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Numerical ranges of companion matrices

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Abstract

We show that an n-by-n companion matrix A can have at most n line segments on the boundary $\partial W(A)$ of its numerical range $W(A)$, and it has exactly n line segments on $\partial W(A)$ if and only if, for n odd, A is unitary, and, for n even, A is unitarily equivalent to the direct sum $A_1 \oplus A_2$ of two $(n/2)$-by-$(n/2)$ companion matrices

$$A_1 = \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ a & & & 0 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ -1/\bar{a} & & & 0 \end{bmatrix}$$

with $1 \leqslant |a| < \tan(\pi/n) + \sec(\pi/n)$.

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For every complex monic polynomial $p(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n$ $(n \geqslant 2)$, there is associated an n-by-n matrix

$$\begin{bmatrix} 0 & 1 & & & & \\ & 0 & \cdot & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \cdot \\ & & & & 0 & 1 \\ -a_n & -a_{n-1} & \cdot & \cdot & -a_2 & -a_1 \end{bmatrix}, \tag{1}$$

called its *companion matrix*. In this paper, we consider properties of the numerical ranges of such matrices. To be more precise, we study the number of line segments on the boundary of such a numerical range. We show that for an n-by-n companion matrix, this number is at most n, and also completely determine all the companion matrices which attain this number "n". In the case of an odd n, this happens exactly when the companion matrix is unitary, while, for even n, the condition is that the matrix be unitarily equivalent to the direct sum of the two $(n/2)$-by-$(n/2)$ companion matrices

$$\begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ a & & & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ -1/\bar{a} & & & 0 \end{bmatrix}$$

for some complex number a satisfying $1 \leqslant |a| < \tan(\pi/n) + \sec(\pi/n)$.

Recall that the *numerical range* $W(A)$ of an n-by-n complex matrix A is by definition the subset $\{\langle Ax, x\rangle : x \in \mathbb{C}^n, \|x\| = 1\}$ of the complex plane, where $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ denote the standard inner product and norm in $\mathbb{C}^n$. The *numerical radius* $w(A)$ of A is $\max\{|z| : z \in W(A)\}$. It is known that the numerical range is always convex. For other properties, the reader can consult [6, Chapter 1].

The study of the numerical ranges of the companion matrices was started in [4]. Among other things, it was shown therein that an n-by-n companion matrix A whose numerical range $W(A)$ is a closed circular disc centered at the origin must be equal to the *Jordan block* of size n:

$$J_n = \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}$$

(cf. [4, Theorem 2.9]). We start with an improvement of this result by weakening the assumption on A to "$W(A)$ contains a closed circular disc D centered at the origin with the boundary $\partial W(A)$ intersecting ∂D at more than n points". For any matrix A, $\operatorname{Re} A$ denotes its *real part* $(A + A^*)/2$.

Theorem 1. *If A is an n-by-n companion matrix with $W(A)$ containing a closed circular disc D centered at the origin and with $\partial W(A) \cap \partial D$ having more than n points, then $A = J_n$.*

Proof. This is done by modifying the proof of [4, Theorem 2.9]. Let A be as in (1) and let r be the radius of D. For $|z| = 1$, consider the expansion of $\det(rI_n - \operatorname{Re}(zA))$ as a trigonometric polynomial $p(z)$ in z. Since zJ_{n-1} is unitarily equivalent to J_{n-1} for all z, $|z| = 1$, the numerical range $W(zJ_{n-1})$ is a circular disc with center the origin and radius $w(\mathrm{Re}(zJ_{n-1}))$. On the other hand, since $\mathrm{Re}(zJ_{n-1})$ is an $(n-1)$-by-$(n-1)$ submatrix of $\mathrm{Re}(zA)$, we infer from our assumption on $W(A)$ that $w(\mathrm{Re}(zJ_{n-1})) \leqslant r \leqslant w(\mathrm{Re}(zA))$ for all z, $|z| = 1$, and $r = w(\mathrm{Re}(zA))$ for more than n values of z. Also, $w(\mathrm{Re}(zJ_{n-1}))$, being the largest eigenvalue of Re (zJ_{n-1}), lies between $w(\mathrm{Re}(zA))$, the largest eigenvalue of $\mathrm{Re}(zA)$, and the second largest eigenvalue of $\mathrm{Re}(zA)$. Thus the same is true for r. Therefore, $p(z) \leqslant 0$ for all z, $|z| = 1$, and $p(z) = 0$ for n values of z. By a classical result of Fejér [7, p. 77, Problem 40], there is a polynomial q of degree n such that $|q(z)|^2 = -p(z)$ for all z. Since $|q(z)|^2 = -p(z) = 0$ for more than n values of z, we conclude that $q \equiv 0$ and thus $p \equiv 0$. In particular, the coefficients of z^j in p for $j = 0, \pm 1, \ldots, \pm n$ are all zero. Since the coefficient of z^n is $a_n/2^n$, we have $a_n = 0$. Then we can proceed as in the second half of the proof of [4, Theorem 2.9] to deduce inductively that $a_j = 0$ for all j. Thus $A = J_n$ as asserted. □

The preceding theorem is analogous to a result of Anderson's: *if A is an n-by-n matrix whose numerical range $W(A)$ is contained in a closed circular disc D such that $\partial W(A) \cap \partial D$ has more than n points*, *then $W(A) = D$*. A proof of this which makes use of Fejér's result on nonnegative trigonometric polynomials can be found in [8, Lemma 6].

An immediate corollary of Theorem 1 is the following:

Theorem 2. *For any n-by-n companion matrix A, there can be at most n points in $\partial W(A) \cap \partial W(J_{n-1})$.*

In this case, Theorem 1 is applicable since J_{n-1} is a submatrix of A and hence $W(A)$ contains the circular disc $W(J_{n-1}) = \{z \in \mathbb{C} : |z| \leqslant \cos(\pi/n)\}$ (cf. [5, Proposition 1]).

Next we give an alternative proof of Theorem 2 based on the following Lemma 3. It is simpler and more direct. Moreover, the techniques involved are useful in the determining of when $\partial W(A) \cap \partial W(J_{n-1})$ contains exactly n points for an n-by-n companion matrix A.

Lemma 3. *Let A be the companion matrix given by* (1). *If $z_0 \cos(\pi/n)$ is a point in $\partial W(A) \cap \partial W(J_{n-1})$*, *where $|z_0| = 1$*, *then z_0 is a zero of the polynomial*

$$p(z) = z^n \sin\frac{\pi}{n} - \sum_{j=2}^{n} z^{n-j} a_j \sin\frac{(n-j+1)\pi}{n}.$$

Proof. It is easily seen that $\cos(\pi/n)$ is an eigenvalue of $\mathrm{Re}(\bar{z}_0 J_{n-1})$ with the corresponding unit eigenvector

$$x_0 = \sqrt{\frac{2}{n}}\left[z_0 \sin\frac{\pi}{n}, z_0^2 \sin\frac{2\pi}{n}, \ldots, z_0^{n-1} \sin\frac{(n-1)\pi}{n}\right]^{\mathrm{T}}$$

in $\mathbb{C}^{n-1}$ (cf. [5, Proposition 1]). Let $y_0 = [x_0^{\mathrm{T}}, 0]^{\mathrm{T}}$ in $\mathbb{C}^n$. Then

$$\langle \mathrm{Re}(\bar{z}_0 A) y_0, y_0\rangle = \langle \mathrm{Re}(\bar{z}_0 J_{n-1}) x_0, x_0\rangle = \cos\frac{\pi}{n}.$$

