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C_{11} CONTRACTIONS ARE REFLEXIVE

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ABSTRACT. It is shown that a completely nonunitary C_{11} contraction defined on a separable Hilbert space with finite defect indices is reflexive.

In this note, only bounded linear operators defined on complex, separable Hilbert spaces will be considered. A contraction T ($\|T\| \leqslant 1$) on a Hilbert space $\mathcal{H}$ is of class C_{11} if $T^n x \nrightarrow 0$ and $T^{*n} x \nrightarrow 0$ for any $x \neq 0$. It is well known that such a contraction is quasi-similar to a unitary operator. Since unitary operators (even normal operators) are reflexive (cf. [3]), the question arises: Is the property of reflexivity preserved under the quasi-similarity? In other words, is a C_{11} contraction reflexive? In the present note we show that the answer is affirmative if the C_{11} contraction is completely nonunitary (c.n.u.) and has finite defect indices. We conjecture that the general case is also true.

Recall that a contraction T is *c.n.u.* if there is no nontrivial reducing subspace on which T is unitary. The *defect indices* of T are, by definition,

$$d_T = \dim\left[(1 - T^*T)^{1/2}\mathcal{H}\right]^-, \qquad d_{T^*} = \dim\left[(1 - TT^*)^{1/2}\mathcal{H}\right]^-.$$

If T is of class C_{11}, then $d_T = d_{T^*}$. In the following discussion we shall make use of the *functional model for contractions* developed by Sz.-Nazy and Foiaş (cf. [4]). More specifically, if T is a c.n.u. contraction with $d_T = d_{T^*} = n < \infty$, then T can be considered as defined on

$$H = \left[H_n^2 \oplus \overline{\Delta L_n^2}\right] \ominus \left\{\Theta_T w \oplus \Delta w: w \in H_n^2\right\}$$

by $T(f \oplus g) = P(e^{it}f \oplus e^{it}g)$ for $f \oplus g \in H$, where L_n^2 and H_n^2 denote the standard Lebesgue and Hardy spaces of $\mathbf{C}^n$-valued functions defined on the unit circle, Θ_T is the characteristic function of T, $\Delta = (1 - \Theta_T^*\Theta_T)^{1/2}$ and P denotes the (orthogonal) projection onto H. Any operator S in $\{T\}'$, the commutant of T, has the form $P\left[\begin{smallmatrix} A & 0 \\ B & C \end{smallmatrix}\right]$, where A is a bounded analytic function while B and C are bounded measurable functions satisfying $A\Theta_T = \Theta_T A_0$ and $B\Theta_T + C\Delta = \Delta A_0$ for some bounded analytic function A_0 (cf. [5]).

For an arbitrary operator T, $\{T\}'$, $\{T\}''$ and Alg T denote the commutant, double commutant and the weakly closed algebra generated by T and I,

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respectively, and Lat T, Lat$''T$ denote the lattice of invariant subspaces and the lattice of bi-invariant subspaces of T, respectively. Let Alg Lat T and Alg Lat$''T$ denote the (weakly closed) algebras of operators which leave all the subspaces in Lat T and Lat$''T$ invariant, respectively. An operator T is *reflexive* if Alg Lat T = Alg T. Two operators T_1, T_2 are *quasi-similar* if there exist one-to-one operators X and Y with dense ranges (called *quasi-affinities*) such that $XT_1 = T_2X$ and $YT_2 = T_1Y$.

We start the proof with the following:

LEMMA 1. *Let T be a normal operator on a separable Hilbert space. Then* $\{T\}' \cap \text{Alg Lat}''T = \{T\}''$.

PROOF. By the spectral theorem, we may assume that $T = M_{E_1} \oplus M_{E_2} \oplus \ldots$ acting on

$$H = L^2(E_1, \mu) \oplus L^2(E_2, \mu) \oplus \cdots,$$

where $E_1 \supseteq E_2 \supseteq \cdots$ are Borel subsets of the complex plane, μ is a finite positive Borel measure and M_{E_j} denotes the operator of multiplication by independent variable on $L^2(E_j, \mu)$, $j = 1,2, \cdots$ (cf. [**2**, p. 916]). Let $S \in \{T\}' \cap \text{Alg Lat}''T$. Since for normal operators bi-invariant subspaces are exactly reducing subspaces, $S \in \{T\}' \in \text{Alg Lat}''T$ implies that $S = \varphi_1 \oplus \varphi_2 \oplus \cdots$, where $\varphi_j \in L^\infty(E_j, \mu)$, $j = 1,2, \cdots$. Consider the reducing (hence bi-invariant) subspace

$$K = \{f_1 \oplus f_2 \oplus \cdots \in H: \chi_{E_j} f_i = f_j \text{ for all } i < j\}.$$

