JOURNAL OF FUNCTIONAL ANALYSIS 31, 1–12 (1979)

Conditions for Completely Nonunitary Contractions to Be Spectral

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Communicated by Peter D. Lax

Received January 5, 1976; revised July, 1977

A necessary and sufficient condition that a contraction of class $C_0(N)$ on a Hilbert space be a spectral operator is given. The condition is in terms of certain factors of the characteristic function of the contraction, which generalizes the known characterization that the compression of the shift on the space $H^2 \ominus mH^2$ is a scalar-type spectral operator if and only if m is a Blaschke product with uniformly separated zeros. Similar conditions for C_0 contractions, C_{11} contractions with characteristic functions admitting scalar multiples, and weak contractions to be spectral are also derived. In these cases the conditions are only sufficient.

For certain contractions T on a Hilbert space, very satisfactory spectral decompositions have been developed recently. Among them are contractions of class C_0, completely nonunitary (c.n.u.) contractions of class C_{11} whose characteristic functions admit scalar multiples and c.n.u. weak contractions. Most of the work along this line was achieved by Sz.-Nagy and Foiaş (cf. [14, Chaps. 3, 7, 8]). More recently, it has also been shown that C_0 operators and weak contractions are decomposable (cf. [4, 10]). A more restricted class than the decomposable operators is the class of spectral operators (in the sense of Dunford [2]). In this paper we are concerned with conditions, necessary or sufficient, for such contractions to be spectral. Our conditions are in terms of certain factors of minimal functions (for C_0 contractions), scalar multiples (for C_{11} contractions and weak contractions), or characteristic functions (for $C_0(N)$ contractions). The proofs depend heavily on the corona theorem (cf. [1] or [3] for the scalar-valued case; [6] for the matrix-valued case). In most of the cases the conditions are only sufficient. For contractions of class $C_0(N)$ we obtain a characterization. In the case of $N = 1$ this generalizes the known fact that the compression of the shift on $H^2 \ominus mH^2$ is a scalar-type spectral operator if and only if m is a Blaschke product with uniformly separated zeros. Other generalizations of this fact can be found in [12, 13].

0022-1236/79/010001-12$02.00/0

In the following only nontrivial, separable, complex Hilbert spaces are considered. The four sections correspond to the contractions of class C_0, class C_{11}, weak contractions, and class $C_0(N)$, respectively. Our basic reference is [14].

1. Contractions of Class C_0

A contraction T on the space H is of class C_0 if it is c.n.u. and $\varphi(T) = 0$ for some inner function φ. Let m_T be the minimal function of such a contraction T. Consider the factorization $m_T(\lambda) = B(\lambda)\, S_\mu(\lambda)$, where $B(\lambda)$ is a Blaschke product and $S_\mu(\lambda)$ is the singular function whose associated singular measure is μ. For any Borel subset ω of the complex plane $\mathbb{C}$, let $m_\omega(\lambda) = B_\omega(\lambda)\, S_\omega(\lambda)$, where $B_\omega(\lambda)$ is the product of those factors of $B(\lambda)$ whose zeros lie in ω and $S_\omega(\lambda)$ is the singular function whose associated measure is $\mu \mid \omega$. Let $H_\omega = \{h: h \in H, m_\omega(T)\, h = 0\}$.

Recall that an operator T is spectral (in the sense of Dunford) if it has a countably additive resolution of the identity defined on the Borel subsets of $\mathbb{C}$ (cf. [2]). For $\omega \subseteq \mathbb{C}$, ω' denotes its complement in $\mathbb{C}$.

THEOREM 1.1. *Let T be a contraction of class C_0 on H with minimal function m_T. For every Borel subset ω of $\mathbb{C}$, let m_ω be defined as above. Assume that there exists an $\epsilon > 0$ such that*

$$\inf_{\omega \subseteq \mathbb{C}} \{|\, m_\omega(\lambda)| + |\, m_{\omega'}(\lambda)|\} \geqslant \epsilon \qquad \text{for all } \lambda, \ |\, \lambda \,| < 1.$$

Then T is a spectral operator.

The following version of the corona theorem is needed in the proof (cf. [1] or [3]).

SCALAR CORONA THEOREM 1.2. *Let $f_1, \ldots, f_n$ be functions in H^∞ with $\| f_k \| \leqslant 1$ ($k = 1, \ldots, n$) and $|\, f_1(\lambda)| + \cdots + |\, f_n(\lambda)| \geqslant \epsilon$ for all λ, $|\, \lambda \,| < 1$, where $0 < \epsilon < \frac{1}{2}$. Then there exist functions $g_1, \ldots, g_n$ in H^∞ such that $f_1(\lambda)\, g_1(\lambda) + \cdots + f_n(\lambda)\, g_n(\lambda) = 1$ for all λ, $|\, \lambda \,| < 1$, and $\| g_k \| \leqslant \epsilon^{-\beta_n}$ ($k = 1, \ldots, n$), where β_n is a constant depending only on n.*