Since $\mathrm{Re}(\bar{z}_0 A) \leqslant \cos(\pi/n) I_n$, we deduce that $\cos(\pi/n)$ is an eigenvalue of $\mathrm{Re}(\bar{z}_0 A)$ with the corresponding eigenvector y_0, that is, it satisfies $(\mathrm{Re}(\bar{z}_0 A) - \cos(\pi/n) I_n) y_0 = 0$. Carrying out the computations, we obtain from the equality of the nth components the equation

$$z_0^n \sin\frac{(n-1)\pi}{n} - \sum_{j=2}^{n} z_0^{n-j} a_j \sin\frac{(n-j+1)\pi}{n} = 0.$$

Hence z_0 is a zero of p as asserted. □

Proof of Theorem 2. If $\partial W(A) \cap \partial W(J_{n-1})$ has more than n points, then the degree-n polynomial in Lemma 3 has more than n zeros. The fundamental theorem of algebra dictates that, in particular, the leading coefficient $\sin(\pi/n)$ be zero, which is a contradiction. □

We next consider the number of line segments on the boundary of the numerical range of a companion matrix. The following theorem says that this number is at most the size of the matrix.

Theorem 4. *An n-by-n companion matrix can have at most n line segments on the boundary of its numerical range.*

This is the consequence of the next lemma and Theorem 2.

Lemma 5. *Let A be an n-by-n matrix and let B be any submatrix of A. Then every line segment of* $\partial W(A)$ *intersects* $\partial W(B)$.

Proof. Let $[a, b]$ be a line segment in $\partial W(A)$ and let $K = \{x \in \mathbb{C}^n : \langle Ax, x\rangle = \lambda\|x\|^2$ for some λ in $[a, b]\}$. It is known that K is a subspace of $\mathbb{C}^n$ with dimension at least two (cf. [2, Lemma 2]). Assume, for convenience, that B is obtained from A by deleting its last row and last column. If $L = \mathbb{C}^{n-1} \oplus \{0\}$, then

$$\dim(K \cap L) = \dim K + \dim L - \dim(K + L) \geqslant 2 + (n-1) - n = 1.$$

Hence there is in K a unit vector $x = x_1 \oplus 0$ with x_1 in $\mathbb{C}^{n-1}$. Thus $\langle Bx_1, x_1\rangle = \langle Ax, x\rangle \in [a, b]$, showing that $[a, b] \cap \partial W(B) \neq \emptyset$. □

Proof of Theorem 4. Let A be an n-by-n companion matrix and let $B = J_{n-1}$. Lemma 5 says that every line segment of $\partial W(A)$ intersects the circle $\partial W(B)$. Our assertion then follows from Theorem 2. □

As the preceding proof shows, for an n-by-n companion matrix A every line segment on $\partial W(A)$ intersects $\partial W(J_{n-1})$. The converse is in general false, namely, not every point in $\partial W(A) \cap \partial W(J_{n-1})$ arises as the intersection of a line segment on $\partial W(A)$ with $\partial W(J_{n-1})$. This is illustrated by the following example.

Example 6. Let A be the 3-by-3 companion matrix associated with the polynomial $p(z) = (z - (1/2))(z - 2\omega)(z - 2\omega^2)$, where $\omega = \left(-1 + \sqrt{3}\mathrm{i}\right)/2$. It can be checked that A is unitarily equivalent to $[1/2] \oplus \begin{bmatrix} 2\omega & 3 \\ 0 & 2\omega^2 \end{bmatrix}$. Thus $W(A)$ is the elliptic disc with foci 2ω and $2\omega^2$ and minor axis of length 3. Hence $\partial W(A) \cap \partial W(J_2)$ consists of the single point 1/2 and there is no line segment on the ellipse $\partial W(A)$.

We remark that via Kippenhahn's result we can show that the number of line segments on $\partial W(A)$ for an n-by-n matrix A is at most $n(n-1)/2$. It was asked in [1, p. 108] whether this number can be further reduced to $2(n-2)$. As of now, nobody knows.

In the remaining part of this paper, we determine when the boundary of the numerical range of an n-by-n companion matrix has exactly n line segments. This is given by the following theorem.

Theorem 7. *The following conditions are equivalent for an n-by-n $(n \geqslant 3)$ companion matrix A:*

(a) $\partial W(A)$ *has n line segments on it*;
(b) $\partial W(A) \cap \partial W(J_{n-1})$ *consists of n points*;
(c) *for n odd*,

$$A = \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ a & & & 0 \end{bmatrix}$$

for some a with $|a| = 1$, and, for n even,

$$A = \begin{bmatrix} 0 & 1 & & & & & & \\ & 0 & 1 & & & & & \\ & & 0 & \cdot & & & & \\ & & & \cdot & \cdot & & & \\ & & & & \cdot & \cdot & & \\ & & & & & \cdot & \cdot & \\ & & & & & & \cdot & 1 \\ a & 0 & \cdots & 0 & b & 0 & \cdots & 0 \end{bmatrix},$$

where $|a| = 1$ and b is in the $(n, (n/2)+1)$-position with $|b| < 2\tan(\pi/n)$ and, for $b \neq 0$, $\arg b = (\arg a \pm \pi)/2$;
(d) *for n odd, A is unitary, and, for n even, A is unitarily equivalent to the direct sum of the two $(n/2)$-by-$(n/2)$ matrices*

$$A_1 = \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ c & & & 0 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ -1/\bar{c} & & & 0 \end{bmatrix},$$

where c is a complex number satisfying $1 \leqslant |c| < \tan(\pi/n) + \sec(\pi/n)$.

The implication (a) $\Rightarrow$ (b) follows from Lemma 5 and Theorem 2. The proofs for the remaining implications (b) $\Rightarrow$ (c), (c) $\Rightarrow$ (d) and (d) $\Rightarrow$ (a) are more laborious. We start with the following lemma on an expression for some determinants associated with the real part of the Jordan block. This is useful in proving the subsequent lemmas.

Lemma 8. *For any k, $1 \leqslant k \leqslant n-1$, we have*

$$\det\left(\left(\cos\frac{\pi}{n}\right) I_k - \operatorname{Re} J_k\right) = \frac{1}{2^k} \cdot \frac{\sin\frac{(k+1)\pi}{n}}{\sin\frac{\pi}{n}}.$$

In particular, if d_k, $1 \leqslant k \leqslant n-1$, *denotes the above determinant and* $d_0 = 1$, *then* $d_k = 2^{n-2k-2} d_{n-k-2}$ *for* $0 \leqslant k \leqslant n-3$.