We have $SK \subseteq K$, which implies that $\chi_{E_j}\varphi_i f_i = \varphi_j f_j$ for all $f_i \in L^2(E_i, \mu)$, $f_j \in L^2(E_j, \mu)$, $i < j$. In particular, if $f_i = \chi_{E_i}$ and $f_j = \chi_{E_j}$, we have $\varphi_i = \varphi_j$ on E_j. Hence

$$S = \varphi_1 \oplus \chi_{E_2}\varphi_1 \oplus \chi_{E_3}\varphi_1 \oplus \cdots = \varphi_1(T) \in \{T\}''.$$

This shows that $\{T\}' \cap \text{Alg Lat}''T \subseteq \{T\}''$. Since the other inclusion is trivial, this completes the proof.

The next lemma characterizes the operators in $\{T\}''$ for a c.n.u. C_{11} contraction T with finite defect indices.

LEMMA 2. *Let T be a c.n.u. C_{11} contraction with $d_T = d_{T^*} = n < \infty$ defined on*

$$H = \left[H_n^2 \oplus \overline{\Delta L_n^2}\right] \ominus \{\Theta_T w \oplus \Delta w: w \in H_n^2\}.$$

Then $\{T\}'' = \{P[\begin{smallmatrix} A & 0 \\ B & C \end{smallmatrix}]: A\Theta_T = \Theta_T A_0,\ B\Theta_T + C\Delta = \Delta A_0$ *for some bounded analytic function A_0, and C is scalar-valued*$\}$.

PROOF. Let $S = P[\begin{smallmatrix} A & 0 \\ B & C \end{smallmatrix}]$ be an operator in $\{T\}''$, where $A\Theta_T = \Theta_T A_0$ and $b\Theta_T + C\Delta = \Delta A_0$ for some bounded analytic function A_0, and let U be the operator of multiplication by e^{it} on $\overline{\Delta L_n^2}$. It was shown in [**6**, Lemma 3.1] that $C \in \{U\}''$. As in the proof of Lemma 1, $C = \varphi(U)$ for some $\varphi \in L^\infty$, that is, C is scalar-valued.

For the other inclusion, let $S = P\left[\begin{smallmatrix} A & 0 \\ B & C \end{smallmatrix}\right] \in \{T\}'$ be such that C is scalar-valued, and let $S' = P\left[\begin{smallmatrix} A' & 0 \\ B' & C' \end{smallmatrix}\right] \in \{T\}'$. Note that the linear manifold $K = \{P(0 \oplus g): g \in \overline{\Delta L_n^2}\}$ is dense in H. Indeed, since Θ_T is an outer function, for any $f \in H_n^2$ there exists a sequence $\{w_j\}$ of elements in H_n^2 such that $\Theta_T w_j \to f$ in norm. Hence

$$P(0 \oplus -\Delta w_j) = P(\Theta_T w_j \oplus 0) \to P(f \oplus 0).$$

It follows that

$$P(f \oplus g) = P(f \oplus 0) + P(0 \oplus g) \in \overline{K}$$

for any $f \in H_n^2$ and $g \in \overline{\Delta L_n^2}$. Thus $\overline{K} = H$, as asserted. Let $Y = S|K$ and $Y' = S'|K$ be operators (not necessarily bounded) defined on K. It is easily seen that $YY' = Y'Y$. By the denseness of K, this implies that $SS' = S'S$ whence $S \in \{T\}''$.

As a preliminary step toward showing that C_{11} contractions are reflexive, the next result says that they satisfy $\{T\}' \cap \operatorname{Alg} \operatorname{Lat} T = \operatorname{Alg} T$.

