4) If $A \neq O, B \neq O$:

$$y_{\mathrm{RIM}} = -\frac{A}{B} x_{\mathrm{RIM}} - \frac{C}{B} \tag{16a}$$

$$x_{\mathrm{RIM}}^2 \left[1 + \left(\frac{A}{B}\right)^2\right] + \left[\frac{2AC}{B^2}\right] x_{\mathrm{RIM}} + \left[\frac{C^2}{B^2} - f(z_{\mathrm{RIM}})\right] = 0. \tag{16b}$$

The descriminant of (16b) is never negative so that two real roots always exist.

As a specific example, let $x_O = x_P = O$ so that both O and P lie in the yz plane. Then, from (11) and (12), $B = C = O$ so that (14a) and (14b) yield $x_{RIM} = O$ and $y_{RIM} = \pm \rho_{RIM}$, as expected.

Having, again, as in the previous section, demonstrated the existence of a specular point for the nonplanar case, many different search techniques are available for locating it. However, should the simple evaluation of $\overline{v}$ at the two points on the reflector ring defined by (9) indicate that no specular point exists, a search procedure has been eliminated.

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Transient and Time-Harmonic Dyadic Green's Functions for a Perfectly Conducting Cone

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***Abstract*—New representations for the time-dependent dyadic Green's functions for a perfectly conducting semi-infinite cone are presented. For the special case of small cone angles and an on-axis source, simplified expressions are given for both the time-dependent and time-harmonic regimes.**

In a separate paper, new representations for the time-dependent scalar Dirichlet and Neumann Green's functions for a semi-infinite cone were developed [1]. It was shown that these formulations can be simplified substantially when the cone angle is small and the source is located on the cone axis. It was also shown that new closed-form time-harmonic solutions can be obtained by Fourier transformation of the transient fields.

Manuscript received October 17, 1977.

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The present communication extends these results to the dyadic Green's functions generated by vector dipole excitation.

Consider an electric dipole of vector strength $\mathbf{J}$ situated at the point $\mathbf{r} = \mathbf{r}'$ outside a perfectly conducting cone defined by the surface $\theta = \theta_0$. The components of $\mathbf{J}$ in the (r, θ, ϕ) directions are J_r, J_θ, and J_ϕ, respectively. The total field can be regarded as a superposition of the separate responses due to J_r, J_θ, and J_ϕ. The response to a radial electric dipole source may be obtained from the scalar Dirichlet Green's function discussed in [1]. The electric field due to a transverse dipole source of vector strength $\mathbf{J}_t$ can be represented in terms of functions S' and S'' as follows [2]:

$$\mathbf{E} = E_r \mathbf{r}_0 + E_\theta \boldsymbol{\theta}_0 + E_\phi \boldsymbol{\phi}_0 \tag{1}$$

$$E_r = \left(\frac{\partial^2}{\partial r^2} + k^2\right)(\mathbf{J}_t \cdot {}_t\nabla') \frac{\partial}{\partial r'} \left(\frac{S'}{-i\omega\epsilon}\right) \tag{1a}$$

$$E_\theta = \frac{-1}{i\omega\epsilon r} \frac{\partial^2}{\partial r \partial\theta} (\mathbf{J}_t \cdot {}_t\nabla') \frac{\partial S'}{\partial r'} - \frac{i\omega\mu}{r \sin\theta} \frac{\partial}{\partial\phi} (\mathbf{J}_t \times \mathbf{r}_0 \cdot {}_t\nabla' S'') \tag{1b}$$

$$E_\phi = -\frac{1}{i\omega\epsilon r \sin\theta} \frac{\partial^2}{\partial r \partial\phi} (\mathbf{J}_t \cdot {}_t\nabla') \frac{\partial S'}{\partial r'} + \frac{i\omega\mu}{r} \frac{\partial}{\partial\theta} (\mathbf{J}_t \times \mathbf{r}_0 \cdot {}_t\nabla' S'') \tag{1c}$$

where

$$\mathbf{J}_t \cdot {}_t\nabla' = \frac{J_\theta}{r'} \frac{\partial}{\partial\theta'} + \frac{J_\phi}{r' \sin\theta'} \frac{\partial}{\partial\phi'} \tag{1d}$$

$$\mathbf{J}_t \times \mathbf{r}_0 \cdot {}_t\nabla' = \frac{J_\phi}{r'} \frac{\partial}{\partial\theta'} - \frac{J_\theta}{r' \sin\theta'} \frac{\partial}{\partial\phi'}. \tag{1e}$$