Proof. For $k = n-1$, the asserted equality is obviously true since $\cos(\pi/n)$ is an eigenvalue of $\operatorname{Re} J_{n-1}$ and thus both sides are equal to zero. We next consider $k = n-2$. Since

$$\begin{bmatrix} \cos\frac{\pi}{n} & -\frac{1}{2} & & & \\ -\frac{1}{2} & \cos\frac{\pi}{n} & \cdot & & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot & -\frac{1}{2} \\ & & & & -\frac{1}{2} & \cos\frac{\pi}{n} \end{bmatrix} \begin{bmatrix} \sin\frac{\pi}{n} \\ \sin\frac{2\pi}{n} \\ \cdot \\ \cdot \\ \cdot \\ \sin\frac{(n-2)\pi}{n} \end{bmatrix} = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ \frac{1}{2}\sin\frac{(n-1)\pi}{n} \end{bmatrix}, \tag{2}$$

applying Cramer's rule to solve for $\sin(\pi/n)$, we obtain

$$\sin\frac{\pi}{n} = \frac{(-1)^{n-1}\frac{1}{2}\sin\frac{(n-1)\pi}{n} \cdot \left(-\frac{1}{2}\right)^{n-3}}{d_{n-2}} = \frac{\left(\frac{1}{2}\right)^{n-2}\sin\frac{(n-1)\pi}{n}}{d_{n-2}}.$$

It follows that $d_{n-2} = 1/2^{n-2}$ as asserted.

Assume now $1 \leqslant k \leqslant n-3$. In this case, we solve $\sin((k+1)\pi/n)$ by Cramer's rule to obtain

$$\begin{aligned} \sin\frac{(k+1)\pi}{n} &= \frac{(-1)^{n-1+k}\frac{1}{2}\sin\frac{(n-1)\pi}{n} \cdot d_k\left(-\frac{1}{2}\right)^{n-3-k}}{d_{n-2}} \\ &= \frac{\left(\frac{1}{2}\right)^{n-2-k}\sin\frac{(n-1)\pi}{n} \cdot d_k}{\left(\frac{1}{2}\right)^{n-2}} = 2^k \sin\frac{\pi}{n} \cdot d_k. \end{aligned}$$

Our asserted expression for d_k follows immediately. □

Note that if A is the n-by-n companion matrix (1), then, for any real θ, $e^{i\theta}A$ is unitarily equivalent to the companion matrix

$$\begin{bmatrix} 0 & 1 & & & \\ & 0 & \ddots & & \\ & & \ddots & 1 & \\ & & & 0 & 1 \\ -a_n e^{in\theta} & -a_{n-1}e^{i(n-1)\theta} & \cdots & -a_2 e^{i2\theta} & -a_1 e^{i\theta} \end{bmatrix}.$$

This will be used in the proofs below.

Lemma 9. *Let A be the n-by-n companion matrix* (1). *If* $\partial W(A) \cap \partial W(J_{n-1})$ *consists of n points, then* $a_j = 0$ *for all* j, $1 \leqslant j \leqslant n-1$, *except possibly, when n is even, for* $j = n/2$.

Proof. Let $z_k \cos(\pi/n)$, $1 \leqslant k \leqslant n$, be the n points in $\partial W(A) \cap \partial W(J_{n-1})$, where the z_k's all have modulus one. Lemma 3 says that every z_k is a zero of the polynomial

$$p(z) = z^n \sin\frac{\pi}{n} - \sum_{j=2}^{n} z^{n-j} a_j \sin\frac{(n-j+1)\pi}{n},$$

which, by Lemma 8, is the same as

$$p(z) = \sin\frac{\pi}{n}\left(z^n - \sum_{j=2}^{n} z^{n-j} a_j 2^{n-j} d_{n-j}\right) \equiv \sin\frac{\pi}{n}\cdot p_1(z), \tag{3}$$

where $d_m = \det((\cos(\pi/n))I_m - \operatorname{Re} J_m)$ for $1 \leqslant m \leqslant n-2$ and $d_0 = 1$. Let $\sigma_0 = 1$ and let

$$\sigma_j = \sum_{k_1<\cdots<k_j} z_{k_1}\cdots z_{k_j},$$

$1 \leqslant j \leqslant n$, be the jth elementary symmetric function of the z_k's. Hence we have

$$p_1(z) = \prod_{k=1}^{n}(z - z_k) = \sum_{j=0}^{n}(-1)^j \sigma_j z^{n-j}. \tag{4}$$

Equating the corresponding coefficients of $p_1(z)$ in (3) and (4) yields $\sigma_1 = 0$, $\sigma_n = (-1)^{n+1}a_n$ and

$$\sigma_j = (-1)^{j+1} a_j 2^{n-j} d_{n-j}, \quad 2 \leqslant j \leqslant n-1. \tag{5}$$

Since $|z_k| = 1$ for all k, we have $\sigma_j = \bar{\sigma}_{n-j}/\bar{\sigma}_n$ and thus

$$a_j 2^{n-j} d_{n-j} = -a_n \bar{a}_{n-j} 2^j d_j, \quad 2 \leqslant j \leqslant n-2. \tag{6}$$

Note that $\sigma_1 = 0$ implies that $\sigma_{n-1} = 0$ and therefore $a_{n-1} = 0$.

To prove that the remaining a_j's are also zero, we consider the $(n-1)$-by-$(n-1)$ matrices

$$A_k = \begin{bmatrix} \cos(\pi/n) & -\bar{z}_k/2 & & & & 0 \\ -z_k/2 & \cdot & \cdot & & & \bar{a}_{n-2}z_k/2 \\ & \cdot & \cdot & \cdot & & \vdots \\ & & \cdot & \cdot & -\bar{z}_k/2 & \bar{a}_3 z_k/2 \\ & & & -z_k/2 & \cos(\pi/n) & (\bar{a}_2 z_k - \bar{z}_k)/2 \\ 0 & a_{n-2}\bar{z}_k/2 & \cdots & a_3\bar{z}_k/2 & (a_2\bar{z}_k - z_k)/2 & \cos(\pi/n) + \operatorname{Re}(a_1\bar{z}_k) \end{bmatrix},$$

$1 \leqslant k \leqslant n$. Since $|z_k| = 1$, the matrices $\bar{z}_k J_m$ and J_m are unitarily equivalent and hence $\det((\cos(\pi/n))I_m - \operatorname{Re}(\bar{z}_k J_m)) = d_m$ for $1 \leqslant m \leqslant n-2$. Expanding $\det A_k$ by cofactors along its last row and then expanding the latter along their last columns, we obtain

$$\begin{aligned} \det A_k = {} & \left(\cos\frac{\pi}{n} + \operatorname{Re}(a_1\bar{z}_k)\right) d_{n-2} - \frac{1}{4}|a_2\bar{z}_k - z_k|^2 d_{n-3} \\ & - 2\operatorname{Re}\left[\sum_{j=3}^{n-2}\left(\frac{a_2\bar{z}_k - z_k}{2}\right)(-1)^j\left(\frac{\bar{a}_j z_k}{2}\right)\left(-\frac{z_k}{2}\right)^{j-2} d_{n-j-1}\right] \\ & - \sum_{j=3}^{n-2}\frac{1}{4}|a_j\bar{z}_k|^2 d_{j-2}d_{n-j-1} \\ & + 2\operatorname{Re}\left[\sum_{l=3}^{n-2}(-1)^{l+1}\left(\frac{a_l\bar{z}_k}{2}\right)\left(\sum_{j=l+1}^{n-2}(-1)^j\left(\frac{\bar{a}_j z_k}{2}\right)\left(-\frac{z_k}{2}\right)^{j-l} d_{l-2}d_{n-j-1}\right)\right] \end{aligned}$$

