Letters to the Editor

Design of Sharp Cutoff Low-Pass Maximally Flat *RC*-Active Filters by Cascading Third-Order Blocks

CHE-HO WEI

***Abstract*—Pairs of coincident $j\omega$-axis zeros are added to the multiple real root maximally flat (MURROMAF) approximation function for designing sharp cutoff low-pass *RC*-active filters. Expressions for transfer function and cutoff slope are derived in terms of the order n, the multiplicity of the real pole μ, the number of pairs of $j\omega$-axis zeros m and their locations. A design example is also given to illustrate the procedure for finding out the best transfer function.**

INTRODUCTION

In the design of *RC*-active filters, it is often very important to minimize the number of operational amplifiers to reduce cost and power consumption. This can be done by taking advantage of the better attenuation performance provided by third-order blocks [1]–[4]. The all-pole multiple real root maximally flat (MURROMAF) approximation function introduced recently by Biey and Premoli [5], [6] can be used to realize low-pass *RC*-active filters with third-order blocks and, if necessary, second-order blocks. However, the cutoff slope of the all-pole MURROMAF approximation function is only slightly steeper than that of the classical Butterworth approximation function for the same number of complex pole pairs.

It is known that sharp cutoff low-pass filters can be obtained by introducing coincident $j\omega$-axis zeros in the approximation function [7]–[9]. In this letter, we introduce a method to generate the approximation function of maximally flat low-pass filters with multiple real poles and finite transmission zeros.

MURROMAF APPROXIMATION WITH FINITE TRANSMISSION ZEROS

The normalized magnitude squared function of the all-pole MURROMAF approximation [5], [6] is expressed by

$$|M(j\omega)|^2=\frac{1}{1+P_{n+\mu}(\omega)}=\frac{1}{(1+\alpha^2\omega^2)^{\mu}\left[\sum_{i=0}^{n}a_i(\alpha\omega)^{2i}\right]} \tag{1}$$

where n is the number of complex poles of the transfer function and is an even number, and μ is the multiplicity of the real pole. The normalization factor α must be chosen to satisfy the condition $P_{n+\mu}(1)=1$. The cutoff slope of $|M(j\omega)|$ in the unit of dB/octave is given by

$$\underset{(\text{dB/octave})}{\text{cutoff slope}}\ S_{n,\mu}=-\frac{3.01}{10}\left.\frac{d[20\log|M(j\omega)|]}{d[\log\omega]}\right|_{\omega=1}$$

$$=6.02\frac{P'_{n+\mu}(1)}{4}. \tag{2}$$

Manuscript received June 29, 1978; revised May 17, 1979.
The author is with the Institute of Electronics, National Chiao Tung University, Hsinchu, Taiwan, Republic of China.

TABLE I
CUTOFF SLOPES OF BUTTERWORTH APPROXIMATION AND ALL-POLE MURROMAF APPROXIMATION WITH $\mu \leq n/2$ FOR $n=2-16$

MURROMAF Approximation				Butterworth Approximation	
n	μ	α	Cutoff Slope (dB/octave)	n	Cutoff Slope (dB/octave)
2	1	1.0000000	9.03	2	6.02
				3	9.03
4	2	0.8009154	16.09	4	12.04
				5	15.05
6	2	0.8332368	22.21	6	18.06
	3	0.7326496	23.00	7	21.07
8	2	0.8555286	28.29	8	24.08
	3	0.7637016	29.17		
	4	0.6974542	29.85	9	27.09
10	2	0.8719795	34.35	10	31.10
	3	0.7874258	35.30		
	4	0.7248115	35.98	11	33.11
	5	0.6757889	36.67		
12	2	0.8846950	40.40	12	36.12
	3	0.8062559	41.40		
	4	0.7470469	42.22		
	5	0.6998954	42.91	13	39.13
	6	0.6610320	43.48		
14	2	0.8948606	46.44		
	3	0.8216304	47.48	14	42.14
	4	0.7654373	48.36		
	5	0.7201270	49.10		
	6	0.6824112	49.74	15	45.15
	7	0.6502968	50.29		
16	2	0.9031994	52.48		
	3	0.8344613	53.56	16	48.16
	4	0.7809850	54.48		
	5	0.7374062	55.26		
	6	0.7008222	55.95		
	7	0.6694524	56.56	17	51.17
	8	0.6421163	57.19		

The cutoff slopes of the classical Butterworth approximation and the all-pole MURROMAF approximation with $\mu \leq n/2$ for $n=2-16$ are listed in Table I. It should be noticed that the number of complex poles in the classical Butterworth approximation of order n is equal to n if n is even and is equal to $n-1$ if n is odd.