THEOREM 3. *Let T be a c.n.u. C_{11} contraction with $d_T = d_{T^*} = n < \infty$ defined on*

$$H = \left[H_n^2 \oplus \overline{\Delta L_n^2} \right] \ominus \left\{ \Theta_T w \oplus \Delta w : w \in H_n^2 \right\}.$$

(1) *If $\Theta_T(e^{it})$ is isometric for t in a set of positive Lebesgue measure, then $\{T\}' \cap \operatorname{Alg} \operatorname{Lat} T = \operatorname{Alg} T = \{T\}''$.*

(2) *If $\Theta_T(e^{it})$ is not isometric for almost all t, then $\{T\}' \cap \operatorname{Alg} \operatorname{Lat} T = \operatorname{Alg} T = \{u(T): u \in H^\infty\}$.*

PROOF. We first show that $\{T\}' \cap \operatorname{Alg} \operatorname{Lat} T \subseteq \{T\}''$. Let $S = P\left[\begin{smallmatrix} A & 0 \\ B & C \end{smallmatrix}\right] \in \{T\}' \cap \operatorname{Alg} \operatorname{Lat} T$. Let U, V be the operators of multiplication by e^{it} on $\overline{\Delta L_n^2}$, $\overline{\Delta_* L_n^2}$, respectively, where $\Delta_* = (1 - \Theta_T \Theta_T^*)^{1/2}$, and let $X: H \to \overline{\Delta_* L_n^2}$ be the quasi-affinity $X(f \oplus g) = -\Delta_* f + \Theta_T g$ (cf. [6, Lemma 3.4]). Since $\Theta_T \Delta = \Delta_* \Theta_T$, we may consider Θ_T as a multiplication operator from $\overline{\Delta L_n^2}$ to $\overline{\Delta_* L_n^2}$. For any $K \in \operatorname{Lat}'' U$, let $H_0 = X^{-1}(\overline{\Theta_T K})$. Since operators in $\{V\}''$ are of the form $\varphi(V)$ where $\varphi \in L^\infty$, it is easily seen that $\overline{\Theta_T K} \in \operatorname{Lat}'' V$. By Corollary 3.6 of [6], $H_0 \in \operatorname{Lat}'' T$. Hence $SH_0 \subseteq H_0$. Thus for any $f \oplus g \in H_0$, $XS(f \oplus g) \in \overline{\Theta_T K}$. As in the proof of Theorem 3.5 in [6], it can be shown that

$$XS(f \oplus g) = \Theta_T C \Theta_T^{-1}\left(-\Delta_* f + \Theta_T g\right).$$

Note that Θ_T admits scalar multiples. Let δ be an outer scalar multiple of Θ_T and let Ω be a contractive analytic function such that $\Theta_T \Omega = \Omega \Theta_T = \delta I_{\mathbf{C}^n}$. Since $\Theta_T^{-1} = \delta^{-1} \Omega$, we conclude that

$$\Theta_T C \Omega\left(-\Delta_* f + \Theta_T g\right) \in \delta \overline{\Theta_T K}$$

for any $f \oplus g \in H_0$. By Corollary 3.6 of [6], XH_0 is dense in $\overline{\Theta_T K}$. Therefore,

$$\Theta_T C \delta K = \Theta_T C \Omega \Theta_T K \subseteq \overline{\delta \Theta_T K}.$$

Hence for any $x \in K$, there exists a sequence $\{x_n\}$ of elements in K such that

$\delta\Theta_T x_n \to \Theta_T C\delta x$, which implies that $\delta^2 x_n = \delta\Omega\Theta_T x_n \to \Omega\Theta_T C\delta x = \delta^2 Cx$. Since $\delta^2 x_n \in K$ for all n, $\delta^2 Cx \in K$. This shows that $\delta^2 CK \subseteq K$, and hence by Lemma 1 we conclude that $\delta^2 C \in \{U\}' \cap \text{Alg Lat}'' U = \{U\}''$. Thus $\delta^2 C$ is scalar-valued, and so is C. By Lemma 2, $S \in \{T\}''$. If $\Theta_T(e^{it})$ is isometric for t in a set of positive Lebesgue measure, then $\{T\}'' = \text{Alg } T$ (cf. [**6**, Theorem 3.8]). This shows that $\{T\}' \cap \text{Alg Lat } T \subseteq \text{Alg } T$. Since the other inclusion is trivial, we have proved (1).