The scalar functions S', S'' can be decomposed into a free-space portion and a perturbation term accounting for the effect of the cone. Since the free-space portion of the solution is known, only the evaluation of the perturbation fields (distinguished by subscript s) need concern us here. Suitable representations for the latter are [3]

$$S_s' = S_{s_1}' + S_{s_2}' \tag{2a}$$

$$S_{s_1}' = -\frac{i}{2\pi k} j_0(kr_<) h_0^{(1)}(kr_>) \sum_{m=1}^{\infty} \frac{\cos m(\phi - \phi')}{m} \cdot \left[\tan\frac{\theta}{2} \tan\frac{\theta'}{2} \tan^2\left(\frac{\pi - \theta_0}{2}\right)\right]^m \tag{2b}$$

$$S_{s_2}' = \frac{1}{8\pi k} \sum_{m=0}^{\infty} \epsilon_m \cos m(\phi - \phi') \int_{-1/2-i\infty}^{-1/2+i\infty} dv\,(2v+1) \cdot \frac{h_v^{(1)}(kr) h_v^{(1)}(kr')}{v(v+1)} \frac{P_v^{-m}(\cos\theta) P_v^{-m}(\cos\theta')}{\sin(v-m)\pi} \cdot \frac{\Gamma(v+m+1)}{\Gamma(v-m+1)} \frac{P_v^{-m}(-\cos\theta_0)}{P_v^{-m}(\cos\theta_0)} \tag{2c}$$

$$S_s'' = S_{s_1}'' + S_{s_2}'' \tag{3a}$$

$$S_{s_1}'' = -S_{s_1}' \tag{3b}$$

$$S_{s_2}'' = \frac{1}{8\pi k}\sum_{m=0}^{\infty} \epsilon_m \cos m(\phi-\phi')\int_{-1/2-i\infty}^{-1/2+i\infty} dv\,(2v+1) \cdot \frac{h_v^{(1)}(kr)h_v^{(1)}(kr')}{v(v+1)}\,\frac{P_v^{-m}(\cos\theta)P_v^{-m}(\cos\theta')}{\sin(v-m)\pi} \cdot \frac{\Gamma(v+m+1)}{\Gamma(v-m+1)}\,\frac{\dfrac{d}{d\theta_0}P_v^{-m}(-\cos\theta_0)}{\dfrac{d}{d\theta_0}P_v^{-m}(\cos\theta_0)} \tag{3c}$$

subject to the restriction $\theta + \theta' < 2\theta_0 - \pi$. The inversion of these results into the time-dependent regime can be accomplished by following the procedure in [1]. Instead of presenting the general formulas, we pass on to the special cases that yield substantial simplification.

For a transverse dipole located on the cone axis $\theta' = 0$, we may assume, without loss of generality, that $\mathbf{J}_t = \boldsymbol{\phi}_0 J_\phi$. It follows that

$$\mathbf{J}_t \cdot {}_t\nabla' = J_\phi \frac{1}{r'\sin\theta'}\frac{\partial}{\partial\phi'} \tag{4a}$$

$$\mathbf{J}_t \times \mathbf{r}_0 \cdot {}_t\nabla' = J_\phi \frac{1}{r'}\frac{\partial}{\partial\theta'}. \tag{4b}$$

In addition, we assume a narrow angle cone so that a closed-form result can be obtained.

In the evaluation of the electric fields, we make use of the following relations:

$$\lim_{\theta'\to 0}\left[\frac{1}{\sin\theta'}P_v^{-m}(\cos\theta')\right] = 0, \qquad m \neq 0, 1 \tag{5a}$$

$$\lim_{\theta'\to 0}\left[\frac{\partial}{\partial\theta'}P_v^{-m}(\cos\theta')\right] = 0, \qquad m \neq 1 \tag{5b}$$

$$\lim_{\theta'\to 0}\left[\frac{1}{\sin\theta'}P_v^{-1}(\cos\theta') \text{ or } \frac{\partial}{\partial\theta'}P_v^{-1}(\cos\theta')\right] = 1/2 \tag{5c}$$

and for small-angle cones ($\theta_0 \approx \pi$) [2, p. 708]

$$\frac{P_v^{-1}(-\cos\theta_0)}{P_v^{-1}(\cos\theta_0)} \sim \frac{v(v+1)\pi}{\sin v\pi}\left(\frac{\pi-\theta_0}{2}\right)^2 \tag{6a}$$

$$\frac{\dfrac{d}{d\theta_0}P_v^{-1}(-\cos\theta_0)}{\dfrac{d}{d\theta_0}P_v^{-1}(\cos\theta_0)} \sim -\frac{v(v+1)\pi}{\sin v\pi}\left(\frac{\pi-\theta_0}{2}\right)^2. \tag{6b}$$