The cutoff slope of the all-pole MURROMAF approximation is only slightly larger than that of the corresponding Butterworth approximation (with the same number of complex poles) as shown in Table I. It can be improved by adding pairs of $j\omega$-axis zeros in the transfer function. If we add m pairs of coincident zeros at $\pm j\omega_0$ to the MURROMAF approximation function, then the normalized magnitude squared function can be written as [7]–[9]

$$|T_a(j\omega)|^2=\frac{(\omega^2-\omega_0^2)^{2m}}{(\omega^2-\omega_0^2)^{2m}+(\omega_0^2-1)^{2m}P_{n+\mu}^{(\omega)}} \tag{3}$$

where $\omega_0>1$, $2m<n+\mu$, and $n/2\geq\mu\geq 2$. The network transfer function $T_a(s)$ can be found from (3) by the usual procedure of

0098-4094/80/0500-0411$00.75 ©1980 IEEE

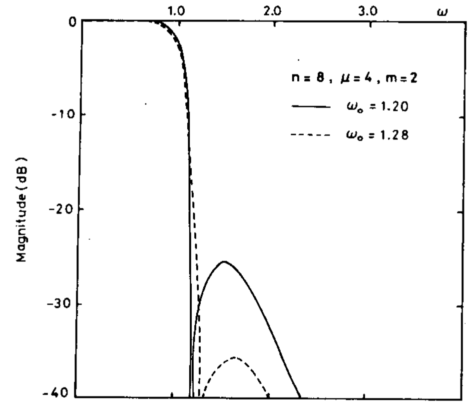

Fig. 1. Transmission characteristics of the modified MURROMAF approximation with $n=8$, $\mu=4$, $m=2$, and $\omega_0=1.20$, 1.28.

analytic continuation. The transfer function will be of the form

$$T_a(s)=\frac{(s^2+\omega_0^2)^m}{(\omega_0^2-1)^m D_a(s)}$$

where $D_a(s)$ is the product of the left-half s-plane factors of the polynomial

$$D_a(s)D_a(-s)=P_{n+\mu}\left(\frac{s}{j}\right)+\left(\frac{s^2+\omega_0^2}{\omega_0^2-1}\right)2m. \tag{4}$$

Unfortunately, the polynomial in (4) does not ensure the existence of multiple real pole in $T_a(s)$. However, if we rewrite (4) as

$$D_a(s)D_a(-s)=1+P_{n+\mu}\left(\frac{s}{j}\right)+\left[\left(\frac{s^2+\omega_0^2}{\omega_0^2-1}\right)^{2m}-1\right]$$

$$=(1-\alpha^2s^2)^\mu\left\{\sum_{i=0}^{n}a_i(-\alpha^2s^2)^i+\frac{1}{(1-\alpha^2s^2)^\mu}\left[\left(\frac{s^2+\omega_0^2}{\omega_0^2-1}\right)^{2m}-1\right]\right\} \tag{5}$$

and substitute $(1+\alpha^2s^2)^\mu$ for $1/(1-\alpha^2s^2)^\mu$ in the bracket { }, then we obtain the new expression

$$D(s)D(-s)=(1-\alpha^2s^2)^\mu\left\{\sum_{i=0}^{n}a_i(-\alpha^2s^2)^i+(1+\alpha^2s^2)^\mu\left[\left(\frac{s^2+\omega_0^2}{\omega_0^2-1}\right)^{2m}-1\right]\right\}. \tag{6}$$

If the number of complex roots of (6) is equal to $2n$, it requires that

$$m\leqslant\frac{n-\mu}{2}. \tag{7}$$

The required transfer function will be of the form

$$T(s)=\frac{(s^2+\omega_0^2)^m}{(\omega_0^2-1)^m D(s)}. \tag{8}$$

The integer m chosen from (7) always satisfies the condition $m\leqslant(n+\mu)/2$ for the function $T(s)$ to be low pass. From (6) and (8), it is clear that the transfer function $T(s)$ always has a multiple real pole at $s=-1/\alpha$. The new magnitude squared function is then given by

