Deterministic Identification of Linear Dynamic Systems

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***Abstract*—This paper considers the problem of identifying multiinput single-output linear time-invariant discrete-time systems from noise-free input/output measurements. The effect of input on identification is studied in detail. It is shown that the correctness associated with an identification is critically dependent on the input sequence used. Furthermore, as long as only finite input/output sequences are used for identification, there is always a possibility that the actual system is of higher order than the identified models reveal. This uncertainty leads to the concept of tentative identification. Sufficient conditions for correct order determination and modeling are also investigated.**

I. Introduction

System identification has been studied extensively, most of the research being devoted to the construction of reliable mathematical models from the noise-corrupted input/output data of systems excited by sufficiently random input (usually the white noise). Identification involving arbitrary input has not been widely studied. Recently, Emre, Silverman, and Glover [1] showed the possibility of multiple models. Rekasius and Brasch [14] observed that false models may be obtained from the input/output data and that false identification is due to the properties of the input. Furthermore, as long as the length of the input/output data is finite, there is always a possibility that the actual system is of higher order than the models reveal. Hence, without the knowledge of the system order, one is not able to tell if a system has been identified successfully. The models obtained in this manner are referred to as tentatively identified [14]. It is shown that the correctness associated with an identification is intimately related to the input used. Sufficient conditions for correct order determination and modeling are investigated.

II. Basic Theory for Order Determination

Consider the following input–output model of an m-input, single-output, linear, time-invariant, discrete-time system:

$$S^0:\quad \sum_{t=0}^{n}\alpha_t z(k+t)=\sum_{j=1}^{m}\sum_{t=0}^{n}\beta_{jt}u_j(k+t),\qquad \alpha_n=1 \tag{1}$$

where k denotes the time instant; $u(\cdot)\equiv \mathrm{col}(u_1(\cdot)\cdots u_m(\cdot))$ is the input; $z(\cdot)$ is the output; and α_t,β_{jt} are constant system parameters. Assume

$$D^{N+1}\triangleq\{u(k)\in R^m,\ z(k)\in R;\ k=0,1,\cdots,N\} \tag{2}$$ [1]

is a noise-free input/output time-series of S^0. Define the matrices

$$Z^p(M)\triangleq\begin{bmatrix} z(0) & \cdots & z(p)\\ z(1) & \cdots & z(p+1)\\ \vdots & & \vdots\\ z(M) & \cdots & z(p+M)\end{bmatrix};$$

$$U_j^{q_j}\triangleq\begin{bmatrix} u_j(0) & \cdots & u_j(q_j)\\ u_j(1) & \cdots & u_j(q_j+1)\\ \vdots & & \vdots\\ u_j(M) & \cdots & u_j(q_j+M)\end{bmatrix} \tag{3}$$

$$U^{q_1\cdots q_m}(M)\triangleq\left[\,U_1^{q_1}(M)\ \vdots\ \cdots\ \vdots\ U_m^{q_m}(M)\,\right] \tag{4}$$

$$R_{z;u}^{p;q_1\cdots q_m}(M)\triangleq\left[\,Z^p(M)\ \vdots\ U^{q_1\cdots q_m}(M)\,\right] \tag{5}$$

$$Q_{z;u}^{p;q_1\cdots q_m}(M)\triangleq\frac{1}{M+1}\left[R_{z;u}^{p;q_1\cdots q_m}(M)\right]^T\left[R_{z;u}^{p;q_1\cdots q_m}(M)\right]. \tag{6}$$

Consider the following function:

$$\begin{aligned}R(p;N)&\triangleq \mathrm{rank}\left[R_{z;u}^{p;p\cdots p}(N-p)\right]-\mathrm{rank}\left[R_{z;u}^{p-1;p\cdots p}(N-p)\right]\\ &\equiv \mathrm{rank}\left[Q_{z;u}^{p;p\cdots p}(N-p)\right]-\mathrm{rank}\left[Q_{z;u}^{p-1;p\cdots p}(N-p)\right].\end{aligned} \tag{7}$$

Note that the only difference between $R_{z;u}^{p;p\cdots p}(N-p)$ and $R_{z;u}^{p-1;p\cdots p}(N-p)$ is that the former contains an extra column,

$$z(p;N)\triangleq \mathrm{col}(z(p)z(p+1)\cdots z(N)). \tag{8}$$

Thus, $R(p;N)$ can only assume the value of either 0 or 1 for all nonnegative integers p. Based on the values of $R(p;N)$, we conclude the following.