Consequently, for $\theta' = 0, \theta_0 \approx \pi$

$$\frac{1}{\sin\theta'}\frac{\partial S_{s2}'}{\partial\phi'} \sim \frac{1}{16}\left(\frac{\pi-\theta_0}{2}\right)^2 \sin(\phi-\phi')\frac{\partial I}{\partial\theta} \tag{7a}$$

$$\frac{\partial S_{s2}''}{\partial\theta'} \sim -\frac{1}{16}\left(\frac{\pi-\theta_0}{2}\right)^2 \cos(\phi-\phi')\frac{\partial I}{\partial\theta} \tag{7b}$$

where

$$I = \int_{-1/2-i\infty}^{-1/2+i\infty} dv\,(2v+1)\frac{h_v^{(1)}(kr)h_v^{(1)}(kr')}{k\sin^2 v\pi}P_v(\cos\theta). \tag{8}$$

Also

$$\lim_{\theta'\to 0}\left[\frac{1}{\sin\theta'}\frac{\partial}{\partial\phi'}S_{s1}'\right] = -\frac{i}{4\pi k}j_0(kr_<)h_0^{(1)}(kr_>)\sin(\phi-\phi')\tan\frac{\theta}{2}\tan^2\left(\frac{\pi-\theta_0}{2}\right) \tag{9a}$$

$$\lim_{\theta'\to 0}\frac{\partial}{\partial\theta'}S_{s1}'' = \frac{i}{4\pi k}j_0(kr_<)h_0^{(1)}(kr_>)\cos(\phi-\phi')\tan\frac{\theta}{2}\tan^2\left(\frac{\pi-\theta_0}{2}\right). \tag{9b}$$

One may, therefore, show that

$$E_{rs} \sim \frac{J_\phi}{\epsilon}\frac{1}{16}\left(\frac{\pi-\theta_0}{2}\right)^2\frac{1}{r'r^2}\sin(\phi-\phi')\frac{\partial}{\partial\theta}L(\theta) \cdot \frac{\partial}{\partial r'}\frac{I}{(-i\omega)} \tag{10a}$$

$$E_{\theta s} \sim \sqrt{\frac{\mu}{\epsilon}}\frac{J_\phi}{8\pi}\left(\frac{\pi-\theta_0}{2}\right)^2\frac{1}{rr'}\sin(\phi-\phi')\sec^2\frac{\theta}{2}e^{ik(r+r')} + \frac{J_\phi}{\epsilon}\frac{1}{16}\left(\frac{\pi-\theta_0}{2}\right)^2\frac{1}{rr'}\sin(\phi-\phi')\left[\frac{1}{(-i\omega)}\frac{\partial}{\partial\theta} \cdot \frac{\partial^2}{\partial r\partial r'} + \frac{1}{c^2}\frac{(-i\omega)}{\sin\theta}\right]\frac{\partial I}{\partial\theta} \tag{10b}$$

$$E_{\phi s} \sim \sqrt{\frac{\mu}{\epsilon}}\frac{J_\phi}{8\pi}\left(\frac{\pi-\theta_0}{2}\right)^2\frac{1}{rr'}\cos(\phi-\phi')\sec^2\frac{\theta}{2}e^{ik(r+r')} + \frac{J_\phi}{\epsilon}\frac{1}{16}\left(\frac{\pi-\theta_0}{2}\right)^2\frac{1}{rr'}\cos(\phi-\phi')\left[\frac{1}{(-i\omega)} \cdot \frac{1}{\sin\theta}\frac{\partial^2}{\partial r\partial r'} + \frac{(-i\omega)}{c^2}\frac{\partial}{\partial\theta}\right]\frac{\partial I}{\partial\theta} \tag{10c}$$

where $L(\theta) = (\sin\theta)^{-1}(\partial/\partial\theta)\sin\theta(\partial/\partial\theta)$. The time-dependent solution can be recovered from (10) as in [1]. The inversion of the first terms in (10b) and (10c) is trivial and yields a delta-function pulse. For the remaining terms one notes that the integral I in (8) can be evaluated by proceeding as in [1, eqs. (4)–(6)], with $\theta' = 0$ and [1, eq. (6a)] approximated in the small cone limit. Following the steps leading to [1, eq. (19)], one finds

$$I \sim \frac{2c}{\pi} \frac{1}{b^2 + d^2} U\left(\tau - \frac{r + r'}{c}\right), \qquad \tau = t - t' \tag{11}$$

where $U(\alpha)$ is the Heaviside unit function and

$$b = \left[\frac{c^2\tau^2 - (r + r')^2}{4rr'}\right]^{1/2} \qquad d = \cos\frac{\theta}{2}. \tag{11a}$$

Furthermore, the factor $(-i\omega)$ in (10) can be replaced by the differential operator $\partial/\partial t$ in the time domain.