Proof of Theorem 1.1. The assumption says that

$$|\, m_\omega(\lambda)| + |\, m_{\omega'}(\lambda)| \geqslant \epsilon \qquad \text{for all } \lambda, \qquad |\, \lambda \,| < 1 \quad \text{and all } \omega.$$

We may assume that $0 < \epsilon < \frac{1}{2}$. It follows from the scalar corona theorem that there exist functions u_ω, $u_{\omega'}$ in H^∞ such that $m_\omega(\lambda)\, u_\omega(\lambda) + m_{\omega'}(\lambda)\, u_{\omega'}(\lambda) = 1$ for all λ, $|\, \lambda \,| < 1$ and $\| u_\omega \|, \| u_{\omega'} \| \leqslant K$, where K is a constant independent of ω. Hence we have $H_\omega \dotplus H_{\omega'} = H$, the sign $\dotplus$ denoting direct (not necessarily orthogonal) sum (cf. [14, Proposition III.6.4]). Let $E(\omega)$ denote the projection (not necessarily orthogonal) along $H_{\omega'}$ onto the subspace H_ω. Then $E(\omega)\,h = m_{\omega'}(T)\,u_{\omega'}(T)\,h$ for $h \in H$. Hence $\|E(\omega)\| \leqslant \|\,m_{\omega'}\,\|\,u_{\omega'}\,\|\,\| \leqslant \|\,u_{\omega'}\,\| \leqslant K$. By virtue of [14, Theorems III.6.3 and III.5.1] it can be easily shown that $E(\cdot)$ is a resolution of the identity for T. To prove the countable additivity we appeal to Mackey's result that for $E(\cdot)$ there exists an invertible operator A such that $A^{-1}E(\omega)\,A$ is self-adjoint for every ω (cf. [16]). Hence for mutually disjoint $\{\omega_n\}$, $\{A^{-1}E(\omega_n)\,A\}$ is a sequence of orthogonal projections mutually orthogonal to each other. Since $H_{\bigcup_n \omega_n} = \bigvee_n H_{\omega_n}$, the range of $A^{-1}E(\bigcup_n \omega_n)\,A$ is the span of the ranges of $A^{-1}E(\omega_n)\,A$. It follows that $A^{-1}E(\bigcup_n \omega_n)\,A = \sum_n \oplus A^{-1}E(\omega_n)\,A$ and $E(\bigcup_n \omega_n) = \sum_n E(\omega_n)$. Thus T is a spectral operator, completing the proof.

COROLLARY 1.3. *Let T be a contraction of class C_0 on H with minimal function m_T. Assume that $m_T(\lambda)$ is a Blaschke product whose zeros $\{\lambda_i\}$, with corresponding multiplicities $\{n_i\}$, satisfy*

$$\prod_{j\neq i} \left| \frac{\lambda_i - \lambda_j}{1 - \bar{\lambda}_j \lambda_i} \right|^{n_i n_j} \geqslant \epsilon \qquad \textit{for all } i,$$

for some $\epsilon > 0$. Then T is a spectral operator.

Proof. Our hypothesis implies that for any decomposition $\{I_1, I_2\}$ of $I = \{1, 2,...\}$ there exists a function f in H^∞ such that

$$\begin{aligned} f(\lambda_i) &= 0 \qquad \text{if} \quad i \in I_1, \\ &= 1 \qquad \text{if} \quad i \in I_2, \end{aligned}$$

$f^{(\ell)}(\lambda_i) = 0$ for $1 \leqslant \ell \leqslant n_i - 1$ and all i, and $\|f\| \leqslant K$, where K is a constant independent of the decomposition $\{I_1, I_2\}$ (cf. [11, Theorem 1.2]). Let $B_1(\lambda)$ be the product of those factors of $m_T(\lambda)$ whose zeros are λ_i with $i \in I_1$ and $B_2(\lambda)$ having zeros λ_i with $i \in I_2$. Let $f_1(\lambda) = B_1(\lambda)^{-1} f(\lambda)$ and $f_2(\lambda) = B_2(\lambda)^{-1}(1 - f(\lambda))$. Note that both $f_1(\lambda)$ and $f_2(\lambda)$ are analytic functions with $\|f_1\| \leqslant K$ and $\|f_2\| \leqslant 1 + K$. Moreover, we have $f_1(\lambda)\,B_1(\lambda) + f_2(\lambda)\,B_2(\lambda) = 1$, for all λ, $|\lambda| < 1$. Thus

$$\prod_{i \in I_1} \left| \frac{\lambda_i - \lambda}{1 - \bar{\lambda}_i \lambda} \right|^{n_i} + \prod_{i \in I_2} \left| \frac{\lambda_i - \lambda}{1 - \bar{\lambda}_i \lambda} \right|^{n_i} \geqslant \frac{1}{1 + K}$$

for all λ, $|\lambda| < 1$. It follows from Theorem 1.1 that T is a spectral operator.