TABLE II
SOME POSSIBLE FILTERS FOR OBTAINING A CUTOFF SLOPE STEEPER THAN 60 dB/OCTAVE BY MODIFIED MURROMAF APPROXIMATION WITH FINITE TRANSMISSION ZEROS

n	μ	m	ω_0	Minimum Attenuation in Stopband	
				ω_p	A_p (dB)
8	2	1	1.157	1.302	21.28
		2	1.295	1.690	33.81
		3	1.419	2.264	38.97
	3	1	1.155	1.295	21.83
		2	1.291	1.656	35.23
	4	1	1.153	1.288	22.20
		2	1.289	1.634	36.45
10	2	1	1.194	1.314	30.24
		2	1.362	1.682	49.85
		3	1.510	2.155	61.35
		4	1.646	2.886	65.95
	3	1	1.196	1.311	31.37
		2	1.364	1.664	52.02
		3	1.513	2.103	64.92
	4	1	1.196	1.306	32.50
		2	1.365	1.645	54.50
		3	1.515	2.050	69.02
	5	1	1.198	1.308	32.95
		2	1.368	1.643	55.42

It is easy to show that

$$|T(j\omega)|^2_{\omega=0}=1,\quad |T(j\omega)|^2_{\omega=1}=\frac{1}{2}$$

and

$$|T(j\omega)|^2>\frac{1}{1+\left(\frac{\omega_0^2-1}{\omega_0^2-\omega^2}\right)^{2m}P_{n+\mu}(\omega)}>\frac{1}{1+P_{n+\mu}(\omega)},$$

for $0<\omega<1$.

Thus the attenuation in the passband of the modified MURROMAF approximation with finite transmission zeros is always less than that of the all-pole MURROMAF approximation.

The cutoff slope of $|T(j\omega)|$ can be obtained from (9) and is given by

$$\underset{(\text{dB/octave})}{S_{n,\mu,m}}=6.02\left\{\frac{P'_{n+\mu}(1)}{4}+\frac{m[2-(1-\alpha^4)^\mu]}{\omega_0^2-1}\right\}. \tag{10}$$

The increase in the cutoff slope due to the added finite transmission zeros is given in the second term of (10). The closer ω_0 is to 1, the steeper is the cutoff slope.

The transmission characteristics of the filter under consideration will be of the form shown in Fig. 1. In the stopband, $\omega>1$, there will be a transmission peak at some frequency $\omega_p>\omega_0$,

$$|T(j\omega)|^2=\frac{1}{1+\left(\frac{\omega_0^2-1}{\omega_0^2-\omega^2}\right)^{2m}\left\{P_{n+\mu}(\omega)-\left[1-(1-\alpha^4\omega^4)^\mu\right]\left[\left(\frac{\omega_0^2-\omega^2}{\omega_0^2-1}\right)^{2m}-1\right]\right\}}. \tag{9}$$

TABLE III
SOME POSSIBLE FILTERS FOR OBTAINING A CUTOFF SLOPE STEEPER THAN 60 dB/OCTAVE BY MODIFIED BUTTERWORTH APPROXIMATION WITH FINITE TRANSMISSION ZEROS

n	m	ω_0	Minimum Attenuation in Stopband	
			ω_{bp}	A_{bp} (dB)
11	1	1.106	1.223	17.45
	2	1.203	1.508	28.56
	3	1.292	1.916	33.57
	4	1.376	2.635	32.38
	5	1.455	4.825	22.59
12	1	1.118	1.225	21.14
	2	1.226	1.501	35.39
	3	1.325	1.874	43.85
	4	1.417	2.454	45.55
	5	1.503	3.682	40.56
13	1	1.135	1.234	25.53
	2	1.255	1.508	42.96
	3	1.365	1.860	54.14
	4	1.467	2.365	59.10
	5	1.562	3.252	58.05
	6	1.652	5.956	48.25
14	1	1.156	1.249	30.58
	2	1.293	1.530	51.78
	3	1.418	1.876	66.07
	4	1.532	2.340	74.09
	5	1.638	3.064	76.13
	6	1.738	4.598	71.18

satisfying the condition $|T(j\omega)|'_{\omega=\omega_p}=0$. The minimum attenuation in the stopband is given by $A_p=-20\log|T(j\omega_p)|$.