$$
\begin{aligned}
&= d_1 d_{n-2} + \mathrm{Re}(a_1 \bar{z}_k) d_{n-2} - \frac{1}{4}(|a_2|^2 - 2\mathrm{Re}(a_2 \bar{z}_k^2) + 1) d_{n-3} \\
&\quad - \frac{1}{4} \sum_{j=3}^{n-2} |a_j|^2 d_{j-2} d_{n-j-1} \\
&\quad + \mathrm{Re}\left[\sum_{j=3}^{n-2} \bar{a}_j \left(\frac{z_k}{2}\right)^j d_{n-j-1} - \frac{1}{4} \sum_{l=2}^{n-3} a_l \left(\sum_{j=l+1}^{n-2} \bar{a}_j \left(\frac{z_k}{2}\right)^{j-l} d_{l-2} d_{n-j-1} \right) \right] \\
&= \left(d_1 d_{n-2} - \frac{1}{4} d_{n-3} \right) + \mathrm{Re}(a_1 \bar{z}_k) d_{n-2} + \frac{1}{2}\mathrm{Re}(a_2 \bar{z}_k^2) d_{n-3} \\
&\quad - \frac{1}{4} \sum_{j=2}^{n-2} |a_j|^2 d_{j-2} d_{n-j-1} \\
&\quad + 2\mathrm{Re}\left[\sum_{j=3}^{n-2} \left(\bar{a}_j \left(\frac{z_k}{2}\right)^j d_{n-j-1} - \frac{1}{4} \sum_{l=2}^{j-1} a_l \bar{a}_j \left(\frac{z_k}{2}\right)^{j-l} d_{l-2} d_{n-j-1} \right) \right] \\
&= \mathrm{Re}\,(a_1 \bar{z}_k)\, d_{n-2} + \frac{1}{2}\mathrm{Re}(a_2 \bar{z}_k^2) d_{n-3} - \frac{1}{4} \sum_{j=2}^{n-2} |a_j|^2 d_{j-2} d_{n-j-1} \\
&\quad + 2\mathrm{Re}\left[\sum_{j=3}^{n-2} \left(\frac{\bar{a}_j}{2^j} \right) d_{n-j-1} \left(z_k^j + \sum_{l=2}^{j-1} (-1)^l \sigma_l z_k^{j-l} \right) \right], \qquad (7)
\end{aligned}
$$

where in the last equality we used the facts that $d_1 d_{n-2} - (1/4) d_{n-3} = d_{n-1} = 0$ since $\cos(\pi/n)$ is an eigenvalue of $\mathrm{Re}\, J_{n-1}$, and

$$
-a_l 2^{l-2} d_{l-2} = -a_l 2^{n-l} d_{n-l} = (-1)^l \sigma_l, \quad 2 \leqslant l \leqslant n-2, \qquad (8)
$$

by Lemma 8 and (5). Since $\cos(\pi/n)$ is the maximum eigenvalue of $\mathrm{Re}(\bar{z}_k J_{n-1})$, we have $A_k \geqslant 0$ and thus $\det A_k \geqslant 0$ for all k. Hence

$$
\begin{aligned}
0 \leqslant \sum_{k=1}^{n} \det A_k &= \mathrm{Re}(a_1 \bar{s}_1) d_{n-2} + \frac{1}{2}\mathrm{Re}(a_2 \bar{s}_2) d_{n-3} - \frac{n}{4} \sum_{j=2}^{n-2} |a_j|^2 d_{j-2} d_{n-j-1} \\
&\quad + 2\mathrm{Re}\left[\sum_{j=3}^{n-2} \left(\frac{\bar{a}_j}{2^j} \right) d_{n-j-1} \left(s_j + \sum_{l=2}^{j-1} (-1)^l \sigma_l s_{j-l} \right) \right], \qquad (9)
\end{aligned}
$$

where $s_j = \sum_{k=1}^{n} z_k^j$ for $1 \leqslant j \leqslant n-1$. Note that $s_1 = \sigma_1 = 0$ and the s_j's and σ_l's are related by Newton's identities:

$$
s_j = \left(\sum_{l=1}^{j-1} (-1)^{l+1} \sigma_l s_{j-l} \right) + (-1)^{j+1} j \sigma_j, \quad 1 \leqslant j \leqslant n.
$$

Hence

$$s_j + \sum_{l=2}^{j-1}(-1)^l \sigma_l s_{j-l} = s_j + \sum_{l=1}^{j-1}(-1)^l \sigma_l s_{j-l}$$
$$= (-1)^{j+1} j\sigma_j = ja_j 2^{j-2} d_{j-2}, \quad 2 \leqslant j \leqslant n-2,$$

by (8). Therefore, (9) becomes

$$0 \leqslant |a_2|^2 d_{n-3} - \frac{n}{4}\sum_{j=2}^{n-2}|a_j|^2 d_{j-2}d_{n-j-1} + 2\mathrm{Re}\left[\sum_{j=3}^{n-2}\left(\frac{\bar{a}_j}{2^j}\right) d_{n-j-1} ja_j 2^{j-2} d_{j-2}\right]$$
$$= \sum_{j=2}^{n-2}\frac{2j-n}{4}|a_j|^2 d_{j-2}d_{n-j-1}. \tag{10}$$

For any real number x, we use $\lfloor x \rfloor$ to denote the largest integer which is less than or equal to x. The second half of the above summation, namely,

$$\sum_{j=\lfloor n/2\rfloor+1}^{n-2}\frac{2j-n}{4}|a_j|^2 d_{j-2}d_{n-j-1},$$

equals

$$\sum_{j=2}^{\lfloor (n-1)/2\rfloor}\frac{2(n-j)-n}{4}|a_{n-j}|^2 d_{n-j-2}d_{j-1}, \tag{11}$$

which we want to express as a linear combination of the $|a_j|^2 d_{j-2}d_{n-j-1}$'s as in the first half. For this purpose, note that $|a_j|2^{n-j}d_{n-j} = |a_{n-j}|2^j d_j$ for $2 \leqslant j \leqslant n-2$ from (6). Therefore,

$$|a_{n-j}|^2 d_{n-j-2}d_{j-1} = |a_j|^2 2^{2n-4j}\frac{d_{n-j}^2}{d_j^2} d_{n-j-2}d_{j-1}$$
$$= |a_j|^2 2^{2n-4j}\frac{(2^{2j-n-2}d_{j-2})^2}{d_j^2}(2^{2j-n+2}d_j)(2^{n-2j}d_{n-j-1})$$
$$= |a_j|^2 d_{j-2}d_{n-j-1}\cdot\frac{1}{4}\frac{d_{j-2}}{d_j}$$
$$= |a_j|^2 d_{j-2}d_{n-j-1}\cdot\frac{\sin\frac{(j-1)\pi}{n}}{\sin\frac{(j+1)\pi}{n}}$$

with the aid of Lemma 8. Plugging this into (11), we obtain from (10) the nonnegativity of

$$-\sum_{j=2}^{\lfloor (n-1)/2\rfloor}\left(\frac{n-2j}{4}\right)\left(1-\frac{\sin\frac{(j-1)\pi}{n}}{\sin\frac{(j+1)\pi}{n}}\right)|a_j|^2 d_{j-2}d_{n-j-1}.$$

Since all the terms except $|a_j|^2$ in the above summation are strictly positive, we conclude that $a_j = 0$ for all j, $2 \leqslant j \leqslant \lfloor (n-1)/2\rfloor$. By (6), we also have $a_j = 0$ for $\lfloor n/2\rfloor + 1 \leqslant j \leqslant n-2$. To complete the proof, we need only show that $a_1 = 0$. Since $|a_n| = 1$, we may assume, by the remark in the paragraph preceding Lemma 9, that $a_n = -1$. Consider the cases of odd and even n separately.

