) = \frac{x_4(t)+S}{t};$$

hence

$$\frac{d}{dt}\left(\frac{x_4(t)+S}{t}\right)=\frac{x_4'(t)}{t}-\frac{x_4(t)+S}{t^2}<0.$$

But it is obvious that

$$\frac{x_4(t)+S}{t}\geqslant f'(0);$$

hence $(x_4(t)+S)/t \to f'(0)$ as $t\to\infty$. And thus $h_1((x_4(t)+S)/t)\to 0$ as $t\to\infty$. Now for t large, we can have

$$x_4'(t)\cong f'(0)+Ah_1^{2n}\left(\frac{x_4(t)+S}{t}\right).$$

But on the other hand, from the definition of h_1,

$$(2n+1)Ah_1^{2n}\left(\frac{x_4(t)+S}{t}\right)+f'(0)\cong f'\left(h_1\left(\frac{x_4(t)+S}{t}\right)\right)=\frac{x_4(t)+S}{t},$$

Letting $X(t)=x_4(t)+S$, we get

$$X'(t)\cong f'(0)+\frac{1}{2n+1}\left(\frac{X}{t}-f'(0)\right) \qquad \text{for } t \text{ large.}$$

Hence

$$X(t)\cong f'(0)t+O(S)\,t^{1/(2n+1)}$$

and

$$h_1\left(\frac{X(t)}{t}\right)\cong\left[\frac{1}{(2n+1)A}\left(\frac{X(t)}{t}-f'(0)\right)\right]^{1/2n}$$
$$=O(S)\,t^{-1/(2n+1)}.$$

Similarly we can consider the case $\alpha>0$. This completes the proof. Q.E.D.

THEOREM 2.11. *If $u_l=u_r=0$ and $f(u)\cong f'(0)u+Au^{2n+1}$, $n\geqslant 1, A>0$, for $|u|$ small, then we have*

$$|u(x,t)|=O(S)\,t^{-1/(2n+1)}.$$

Proof. This theorem is a consequence of Lemma 2.10 and the Ordering Principle.

THEOREM 2.12. *If $u_l = 0 < u_r$ and $f(u) \cong f'(0)u + Au^{2n+1}$, $n \geqslant 1$, $A > 0$, for $|u|$ small, define*

$$p(t) = \min_x \int_{-\infty}^{x} u(y, t)\, dy,$$

$$q(t) = \max_x \int_{x}^{\infty} (u(y, t) - u_r)\, dy;$$

then

(i) *if $p(0) = 0$, then $p(t) = 0$ for all t,*

(ii) *if $p(0) < 0$, then $p'(t) > 0$ for all t and $p(t) \to 0$ as $t \to \infty$,*

(iii) *if $p(t) = \min_x \int_{-\infty}^{x} u(y, t)\, dy = \int_{-\infty}^{x_p(t)} u(y, t)\, dy$, then*

$$x_p(t) \cong f'(0)t + O(S)\, t^{\xi^{2n}} - S \qquad \text{for } t \text{ large},$$

where $2n\xi^{2n} - \xi^{2n-1} - \cdots - \xi - 1 = 0$, $-1 < \xi < 0$,

(iv) *$q'(t) \leqslant 0$ for all $t \geqslant 0$,*

(v) *there exists $t_0 \geqslant 0$, such that for all $t \geqslant t_0$, $q(t) = q(t_0)$.*