The cases $\mu=0$ and $\mu=1$ in (3) coincide with the modified Butterworth approximation with m pairs of $j\omega$-axis zeros [7], [8]. The cutoff slope is given by

$$\underset{(\text{dB/octave})}{S_{n,m}}=6.02\left[\frac{2n}{4}+\frac{m}{\omega_0^2-1}\right] \tag{11}$$

where n is the order of the Butterworth approximation and is an integer. The minimum stopband attenuation is expressed by

$$A_{bp}=1+\left(\frac{\omega_0^2-1}{\omega_{bp}^2-\omega_0^2}\right)^{2m}\omega_{bp}^{2n} \tag{12}$$

where

$$\omega_{bp}^2=\left(\frac{n}{n-2m}\right)\omega_0^2.$$

AN EXAMPLE OF DESIGN

Suppose it is required to design a sharp cutoff maximally flat RC-active filter with a cutoff slope steeper than 60 dB/octave. The required minimum order of the classical Butterworth approximation function is 20. If the MURROMAF approximation is used, the required n estimated is 18. With the modified MURROMAF approximation with finite transmission zeros, several solutions are possible by solving the inequality

$$6.02\left\{\frac{P'_{n+\mu}(1)}{4}+\frac{m\left[2-(1-\alpha^4)^\mu\right]}{\omega_0^2-1}\right\}>60.$$

Some of the solutions are given in Table II. The values of n, μ, ω_0, and m to be selected depend on the minimum stopband-attenuation required. For a given set of (n,μ,m,ω_0), the frequency ω_p and A_p are also given in Table II. If the minimum stopband-attenuation required is 60 dB, then $n=10$, $\mu=2$, $m=3$, and $\omega_0=1.510$ are a good choice. Finally, the transfer function $T(s)$ can be obtained from (6) and (8).

By the same procedure, if the modified Butterworth approximation is used, some of the possible solutions for cutoff slope steeper than 60 dB/octave are given in Table III. If the minimum stopband attenuation required is also 60 dB, then the optimum choice is $n=14$, $m=3$, and $\omega_0=1.875$. Thus if we realize the required filter with the modified MURROMAF approximation with finite $j\omega$-axis zeros, two operational amplifiers can be saved.

REFERENCES

[1] L. P. Huelsman, "Equal-valued-capacitor active-RC-network realization of a 3rd-order lowpass Butterworth characteristic," *Electron. Lett.*, vol. 7, no. 10, pp. 271–272, May 1971.
[2] S. Tirtoprodjo, "Constraint removal for Huelsman's equal-valued capacitor active RC circuit," *Electron. Lett.*, vol. 7, no. 16, pp. 448–449, Aug. 1971.
[3] G. S. Moschytz, *Linear Integrated Networks Design.* New York: Van Nostrand-Reinhold, 1975, pp. 247–253.
[4] S. A. Boctor, "Design of a third-order single amplifier filter," *IEEE Trans. Circuits Syst.*, vol. CAS-22, pp. 329–334, Apr. 1975.
[5] M. Biey and A. Premoli, "Maximally flat approximation of low-pass RC active filters with multiple real pole," *Electron. Lett.*, vol. 13, no. 3, pp. 76–78, Feb. 1977.
[6] M. Biey and A. Premoli, "Design of low-pass maximally flat RC-active filters with multiple real pole: The MURROMAF polynomials," *IEEE Trans. Circuits Syst.*, vol. CAS-25, pp. 196–200, Apr. 1978.
[7] A. Budak and P. Aronhime, "Maximally flat low-pass filters with steeper slopes at cutoff," *IEEE Trans. Audio Electroacoust.* vol AU-18, pp. 63–66, Mar. 1970.
[8] S. C. Dutta Roy, "On maximally flat sharp cutoff low-pass filters," *IEEE Trans. Audio Electroacoust.*, vol. AU-19, pp. 58–63, Mar. 1971.
[9] M. C. Agarwal and A. S. Sedra, "On designing sharp cutoff low-pass filters," *IEEE Trans. Audio Electroacoust.*, vol. AU-20, pp. 138–141, June 1972.

Padé-Type Order Reduction of Two-Dimensional Systems

P. N. PARASKEVOPOULOS

***Abstract*—The problem of reducing the order of the transfer function of a given two-dimensional system to a lower order model using Padé-type approximation is considered. Two computational algorithms are proposed. The first is a generalization of the known one-dimensional Padé-type approximation algorithm. The second is a computationally improved version of the first since it involves the parameters of the transfer function of the given system directly, thus circumventing the task of determining the power series expansion coefficients of the given transfer function required in the first algorithm.**

I. INTRODUCTION

It is usually desirable (for realization, control, computation and other purposes) to be able to represent "adequately" a high-order system by a lower order model. To this end, several

Manuscript received May 5, 1978; revised June 1, 1979.
The author is with the Department of Electrical Engineering, School of Engineering, Democritus University of Thrace, Xanthi, Greece.

0098-4094/80/0500-0413$00.75 ©1980 IEEE