Assume first that n is odd. Then, from (3),

$$p_1(z) = z^n - 2a_{n-1}d_1 z - a_n = z^n + 1.$$

We assume that the zeros of p_1 are given by $z_k = \mathrm{e}^{(2k-1)\pi\mathrm{i}/n}$, $1 \leqslant k \leqslant n$. Now we obtain from (7) that $\det A_k = \mathrm{Re}(a_1\bar{z}_k)d_{n-2}$. Hence

$$0 \leqslant \mathrm{Re}(a_1\bar{z}_k) = \cos\frac{(2k-1)\pi}{n}\mathrm{Re}\, a_1 + \sin\frac{(2k-1)\pi}{n}\mathrm{Im}\, a_1$$

for all k, $1 \leqslant k \leqslant n$. Replacing k by $n-k+1$ in the above, we also have

$$\cos\frac{(2k-1)\pi}{n}\mathrm{Re}\, a_1 - \sin\frac{(2k-1)\pi}{n}\mathrm{Im}\, a_1 \geqslant 0.$$

Thus $\cos((2k-1)\pi/n)\mathrm{Re}\, a_1 \geqslant 0$ for all k. Since $\cos((2k-1)\pi/n)$ can be positive or negative for different values of k, we infer that $\mathrm{Re}\, a_1 = 0$. Then, from above, $\pm\sin((2k-1)\pi/n)\mathrm{Im}\, a_1 \geqslant 0$ for all k, which implies that $\mathrm{Im}\, a_1 = 0$. Hence, as asserted, $a_1 = 0$ for odd n.

Finally, assume that n is even. In this case, we deduce from (6) that $a_{n/2} = -a_n\bar{a}_{n/2} = \bar{a}_{n/2}$, that is, $a_{n/2}$ is real, and from (3) that

$$p_1(z) = z^n - 2^{n/2}a_{n/2}d_{n/2}z^{n/2} + 1 = (z^{n/2} - z_+)(z^{n/2} - z_-),$$

where $z_\pm = \left(2^{n/2}a_{n/2}d_{n/2} \pm \left(2^n a_{n/2}^2 d_{n/2}^2 - 4\right)^{1/2}\right)\Big/ 2$. Since the zeros z_k of p_1 have modulus one, we have $|z_\pm| = 1$, which is equivalent to $|2^{n/2}a_{n/2}d_{n/2}| \leqslant 2$. Hence, in particular, $\mathrm{Re}\, z_\pm = 2^{(n/2)-1}a_{n/2}d_{n/2}$. On the other hand, from (7) we have

$$\det A_k = \mathrm{Re}(a_1\bar{z}_k)d_{n-2} - \frac{1}{4}a_{n/2}^2 d_{(n/2)-2}d_{(n/2)-1} + 2\mathrm{Re}\left(\frac{a_{n/2}}{2^{n/2}}d_{(n/2)-1}z_k^{n/2}\right),$$

where, since $z_k^{n/2} = z_\pm$, the last term can be simplified as

$$\begin{aligned} 2\mathrm{Re}\left(\frac{a_{n/2}}{2^{n/2}}d_{(n/2)-1}z_k^{n/2}\right) &= 2\frac{a_{n/2}}{2^{n/2}}d_{(n/2)-1}\mathrm{Re}\, z_\pm \\ &= 2\frac{a_{n/2}}{2^{n/2}}d_{(n/2)-1}2^{(n/2)-1}a_{n/2}d_{n/2} \\ &= a_{n/2}^2 d_{(n/2)-1}d_{n/2}. \end{aligned}$$

Hence

$$0 \leqslant \det A_k = \mathrm{Re}(a_1\bar{z}_k)d_{n-2} - a_{n/2}^2 d_{(n/2)-1}\left(\frac{1}{4}d_{(n/2)-2} - d_{n/2}\right) = \mathrm{Re}(a_1\bar{z}_k)d_{n-2}$$

by Lemma 8. Because $d_{n-2} > 0$, we have $\mathrm{Re}(a_1\bar{z}_k) \geqslant 0$ for all k, $1 \leqslant k \leqslant n$. If $z_+ = \mathrm{e}^{\mathrm{i}\theta_0}$ for some real θ_0, then $z_- = \mathrm{e}^{-\mathrm{i}\theta_0}$ and the z_k's are equal to $u_j \equiv \mathrm{e}^{(2\theta_0+4j\pi)/n}$ and $v_j \equiv \mathrm{e}^{(-2\theta_0+4j\pi)/n}$, $0 \leqslant j \leqslant (n/2)-1$. Since $u_j = \bar{v}_{(n/2)-j}$, both $\mathrm{Re}(a_1\bar{u}_j)$ and $\mathrm{Re}(a_1 u_j)(= \mathrm{Re}(a_1\bar{v}_{(n/2)-j}))$ are nonnegative. Hence $(\mathrm{Re}\, a_1)\cos((2\theta_0+4j\pi)/n) \geqslant 0$ for all j. Since different values of j yield positive and negative values of $\cos((2\theta_0+4j\pi)/n)$, we infer that $\mathrm{Re}\, a_1 = 0$. Then

$$\mathrm{Re}(a_1\bar{u}_j) = (\mathrm{Im}\, a_1)\sin\frac{2\theta_0+4j\pi}{n} \geqslant 0$$

and

$$\operatorname{Re}(a_1 u_j) = -(\operatorname{Im} a_1) \sin \frac{2\theta_0 + 4j\pi}{n} \geqslant 0$$

for all j. Hence $\operatorname{Im} a_1 = 0$ and, therefore, $a_1 = 0$. This completes the proof. □

We now resume the proof of Theorem 7.

Proof of Theorem 7. (b) $\Rightarrow$ (c). If n is odd, then, as proved in Lemma 9,

$$A = \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ -a_n & & & 0 \end{bmatrix}$$

with $|a_n| = 1$ as required.

Now assume that n is even. From Lemma 9, we have

$$A = \begin{bmatrix} 0 & 1 & & & & & & \\ & 0 & \cdot & & & & & \\ & & \cdot & \cdot & & & & \\ & & & \cdot & \cdot & & & \\ & & & & \cdot & \cdot & & \\ & & & & & \cdot & \cdot & \\ & & & & & & \cdot & 1 \\ -a_n & 0 & \cdots & 0 & -a_{n/2} & 0 & \cdots & 0 \end{bmatrix}$$

with $|a_n| = 1$. Let $a_n = e^{i\theta_0}$ with θ_0 real and let $\theta = (\pi - \theta_0)/n$. Then $e^{i\theta} A$ is unitarily equivalent to

$$A' = \begin{bmatrix} 0 & 1 & & & & & & \\ & 0 & \cdot & & & & & \\ & & \cdot & \cdot & & & & \\ & & & \cdot & \cdot & & & \\ & & & & \cdot & \cdot & & \\ & & & & & \cdot & \cdot & \\ & & & & & & \cdot & 1 \\ 1 & 0 & \cdots & 0 & -ia_{n/2}e^{-i\theta_0/2} & 0 & \cdots & 0 \end{bmatrix}$$

(cf. the paragraph before Lemma 9). If $b' = -ia_{n/2}e^{-i\theta_0/2}$, then Lemma 3 as applied to A' yields that the zeros of the polynomial $p_1(z) = z^n + z^{n/2}b' \cot(\pi/n) + 1$ are distinct and have modulus one. However, the zeros of p_1 are the $(n/2)$th roots of $(-b' \cot(\pi/n) \pm (b'^2 \cot^2(\pi/n) - 4)^{1/2})/2$. Thus we must have $|b' \cot(\pi/n)| < 2$ or $|b'| < 2\tan(\pi/n)$. On the other hand, (6) as applied to A' with $j = n/2$ yields that $b' (= -ia_{n/2}e^{-i\theta_0/2})$ is real. Hence for nonzero b' we have $\arg a_{n/2} = (\theta_0 \pm \pi)/2$. Letting $a = -a_n$ and $b = -a_{n/2}$, we conclude that $|a| = 1$, $|b| < 2\tan(\pi/n)$ and, for $b \neq 0$, $\arg b = (\theta_0 \pm \pi)/2$. □

We next prove the implication (c) $\Rightarrow$ (d) of Theorem 7.