1) If $R(p;N)=0$, then $z(p;N)$ is linearly dependent on the columns of $R_{z;u}^{p-1;p\cdots p}(N-p)$. Thus, D^{N+1} satisfies a pth order equation,

$$\sum_{t=0}^{p}\alpha_t z(k+t)=\sum_{j=1}^{m}\sum_{t=0}^{p}\beta_{jt}u_j(k+t),\qquad \alpha_p\triangleq 1. \tag{9}$$

2) If $R(p;N)=1$, then $z(p;N)$ is linearly independent of the columns of $R_{z;u}^{p-1;p\cdots p}(N-p)$. Since this condition also implies that $R(p';N)=1$ for all $p'<p$, it is clear that D^{N+1} satisfies no models of the form (9) for all orders $p'<p$. Hence, $n>p$.

Now start with $p=0$ and check the value of $R(p;N)$. If $R(p;N)=1$, then $n>p$. Increase p by 1 and repeat the procedure until $R(p_1;N)=0$ is obtained. Then p_1 is the minimal possible order of S^0 based on the data D^{N+1}. We summarize the above conclusions as follows:

$$\text{if } R(p;N)=\begin{cases}1, & \text{then } n>p\\ 0, & \text{then } n=p_1.\end{cases} \tag{10}$$

Finally, some remarks about the relation of S^0 to its state-space model are in order. Let the equations

$$S^0:\quad x(k+1)=Ax(k)+Bu(k) \tag{11a}$$

$$y(k)=Cx(C)+Dx(k) \tag{11b}$$

be a state-space model of a system S^0. Here $x(\cdot)$ is the n'-dimensional state vector. If the system is completely controllable and completely observable,

$$n'=n.$$

Otherwise,

$$n'>n.$$

Therefore, we will assume throughout this paper that the system S^0, which we want to identify is completely controllable and completely observable, i.e., we assume that there are no pole-zero cancellations in (1), in the sense that the McMillan degree of the transfer matrix is n.

III. Order Determinability

In this section, sufficient conditions for correct determination of the system order are investigated. The question that is of interest here is the following: based on the data D^{N+1}, under what conditions can the order of S^0 be determined correctly by means of the algorithm (10)?

Definition 1: The order n of the input–output model (1) of the system S^0 is said to be correctly determined if the inequality

$$n\leqslant p \tag{12}$$

holds for all p, for which the input–output sequence D^{N+1} satisfies (9).

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[1] $\mathfrak{R}$ denotes the field of real numbers.

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Next we investigate the sufficient conditions for correct order determination. For convenience, define the advance operator Δ by

$$\Delta^t F(k) \triangleq F(k+t) \tag{13}$$

for all time functions $F(k)$. Then (1) can be written as

$$S^0: \quad L(\Delta)z(k) = \sum_{j=1}^{m} M_j(\Delta)u_j(k) \tag{14}$$

where $L(\Delta)$ and $M_j(\Delta)$ are polynomials in Δ defined as

$$L(\Delta) \triangleq \sum_{t=0}^{n} \alpha_t \Delta^t, \qquad \alpha_n = 1 \quad \text{and}$$

$$M_j \triangleq \sum_{t=0}^{n} \beta_{jt}\Delta^t, \qquad j=1,\cdots,m. \tag{15}$$

Theorem 1: The order n of the m-input, single-output system S^0 can be determined correctly from the noise-free input/output time-series if

1) S^0 is completely controllable, and
2) the input $\{u(k) \in R^m;\ k=0,1,\cdots,N\}$ satisfies the inequality

$$\operatorname{rank}[U^{2n-1\cdots 2n-1}(N-2n+1)] > 2nm$$

where $U^{2n-1\cdots 2n-1}(N-2n+1)$ is an $(N-2n+2)\times 2nm$ matrix defined by (4).

