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ON THE REFLEXIVITY OF $C_0(N)$ CONTRACTIONS

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ABSTRACT. Let T be a $C_0(N)$ contraction on a separable Hilbert space and let $J = S(\varphi_1) \oplus S(\varphi_2) \oplus \cdots \oplus S(\varphi_k)$ be its Jordan model, where $\varphi_1, \varphi_2, \ldots, \varphi_k$ are inner functions satisfying $\varphi_j | \varphi_{j-1}$ for $j = 2, 3, \ldots, k$, and $S(\varphi_j)$ denotes the compression of the shift on $H^2 \ominus \varphi_j H^2, j = 1, 2, \ldots, k$. In this note we show that T is reflexive if and only if $S(\varphi_1/\varphi_2)$ is.

In this note we only consider bounded linear operators defined on complex, separable Hilbert spaces. For each operator T, let $\{T\}'$, $\{T\}''$ and Alg T denote the commutant, double commutant and the weakly closed algebra generated by T and I, respectively. Let Lat T denote the lattice of invariant subspaces of T and Alg Lat T denote the (weakly closed) algebra of operators which leave all the subspaces in Lat T invariant. Recall that T is *reflexive* if and only if Alg Lat $T =$ Alg T. In **[1]** Deddens and Fillmore characterized reflexive operators on finite-dimensional spaces in terms of their Jordan canonical forms. Now we generalize their result to $C_0(N)$ contractions. More specifically, we prove the following

THEOREM 1. *If T is a $C_0(N)$ contraction and $J = S(\varphi_1) \oplus S(\varphi_2) \oplus \cdots \oplus S(\varphi_k)$ is its Jordan model, then T is reflexive if and only if $S(\varphi_1/\varphi_2)$ is.*

A contraction T ($\|T\| \leqslant 1$) on a Hilbert space is *of class* $C_0(N)$ for some integer $N \geqslant 1$ if there exists an inner function φ such that $\varphi(T) = 0$ and the *defect indices* of T, $d_T \equiv \operatorname{rank}(I - T^*T)^{1/2}$ and $d_{T^*} \equiv \operatorname{rank}(I - TT^*)^{1/2}$, are both equal to some $M \leqslant N$. A $C_0(N)$ contraction is unitarily equivalent to the operator T defined on $H = H_N^2 \ominus \Theta_T H_N^2$ by $Tf = P(e^{it}f)$ for $f \in H$, where H_N^2 denotes the standard Hardy space of $\mathbf{C}^N$-valued functions defined on the unit circle, Θ_T is the characteristic function of T, and P denotes the (orthogonal) projection from H_N^2 onto H (cf. **[5**, Chapter VI**]**). Two operators T_1, T_2 are *quasi-similar* if there exist one-to-one operators X and Y with dense ranges (called *quasi-affinities*) such that $XT_1 = T_2X$ and $YT_2 = T_1Y$. A $C_0(N)$ contraction is quasi-similar to a uniquely determined Jordan operator (called its *Jordan model*) $J = S(\varphi_1) \oplus S(\varphi_2) \oplus \cdots \oplus S(\varphi_k)$, where $\varphi_1, \varphi_2, \ldots, \varphi_k$ are inner functions satisfying $\varphi_j | \varphi_{j-1}$, $j = 2, 3, \ldots, k$, and $S(\varphi_j)$ denotes the operator defined on $H^2 \ominus \varphi_j H^2$ by $S(\varphi_j)f = P_j(e^{it}f)$ for $f \in H^2 \ominus \varphi_j H^2$, P_j being the (orthogonal) projection from H^2 onto $H^2 \ominus \varphi_j H^2$, $j = 1, 2, \ldots, k$ (cf. **[4]**). For ξ and η in H^∞, $\xi \wedge \eta = 1$ denotes that ξ and η have no nontrivial common inner divisor.

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We start the proof of Theorem 1 by showing that for $C_0(N)$ contractions, the property of reflexivity is preserved under quasi-similarities. This generalizes Corollary 4.5 in [7].