Proof. From Liu [9, Theorem 1(i)], $p'(t) \geqslant 0$ and $p(t) \leqslant 0$; thus if $p(0) = 0$, then $p(t) = 0$ for all t which proves (i). If $p(0) < 0$, then we want to prove that the maximum point $x_p(t)$ as defined in (iii) is a shock curve with $u(x_p(t) - 0, t) < 0$ and $0 < u(x_p(t) + 0, t) \leqslant u(x_p(t) - 0, t)^*$. If $u(x, t)$ is continuous at $x_p(t)$, then $u(x_p(t), t) = 0$ and $u(x_p(t) - \varepsilon, t) < 0$, $u(x_p(t) + \varepsilon, t) > 0$ for sufficiently small $\varepsilon > 0$. But this is impossible, because the characteristics from the immediate left-hand side of $x_p(t)$ will intersect the characteristics from $x_p(t)$ immediately. Thus $x_p(t)$ must be a shock curve. It is then obvious that $u(x_p(t) - 0, t) < 0$ and $0 < u(x_p(t) + 0, t) \leqslant u(x_p(t) - 0, t)^*$. Thus

$$\begin{aligned} p'(t) &= u(x_p(t) - 0, t)\, x_p'(t) - [f(u(x_p(t) - 0, t)) - f(0)] \\ &= u(x_p(t) - 0, t) \left[\frac{f(u(x_p(t) - 0, t)) - f(u(x_p(t) + 0, t))}{u(x_p(t) - 0, t) - u(x_p(t) + 0, t)} \right. \\ &\qquad \left. - \frac{f(u(x_p(t) - 0, t)) - f(0)}{u(x_p(t) - 0, t) - 0} \right] > 0; \end{aligned}$$

this proves (ii). Now we would like to estimate the order of $x_p(t)$. From (R–H) and the above arguments, we have

$$x_p'(t) = \frac{f(u(x_p(t) - 0, t)) - f(u(x_p(t) + 0, t))}{u(x_p(t) - 0, t) - u(x_p(t) + 0, t)} < f'(u(x_p(t) - 0, t)).$$

For t large, $u(x_p(t)-0,t)\cong h_1((x_p(t)+S)/t)$; hence

$$\frac{d}{dt}\left(\frac{x_p(t)+S}{t}\right)<0 \quad\text{and}\quad \frac{x_p(t)+S}{t}\to f'(0) \quad\text{as}\quad t\to\infty.$$

Thus for t large, we have $u(x_p(t)+0,t)=u(x_p(t)-0,t)^*$ and

$$\begin{aligned} x_p'(t)&\cong\frac{f(h_1((x_p(t)+S)/t))-f(h_1((x_p(t)+S)/t)^*)}{h_1((x_p(t)+S)/t)-h_1((x_p(t)+S)/t)^*}\\ &=f'(0)+A[h_1^{2n}+h_1^{2n-1}h_1^*+\cdots+h_1^{*2n}]\\ &=f'(0)+Ah_1^{2n}\left(\frac{x_p(t)+S}{t}\right)\cdot[1+\xi+\xi^2+\cdots+\xi^{2n}],\end{aligned}$$

where ξ satisfies

$$2n\xi^{2n}-\xi^{2n-1}-\xi^{2n-2}-\xi\cdots-\xi-1=0, \qquad -1<\xi<0.$$

Thus

$$\begin{aligned} x_p'(t)&\cong f'(0)+A(2n+1)\,h_1^{2n}\left(\frac{x_p(t)+S}{t}\right)\cdot\xi^{2n}\\ &\cong f'(0)+\xi^{2n}\left(\frac{x_p(t)+S}{t}-f'(0)\right).\end{aligned}$$

Hence

$$x_p(t)+S\cong f'(0)t+O(S)\,t^{\xi^{2n}}.$$

This proves (iii); (iv) follows from Liu [9, Theorem 1(i)]. To prove (v), first, we use the solution $u_m(x,t)$ to prove that $u(x,t)\geqslant u_m(x,t)>(u_r)_{**}$ for all $t\geqslant t_1$, where m is the infimum of $u(x,0)$ and t_1 is some constant greater than zero. This is by direct construction of the solution $u_m(x,t)$. We do not want to repeat it here. Assume that the maximum point in the definition of $q(t)$ is taken place at $X(t)$. Since $u(x,t)>(u_r)_{**}$, for all $t\geqslant t_1$, $X(t)$ cannot be a shock curve. Hence $u(x,t)$ is continuous at the point $X(t)$ and $u(X(t),t)=u_r$. Direct calculation of $q'(t)$ will prove (v). Q.E.D.