Proof: Since S^0 is assumed to be completely controllable and completely observable, it is in minimal form; hence, it cannot be reduced to a lower order model by common factor cancellation of the polynomials $L(\Delta)$ and $M_j(\Delta)$, $j=1,\cdots,m$. Assume the noise-free input/output sequence D^{N+1} also satisfies a model $\bar{S}^0$ of order $\bar{n}<n-1$ described by the equation

$$\bar{S}^0: \quad \bar{L}(\Delta)z(k) = \sum_{j=1}^{m} \bar{M}_j(\Delta)u_j(k) \tag{16}$$

where

$$\bar{L}(\Delta) \triangleq \sum_{t=0}^{\bar{n}} \bar{\alpha}_t\Delta^t \quad \text{and} \quad \bar{M}_j(\Delta) \triangleq \sum_{t=0}^{\bar{n}} \bar{\beta}_{jt}\Delta^t, \qquad j=1,\cdots,m.$$

From (14) and (16), one obtains

$$\sum_{j=1}^{m} M_j^*(\Delta)u_j(k) = 0 \tag{17}$$

where polynomials $M_j^*(\Delta)$ of degrees less than $2n$ are defined as

$$M_j^*(\Delta) \triangleq \bar{L}(\Delta)M_j(\Delta) - L(\Delta)\bar{M}_j(\Delta) = \sum_{t=0}^{2n-1} \gamma_{jt}\Delta^t.$$

Evaluating (17) for $k=0,1,\cdots,N-2n+1$, we obtain

$$[U^{2n-1\cdots 2n-1}(N-2n+1)]\gamma = 0.$$

where γ is a $2nm\times 1$ vector defined as

$$\gamma \triangleq \operatorname{col}(\gamma_{10}\cdots\gamma_{1,2n-1}\cdots\gamma_{m0}\cdots\gamma_{m,2n-1}).$$

Now if $\gamma = 0$, then above implies that

$$\frac{M_j(\Delta)}{L(\Delta)} = \frac{\bar{M}_j(\Delta)}{\bar{L}(\Delta)}, \qquad \text{for all } j=1,\cdots,m. \tag{18}$$

This means that S^0 can be reduced to the lower order model $\bar{S}^0$ by common factor cancellation of $L(\Delta)$ and $M_j(\Delta)$, hence contradicting assumption 2) of Theorem 1. Thus, $\gamma \neq 0$ and the $2nm$ columns of $U^{2n-1\cdots 2n-1}(N-2n+1)$ are linearly dependent. That is,

$$\operatorname{rank}[U^{2n-1\cdots 2n-1}(N-2n+1)] < 2nm. \tag{19}$$

However, (19) contradicts the assumption 2) of Theorem 1. Q.E.D.

Define the following $(q+1)m\times(q+1)m$ nonnegative definite matrix:

$$\mathcal{U}^{q\cdots q}(M) \triangleq \frac{1}{m+1}[U^{q\cdots q}(M)]^T[U^{q\cdots q}(M)]. \tag{20}$$

Since $\operatorname{rank}\mathcal{U}^{q\cdots q}(M) \equiv \operatorname{rank} U^{q\cdots q}(M)$, we can also state Theorem 1 as the following corollary.