Proof of Theorem 7. (c) $\Rightarrow$ (d). We need only prove the case for even n. Considering $e^{i\theta} A$ with $\theta = (\pi - \arg a)/n$ instead of A, we may assume that $a = 1$ and b is real (cf. the paragraph before Lemma 9). Let $c = (b \pm (b^2 + 4)^{1/2})/2$ with the "+" sign if $b \geqslant 0$ and "−" sign if $b < 0$. Then

$$1 \leqslant |c| = \frac{1}{2}|b \pm (b^2+4)^{1/2}| \leqslant \frac{1}{2}(|b| + |b^2+4|^{1/2}) < \tan\frac{\pi}{n} + \sec\frac{\pi}{n}$$

and $b = c - (1/c)$. Let $d = 1/(1+c^2)^{1/2}$ and

$$U = d\begin{bmatrix} I_{n/2} & cI_{n/2} \\ cI_{n/2} & -I_{n/2} \end{bmatrix}.$$

Then U is unitary and $UA = (A_1 \oplus A_2)U$, completing the proof. □

To prove (d) ⇒ (a) of Theorem 7, we need the following lemma for even n.

Lemma 10. *Let*

$$A_1 = \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ c & & & 0 \end{bmatrix} \quad \textit{and} \quad A_2 = \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ -1/c & & & 0 \end{bmatrix}$$

be $(n/2)$*-by-*$(n/2)$ *matrices, where* n $(\geqslant 4)$ *is even and* c *is real satisfying* $1 \leqslant c < \tan(\pi/n) + \sec(\pi/n)$. *Let* z_0 *be a zero of* $p_1(z) = z^n + z^{n/2}(c - (1/c))\cot(\pi/n) + 1$ *and let*

$$x = \left[z_0 \sin\frac{\pi}{n}, z_0^2 \sin\frac{2\pi}{n}, \ldots, z_0^{n/2}\sin\frac{\frac{n}{2}\pi}{n}\right]^{\mathrm{T}},$$

$$y = \left[z_0^{(n/2)+1}\cos\frac{\pi}{n}, z_0^{(n/2)+2}\cos\frac{2\pi}{n}, \ldots, z_0^{n-1}\cos\frac{\left(\frac{n}{2}-1\right)\pi}{n}, 0\right]^{\mathrm{T}},$$

$u = (x+cy)/\|x+cy\|$ *and* $v = (cx-y)/\|cx-y\|$ *be vectors in* $\mathbb{C}^{n/2}$. *Then*

$$\langle \bar{z}_0 A_1 u, u\rangle = \cos\frac{\pi}{n} - \mathrm{i}\frac{nc\,\mathrm{Im}(z_0^{n/2})\sin\frac{\pi}{n}}{\frac{n}{2}(1+c^2) + (1-c^2)\csc^2\left(\frac{\pi}{n}\right)}$$

and

$$\langle \bar{z}_0 A_2 v, v\rangle = \cos\frac{\pi}{n} + \mathrm{i}\frac{nc\,\mathrm{Im}(z_0^{n/2})\sin\frac{\pi}{n}}{\frac{n}{2}(1+c^2) + (c^2-1)\csc^2\left(\frac{\pi}{n}\right)}.$$

Proof. Since $1 \leqslant c < \tan(\pi/n) + \sec(\pi/n)$, we have $0 \leqslant c - \tan(\pi/n) < \sec(\pi/n)$ and therefore $c^2 - 2c\tan(\pi/n) + \tan^2(\pi/n) < \sec^2(\pi/n)$ or $c^2 - 2c\tan(\pi/n) < 1$. Hence $(c - (1/c))\cot(\pi/n) < 2$. Thus

$$z_0^{n/2} = -\frac{1}{2}\left(c - \frac{1}{c}\right)\cot\frac{\pi}{n} \pm \frac{1}{2}\mathrm{i}\left(4 - \left(c - \frac{1}{c}\right)^2\cot^2\frac{\pi}{n}\right)^{1/2}$$

and, in particular, z_0 has modulus one. Since

$$\langle \bar{z}_0 A_1 u, u\rangle = \frac{1}{\|x+cy\|^2}(\langle \bar{z}_0 A_1 x, x\rangle + c\langle \bar{z}_0 A_1 x, y\rangle + c\langle \bar{z}_0 A_1 y, x\rangle + c^2\langle \bar{z}_0 A_1 y, y\rangle),$$

we need compute the values of $\|x+cy\|$ and the four inner products above. To obtain the former, note that

$$
\begin{aligned}
\|x\|^2 &= \sum_{j=1}^{n/2} |z_0|^{2j} \sin^2\left(\frac{j\pi}{n}\right) \\
&= \frac{1}{2}\sum_{j=1}^{n/2}\left(1-\cos\frac{2j\pi}{n}\right) = \frac{n}{4} - \frac{1}{2}\mathrm{Re}\left(\frac{1-\mathrm{e}^{(1+(2/n))\pi\mathrm{i}}}{1-\mathrm{e}^{2\pi\mathrm{i}/n}} - 1\right) \\
&= \frac{n}{4} - \frac{1}{2}(-1) = \frac{1}{4}(n+2), \\
\|y\|^2 &= \sum_{j=1}^{(n/2)-1} |z_0|^{n+2j}\cos^2\left(\frac{j\pi}{n}\right) \\
&= \frac{1}{2}\sum_{j=1}^{(n/2)-1}\left(1+\cos\frac{2j\pi}{n}\right) = \frac{1}{4}(n-2), \\
\langle x, y\rangle &= \bar{z}_0^{n/2}\sum_{j=1}^{(n/2)-1}\sin\frac{j\pi}{n}\cos\frac{j\pi}{n} \\
&= \frac{1}{2}\bar{z}_0^{n/2}\sum_{j=1}^{(n/2)-1}\sin\frac{2j\pi}{n} = \frac{1}{2}\bar{z}_0^{n/2}\mathrm{Im}\left(\frac{1-\mathrm{e}^{\pi\mathrm{i}}}{1-\mathrm{e}^{2\pi\mathrm{i}/n}} - 1\right) \\
&= \frac{1}{2}\bar{z}_0^{n/2}\cot\frac{\pi}{n},
\end{aligned}
$$