THEOREM 2. *Let T_1 and T_2 be $C_0(N)$ contractions on H_1 and H_2, respectively. Assume that T_1 is quasi-similar to T_2. Then T_1 is reflexive if and only if T_2 is.*

PROOF. We may assume that T_1 and T_2 are defined on $H_1 = H_N^2 \ominus \Theta_1 H_N^2$ and $H_2 = H_N^2 \ominus \Theta_2 H_N^2$ by $T_1 f = P_1(e^{it}f)$ and $T_2 g = P_2(e^{it}g)$, respectively, where $f \in H_1$ and $g \in H_2$. Since T_1 and T_2 are quasi-similar to each other, there exist bounded analytic functions Φ and Ψ such that $\Phi\Theta_1 = \Theta_2\Psi$ and $(\det \Phi)(\det \Psi) \wedge (\det \Theta_1)(\det \Theta_2) = 1$ (cf. [3] and [2]). Let Φ^A denote the algebraic adjoint of Φ. It can be easily verified that the operators X: $H_1 \to H_2$ and Y: $H_2 \to H_1$ defined by $Xf = P_2(\Phi f)$ for $f \in H_1$ and $Yg = P_1((\det \Psi)\Phi^A g)$ for $g \in H_2$ implement the quasi-affinities intertwining T_1 and T_2 (cf. [2, Theorem 2]). Moreover, we have $YX = \eta(T_1)$ and $XY = \eta(T_2)$, where $\eta = (\det \Phi)(\det \Psi)$. Let m_1 and m_2 denote the minimal functions of T_1 and T_2, respectively. From the quasi-similarity of T_1 and T_2 we have $m_1 = m_2$.

Assume that T_1 is reflexive. Let $S \in \text{Alg Lat } T_2$ and $K \in \text{Lat } T_1$. Then $YSXK \subseteq \overline{YXK} = \overline{\eta(T_1)K}$. $\eta \wedge (\det \Theta_1) = 1$ implies that $\eta \wedge m_1 = 1$ (cf. [5, Theorem VI.5.2]). In particular, η and the minimal function of $T_1|K$ have no nontrivial common inner divisor. Thus $\eta(T_1|K)$ is a quasi-affinity (cf. [7, Theorem 2.3]) and therefore $\overline{\eta(T_1)K} = \overline{\eta(T_1|K)K} = K$. We have $YSXK \subseteq K$ for any $K \in \text{Lat } T_1$, which shows that $YSX \in \text{Alg Lat } T_1 = \text{Alg } T_1$. Hence $YSX = v(T_1)^{-1}u(T_1)$ for some $u, v \in H^\infty$, where $v \wedge m_1 = 1$ (cf. [7, Theorem 3.2]). So $v(T_1)YSX = u(T_1)$ and we have $\eta(T_2)v(T_2)S\eta(T_2) = XYv(T_2)SXY = X(v(T_1)YSX)Y = Xu(T_1)Y = u(T_2)XY = u(T_2)\eta(T_2)$. Since as above $\eta(T_2)$ is a quasi-affinity, this implies that $\eta(T_2)v(T_2)S = u(T_2)$. Note that $(\eta v) \wedge m_2 = 1$. We obtain $S = (\eta v)(T_2)^{-1}u(T_2) \in \text{Alg } T_2$. This shows that T_2 is reflexive, completing the proof.

As a by-product of the preceding proof, we have the following

THEOREM 3. *Let T_1 and T_2 be $C_0(N)$ contractions on H_1 and H_2, respectively. If T_1 is quasi-similar to T_2, then* Lat $T_1 \cong$ Lat T_2.

PROOF. Let X: $H_1 \to H_2$ and Y: $H_2 \to H_1$ be the intertwining quasi-affinities given in the proof of Theorem 2. For $K_1 \in \text{Lat } T_1$ and $K_2 \in \text{Lat } T_2$ consider the mappings $K_1 \to \overline{XK_1}$ and $K_2 \to \overline{YK_2}$. As before we have

$$\overline{Y\overline{XK}_1} = \overline{\eta(T_1)K_1} = \overline{\eta(T_1|K_1)K_1} = K_1.$$

Similarly, $\overline{X\overline{YK_2}} = K_2$. We infer that these mappings implement the lattice isomorphisms between Lat T_1 and Lat T_2 and hence Lat $T_1 \cong$ Lat T_2.