Corollary 1.1: The order n of the m-input, single-output system S^0 can be determined correctly from the noise-free input/output time-series if

1) S^0 is completely controllable, and
2) the input $\{u(k) \in \mathfrak{R}^m,\ k=0,1,\cdots,N\}$ satisfies the inequality

$$\det[\mathcal{U}^{2n-1\cdots 2n-1}(N-2n+1)] > 0. \tag{21}$$

Remark: To establish assumption 2) of Theorem 1 or (21), the input sequence needs a length

$$N+1 > 2n(m+1)-1. \tag{22}$$

Remark: Theorem 1 and Corollary 1.1 represent only the sufficient conditions. Example 3 illustrates the possibility of correct order determination from an input/output sequence for which the input does not satisfy assumption 2) of Theorem 1 or (21).

Remark: If an input yields a correct order determination, it does not necessarily mean that the system can also be modeled correctly. Correct modeling also requires the uniqueness of the system parameters. This problem will be discussed in Section IV.

Based on the matrix $\mathcal{U}^{q\cdots q}(M)$ defined by (20), it is obvious that if $\det[\mathcal{U}^{q\cdots q}(N-q)]=0$ for some $q=\hat{q}$, then it is true for all $q>\hat{q}$. Thus, if an input is "unsuitable" for order determination of an nth order system, it is also "unsuitable" for all systems of order $n'>n$. The vanishing of $\det[\mathcal{U}^{q\cdots q}(N-q)]$ is a critical property of an input for order determination.

One may finally observe that order determination is not a goal in itself. However, once the order of the system has been determined, one has its input–output model with a fixed number of unknown coefficients.

To complete the identification one has to compute these unknown coefficients. This relatively straightforward problem is discussed in the next section.

IV. System Identifiability

In this section, sufficient conditions for correct system identification is investigated.

Theorem 2: The nth order, m-input, single-output system S^0 can be identified correctly from the noise-free input/output time-series if

1) S^0 is completely controllable, and
2) the input $\{u(k) \in \mathfrak{R}^m,\ k=0,1,\cdots,N\}$ satisfies the inequality

$$\operatorname{rank}[U^{2n\cdots 2n}(N-2n)] > (2n+1)m, \tag{23}$$

or equivalently,

$$\det[\mathcal{U}^{2n\cdots 2n}(N-2n)] > 0 \tag{24}$$

where $U^{2n\cdots 2n}(N-2n)$ and $\mathcal{U}^{2n\cdots 2n}(N-2n)$ are $(N-2n+1)\times(2n+1)m$ and $(2n+1)m\times(2n+1)m$ matrices defined by (4) and (20), respectively.

Proof: Let D^{N+1} be a sequence of noise-free input/output measurements of S^0. Assume that D^{N+1} also satisfies a different model $\bar{S}^0$ as described by (16) of order $\bar{n}\leq n$. From (14) and (16) we obtain

$$0 = \sum_{j=1}^{m} M_j^*(\Delta)u_j(k) \tag{25}$$

where polynomials $M_j^*(\Delta)$ are of degrees less than $2n+1$ defined as

$$M_j^*(\Delta) \triangleq \bar{L}(\Delta)M_j(\Delta) - L(\Delta)\bar{M}_j(\Delta) \triangleq \sum_{t=0}^{2n} \gamma_{jt}\Delta^t. \tag{26}$$

Evaluating (25) for $k=0,1,\cdots,N-2n$, we obtain

$$[U^{2n\cdots 2n}(N-2n)]\gamma = 0 \tag{27}$$

where γ is a $(2n+1)m\times 1$ vector defined as

$$\gamma \triangleq \mathrm{col}(\gamma_{10}\cdots\gamma_{1,2n}\cdots\gamma_{m0}\cdots\gamma_{m,2n}). \tag{28}$$

Since S and $\bar{S}^0$ are different models, we have $\gamma\neq 0$; hence, (27) implies

$$\mathrm{rank}[U^{2n\cdots 2n}(N-2n)]\equiv\mathrm{rank}[\mathcal{U}^{2n\cdots 2n}(N-2n)]<(2n+1)m. \tag{29}$$

However, (29) contradicts the assumption (23) or (24). Q.E.D.