THEOREM 2.13. *Under the same assumptions of Theorem* 2.12, *define the one-sided* ε–*N-wave* $N_\varepsilon(x,t)$ *as*

$$\begin{aligned} N_\varepsilon(x,t)&=u_r &&\textit{for}\quad x>f'(u_r)t+\sqrt{2q(\infty)f''(u_r)t},\\ &=h_2\left(\frac{x}{t}\right) &&\textit{for}\quad f'(\varepsilon)t<x\leqslant f'(u_r)t+\sqrt{2q(\infty)f''(u_r)t};\end{aligned}$$

then

(i) $|x_r(t) - f'(u_r)t - \sqrt{2q(\infty)f''(u_r)t}| = O(S)$ *for all* t,

(ii) $|N_\epsilon(x, t) - u(x, t)| \leqslant \Lambda^{-1}(\varepsilon)\, O(S)\, t^{-1}$ *for* x *between* $f'(\varepsilon)t$ *and* $\min(x_l(t), f'(u_r)t + \sqrt{2q(\infty)f''(u_r)t})$,

(iii) $|N_\epsilon(x, t) - u(x, t)| \leqslant O(S)\, t^{-1/2}$ *for* x *between* $x_r(t)$ *and* $f'(u_r)t + \sqrt{2q(\infty)f''(u_r)t}$, *where* ε *is a small fixed number with* $0 < \varepsilon < u_r$ *and* $q(\infty)$ *is the constant* $q(t_0)$ *in Theorem* 2.12(v) *and* $\Lambda(\varepsilon) = \min_{\epsilon \leqslant u \leqslant B} f''(u)$.

Remark. We can have a similar theorem for the case $u_l = 0 > u_r$.

Proof. From Theorem 2.12(iii), we find a time $t_1 \geqslant 0$, such that $f'(\varepsilon)t > x_p(t)$ for $t \geqslant t_1$. Thus for $t \geqslant t_1$, $u(x, t) \geqslant \varepsilon$ for all $x \geqslant f'(\varepsilon)t$. Then (i), (ii) and (iii) follows from Theorem 4 of Liu [9]. Q.E.D.

3. The Case When f'' Vanishes at n Points and Changes Sign at These Points

Without loss of generality, we assume that f'' vanishes at $a_1, a_2, \ldots, a_N$, where $a_1 < a_2 < \cdots < a_N$, and $f''(u) < 0$ for $u < a_1$, $f''(u) > 0$ for $a_1 < u < a_2, \ldots$, etc. We also adopt the definitions of $u_\alpha(x, t)$, M, m, $x_l(t)$ and $x_r(t)$ of Section 2. For convenience, we put $a_0 = -\infty$ and $a_{N+1} = +\infty$. In this section, we use $u(x, t)$ to denote the solution of (1.1), (1.2) with initial condition (1.3), where f is under the assumption of this section.

We may need direct construction of solution $u(x, t)$ in the proof of the following lemmas and theorems. We will give only some indications and omit the details. These constructions are similar to the constructions in Section 2.

Lemma 3.1. *If* $u_l \in (a_{i-1}, a_i)$, $u_r \in (a_{j-1}, a_j)$, *where* $1 \leqslant i \leqslant j \leqslant N + 1$, *then there exists* $t_\alpha \geqslant 0$, *such that for all* $t \geqslant t_\alpha$, $u_\alpha(x, t) \in (A_l, A_r)$, *where* A_l, A_r *are two fixed constants with* $A_l \in (a_{i-1}, u_l)$ *and* $A_r \in (u_r, a_j)$.