As a consequence of Theorem 2, to prove Theorem 1 it suffices to consider Jordan operators. The next lemma will be needed in the proof of the necessity part.

LEMMA 4. *Let T be an operator on a Hilbert space H. Let $S \in$* Alg Lat $T \cap \{T\}'$ *and $T_1 = T|\overline{SH}$. Assume that* Alg Lat $T_1 \cap \{T_1\}' =$ Alg T_1. *If T is reflexive, so is T_1.*

PROOF. Let $S_1 \in \operatorname{Alg} \operatorname{Lat} T_1$. Consider S_1S as an operator on H. For any $K \in \operatorname{Lat} T$, $SK \subseteq K \cap \overline{SH}$. Since $K \cap \overline{SH} \in \operatorname{Lat} T_1$, we have $S_1SK \subseteq S_1(K \cap \overline{SH}) \subseteq K \cap \overline{SH} \subseteq K$. This shows that $S_1S \in \operatorname{Alg} \operatorname{Lat} T = \operatorname{Alg} T$. Hence $S_1TS = S_1ST = TS_1S$. It follows that $S_1T_1 = T_1S_1$ on $\overline{SH}$, that is, $S_1 \in \{T_1\}'$. We conclude that $S_1 \in \operatorname{Alg} \operatorname{Lat} T_1 \cap \{T_1\}' = \operatorname{Alg} T_1$ and hence T_1 is reflexive.

To prove the sufficiency part, we essentially follow the same line of arguments as given by Deddens and Fillmore **[1]** for reflexive linear transformations. The next two lemmas are analogous to part of Theorem 2 and its Corollary in **[1]**, respectively.

LEMMA 5. *Let $T = S(\varphi_1) \oplus \cdots \oplus S(\varphi_k)$ be a Jordan operator defined on $H = (H^2 \ominus \varphi_1 H^2) \oplus \cdots \oplus (H^2 \ominus \varphi_k H^2)$ and let $\psi = \varphi_1/\varphi_2$. If $S \in \operatorname{Alg} \operatorname{Lat} T$, then there exist an outer $\eta \in H^\infty$ and $\delta \in H^\infty$ such that $\eta(T)S = \delta(T) + D$, where D is an operator on H satisfying*

$$D\big[(\xi H^2 \ominus \varphi_1 H^2) \oplus (H^2 \ominus \varphi_2 H^2) \oplus \cdots \oplus (H^2 \ominus \varphi_k H^2)\big] \subseteq (\xi \varphi_2 H^2 \ominus \varphi_1 H^2) \oplus 0 \oplus \cdots \oplus 0 \quad \textit{for any } \xi | \psi.$$

PROOF. Let $T_j = S(\varphi_j)$, $H_j = H^2 \ominus \varphi_j H^2$ and let P_j denote the (orthogonal) projection from H^2 onto H_j, $j = 1, 2, \ldots, k$. For brevity of notation, we identify H_j as a subspace of H in the natural way. Let $e = P_1(1) \in H_1$ and $h = Se \in H_1$, since S leaves H_1 invariant. Let

$$\begin{aligned} h(\lambda) &= h_i(\lambda) h_e(\lambda) \\ &= h_i(\lambda) \exp\left[\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + \lambda}{e^{it} - \lambda} k(t)\, dt\right] \quad \text{for } |\lambda| < 1, \end{aligned}$$

where h_i and h_e are the inner and outer parts of h, and $k(t) = \log|h_e(t)|$ a.e. Fix $M > 0$ and let $\alpha = \{t\colon |h_e(t)| \geqslant M\}$. Let

$$\eta(\lambda) = \exp\left[\frac{1}{2\pi} \int_\alpha \frac{e^{it} + \lambda}{e^{it} - \lambda} (-k(t))\, dt\right] \quad \text{for } |\lambda| < 1,$$

and $\delta = \eta h$. Then it is easily seen that $\eta, \delta \in H^\infty$ and $\eta(T)Se = \delta(T)e$. Let $D = \eta(T)S - \delta(T)$. Then $De = 0$.