As a result of these considerations, one obtains for the time-dependent electric fields initiated by the impulse $\delta(t - t')$

$$\hat{E}_{rs} \sim c^2 K \frac{1}{r'r^2} \sin(\phi - \phi') \frac{\partial}{\partial r'} \frac{\partial}{\partial \theta} L(\theta) \int_0^\tau dt\, F(\mathbf{r}, \mathbf{r}'; t) \tag{12a}$$

$$\hat{E}_{\theta s} \sim K \frac{1}{rr'} \sin(\phi - \phi') \sec^2 \frac{\theta}{2} \delta\left(\tau - \frac{r + r'}{c}\right) + K \frac{1}{rr'} \sin(\phi - \phi') \frac{\partial^2}{\partial\tau\partial\theta} F(\mathbf{r}, \mathbf{r}'; \tau) + c^2 K \frac{1}{rr'} \cdot \sin(\phi - \phi') \frac{\partial^2}{\partial\theta^2} \frac{\partial^2}{\partial r \partial r'} \int_0^\tau dt\, F(\mathbf{r}, \mathbf{r}'; t) \tag{12b}$$

$$\hat{E}_{\phi s} \sim K \frac{1}{rr'} \cos(\phi - \phi') \sec^2 \frac{\theta}{2} \delta\left(\tau - \frac{r + r'}{c}\right) + K \frac{1}{rr'} \cos(\phi - \phi') \frac{\partial}{\partial\tau} \frac{\partial^2}{\partial\theta^2} F(\mathbf{r}, \mathbf{r}'; \tau) + c^2 K \frac{1}{rr'} \cdot \cos(\phi - \phi') \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \frac{\partial^2}{\partial r \partial r'} \int_0^\tau dt\, F(\mathbf{r}, \mathbf{r}'; t) \tag{12c}$$

where

$$K = \sqrt{\frac{\mu}{\epsilon}} \frac{J_\phi}{8\pi} \left(\frac{\pi - \theta_0}{2}\right)^2 \tag{12d}$$

$$F(\mathbf{r}, \mathbf{r}'; \tau) = \frac{1}{b^2 + d^2} U\left(\tau - \frac{r + r'}{c}\right). \tag{12e}$$

The differentiations to be performed in (12a)–(12c) are straightforward but cumbersome. Because of the occurrence of the unit step function in (12e), one generates thereby a delta-function singularity across the wavefront. When the dominant term near the wavefront, so obtained, is compared with the Laplace transform of the high-frequency asymptotic solution [3], one verifies agreement of these results.

By Fourier inversion of F in (12e), one may generate a time-harmonic solution F_ω valid for all frequencies. Referring to [1, eq. (32)] one finds

$$F_\omega(\mathbf{r}, \mathbf{r}') = \frac{2\pi rr'}{c} \frac{e^{ik(r+r')}}{|\mathbf{r} - \mathbf{r}'|} [e^{Q_1} E_1(Q_1) - e^{Q_2} E_1(Q_2)] \tag{13}$$

where

$$E_1(x) = \int_x^\infty e^{-t} \frac{dt}{t} \tag{13a}$$

and

$$Q_1 = -ik(r + r' - |\mathbf{r} - \mathbf{r}'|) \qquad Q_2 = -ik(r + r' + |\mathbf{r} - \mathbf{r}'|). \tag{13b}$$

The time-harmonic dyadic Green's function for $\theta' = 0, \theta_0 \approx \pi$, $\theta < 2\theta_0 - \pi$ is then inferred from the inverted form of (12a)–(12c).

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Circumferential Distribution of Scattering Current and Small Hole Coupling for Thin Finite Cylinders

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Abstract—The scattering current induced on a thin finite conducting cylinder immersed in a "θ"-polarized E-field is studied. Particular attention is paid to the circumferentially nonuniform mode as the E-field angle of incidence varies. This nonuniformity is shown significant (peak-to-average ratio of 3 dB at cylinder midlength) at certain incidence angles for wavelength long cylinders with diameters as small as 0.067λ. Also investigated is the relationship between scattering current and cavity response patterns for narrow thin-walled cylindrical cavities with small holes through which energy is coupled. It is demonstrated theoretically, with experimental verification, that the circumferential variation of scattering current strongly affects the fields within thin cylindrical cavities having apertures with small circumferential extents. It is noted, however, that for most thin-body radiation and scattering problems (in contrast with aperature coupling) only the uniform current mode is significant.

I. INTRODUCTION

The penetration of electromagnetic fields through small apertures in rotationally symmetric cavities (approximating cables, missiles, etc.) presents a problem of considerable interest. Theoretical predictions of the cavity fields are usually obtained with variations of "Bethe" Small-Hole Theory [3]–[6]. In these methods an equivalent cavity excitation, com-

Manuscript received November 22, 1977; revised July 26, 1978. This work was supported by the Rome Air Development Center Post-Doctoral Program under Contract F30602-75-C-0121.

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