The preceding result appeared in [11], but the proof there is different. In particular, we have

COROLLARY 1.4. *Let T be a contraction of class C_0 on H with minimal function*

m_T. Assume that $m_T(\lambda)$ is a Blaschke product whose zeros $\{\lambda_i\}$ are uniformly separated; that is, there exists an $\epsilon > 0$ such that

$$\prod_{j \neq i} \left| \frac{\lambda_i - \lambda_j}{1 - \bar{\lambda}_j \lambda_i} \right| \geqslant \epsilon \qquad \textit{for all } i.$$

Then T is a scalar-type spectral operator.

Proof. In this case each λ_i must have multiplicity 1. That T is a spectral operator follows from Corollary 1.3. If we let $\omega_i = \{\lambda_i\}$, then $H_{\omega_i} = \{h: h \in H, Th = \lambda_i h\}$ and $H = \bigvee_i H_{\omega_i}$. By Mackey's result we have $H = \sum_i H_{\omega_i}$, and T is similar to a normal operator. Hence T is of scalar type.

2. Contractions of Class C_{11}

A contraction T on H is of class C_{11} if $T^n h \nrightarrow 0$ and $T^{*n} h \nrightarrow 0$ for all $h \in H$, $h \neq 0$. The characteristic function of a c.n.u. C_{11} contraction is outer from both sides (cf. [14, Proposition VI. 3.5]).

Recall that a contractive analytic function $\{\mathscr{D}_1, \mathscr{D}_2, \Theta(\lambda)\}$ is said to have the scalar multiple $\delta(\lambda)$, if $\delta(\lambda)$ is a scalar-valued analytic function, $\delta(\lambda) \not\equiv 0$, and there exists a contractive analytic function $\{\mathscr{D}_2, \mathscr{D}_1, \Omega(\lambda)\}$ such that $\Omega(\lambda)\, \Theta(\lambda) = \delta(\lambda) I_{\mathscr{D}_1}$, $\Theta(\lambda)\, \Omega(\lambda) = \delta(\lambda) I_{\mathscr{D}_2}$ for all λ, $|\lambda| < 1$. For an outer function $\Theta(\lambda)$, $\delta(\lambda)$ may be chosen to be outer (cf. [14, Theorem V.6.2]).

Consider a c.n.u. contraction T of class C_{11} whose characteristic function $\Theta_T(\lambda)$ admits the outer scalar multiple

$$\delta(\lambda) = \exp\left[\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + \lambda}{e^{it} - \lambda} \log |\delta(e^{it})|\, dt\right].$$

Then $\sigma(T) \subseteq C$, the unit circle (cf. [14, Proposition VI.4.4]). In [14, Theorems VII.5.2 and VII.6.2], a spectral decomposition for such a contraction is given. For a Borel subset α of C, let H_α be the spectral subspace defined there, and let $\Theta_T(\lambda) = \Theta_{2\alpha}(\lambda)\, \Theta_{1\alpha}(\lambda)$ be the corresponding regular factorization. If we let

$$\delta_\alpha(\lambda) = \exp\left[\frac{1}{2\pi} \int_\alpha \frac{e^{it} + \lambda}{e^{it} - \lambda} \log |\delta(e^{it})|\, dt\right], \qquad |\lambda| < 1,$$

then $\Theta_{1\alpha}(\lambda)$ an $\Theta_{2\alpha}(\lambda)$ have the functions $\delta_\alpha(\lambda)$ and $\delta_{\alpha'}(\lambda)$ as scalar multiples, respectively (cf. [14, Proposition V.7.2]), where $\alpha' \equiv C \backslash \alpha$.

Theorem 2.1. *Let T be a c.n.u. contraction of class C_{11}, whose characteristic* *function $\Theta_T(\lambda)$ admits the outer scalar multiple $\delta(\lambda)$. For any Borel subset α of C, let δ_α be defined as above. Assume that there exists an $\epsilon > 0$ such that*

$$\inf_{\alpha \subseteq C} \{| \delta_\alpha(\lambda)| + | \delta_{\alpha'}(\lambda)|\} \geqslant \epsilon \qquad \textit{for all} \quad \lambda, | \lambda | < 1.$$

Then T is a spectral operator.

The following lemma is needed.

LEMMA 2.2. *Let T be an operator on the Hilbert space H, and let $H_1 \subseteq H$ be an invariant subspace for T. Assume that there exists a hyperinvariant subspace $H_2 \subseteq H$ for T such that $H_1 \vee H_2 = H$ and $H_1 \cap H_2 = \{0\}$. If $K \subseteq H$ is any invariant subspace for T such that $H = H_1 \dotplus K$, then $K = H_2$.*

Proof. Let P be the projection (not necessarily orthogonal) along H_1 onto the subspace K. Since both H_1 and K are invariant for T, we have $PT = TP$. Hence $PH_2 \subseteq H_2$. It follows that $PH = P(H_1 \vee H_2) \subseteq \overline{PH_2} \subseteq H_2$; that is, $K \subseteq H_2$. Since $H_1 \cap H_2 = \{0\}$ and $H = H_1 \dotplus K$, it can be easily seen that $K = H_2$.

We also need the following theorem due to Teodorescu [15].