and

$$
\begin{aligned}
\|x+cy\|^2 &= \|x\|^2 + 2c\,\mathrm{Re}\langle x, y\rangle + c^2\|y\|^2 \\
&= \frac{1}{4}(n+2) + c\cot\frac{\pi}{n}\cdot\mathrm{Re}(\bar{z}_0^{n/2}) + \frac{1}{4}(n-2)c^2 \\
&= \frac{n}{4}(1+c^2) + \frac{1}{2}(1-c^2) + c\cot\frac{\pi}{n}\left(-\frac{1}{2}\left(c-\frac{1}{c}\right)\cot\frac{\pi}{n}\right) \\
&= \frac{n}{4}(1+c^2) + \frac{1}{2}(1-c^2)\csc^2\left(\frac{\pi}{n}\right).
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
\langle\bar{z}_0A_1x, x\rangle &= \left(\sum_{j=1}^{(n/2)-1}\sin\frac{j\pi}{n}\sin\frac{(j+1)\pi}{n}\right) + c\bar{z}_0^{n/2}\sin\frac{\pi}{n}\sin\frac{\pi}{2} \\
&= c\bar{z}_0^{n/2}\sin\frac{\pi}{n} - \frac{1}{2}\sum_{j=1}^{(n/2)-1}\left(\cos\frac{(2j+1)\pi}{n} - \cos\frac{\pi}{n}\right) \\
&= c\bar{z}_0^{n/2}\sin\frac{\pi}{n} - \frac{1}{2}\mathrm{Re}\left(\mathrm{e}^{3\pi\mathrm{i}/n}\cdot\frac{1-\mathrm{e}^{(2\pi\mathrm{i}/n)(n-2)/2}}{1-\mathrm{e}^{2\pi\mathrm{i}/n}}\right) + \frac{1}{2}\left(\frac{n}{2}-1\right)\cos\frac{\pi}{n}
\end{aligned}
$$

$$
\begin{aligned}
&= c\bar{z}_0^{n/2}\sin\frac{\pi}{n} + \frac{n}{4}\cos\frac{\pi}{n},\\
\langle \bar{z}_0 A_1 x, y\rangle &= \bar{z}_0^{n/2}\sum_{j=1}^{(n/2)-1}\sin\frac{(j+1)\pi}{n}\cos\frac{j\pi}{n}\\
&= \frac{1}{2}\bar{z}_0^{n/2}\sum_{j=1}^{(n/2)-1}\left(\sin\frac{(2j+1)\pi}{n} + \sin\frac{\pi}{n}\right)\\
&= \frac{1}{2}\bar{z}_0^{n/2}\left(\mathrm{Im}(\mathrm{e}^{3\pi\mathrm{i}/n}\cdot\frac{1-\mathrm{e}^{(2\pi\mathrm{i}/n)(n-2)/2}}{1-\mathrm{e}^{2\pi\mathrm{i}/n}}) + \left(\frac{n}{2}-1\right)\sin\frac{\pi}{n}\right)\\
&= \frac{1}{2}\bar{z}_0^{n/2}\left(\csc\frac{\pi}{n} + \left(\frac{n}{2}-2\right)\sin\frac{\pi}{n}\right),\\
\langle \bar{z}_0 A_1 y, x\rangle &= z_0^{n/2}\left(\sum_{j=1}^{(n/2)-2}\cos\frac{(j+1)\pi}{n}\sin\frac{j\pi}{n}\right) + c\cos\frac{\pi}{n}\\
&= c\cos\frac{\pi}{n} + \frac{1}{2}z_0^{n/2}\sum_{j=1}^{(n/2)-2}\left(\sin\frac{(2j+1)\pi}{n} - \sin\frac{\pi}{n}\right)\\
&= c\cos\frac{\pi}{n} + \frac{1}{2}z_0^{n/2}\left(\left(\csc\frac{\pi}{n} - \sin\frac{\pi}{n} - \sin\frac{(n-1)\pi}{n}\right) - \left(\frac{n}{2}-2\right)\sin\frac{\pi}{n}\right)\\
&= c\cos\frac{\pi}{n} + \frac{1}{2}z_0^{n/2}\left(\csc\frac{\pi}{n} - \frac{n}{2}\sin\frac{\pi}{n}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
\langle \bar{z}_0 A_1 y, y\rangle &= \sum_{j=1}^{(n/2)-2}\cos\frac{(j+1)\pi}{n}\cos\frac{j\pi}{n}\\
&= \frac{1}{2}\sum_{j=1}^{(n/2)-2}\left(\cos\frac{(2j+1)\pi}{n} + \cos\frac{\pi}{n}\right)\\
&= \left(\frac{n}{4}-1\right)\cos\frac{\pi}{n}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\langle \bar{z}_0 A_1 u, u\rangle &= \frac{1}{\|x+cy\|^2}\left[\left(c\bar{z}_0^{n/2}\sin\frac{\pi}{n} + \frac{n}{4}\cos\frac{\pi}{n}\right) + \frac{1}{2}c\bar{z}_0^{n/2}\left(\csc\frac{\pi}{n} + \left(\frac{n}{2}-2\right)\sin\frac{\pi}{n}\right)\right.\\
&\quad\left. + c\left(c\cos\frac{\pi}{n} + \frac{1}{2}z_0^{n/2}\left(\csc\frac{\pi}{n} - \frac{n}{2}\sin\frac{\pi}{n}\right)\right) + c^2\left(\frac{n}{4}-1\right)\cos\frac{\pi}{n}\right]\\
&= \frac{1}{\|x+cy\|^2}\left(\frac{n}{4}(1+c^2)\cos\frac{\pi}{n} + \frac{1}{2}c\left(\bar{z}_0^{n/2} + z_0^{n/2}\right)\csc\frac{\pi}{n}\right.\\
&\quad\left. + \frac{n}{4}c\left(\bar{z}_0^{n/2} - z_0^{n/2}\right)\sin\frac{\pi}{n}\right)
\end{aligned}
$$

$$= \frac{1}{\|x+cy\|^2}\left(\frac{n}{4}(1+c^2)\cos\frac{\pi}{n} + c\,\mathrm{Re}(z_0^{n/2})\csc\frac{\pi}{n} - \frac{1}{2}nc\mathrm{i}\,\mathrm{Im}(z_0^{n/2})\sin\frac{\pi}{n}\right)$$
$$= \frac{1}{\|x+cy\|^2}\left(\frac{n}{4}(1+c^2)\cos\frac{\pi}{n} - \frac{1}{2}c\left(c-\frac{1}{c}\right)\cot\frac{\pi}{n}\csc\frac{\pi}{n}\right.$$
$$\left.-\frac{1}{2}nc\mathrm{i}\,\mathrm{Im}\left(z_0^{n/2}\right)\sin\frac{\pi}{n}\right)$$
$$= \frac{1}{\|x+cy\|^2}\left(\left(\frac{n}{4}(1+c^2) + \frac{1}{2}(1-c^2)\csc^2\left(\frac{\pi}{n}\right)\right)\cos\frac{\pi}{n}\right.$$
$$\left.-\ \frac{1}{2}nc\mathrm{i}\,\mathrm{Im}\left(z_0^{n/2}\right)\sin\frac{\pi}{n}\right)$$
$$= \cos\frac{\pi}{n} - \mathrm{i}\frac{nc\,\mathrm{Im}(z_0^{n/2})\sin\frac{\pi}{n}}{\frac{n}{2}(1+c^2)+(1-c^2)\csc^2\left(\frac{\pi}{n}\right)}$$

as asserted.