Lemma 3.2. *Under the assumptions of Lemma* 3.1, *there exists* $t_0 \geqslant 0$, *such that for all* $t \geqslant t_0$, $u(x, t) \in (A_l, A_r)$, *where* A_l, A_r *are as in Lemma* 3.1.

Proof of Lemmas 3.1 *and* 3.2. Using the Ordering Principle, we can easily establish Lemma 3.2 from Lemma 3.1 if $u(x, 0) \in (\alpha, a_j)$ for all x. To prove Lemma 3.1, we use induction. If $\alpha \in (a_{i-1}, a_j)$, then Lemma 3.1 is obviously true. Now assume that when $\alpha \in (a_{i-k}, a_{i-k+1})$, Lemma 3.1 is true, and hence Lemma 3.2 is also true when $u(x, 0) \in (\alpha, a_j)$. We would like to establish that when $\alpha \in (a_{i-k-1}, a_{i-k})$, Lemma 3.1 is true. The solutions for the Riemann problems (u_l, α) and (α, u_r) are combinations of shock waves and rarefaction waves. Let us denote these simple wave resolutions of (u_l, α) and (α, u_r) by $(u_l, v_1), (v_1, v_2), ..., (v_m, \alpha)$ and $(\alpha, w_1), (w_1, w_2), ...,$ (w_n, u_r). It is easy to see that at least one of the simple waves (v_m, α) and (α, w_1) must be a shock wave. It is this simple consequence of entropy condition (E) that causes the cancellation of waves. Now it is easy to see that shock wave (v_m, α) or (α, w_1) will kill the α-states in a finite time. After that, the remaining rarefaction wave, (v_m, α) or (α, w_1), will be killed by a combination of type I and type II shocks in a finite time. Thus there exists $t_1 \geqslant 0$, such that for $t \geqslant t_1$, $u_\alpha(x, t) \in (a_{i-k}, a_j)$. Using induction hypotheses, we prove that when $\alpha \in (a_{i-k-1}, a_{i-k})$, Lemma 3.1 is true. Similarly we can consider the case $\alpha > a_j$. This completes the proof of Lemmas 3.1 and 3.2.

Q.E.D.

Remark. We have two similar lemmas when $1 \leqslant j \leqslant i \leqslant N+1$.

DEFINITION. If the solution of the Riemann problem (u_l, u_r) consists of a simple shock wave with $f'(u_l) > \sigma(u_l, u_r) > f'(u_r)$ and $\sigma(u_l, u) > \sigma(u_l, u_r) >$ $\sigma(u, u_r)$ for all u between u_l and u_r, then we call (u_l, u_r) a strict shock.

THEOREM 3.3. *If (u_l, u_r) is a strict shock, then there exists x_0 and t_0, $t_0 \geqslant 0$, such that for all $t \geqslant t_0$,*

$$\begin{aligned} u(x,t) &= u_l \qquad \textit{for} \quad x \leqslant x_0 + \sigma(u_l, u_r)t, \\ &= u_r \qquad \textit{for} \quad x > x_0 + (u_l, u_r)t, \end{aligned}$$

where

$$x_0 = \frac{1}{u_r - u_l} \int_{-N}^{N} \left(\frac{u_r + u_l}{2} - u^0(x) \right) dx, \qquad N \geqslant S.$$

Proof. From Lemma 3.2, if $u_l \in [a_{i-1}, a_i]$, $u_r \in [a_{j-1}, a_j]$ and $i > j$ (note that the case $i \leqslant j$ can be similarly considered), then we can choose $A_l \in$ (a_{i-1}, u_l), $A_r \in (u_r, a_j)$ such that (A_l, u_r) and (u_l, A_r) are all strict shocks. It is easy to construct the solutions $u_{A_l}(x, t)$ and $u_{A_r}(x, t)$ directly and find t_{A_l}, t_{A_r} and x_{A_l}, x_{A_r}, such that