Remark: To establish (23), the input sequence needs a length

$$N+1>2n(m+1)+m. \tag{30}$$

$\mathcal{D}_2^{14}$:

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$u_1(k)$	1	1	−1	−1	−1	0	1	2	1	−2	−4	−1	4	2
$u_2(k)$	0	−1	1	1	1	−2	−3	0	4	2	−3	−3	0	3
$z(k)$	0	0	1	−1	0	0	−1	−2	1	4	1	−6	−7	4

Remark: Theorem 2 represents only the sufficient conditions. In certain cases, the system may still be identified correctly and uniquely even if the input does not satisfy (23) [see example 3].

V. Modeling

In this section, modeling of the m-input, single-output system S^0 from the noise-free input/output measurements is considered.

Assume $\hat{n}$ is the order of S^0 determined from the procedure described in Section IV based on the noise-free input/output data D^{N+1} given by (2). Then the vector $z(\hat{n};N)$, defined by (8), is linearly dependent upon the columns of the matrix $R_{z;u}^{\hat{n}-1;\hat{n}\cdots\hat{n}}(N-\hat{n})$. Thus, we can write

$$z(\hat{n};N)=\left[R_{z;u}^{\hat{n}-1;\hat{n}\cdots\hat{n}}(N-\hat{n})\right]\hat{a} \tag{31}$$

where $\hat{a}$ is a vector comprising $\hat{n}+\hat{n}m+m$ parameters. By defining

$$\hat{a}\triangleq\mathrm{col}\left(-\hat{\alpha}_0\cdots-\hat{\alpha}_{\hat{n}-1}\hat{\beta}_{10}\cdots\hat{\beta}_{1\hat{n}}\cdots\hat{\beta}_{m0}\cdots\hat{\beta}_{m\hat{n}}\right), \tag{32}$$

we obtain an $\hat{n}$th order model

$$\sum_{t=0}^{\hat{n}}\hat{\alpha}_t z(k+t)=\sum_{j=1}^{m}\sum_{t=0}^{\hat{n}}\hat{\beta}_{jt}u_j(k+t), \qquad \hat{\alpha}_n\triangleq 1. \tag{33}$$

Premultiplying (31) by $1/N-\hat{n}+1[R_{z;u}^{\hat{n}-1;\hat{n}\cdots\hat{n}}(N-\hat{n})]^T$ and using (6) we obtain

$$\left[Q_{z;u}^{\hat{n}-1;\hat{n}\cdots\hat{n}}(N-\hat{n})\right]\hat{a}=\frac{1}{N-\hat{n}+1}\left[R_{z;u}^{\hat{n}-1;\hat{n}\cdots\hat{n}}(N-\hat{n})\right]^T z(\hat{n};N). \tag{34}$$

The parameter vector $\hat{a}$ can be solved from (34) as follows:

1) if $\det[Q_{z;u}^{\hat{n}-1;\hat{n}\cdots\hat{n}}(N-\hat{n})]\neq 0$, we obtain a unique solution,

$$\hat{a}=\left[Q_{z;u}^{\hat{n}-1;\hat{n}\cdots\hat{n}}(N-\hat{n})\right]^{-1}\left[R_{z;u}^{\hat{n};\hat{n}\cdots\hat{n}}(N-\hat{n})\right]^T z(\hat{n};N); \tag{35}$$

2) if $\det[Q_{z;u}^{\hat{n}-1;\hat{n}\cdots\hat{n}}(N-\hat{n})]=0$, there are multiple solutions for $\hat{a}$. For each solution of $\hat{a}$, we obtain an $\hat{n}$th order model of the form (33).