THEOREM 2.3. *Let T be a c.n.u. contraction on H with the characteristic function $\{\mathscr{D}_T, \mathscr{D}_{T^*}, \Theta_T(\lambda)\}$. Let $H_1 \subseteq H$ be an invariant subspace for T and $\Theta_T(\lambda) = \Theta_2(\lambda)\, \Theta_1(\lambda)$ the corresponding regular factorization with the intermediate space $\mathscr{F}$. Then there exists a subspace H', invariant for T, such that $H = H' \dotplus H_1$ if and only if there exist bounded analytic functions $\{\mathscr{F}, \mathscr{D}_T, \Psi(\lambda)\}$ and $\{\mathscr{D}_{T^*}, \mathscr{F}, \Phi(\lambda)\}$ such that $\Phi(\lambda)\, \Theta_2(\lambda) + \Theta_1(\lambda)\, \Psi(\lambda) = I_{\mathscr{F}}$ for all λ, $| \lambda | < 1$.*

We remark that from the proof of the sufficient part in [15] we can easily check that if P denotes the projection (not necessarily orthogonal) along H_1 onto the subspace H' and $\| \Psi \|, \| \Phi \| \leqslant K$, then $\| P \| \leqslant K'$, where K' is a constant depending only on K.

Proof of Theorem 2.1. Note that a scalar multiple of a contractive analytic function must be contractive. Hence we have $\| \delta_\alpha \|, \| \delta_{\alpha'} \| \leqslant 1$ for any α. Assuming that $0 < \epsilon < \frac{1}{2}$, it follows from the scalar corona theorem that there exist functions u_α, $u_{\alpha'}$ in H^∞ such that $\delta_\alpha(\lambda)\, u_\alpha(\lambda) + \delta_{\alpha'}(\lambda)\, u_{\alpha'}(\lambda) = 1$ for all λ, $| \lambda | < 1$ and $\| u_\alpha \|, \| u_{\alpha'} \| \leqslant K$, where K is independent of α. Assume that $\Omega_{1\alpha}(\lambda)$ and $\Omega_{2\alpha}(\lambda)$ are contractive analytic functions such that $\Omega_{1\alpha}(\lambda)\, \Theta_{1\alpha}(\lambda) = \delta_\alpha(\lambda)\, I$, $\Theta_{1\alpha}(\lambda)\, \Omega_{1\alpha}(\lambda) = \delta_\alpha(\lambda)\, I$, $| \lambda | < 1$ and $\Omega_{2\alpha}(\lambda)\, \Theta_{2\alpha}(\lambda) = \delta_{\alpha'}(\lambda)\, I$, $\Theta_{2\alpha}(\lambda)\, \Omega_{2\alpha}(\lambda) = \delta_{\alpha'}(\lambda)\, I$, $| \lambda | < 1$. Then we have

$$[u_{\alpha'}(\lambda)\, \Omega_{2\alpha}(\lambda)]\, \Theta_{2\alpha}(\lambda) + \Theta_{1\alpha}(\lambda)[u_\alpha(\lambda)\, \Omega_{1\alpha}(\lambda)] = I, \qquad | \lambda | < 1,$$

where $\| u_{\alpha'} \Omega_{2\alpha} \|, \| u_\alpha \Omega_{1\alpha} \| \leqslant K$. It follows from Theorem 2.3 that $H = H'_\alpha \dotplus H_\alpha$

for some invariant subspace H'_α. By virtue of Lemma 2.2 we have $H'_\alpha = H_{\alpha'}$, since $H_{\alpha'}$ is hyperinvariant for T, $H_\alpha \vee H_{\alpha'} = H$ and $H_\alpha \cap H_{\alpha'} = \{0\}$ (cf. [14, Theorems VII, 5.2 and VII.6.2]). Hence $H = H_{\alpha'} \dotplus H_\alpha$. Let $E(\alpha)$ denote the projection from H onto H_α along the subspace $H_{\alpha'}$. By the remark following Theorem 2.3 we conclude that $\| E(\alpha)\| \leqslant 1 + K'$, where K' is independent of α. Using [14, Theorems VII.5.2 and VII.6.2] and an argument similar to the one in the proof of Theorem 1.1, it can be easily shown that $E(\cdot)$ is a resolution of the identity for T. This shows that T is a spectral operator and completes the proof.

3. Weak Contractions

A contraction T on the space H is a weak contraction if $\sigma(T)$ does not fill the unit disk $D = \{\lambda: |\lambda| < 1\}$ and $I - T^*T$ is of finite trace. The characteristic function of a weak contraction admits a scalar multiple. For a c.n.u. weak contraction we may consider its $C_0 - C_{11}$ decomposition. Let H_0 and H_1 be the invariant subspaces for such a contraction T for which $T_0 = T|_{H_0}$ and $T_1 = T|_{H_1}$ are the C_0 and C_{11} parts of T. Let $m_0(\lambda)$ be the minimal function of T_0 and $\delta_1(\lambda)$ an outer scalar multiple of the characteristic function $\Theta_1(\lambda)$ of T_1. As in Sections 1 and 2, for every Borel subset ω of $\mathbb{C}$ we consider the divisors $m_\omega(\lambda)$ and $\delta_{1,\omega\cap C}(\lambda)$ of $m_0(\lambda)$ and $\delta_1(\lambda)$, respectively, and the invariant subspaces $H_0(\omega) = \{h: h \in H_0,\ m_\omega(T_0)\, h = 0\}$ and $H_1(\omega) =$ the spectral subspace of H_1 associated with T_1 and the Borel subset $\alpha = \omega \cap C$. Thus to every ω there corresponds an invariant subspace $H(\omega) = H_0(\omega) \bigvee H_1(\omega)$ of H. Such subspaces give a spectral decomposition of the contraction T (cf. [14, Theorem VIII.3.1]).