We first check that $\overline{D(H_2 \oplus \cdots \oplus H_k)} = \{0\}$. Let $f \in H^\infty$ and consider the element $P_j(f)$ in H_j, $j = 2, 3, \ldots, k$. Let W and U be the invariant subspaces for T generated by $P_j(f)$ and $e \oplus P_j(f) \in H_1 \oplus H_j$, respectively. Let $g \in W \cap U \subseteq H_j$. Then there exists a sequence of polynomials $\{p_n\}$ such that $p_n(T)(e \oplus P_j(f)) \to 0 \oplus g$ as $n \to \infty$. Hence $P_1(p_n) = p_n(T)e \to 0$ and $P_j(p_n f) = p_n(T)P_j(f) \to g$, which imply that $P_j(p_n) = P_jP_1(p_n) \to 0$ and $f(T_j)P_j(p_n) \to g$. It follows that $g = 0$, whence $W \cap U = \{0\}$. Since $De = 0$, we have $D(P_j(f)) = D(e \oplus P_j(f)) \in W \cap U = \{0\}$. Therefore $D(P_j(f)) = 0$. Note that $\{P_j(f)\colon f \in H^\infty\}$ is dense in H_j. We conclude that $\overline{DH_j} = \{0\}$ for $j = 2, 3, \ldots, k$. Hence $\overline{D(H_2 \oplus \cdots \oplus H_k)} = \{0\}$, as asserted.

Next we show that $\underline{D(\xi H^2 \ominus \varphi_1 H^2) \subseteq \xi\varphi_2 H^2 \ominus \varphi_1 H^2}$ for any $\xi|\psi$. Let $W_1 = \xi H^2 \ominus \varphi_1 H^2$ and $U_1 = \{P_1(\xi f) \oplus P_2(f): f \in H^2\}$. For $g = \xi f \in W_1$, $Dg = D(P_1(\xi f) \oplus P_2(f)) \in W_1 \cap U_1$. Thus to complete the proof it suffices to show that $W_1 \cap U_1 \subseteq \xi\varphi_2 H^2 \ominus \varphi_1 H^2$. Let $w \in W_1 \cap U_1$. There exists a sequence $\{f_n\} \subseteq H^2$ such that $P_1(\xi f_n) \oplus P_2(f_n) \to w \oplus 0$ as $n \to \infty$. Assume that $f_n = g_n + \varphi_2 h_n$, where $g_n \in H^2 \ominus \varphi_2 H^2$ and $h_n \in H^2$ for each n. We infer that $P_1(\xi g_n + \xi\varphi_2 h_n) \to w$ and $g_n \to 0$. Thus $w - P_1(\xi\varphi_2 h_n) = (w - P_1(\xi g_n + \xi\varphi_2 h_n)) + P_1(\xi g_n) \to 0$. It follows that $w \in \xi\varphi_2 H^2 \ominus \varphi_1 H^2$ completing the proof.

LEMMA 6. *Let $T = S(\varphi_1) \oplus \cdots \oplus S(\varphi_k)$ be a Jordan operator defined on $H = (H^2 \ominus \varphi_1 H^2) \oplus \cdots \oplus (H^2 \ominus \varphi_k H^2)$ and let $\psi = \varphi_1/\varphi_2$. Then T is reflexive if and only if $S(\psi)$ is.*

PROOF. *Necessity*. Note that $T|\overline{\varphi_2(T)H}$ is unitarily equivalent to $S(\psi)$. (An explicit proof can be found in [6, pp. 315–316].) Since $\varphi_2(T) \in \text{Alg Lat } T \cap \{T\}'$ and $\text{Alg Lat } S(\psi) \cap \{S(\psi)\}' = \text{Alg } S(\psi)$, the reflexivity of T implies that of $S(\psi)$ by Lemma 4.