$$\begin{aligned} u_{A_l}(x,t) &= u_l \qquad \text{for} \quad x \leqslant x_{A_l} + \sigma(u_l, u_r)t, \\ &= u_r \qquad \text{for} \quad x > x_{A_l} + \sigma(u_l, u_r)t, \end{aligned} \qquad t \geqslant t_{A_l},$$

$$\begin{aligned} u_{A_r}(x,t) &= u_l \qquad \text{for} \quad x \leqslant x_{A_r} + \sigma(u_l, u_r)t, \\ &= u_r \qquad \text{for} \quad x > x_{A_r} + \sigma(u_l, u_r)t, \end{aligned} \qquad t \geqslant t_{A_r}.$$

Thus for t sufficiently large, say $t \geqslant t_0'$, we have

$$\begin{aligned} u(x, t) &= u_l \qquad \text{for} \quad x \leqslant x_{A_r} + \sigma(u_l, u_r)t, \\ &= u_r \qquad \text{for} \quad x > x_{A_l} + \sigma(u_l, u_r)t, \end{aligned}$$

and $u_l = u_{A_l}(x, t) \leqslant u(x, t) \leqslant u_{A_r}(x, t) = u_r$ for $x_{A_r} + \sigma(u_l, u_r)t \leqslant x \leqslant x_{A_l} + \sigma(u_l, u_r)t$. Using the strick shock properties of (u_l, u_r), we can prove this theorem by using the same arguments as in the proof of Theorem 2.4.

Q.E.D.

DEFINITION. If the solution of the Riemann problem (u_l, u_r) is a simple rarefaction wave and if $u_l \neq a_i \neq u_r$ for all i, then we call (u_l, u_r) a strict rarefaction wave.

Remark. From this definition, a strict rarefaction wave can have two possibilities only. Either $a_{i-1} < u_l \leqslant u_r < a_i$ and $f''(u) > 0$ for all $u \in (a_{i-1}, a_i)$ or $a_{j-1} < u_r \leqslant u_l < a_j$ and $f''(u) < 0$ for all $u \in (a_{j-1}, a_j)$.

THEOREM 3.4. *If (u_l, u_r) is a strict rarefaction wave, with the proper definitions of $p(t)$ and $q(t)$ in Theorem* 2.5 *and the definition of $N(x, t)$ in Theorem* 2.6, *where we have to replace h_2 by some proper h_i and h_i is the inverse function of $f'(u)$ restricted in (a_{i-1}, a_i), then the proper statements of Theorems* 2.5 *and* 2.6 *hold.*

Proof. In view of the Ordering Principle and Lemmas 3.1 and 3.2, we can push the solution $u(x, t)$ at a finite time into the interval (a_{i-1}, a_i) which contains u_l and u_r. Then the whole story of Liu [9] goes and the theorem is proved. Q.E.D.

For nonstrict shocks and nonstrict rarefaction waves, they can be treated as in Theorems 2.9, 2.11, 2.12, and 2.13. We do not treat them here. Similarly we can treat the case of the combination of shocks and rarefaction waves. For example, if the resolutions of (u_l, u_r) to simple waves are (u_l, v_1), (v_1, v_2), (v_2, u_r), where (u_l, v_1) is a shock with $f'(u_l) > \sigma(u_l, v_1) = f'(v_1)$, (v_1, v_2) is a strict rarefaction wave, (v_2, v_r) is a shock with $f'(v_2) = \sigma(v_2, u_r) > f'(u_r)$, then we can prove that after a finite time, $x_l(t) = X_l + \sigma(u_l, v_1)t$, $x_r(t) = X_r + \sigma(v_2, u_r)t$ and between these two shocks is rarefaction wave (v_1, v_2). The proof is similar to the proof of Theorem 2.9. For $u_l = u_r = a_i$, the treatment is almost identical to the treatment of Theorem 2.11. Although we did not consider the case $f''(a_i) = 0$ and f'' does not change sign at a_i, it is obvious that we can apply our technique to this case as well.

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