Let $\delta_\omega(\lambda) = m_\omega(\lambda)\, \delta_{1,\omega\cap C}(\lambda)$. We begin by giving the following technical lemma, the proof of which can be found in [17].

Lemma 3.1. *Let T be a c.n.u. weak contraction on H with characteristic function $\Theta_T(\lambda)$. Let $\Theta_T(\lambda) = \Theta'_\omega(\lambda)\, \Theta_\omega(\lambda)$ be the regular factorization corresponding to the invariant subspace $H(\omega)$. Then $\delta_\omega(\lambda)$ and $\delta_{\omega'}(\lambda)$ are scalar multiples of $\Theta_\omega(\lambda)$ $\Theta'_\omega(\lambda)$, respectively.*

Theorem 3.2. *Let T be a c.n.u. weak contraction on H with characteristic function $\Theta_T(\lambda)$. Let $m_0(\lambda)$ be the minimal function of the C_0 part T_0 of T and $\delta_1(\lambda)$ an outer scalar multiple of the characteristic function of the C_{11} part T_1 of T. For any Borel subset ω of $\mathbb{C}$, let $m_\omega(\lambda)$, $\delta_{1,\omega\cap C}(\lambda)$ be defined as above and $\delta_\omega(\lambda) = m_\omega(\lambda)\, \delta_{1,\omega\cap C}(\lambda)$. Assume that there exists an $\epsilon > 0$ such that*

$$\inf_{\omega\subseteq\mathbb{C}} \{|\delta_\omega(\lambda)| + |\delta_{\omega'}(\lambda)|\} \geqslant \epsilon \qquad \text{for all} \quad \lambda, |\lambda| < 1.$$

Then T is a spectral operator. Moreover, T_0 and T_{11} are also spectral.

The proof proceeds as that of Theorem 2.1. The assertions concerning T_0 and T_{11} follow from the following observations,

$$\inf_{\omega\subseteq\mathbb{C}} \{| m_\omega(\lambda)| + | m_{\omega'}(\lambda)|\} \geqslant \inf_{\omega\subseteq\mathbb{C}} \{| \delta_\omega(\lambda)| + | \delta_{\omega'}(\lambda)|\} \geqslant \epsilon,$$

$$\inf_{\omega\subseteq\mathbb{C}} \{ \delta_{1,\omega\cap C}(\lambda)| + | \delta_{1,\omega'\cap C}(\lambda)|\} \geqslant \inf_{\omega\subseteq\mathbb{C}} \{| \delta_\omega(\lambda)| + | \delta_{\omega'}(\lambda)|\} \geqslant \epsilon,$$

for $| \lambda | < 1$, and Theorems 1.1 and 2.1.

Remark. Theorem 1.1 can also be proved along the same line as that of Theorems 2.1 and 3.2, since the minimal function of a C_0 contraction is a scalar multiple of its characteristic function.

4. Contractions of Class $C_0(N)$

Matrix Corona Theorem 4.1. *Let $\Theta_1, ..., \Theta_m$ be functions in H_N^∞ with $\| \Theta_i \| \leqslant 1$ $(i = 1,..., m)$ and $\inf\{\sum_{i=1}^m \| \Theta_i(\lambda)^* \xi \|: \xi \in \mathbb{C}^N, \ \| \xi \| = 1\} \geqslant \epsilon$ for all λ, $| \lambda | < 1$, for some $\epsilon > 0$. Then there exist functions $\Omega_1, ..., \Omega_m$ in H_N^∞ such that $\sum_{i=1}^m \Theta_i(\lambda)\, \Omega_i(\lambda) = I$ for all λ, $| \lambda | < 1$ and $\| \Omega_i \| \leqslant K$ $(i = 1,..., m)$, where K is a constant depending only on m and N.*

The proof of this theorem, except the last assertion, is in [6]. By following the proof there, it can be shown that $\Omega_1, ..., \Omega_m$ can be chosen in this particular way.

Recall that a contraction T on H is of class $C_0(N)$, $N \geqslant 1$, if it is of class C_0 and has defect indices N. Such a T can be considered in its functional model as defined on the space $H = H_N^2 \ominus \Theta_T H_N^2$ by $T^*u = e^{-it}[u(e^{it}) - u(0)]$, $u \in H$. If $H_1 \subseteq H$ is an invariant subspace for T and $\Theta_T(\lambda) = \Theta_2(\lambda)\, \Theta_1(\lambda)$ is the corresponding regular factorization, then $H_1 = \Theta_2 H_N^2 \ominus \Theta_T H_N^2$ and $H \ominus H_1 = H_N^2 \ominus \Theta_2 H_N^2$.