Sufficiency. Let T_j, H_j and P_j be as in the proof of Lemma 5 and let $S \in \text{Alg Lat } T$. By Lemma 5, there exist an outer $\eta \in H^\infty$ and $\delta \in H^\infty$ such that $\eta(T)S = \delta(T) + D$, where D satisfies

$$D\left[(\xi H^2 \ominus \varphi_1 H^2) \oplus H_2 \oplus \cdots \oplus H_k\right] \subseteq (\xi\varphi_2 H^2 \ominus \varphi_1 H^2) \oplus 0 \oplus \cdots \oplus 0 \quad \text{for any } \xi|\psi.$$

Let $D_1 = D|H^2 \ominus \psi H^2$ and $D_2 = D|(\psi H^2 \ominus \varphi_1 H^2) \oplus H_2 \oplus \cdots \oplus H_k$. Since $D(\psi H^2 \ominus \varphi_1 H^2) \subseteq \psi\varphi_2 H^2 \ominus \varphi_1 H^2 = \{0\}$ and $\overline{D(H_2 \oplus \cdots \oplus H_k)} = \{0\}$, we have $D_2 = 0$. On the other hand, for any $\xi|\psi$ consider the subspace $\xi H^2 \ominus \psi H^2$ in Lat $S(\psi)$. Note that $\xi H^2 \ominus \psi H^2 \subseteq \xi H^2 \ominus \varphi_1 H^2$. Hence from the property of D we infer that $D_1(\xi H^2 \ominus \psi H^2) \subseteq \xi\varphi_2 H^2 \ominus \varphi_1 H^2$. Thus the operator D' defined on $H^2 \ominus \psi H^2$ by $D'f = \bar{\varphi}_2 D_1 f$ for $f \in H^2 \ominus \psi H^2$ is in Alg Lat $S(\psi)$. By the reflexivity of $S(\psi)$, there exists ρ in H^∞ such that $D'f = \rho(S(\psi))f$ for all $f \in H^2 \ominus \psi H^2$. It follows that $D_1 f = \varphi_2(P(\rho f)) = P_1(\varphi_2\rho f)$, where P denotes the projection from H^2 onto $H^2 \ominus \psi H^2$. For any $h \in H$, $h = f + g$ where $f \in H^2 \ominus \psi H^2$ and $g = g_1 \oplus \cdots \oplus g_k \in (\psi H^2 \ominus \varphi_1 H^2) \oplus H_2 \oplus \cdots \oplus H_k$. We deduce that $(\varphi_2\rho)(T)h = (\varphi_2\rho)(T_1)(f + g_1) = P_1(\varphi_2\rho f + \varphi_2\rho g_1) = P_1(\varphi_2\rho f) = D_1 f$. Consequently, $Dh = D_1 f + D_2 g = (\varphi_2\rho)(T)h$. This shows that $D = (\varphi_2\rho)(T)$ and hence $\eta(T)S = \delta(T) + (\varphi_2\rho)(T) = (\delta + \varphi_2\rho)(T)$. Since η is outer, we conclude that $S \in \{T\}'' = \text{Alg } T$ (cf. [7, Theorem 3.2]). Thus T is reflexive, completing the proof.

Now Theorem 1 follows from Theorem 2 and Lemma 6. The condition in Theorem 1 was first formulated by C. Foiaş for general C_0 contractions in a private communication to the author. He also proved the necessity part. However our presentation here is more simplified.

REFERENCES

1. J. A. Deddens and P. A. Fillmore, *Reflexive linear transformations*, Linear Algebra and Appl. **10** (1975), 89–93.

2. B. Moore III and E. A. Nordgren, *On quasi-equivalence and quasi-similarity*, Acta Sci. Math. (Szeged) **34** (1973), 311–316.

3. E. A. Nordgren, *On quasi-equivalence of matrices over* H^∞, Acta Sci. Math. (Szeged) **34** (1973), 301–310.

4. B. Sz.-Nagy and C. Foiaş, *Modèle de Jordan pour une classe d'opérateurs de l'espace de Hilbert*, Acta Sci. Math. (Szeged) **31** (1970), 91–115.

5. ______, *Harmonic analysis of operators on Hilbert space*, Akadémiai Kiadó, Budapest, 1970.

6. ______, *Jordan model for contractions of class* $C_{\cdot 0}$, Acta Sci. Math. (Szeged) **36** (1974), 305–322.

7. P. Y. Wu, *Commutants of* $C_0(N)$ *contractions*, Acta Sci. Math. (Szeged) **38** (1976), 193–202.

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