VI. Examples

In this section, examples are presented for illustrating the effect of the input on identification. The noise-free input/output data are obtained from the following two-input single-output second-order model:[2]

$$z(k+2)=z(k+1)-z(k)+u_1(k+1)-u_1(k)+u_2(k+2)+u_2(k). \tag{36}$$

Example 1: The possibility of false order determination is illustrated. The input/output time-series[3]

D_1^{14}:

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$u_1(k)$	1	0	0	−1	−1	2	2	−2	−1	0	1	1	−1	−2
$u_2(k)$	0	1	0	0	1	−1	−1	4	1	0	0	0	1	0
$z(k)$	0	2	1	0	−1	−2	2	7	1	−1	0	2	3	−1

yields

$$R(p;13)=\begin{cases}1, & p=0\\ 0, & p=1.\end{cases}$$

Hence, the determined order is $\hat{n}=1$ and we find a unique model,

$$z(k+1)=z(k)+u_1(k+1)+u_1(k)+u_2(k+1)-u_2(k).$$

Example 2: The possibility of false modeling is illustrated. The input/output time-series $\mathcal{D}_2^{14}$ yields

$$R(p;13)=\begin{cases}1, & p=0,1\\ 0, & p=2.\end{cases}$$

Hence, the determined order is $\hat{n}=2$. From (36)–(38), we find that D_2^{14} satisfies both the model (36) and the following model:[4]

$$z(k+2)=-z(k)+u_1(k+2)+u_1(k)+u_2(k+2)+2u_2(k).$$

Example 3: This example illustrates the possibility of correct modeling from a theoretically "unsuitable" input sequence and the existence of higher order models. The input/output time-series[5]

D_3^{10}:

k	0	1	2	3	4	5	6	7	8	9
$u_1(k)$	0	1	0	0	1	−1	−1	4	1	0
$u_2(k)$	1	0	0	−1	−1	2	2	−2	−1	0
$z(k)$	1	1	2	−1	−4	−1	2	3	7	−1

yields

$$R(p;9)=\begin{cases}1, & p=0,1\\ 0, & p=2.\end{cases}$$

Hence, the determined order is $\hat{n}=2$ and we find a unique model (36). However, D_3^{10} also satisfies the following fourth-order system:

$$\begin{aligned}z(k+4)=&\,z(k+3)-15.58z(k+2)+23.56z(k+1)\\&-22.78z(k)-u_1(k+4)-0.65u_1(k+3)\\&+0.15u_1(k+2)-0.62u_1(k+1)+4.73u_1(k)\\&+11u_2(k+3)+31u_2(k+2)-17u_2(k+1)+40u_2(k).\end{aligned}$$

VII. Conclusions

In this paper, identification of multiinput single-output systems is considered. Order determination is based on the rank difference between two matrices constructed from the input/output data. It has been shown that the correctness of an order determined or a model identified is critically dependent on the input used. Also, whenever the input/output sequence is finite, there is always a possibility that the actual system is of higher order than the one identified. Thus, in general, one is not able to tell if the order determined or the model identified is correct. For this reason, identification is only tentative. Sufficient conditions for correct order determination and modeling are also derived.

[2]Here an unstable model is used for the convenience of computation.

[3]The model is satisfied by D_1^{14} at $k=0$ with initial condition $z(-1)=-1$.

[4]The model is satisfied by D_2^{14} at $k=0,1$ with $z(-1)=0$ and $z(-2)=1$.

[5]The model is satisfied by D_3^{10} at $k=0,1,2,3$ with $z(-1)=-0.2576821773$, $z(-2)=0.3379722520$, $z(-3)=0.9647635901$, and $z(-4)=0.7114372396$.

The order determination and the modeling procedures presented in this paper can be extended to stable stochastic systems under appropriate assumptions on the noise statistics. This problem will be considered in a separate paper.