Theorem 4.2. *Let T be a contraction of class $C_0(N)$ on the space H with characteristic function $\Theta_T(\lambda)$ and minimal function $m_T(\lambda)$. For any Borel subset ω, let m_ω and H_ω be defined as in Section 1 and let $\Theta_T(\lambda) = \Theta_{2\omega}(\lambda)\, \Theta_{1\omega}(\lambda)$ be the corresponding regular factorization. Assume that there exists an $\epsilon > 0$ such that*

$$\inf_{\omega\subseteq\mathbb{C}} \inf_{\substack{\xi\in\mathbb{C}^N \\ \|\xi\|=1}} \{\| \Theta_{2\omega}(\lambda)^* \xi \| + \| \Theta_{2\omega'}(\lambda)^* \xi \|\} \geqslant \epsilon \qquad \textit{for all} \quad \lambda, | \lambda | < 1. \quad (1)$$

Then T is a spectral operator.

The proof follows the same idea as of Theorem 1.1.

Proof. By the assumption and Theorem 4.1, there exist functions $\Omega_{2\omega}$,

$\Omega_{2\omega'}$ in H_N^∞ such that $\Theta_{2\omega}(\lambda)\,\Omega_{2\omega}(\lambda) + \Theta_{2\omega'}(\lambda)\Omega_{2\omega'}(\lambda) = I$, $|\lambda| < 1$, and $\|\Omega_{2\omega}\|$, $\|\Omega_{2\omega'}\| \leqslant K$, where K is independent of ω. Hence for $u \in H_N^2$ we have $\Theta_{2\omega}\Omega_{2\omega}u + \Theta_{2\omega'}\Omega_{2\omega'}u = u$. where $\Theta_{2\omega}\Omega_{2\omega}u \in \Theta_{2\omega}H_N^2$ and $\Theta_{2\omega'}\Omega_{2\omega'}u \in \Theta_{2\omega'}H_N^2$; this shows that $\Theta_{2\omega}H_N^2 + \Theta_{2\omega'}H_N^2 = H_N^2$. It follows that $[\Theta_{2\omega}H_N^2 \ominus \Theta_T H_N^2] \dotplus [\Theta_{2\omega'}H_N^2 \ominus \Theta_T H_N^2] = H_N^2 \ominus \Theta_T H_N^2$, that is, $H_\omega \dotplus H_{\omega'} = H$; indeed, $H_\omega \cap H_{\omega'} = \{0\}$ follows from the fact that m_ω and $m_{\omega'}$ have no nontrivial common inner divisor. If $E(\omega)$ denotes the projection along $H_{\omega'}$ onto the subspace H_ω, then $E(\omega)\,u = P(\Theta_{2\omega}\Omega_{2\omega}u)$, $u \in H$, where P denotes the (orthogonal) projection from H_N^2 onto $H_N^2 \ominus \Theta_T H_N^2$. Hence $\|E(\omega)\| \leqslant \|\Theta_{2\omega}\|\,\|\Omega_{2\omega}\| \leqslant K$. Then we can proceed as in the proof of Theorem 1.1 to show that T is a spectral operator.

In this case, condition (1) turns out to be also necessary. In fact, we have the following:

THEOREM 4.3. *Let T be as in Theorem 4.2. Assume that T is spectral with the resolution of the identity $E(\cdot)$. For any Borel subset ω of $\mathbb{C}$, let $H_\omega = E(\omega)\,H$ denote the corresponding spectral subspace and $\Theta_T(\lambda) = \Theta_{2\omega}(\lambda)\,\Theta_{1\omega}(\lambda)$ the regular factorization associated with H_ω. Then there exists an $\epsilon > 0$ such that condition* (1) *holds.*

We list below the definition and some basic properties of the angle between two subspaces of H, which are needed in the proof of Theorem 4.3 (cf. [9, pp. 339–340]).

DEFINITION 4.4. *By the angle between two subspaces H_1 and H_2 is meant the angle $\varphi(H_1, H_2)$ $(0 \leqslant \varphi \leqslant \pi/2)$, defined by* $\cos\varphi(H_1, H_2) = \sup|(x, y)|$, *where the supremum is taken over all $x \in H_1$, $y \in H_2$ with $\|x\| = \|y\| = 1$.*

PROPOSITION 4.5. *If $H_1 \dotplus H_2 = H$ and P is the projection from H onto H_1 along the subspace H_2, then* $\sin\varphi(H_1, H_2) = \|P\|^{-1}$.