For the rest of the proof we assume that $\Theta_T(e^{it})$ is not isometric for almost all t. Let

$$Y\colon \overline{\Delta_* L_n^2} \to L^2 \oplus L^2(E_2) \oplus \cdots \oplus L^2(E_k)$$

be the unitary transformation which intertwines V and $M \oplus M_{E_2} \oplus \cdots \oplus M_{E_k}$, where $E_2 \supseteq \cdots \supseteq E_k$ are Borel subsets of the unit circle and $M, M_{E_2}, \ldots, M_{E_k}$ denote the operators of multiplication by e^{it} on L^2, $L^2(E_2), \ldots, L^2(E_k)$, respectively (cf. [**4**, pp. 272–273]). For any $x \in K \equiv Y^{-1}(H^2 \oplus 0 \oplus \cdots \oplus 0)$, consider the element

$$f \oplus g \equiv P(0 \oplus \Omega x) = (0 \oplus \Omega x) - (\Theta_T w \oplus \Delta w)$$

in H, where $w \in H_n^2$. Since $\delta \in H^\infty$,

$$\begin{aligned} -\Delta_* f + \Theta_T g &= -\Delta_*(-\Theta_T w) + \Theta_T(\Omega x - \Delta w) = \Theta_T \Omega x \\ &= \delta x \in \delta K = Y^{-1}(\delta H^2 \oplus 0 \oplus \cdots \oplus 0) \subseteq K. \end{aligned}$$

It follows that $f \oplus g \in X^{-1}K$. Since $X^{-1}K \in \text{Lat } T$, $S(f \oplus g) \in X^{-1}K$. Hence

$$\begin{aligned} XS(f \oplus g) &= XP\begin{bmatrix} A & 0 \\ B & C \end{bmatrix}\begin{bmatrix} f \\ g \end{bmatrix} = XP\begin{bmatrix} A & 0 \\ B & C \end{bmatrix}\begin{bmatrix} 0 \\ \Omega x \end{bmatrix} = XP(0 \oplus C\Omega x) \\ &= \Theta_T C\Omega x = C\delta x \in K \end{aligned}$$

for any $x \in K$. In particular, for $h \in H^2$ consider

$$x = Y^{-1}(h \oplus 0 \oplus \cdots \oplus 0).$$

Then

$$C\delta x = Y^{-1}(C\delta h \oplus 0 \oplus \cdots \oplus 0) \in K = Y^{-1}(H^2 \oplus 0 \oplus \cdots \oplus 0),$$

which implies that $C\delta h \in H^2$ for any $h \in H^2$. Since δ is outer, δH^2 is dense in H^2. From above we conclude that $CH^2 \subseteq H^2$ whence $C \in H^\infty$.

Note that the linear manifold $\{P(0 \oplus g)\colon g \in \overline{\Delta L_n^2}\}$ is dense in H (cf. the proof of Lemma 2). Hence

$$SP(0 \oplus g) = P(0 \oplus Cg) = C(T)P(0 \oplus g)$$

for any $g \in \overline{\Delta L_n^2}$ implies that $S = C(T)$ on H. Thus $S = C(T) \in \text{Alg } T$, which proves (2).

LEMMA 4. *Let T be an operator on H satisfying $\{T\}' \cap \text{Alg Lat } T = \text{Alg } T$. If T is quasi-similar to a normal operator, then T is reflexive.*

PROOF. It was proved by Apostol [1] that if T is quasi-similar to a normal operator, then there exists a sequence $\{H_n\}$ of invariant subspaces for T which are basic and such that $T_n \equiv T|H_n$ is similar to a normal operator for each n. Let $S \in \operatorname{Alg} \operatorname{Lat} T$. Then $SH_n \subseteq H_n$. Let $S_n = S|H_n$ for each n. Since T_n, being similar to a normal operator, is reflexive, we have $S_n \in \operatorname{Alg} \operatorname{Lat} T_n = \operatorname{Alg} T_n$. It follows that $ST = TS$ on H_n for all n. Since $\{H_n\}$ spans H, $ST = TS$ on H. Thus $S \in \{T\}' \cap \operatorname{Alg} \operatorname{Lat} T = \operatorname{Alg} T$. This shows that T is reflexive.

THEOREM 5. *If T is a c.n.u. C_{11} contraction with finite defect indices, then T is reflexive.*

PROOF. This follows from Theorem 3 and Lemma 4.

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