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On the Stabilization of Nonlinear Systems Using State Detection

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***Abstract*—In this paper, we study the problem of stabilizing a nonlinear control system by means of a feedback control law, in cases where the entire state of the system is not available for measurement. The proposed method of stabilization consists of three parts: 1) determine a stabilizing control law based on state feedback, assuming the state vector $x(t)$ can be measured; 2) construct a state detection mechanism, which generates a vector $z(t)$ such that $z(t)-x(t)\to 0$ as $t\to\infty$; and 3) apply the previously determined control law to $z(t)$. This scheme is well established for linear time-invariant systems, and its global convergence has previously been studied in the case of nonlinear systems. Hence, the contribution of this paper is in showing that such a scheme works *in the absence of any linearity assumptions*, and in studying both local asymptotic stability and global asymptotic stability.**

I. Introduction

In this paper, we consider nonlinear control systems described by equations of the type

$$\dot{x}(t)=f(t,x(t),u(t)) \tag{1.1}$$

$$y(t)=r(t,x(t)) \tag{1.2}$$

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where $x(t)$, $y(t)$, and $u(t)$ denote the state, output, and input of the system, respectively. We assume that $x(t)\in R^n$, $y(t)\in R^m$, and $u(t)\in R^l$, $\forall t>0$, and that f, r are continuously differentiable functions that vanish when all of their arguments except t vanish. The problem under study here is that of finding a stabilizing control law for the system (1.1)–(1.2), in the case where only $y(t)$ can be measured, but not necessarily $x(t)$.

In the special case of linear time-invariant systems, (1.1) and (1.2) assume the form

$$\dot{x}(t)=Ax(t)+Bu(t) \tag{1.3}$$

$$y(t)=Cx(t). \tag{1.4}$$

For this case, it is well known [1] that a stabilizing dynamic feedback compensator can be constructed if the system (1.3)–(1.4) is both stabilizable and detectable. Specifically, suppose

1) there exists a matrix K such that $A-BK$ is Hurwitz (i.e., all eigenvalues of $A-BK$ have negative real parts), and

2) there exists a matrix F such that $A-FC$ is Hurwitz. Assumption 1 implies that the system (1.3) is stabilized by the control law

$$u(t)=-Kx(t). \tag{1.5}$$

However, in general, the control law (1.5) cannot be implemented, because only $y(t)$ can be measured. To circumvent this difficulty, we set up a "detector" described by[1]

$$\dot{z}(t)=(A-FC)z(t)+Fy(t)+Bu(t) \tag{1.6}$$

which has the property that

$$z(t)-x(t)\to 0 \qquad \text{as } t\to\infty, \text{ for all } x(0), z(0). \tag{1.7}$$

Finally, we apply the control law

$$u(t)=-Kz(t). \tag{1.8}$$

Now, it is easy to show that $x=0$, $z=0$ is a (globally) asymptotically stable equilibrium point of the resulting system

$$\dot{x}(t)=Ax(t)-BKz(t) \tag{1.9}$$

$$\dot{z}(t)=(A-FC-BK)z(t)+FCx(t). \tag{1.10}$$

The usual proof of the above special result is very easy, but depends in a crucial way on the fact that the system at hand is linear and time invariant. The objective of this paper is to state and prove results analogous to the above, *without making any assumption about linearity*. The tool that we use to achieve this is a collection of converse theorems from Lyapunov theory, plus some ideas from [5]. The results given here pertain to both local and global asymptotic stability. The results on global asymptotic stability are essentially equivalent to those in [8], and generalize those in [9]. However, the techniques used here are quite different from those in [8]. The local asymptotic stability results do not appear to have any parallel in the literature.

The paper is organized as follows. In Section II, we present some preliminary results, including definitions and converse Lyapunov theorems. In Section III, we present the main theorems concerning asymptotic stability and exponential stability, while in Section IV, we present the main theorems concerning global exponential stability. Section V contains some illustrative examples, while Section VI contains the conclusions.

II. Preliminaries

In this section, we briefly summarize some results from the Lyapunov theory that are needed in the sequel, and introduce a few definitions.

First of all, following Hahn [2], we say that a function ϕ: $R_+\to R_+$ belongs to *class K* if ϕ is continuous, strictly increasing, and $\phi(0)=0$. If, in addition, $\lim_{\sigma\to\infty}\phi(\sigma)=\infty$, we say that ϕ belongs to *class KR*. Finally,

[1]To keep the exposition simple, we do not discuss the minimum order observer.

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