PROPOSITION 4.6. *If $H_1 \dotplus H_2 = H$, $H_1^\perp = H \ominus H_1$ and $H_2^\perp = H \ominus H_2$, then $H_1^\perp \dotplus H_2^\perp = H$ and $\varphi(H_1, H_2) = \varphi(H_1^\perp, H_2^\perp)$.*

Proof of Theorem 4.3. Since T is spectral, for any Borel subset ω we have $H_\omega \dotplus H_{\omega'} = H$. By virtue of Proposition 4.6, this implies that $H_\omega^\perp \dotplus H_{\omega'}^\perp = H$, that is, $[H_N^2 \ominus \Theta_{2\omega}H_N^2] \dotplus [H_N^2 \ominus \Theta_{2\omega'}H_N^2] = H_N^2 \ominus \Theta_T H_N^2$. Using Propositions 4.5 and 4.6, we have $\sin\varphi(H_\omega^\perp, H_{\omega'}^\perp) = \sin\varphi(H_\omega, H_{\omega'}) = \|E(\omega)\|^{-1}$. Since $\|E(\omega)\| \leqslant K$ for some constant K, $\varphi(H_\omega^\perp, H_{\omega'}^\perp) \geqslant K_1$, where $K_1 = \arcsin(1/K)$.

Now assume that no $\epsilon > 0$ exists for which condition (1) holds. Then there exist a sequence of Borel subsets $\{\omega_n\}$, a sequence of points $\{\lambda_n\}$ in $D = \{|\lambda| < 1\}$, and a sequence of unit vectors $\{\xi_n\}$ in $\mathbb{C}^N$ satisfying $\lim_n \|\Theta_{2\omega_n}(\lambda_n)^*\,\xi_n\| = \lim_n \|\Theta_{2\omega_n'}(\lambda_n)^*\,\xi_n\| = 0$. We show the existence of vectors $\{u_n\}$ in $H_N{}^2 \ominus \Theta_{2\omega_n} H_N{}^2$ and vectors $\{u_n'\}$ in $H_N{}^2 \ominus \Theta_{2\omega_n'} H_N{}^2$ such that $\lim_n \| u_n \| = \lim_n \| u_n' \| = 1$ and $\lim_n (u_n , u_n') = 1$, where (,) denotes the inner product in H. This implies that $\varphi(H_{\omega_n}^{\perp}, H_{\omega_n'}^{\perp})$ approaches zero, contradicting $\varphi(H_{\omega_n}^{\perp}, H_{\omega_n'}^{\perp}) \geqslant K_1$.

Note that we have the following orthogonal decomposition of $H_N{}^2$:

$$[H_N{}^2 \ominus \Theta_{2\omega_n} H_N{}^2] \oplus \Theta_{2\omega_n} H_N{}^2 = H_N{}^2. \tag{2}$$

Consider the normalized eigenfunctions of the left shift in $H_N{}^2$ given by

$$w_n(\lambda) = \frac{(1 - |\lambda_n|^2)^{1/2} \xi_n}{1 - \bar{\lambda}_n \lambda},$$

and decompose them with respect to the decomposition (2), say, $w_n = u_n + v_n$ with $u_n \in H_N{}^2 \ominus \Theta_{2\omega_n} H_N{}^2$ and $v_n \in \Theta_{2\omega_n} H_N{}^2$. A simple computation yields

$$u_n(\lambda) = (1 - |\lambda_n|^2)^{1/2} \frac{(I - \Theta_{2\omega_n}(\lambda)\, \Theta_{2\omega_n}(\lambda_n)^*) \xi_n}{1 - \bar{\lambda}_n \lambda}$$

and

$$v_n(\lambda) = (1 - |\lambda_n|^2)^{1/2} \frac{\Theta_{2\omega_n}(\lambda)\, \Theta_{2\omega_n}(\lambda_n)^* \xi_n}{1 - \bar{\lambda}_n \lambda}$$

(cf. [7]). Since $\Theta_{2\omega_n}(\lambda)$ is inner, we have $\| v_n \| = \| \Theta_{2\omega_n}(\lambda_n)^* \xi_n \|$ and hence $\lim_n \| v_n \| = 0$. Thus $\lim_n \| u_n \| = 1$. Similarly, we obtain u_n' in $H_N{}^2 \ominus \Theta_{2\omega_n'} H_N{}^2$ and v_n' in $\Theta_{2\omega_n'} H_N{}^2$ satisfying $\lim_n \| v_n' \| = 0$ and $\lim_n \| u_n' \| = 1$. We have $1 = (w_n , w_n) = (u_n + v_n , u_n' + v_n') = (u_n , u_n') + (u_n , v_n') + (v_n , u_n') + (v_n , v_n')$. Since the last three terms of this equation tend to zero, we have $\lim_n (u_n , u_n') = 1$ as asserted. This completes the proof.

The latter part of the proof is modified from the one given by Furhmann for Theorem 3.5 in [8].

By virtue of the next lemma we are able to combine Theorems 4.2 and 4.3.