In a similar fashion, we derive that

$$\|cx-y\|^2 = \frac{n}{4}(1+c^2) + \frac{1}{2}(c^2-1)\csc^2\left(\frac{\pi}{n}\right),$$
$$\langle \bar{z}_0 A_2 x, x\rangle = -\frac{1}{c}\bar{z}_0^{n/2}\sin\frac{\pi}{n} + \frac{n}{4}\cos\frac{\pi}{n},$$
$$\langle \bar{z}_0 A_2 x, y\rangle = \frac{1}{2}\bar{z}_0^{n/2}\left(\csc\frac{\pi}{n} + \left(\frac{n}{2}-2\right)\sin\frac{\pi}{n}\right),$$
$$\langle \bar{z}_0 A_2 y, x\rangle = -\frac{1}{c}\cos\frac{\pi}{n} + \frac{1}{2}z_0^{n/2}\left(\csc\frac{\pi}{n} - \frac{n}{2}\sin\frac{\pi}{n}\right)$$

and

$$\langle \bar{z}_0 A_2 y, y\rangle = \left(\frac{n}{4}-1\right)\cos\frac{\pi}{n}.$$

The asserted expression for $\langle \bar{z}_0 A_2 v, v\rangle$ can be proved analogously as before. □

Finally, we are ready for the proof of (d) ⇒ (a) in Theorem 7.

Proof of Theorem 7. (d) ⇒ (a). If A is unitary, then

$$A = \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ -a_n & & & 0 \end{bmatrix}$$

with $|a_n| = 1$ and $\partial W(A)$ is a regular n-gon (cf. [4, Corollary 1.2]). For the remaining part of the proof, we assume that n is even and $A = A_1 \oplus A_2$, where A_1 and A_2 are as in (d). Multiplying A by an $\mathrm{e}^{\mathrm{i}\theta}$ with $\theta = -\arg c$, we may further assume that c is positive. If $c = 1$, then A_1 and A_2, and hence A, are all unitary, in which case $\partial W(A)$ has n line segments. Under the hypotheses that $n \geqslant 4$ and $1 < c < \tan(\pi/n) + \sec(\pi/n)$, we have $0 < (c - (1/c))\cot(\pi/n) < 2$. Since the zeros of the polynomial $p_1(z) = z^n + z^{n/2}(c - (1/c))\cot(\pi/n) + 1$ are the $(n/2)$th roots of

$(-(c-(1/c))\cot(\pi/n) \pm ((c-(1/c))^2\cot^2(\pi/n)-4)^{1/2})/2$, we infer that they are all distinct and have modulus one. These we denote by z_k, $1 \leqslant k \leqslant n$.

We now show that $\cos(\pi/n)$ is a multiple eigenvalue of $\mathrm{Re}(\bar{z}_k A)$ for any k. Indeed, if

$$x_k = \left[z_k \sin\frac{\pi}{n}, z_k^2 \sin\frac{2\pi}{n}, \ldots, z_k^{n/2}\sin\frac{\frac{n}{2}\pi}{n}\right]^{\mathrm{T}},$$

$$y_k = \left[z_k^{(n/2)+1}\cos\frac{\pi}{n}, z_k^{(n/2)+2}\cos\frac{2\pi}{n}, \ldots, z_k^{n-1}\cos\frac{\left(\frac{n}{2}-1\right)\pi}{n}, 0\right]^{\mathrm{T}},$$

$u_k = (x_k + cy_k)/\|x_k + cy_k\|$ and $v_k = (cx_k - y_k)/\|cx_k - y_k\|$, then it is easily checked that $\mathrm{Re}(\bar{z}_k A_1)u_k = \cos(\pi/n)u_k$ and $\mathrm{Re}(\bar{z}_k A_2)v_k = \cos(\pi/n)v_k$, where for the equality of the $(n/2)$th components we need that z_k be a zero of p_1. Hence $\cos(\pi/n)$ is a multiple eigenvalue of $\mathrm{Re}(\bar{z}_k A)$.

Next note that $\cos(\pi/n)$ is the maximum eigenvalue of $\mathrm{Re}(\bar{z}_k A)$. To prove this, let $c_1 \geqslant c_2 \geqslant \cdots \geqslant c_n$ and $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_{n-1}$ be the eigenvalues of $\mathrm{Re}(\bar{z}_k A)$ and $\mathrm{Re}(\bar{z}_k J_{n-1})$, respectively. Since $\mathrm{Re}(\bar{z}_k J_{n-1})$ is unitarily equivalent to $\mathrm{Re}\, J_{n-1}$, the d_j's are all distinct and $d_1 = \cos(\pi/n)$ (cf. [3, Corollary 2.7]). On the other hand, we proved in the preceding paragraph that $\cos(\pi/n) = c_{j_0} = c_{j_0+1}$ for some j_0. If $j_0 > 1$, then from the interlacing of the c_j's and the d_j's: $c_1 \geqslant d_1 \geqslant c_2 \geqslant d_2 \geqslant \cdots \geqslant c_{n-1} \geqslant d_{n-1} \geqslant c_n$, we obtain $d_1 = c_2 = d_2 = \cdots = c_{j_0+1} = \cos(\pi/n)$, which contradicts the distinctness of the d_j's. Hence $j_0 \leqslant 1$ and therefore $c_1 = \cos(\pi/n)$ as required. In particular, we have $\cos(\pi/n) = \max W(\mathrm{Re}(\bar{z}_k A)) = \max \mathrm{Re}\, W(\bar{z}_k A)$.

Finally, we check that $W(A)$ has n line segments on its boundary. For this, consider $u'_k = u_k \oplus 0$ and $v'_k = 0 \oplus v_k$ as vectors in $\mathbb{C}^n$. Then

$$\langle \bar{z}_k A u'_k, u'_k\rangle = \langle \bar{z}_k A_1 u_k, u_k\rangle = \cos\frac{\pi}{n} - \mathrm{i}\frac{nc\,\mathrm{Im}(z_k^{n/2})\sin\frac{\pi}{n}}{\frac{n}{2}(1+c^2) + (1-c^2)\csc^2\left(\frac{\pi}{n}\right)}$$

and

$$\langle \bar{z}_k A v'_k, v'_k\rangle = \langle \bar{z}_k A_2 v_k, v_k\rangle = \cos\frac{\pi}{n} + \mathrm{i}\frac{nc\,\mathrm{Im}(z_k^{n/2})\sin\frac{\pi}{n}}{\frac{n}{2}(1+c^2) + (c^2-1)\csc^2\left(\frac{\pi}{n}\right)}$$

by Lemma 10. Hence

$$\mathrm{Re}\langle \bar{z}_k A u'_k, u'_k\rangle = \mathrm{Re}\langle \bar{z}_k A v'_k, v'_k\rangle = \cos\frac{\pi}{n} = \max \mathrm{Re}\, W(\bar{z}_k A)$$

and

$$\mathrm{Im}\langle \bar{z}_k A u'_k, u'_k\rangle \neq \mathrm{Im}\langle \bar{z}_k A v'_k, v'_k\rangle.$$

Therefore, the vertical line $\mathrm{Re}\, z = \cos(\pi/n)$ yields a line segment on $\partial W(\bar{z}_k A)$. Thus $\partial W(A)$ has n line segments given by $\mathrm{Re}(\bar{z}_k z) = \cos(\pi/n)$, $1 \leqslant k \leqslant n$. This completes the proof. □

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