LEMMA 4.7. *Let T be a contraction of class C_0 on H with minimal function m_T. Assume that T is a spectral operator with the resolution of the identity $E(\cdot)$. For any Borel subset ω, let H_ω be the subspace as defined in Section 1 and $H(\omega) = E(\omega) H$. Then $H_\omega = H(\omega)$ for any ω.*

Proof. Let $\mathscr{S}$ be the class of those Borel subsets ω for which $H_\omega = H(\omega)$ holds. Then $\mathscr{S}$ contains all closed subsets of $\mathbb{C}$ (cf. [4]). Assume that $\omega \in \mathscr{S}$. We want to show that $\omega' \in \mathscr{S}$. We have $H = H(\omega) \dotplus H(\omega') = H_\omega \dotplus H(\omega')$ and $H_\omega \vee H_{\omega'} = H$, $H_\omega \cap H_{\omega'} = \{0\}$. By virtue of Lemma 2.2, $H_{\omega'} = H(\omega')$, and hence $\omega' \in \mathscr{S}$ as asserted. It can also be easily seen that $\mathscr{S}$ is closed under countable union. This shows that $\mathscr{S}$ is a σ-algebra. Thus $\mathscr{S}$ must contain all Borel subsets.

Now we have the following main theorem.

THEOREM 4.8. *Let T be as in Theorem 4.2. For any Borel subset ω, let H_ω be defined as in Section 1, and let $\Theta_T(\lambda) = \Theta_{2\omega}(\lambda)\,\Theta_{1\omega}(\lambda)$ be the corresponding regular factorization. Then the following are equivalent to each other:*

(i) *T is a spectral operator;*

(ii) *there exists an $\epsilon_1 > 0$ such that*

$$\inf_{\omega \subseteq \mathbb{C}} \inf_{\substack{\xi \in \mathbb{C}^N \\ \|\xi\|=1}} \{\| \Theta_{1\omega}(\lambda)\xi \| + \| \Theta_{1\omega'}(\lambda)\xi \|\} \geqslant \epsilon_1 \qquad \textit{for all} \quad \lambda, |\lambda| < 1;$$

(iii) *there exists an $\epsilon_2 > 0$ such that*

$$\inf_{\omega \subseteq \mathbb{C}} \inf_{\substack{\xi \in \mathbb{C}^N \\ \|\xi\|=1}} \{\| \Theta_{2\omega}(\lambda)^*\xi \| + \| \Theta_{2\omega'}(\lambda)^*\xi \|\} \geqslant \epsilon_2 \qquad \textit{for all} \quad \lambda, |\lambda| < 1.$$

The equivalence of (i) and (iii) follows from Theorems 4.2, 4.3, and Lemma 4.7. By considering T^*, instead of T, in Theorems 4.2 and 4.3, we obtain the equivalence of (i) and (ii).

If T is of class $C_0(1)$, then Theorem 4.8 is reduced to:

COROLLARY 4.9. *Let $m(\lambda) = B(\lambda)\,S(\lambda)$ be an inner function and let $T = S(m)$ be the operator on the space $H = H^2 \ominus mH^2$ defined by $(T^*u)(\lambda) = (1/\lambda)[u(\lambda) - u(0)]$ for $u \in H$, $\lambda \in D$. For any Borel subset ω, let $m_\omega(\lambda) = B_\omega(\lambda)\,S_\omega(\lambda)$. Then the following are equivalent to each other:*

(i) *T is a spectral operator;*

(ii) *there exists an $\epsilon > 0$ such that*

$$\inf_{\omega \subseteq \mathbb{C}} \{| m_\omega(\lambda)| + | m_{\omega'}(\lambda)|\} \geqslant \epsilon \qquad \textit{for all} \quad \lambda, |\lambda| < 1.$$

This generalizes the following well-known result (cf. [5]).

COROLLARY 4.10. *If $m(\lambda)$ is a Blaschke product with simple zeros $\{\lambda_i\}$, then $T = S(m)$ is a spectral operator if and only if $\{\lambda_i\}$ is uniformly separated. In this case, T is of scalar type.*

Indeed, the assertion follows from Corollary 4.9 by noting that $\{\lambda_i\}$ is uniformly separated if and only if there exists an $\epsilon > 0$ such that for any decomposition $\{I_1, I_2\}$ of I,

$$\prod_{i \in I_1} \left| \frac{\lambda_i - \lambda}{1 - \bar{\lambda}_i \lambda} \right| + \prod_{i \in I_2} \left| \frac{\lambda_i - \lambda}{1 - \bar{\lambda}_i \lambda} \right| \geqslant \epsilon \qquad \text{for all} \quad \lambda, |\lambda| < 1$$

(cf. Corollaries 1.3 and 1.4).

As another application, we can use Corollary 4.9 to show that if

$$m(\lambda) = \exp\left[-\int_0^{2\pi} \frac{e^{it} + \lambda}{e^{it} - \lambda}\, d\mu(t)\right],$$

where μ is singular and continuous (i.e., $\mu(\{t\}) = 0$ for all $t \in [0, 2\pi)$), then $T = S(m)$ is never spectral (cf. [5]).

Acknowledgment

This work is a part of the author's Ph.D. thesis at Indiana University. The author wishes to express his profound gratitude to his advisor John B. Conway for suggesting this topic and constant guidance in the preparation of the thesis.

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